Sequential Best-Arm Identification with Application to P300 Speller

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Paper under double-blind review

Abstract

A brain-computer interface (BCI) is an advanced technology that facilitates direct communication between the human brain and a computer system, by enabling individuals to interact with devices using only their thoughts. The P300 speller is a primary type of BCI system, which allows users to spell words without using a physical keyboard, but instead by capturing and interpreting brain electroencephalogram (EEG) signals under different stimulus presentation paradigms. Traditional non-adaptive presentation paradigms, however, treat each word selection as an isolated event, resulting in a lengthy learning process. To enhance efficiency, we cast the problem as a sequence of best-arm identification tasks within the context of multi-armed bandits, where each task corresponds to the interaction between the user and the system for a single character or word. Leveraging large language models, we utilize the prior knowledge learned from previous tasks to inform and facilitate subsequent tasks. We propose a sequential top-two Thompson sampling algorithm under two scenarios: the fixed-confidence setting and the fixed-budget setting. We study the theoretical property of the proposed algorithm, and demonstrate its substantial empirical improvement through both simulations as well as the data generated from a P300 speller simulator that was built upon the real BCI experiments.

1 Introduction

A brain-computer interface (BCI) is a groundbreaking technology that enables direct communication between the brain and an external device or computer system. It involves the use of various sensors, such as electroencephalography (EEG), electrocorticography (ECoG), or implantable neural electrodes, which detect and record the electrical signals produced by the brain. The brain signals are then processed by machine learning algorithms to interpret and extract meaningful commands and intentions. BCI holds immense potential for a wide range of applications in medicine, rehabilitation, and human augmentation. For instance, it provides a valuable communication aid for individuals with disabilities (Wolpaw et al., 2018).

The P300 speller is a primary type of BCI system that allows users to select characters or spell words on a computer screen without using a physical keyboard but instead the brain signals. It is based on the P300 event-related potential (ERP), which is a brain response, in the form of a specific pattern of voltage fluctuation, that occurs approximately 300 milliseconds after a relevant stimulus is presented. The stimuli are typically individual characters or symbols flashed on a computer screen in a grid-like layout, and ERP is detected and recorded by a scalp EEG cap or a similar device. After a stimulus is presented, the EEG signals captured by the electrodes are analyzed within a fixed time window by signal processing and machine learning algorithms, which detect the occurrence of the P300 response and determine the target character. Figure 1 gives a graphical illustration of the system.

A key limitation of the existing system is that the stimuli are usually presented in a fixed and predetermined fashion. In addition, when presenting a word or a sentence, the system treats each character or each word independently, and totally ignores the inherent relations among the characters or words. As a result, a large number of stimulus flashes are usually required to achieve a certain level of accuracy of character or word identification. A language model essentially defines a collection of conditional probability distributions

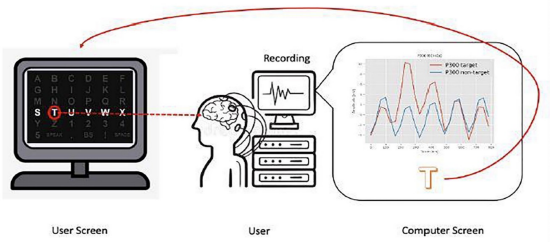

Figure 1: An illustration of the P300 speller system (Ma et al., 2022). At the beginning of the experiment, the user focuses the attention on the desired character or word they wish to type. The system presents a sequence of flashes on a virtual screen to the user, who responds to different flashes eliciting different brain signals (target or non-target). These EEG signals are captured by the electrodes and analyzed in a fixed time window after each flash to make a binary decision whether a P300 ERP response is produced. The binary classification results are then converted into the character or word level probabilities, and the one with the highest probability is shown on the screen.

over the next token given the past tokens. Recently, large language models (LLMs) such as GPT-3 (Brown et al., 2020) have achieved striking success in natural language processing, and can produce coherent and human-like text. In this article, we aim to utilize language models as the prior information to improve the sampling efficiency of the P300 BCI system through an adaptive stimulus design.

Contributions Our contributions are three-fold.

- We introduce a novel sequential best-arm identification formulation for the P300 speller BCI application. We treat each word the user wishes to type as an optimal arm, and formulate stimulus presentation as an adaptive selection problem. We leverage language models as an informative prior, and aim to identify the target sequence of words as soon as possible in the fixed-confidence setting, or to make as fewer mistakes as possible given a fixed number of flashes in the fixed-budget setting.
- We propose a sequential top-two Thompson sampling (`STTS`) algorithm that utilizes the prior information in a coherent way. We derive the probability error bound in the fixed-budget setting that explicitly quantifies the prior effect through the conditional entropy of the prior distribution of the optimal arms. We also study the probability error bound in the fixed-confidence setting.
- We conduct intensive experiments using both simulations and data generated from a P300 speller simulator that was built based on real BCI experiments (Ma et al., 2022). We use GPT-3 (Brown et al., 2020) to generate the sentence of words that a user wishes to type, and use GPT-2 (Radford et al., 2019) to inform the prior distribution. We demonstrate the substantial improvement over several state-of-the-art baseline algorithms that do not utilize the prior information.

Related work We first review the literature on multi-armed bandits, then the literature on P300 BCI.

For learning a single task, Even-Dar et al. (2002) first introduced best-arm identification in the fixed-confidence setting, while Audibert et al. (2010) studied the fixed-budget setting. Kaufmann et al. (2016) investigated the optimal sample complexity, and Jun et al. (2016) explored the batch arm pulls setting. Russo (2016) proposed the top-two Thompson sampling as an effective anytime sampling rule that does not depend on the confidence parameter. Its theoretical properties were studied in Russo (2016); Qin et al. (2017); Shang et al. (2020); Qin & Russo (2022); Jourdan et al. (2022). However, most existing asymptotic analyses did not quantify the prior effect. One exception is Qin & Russo (2022), who showed the prior effect through the entropy in the simple regret setting.

For learning multiple tasks sequentially, Boutilier et al. (2020); Simchowitz et al. (2021); Kveton et al. (2021); Azizi et al. (2022) studied meta-learning in the context of Bayesian bandits for cumulative and simple regret minimization. They assumed that an unknown instance prior is drawn from a known meta-prior. Then each

task is sampled indenpendently from this instance prior. In contrast, we assume the sequence of tasks is sampled from a joint prior distribution such that each task is *not independent* of each other. This necessitates the prior-dependent analysis that has only been studied in the regret minimization setting (Russo & Van Roy, 2016; Hao et al., 2023).

For P300 BCI studies, there have recently emerged a number of proposals for adaptive stimulus selection. Speier et al. (2011) used language models to weigh the output of stepwise linear discriminant analysis (LDA) for classification confidence. Park & Kim (2012) framed the problem as a a partially observable Markov decision process (POMDP). However, as the number of states grows, the belief states grow exponentially for POMDP, and the problem becomes intractable for a real-time system with a large search space. Koçanaoğulları et al. (2018) formulated the problem as query selection optimization and utilized language models to obtain the prior. However, they focused on a single task setting. Naively extending their solution to multiple tasks in effect requires the prior information of previous tasks must all be correct. If an incorrect selection is made in one task, then the subsequent prior information would be incorrect, which would in turn derail the subsequent stimulus selection. Besides, they did not establish any theoretical guarantee. Ma et al. (2021) used Beta-Bernoulli Thompson sampling for adaptive stimulus selection, but did not formulate the problem as best-arm identification, and did not incorporate any prior from language models. We refer to Heskebeck et al. (2022) for a comprehensive review of the multi-armed bandits approaches in the BCI setting.

2 Sequential best-arm identification

2.1 Problem formulation

We formulate the problem as an agent sequentially interacting with M bandit environments, with each interaction referred to as a *task*. In a P300 speller experiment, each task corresponds to a single character or word. Without loss of generality, in this article, we focus on the setting where each task is a single *word*. Each environment, indexed by $m \in [M] = \{1, \dots, M\}$, is characterized by a random vector $\theta_m \in \mathbb{R}^J$ with a prior distribution $\nu_m(\cdot)$, which will be discussed in detail in Section 2.2. The action set is $\mathcal{A} = \{a_1, \dots, a_J\} \subseteq \mathbb{R}^J$, where a_j is the standard basis vector with the jth element equal to one and the rest all zero. At each task m and step t, the agent selects an action $A_{t,m} \in \mathcal{A}$, and receives a reward $R_{t,m} = \langle A_{t,m}, \theta_m \rangle + \eta_{t,m}$, where $\langle \cdot, \cdot \rangle$ is the vector inner product, and $(\eta_{t,m})$'s are a sequence of independent standard Gaussian variables. The optimal arm, $A^*_m = \text{argmax}_{a \in \mathcal{A}} \theta_m^\top a$, is also a random variable. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a measurable space, $\mathcal{H}_{t,m}$ the history of task m up to step t, $\mathcal{D}_m = (\mathcal{H}_{\tau_1,1}, \dots, \mathcal{H}_{\tau_{m-1},m-1})$, and $\mathbb{P}_{t,m}(\cdot) = \mathbb{P}(\cdot | \mathcal{H}_{t,m}, \mathcal{D}_m)$.

We also comment that, we formulate the problem as a series of best-arm identification tasks within the context of multi-armed bandits, rather than reinforcement learning, because in our setting, there is no obvious definition of the state. If we define the underlying true word as a state, then the action we take would not affect the future state. Therefore, we feel the bandits formulation is more suitable for our problem.

We consider two distinct scenarios.

In the fixed-confidence scenario (Even-Dar et al., 2002), the agent chooses a policy $\pi^m = (\pi_{t,m})_{t=1}^\infty$. The horizon is not fixed in advance, as the agent decides a stopping time τ_m adapted to filtration, $\mathbb{F}^m = (\mathcal{F}_{t,m})_{t=0}^\infty$, with $\mathcal{F}_{t,m} = \sigma(A_{1,m}, R_{1,m}, \dots, A_{t,m}, R_{t,m})$, where σ is the Borel σ-algebra. At the end of the task, the agent takes an action, or say a decision ψ_m. In a P300 speller experiment, this decision is the word the system believes the user intends to type. For a given confidence level $\delta \in (0, 1)$, the objective is to output a sequence of arms that are optimal for each task with probability at least $1 - \delta$ as soon as possible.

In the fixed-budget scenario (Bubeck et al., 2009), the agent is given a budget n for each task, choose a policy $\pi^m = (\pi_{t,m})_{t=1}^n$, and takes an action ψ_m at the end of the task. The objective is to make the cumulative probability that ψ_m is sub-optimal as small as possible.

2.2 Prior specification

A language model ρ defines a collection of conditional probability distributions $(\rho_1, \ldots, \rho_M)$, where ρ_m denotes a probability distribution over the mth word given the first $m-1$ words, $m \in [M]$. In a P300 speller experiment, the word that an individual attempts to type is viewed as the optimal arm. The joint distribution over the collection of optimal arms $(A_1^*, \ldots, A_M^*)$ can be written through the chain rule:

$$\rho(A_1^*, \ldots, A_M^*) = \prod_{m=1}^{M} \rho_m = \prod_{m=1}^{M} \mathbb{P}\left(A_m^* = \cdot | A_1^*, \ldots, A_{m-1}^*\right) . \tag{2.1}$$

For each task, the prior distribution of θ_m can be specified through

$$\nu_m = \mathbb{P}\left(\theta_m \in \cdot | A_1^*, \ldots, A_{m-1}^*\right) = \sum_{j=1}^{J} \Big\{ \underbrace{\mathbb{P}\left(\theta_m \in \cdot | A_m^* = j, A_1^*, \ldots, A_{m-1}^*\right)}_{\text{prior of the conditional mean reward}} \times \underbrace{\mathbb{P}\left(A_m^* = j | A_1^*, \ldots, A_{m-1}^*\right)}_{\text{prior of the optimal arm}} \Big\}, \tag{2.2}$$

which is a mixture distribution. While the prior of the optimal arm can be defined by (2.1), there are several ways to specify the prior of the conditional mean reward, depending on the problem setting. For the P300 speller example, the reward only differs upon whether the stimulus is a target or a non-target. Consequently, all sub-optimal arms share the same mean reward. It is thus natural to assume that,

$$\theta_m | A_m^* = j, A_1^*, \ldots, A_{m-1}^* \stackrel{d}{=} (\mu, \ldots, \underbrace{\mu + \Delta}_{j\text{th}}, \ldots, \mu),$$

where $\mu \sim \mathcal{N}(0, \sigma_0^2)$, and $\Delta \sim \exp(\sigma_1)$, for some $\sigma_0, \sigma_1 > 0$. We comment that, for the P300 speller experiment, there are often offline data available so that Δ can be accurately estimated (Ma et al., 2021).

3 Sequential top-two Thompson sampling procedure

We propose a sequential top-two Thompson sampling (`STTS`) algorithm that utilizes the prior information in a coherent way. We also develop the corresponding stopping rule and the decision rule for both the fixed-confidence setting and the fixed-budget setting.

3.1 Sampling procedure

Our proposed `STTS` algorithm requires a posterior sampling oracle that can be obtained exactly when a conjugate prior is used, or through various approximation methods, such as Markov chain Monte Carlo.

Definition 3.1 (Posterior sampling oracle)**.** *Given a prior ν_m over θ_m and history $\mathcal{H}_{t,m}, \mathcal{D}_m$, the posterior sampling oracle, abbreviated as `SAMP`, is a subroutine that returns a sample from the posterior distribution $\mathbb{P}_{t,m}(\theta_m \in \cdot)$. Multiple calls to the procedure result in independent samples.*

Our `STTS` algorithm extends the top-two Thompson sampling algorithm of Qin & Russo (2022); Russo (2016) that was designed for a single task setting now to the sequential multiple tasks setting, where we sequentially call the language model ρ to construct an informative prior. At task m and step t, `STTS` first draws a posterior sample $\widetilde{\theta}_{t,m}$ using `SAMP` as well as the language model ρ, and sets $A_{t,m,1} = \text{argmax}_{a \in \mathcal{A}} \widetilde{\theta}_{t,m}^\top a$. Then `STTS` repeatedly samples from `SAMP` to obtain $\widetilde{\theta}_{t,m}$, and sets $A_{t,m,2} = \text{argmax}_{a \in \mathcal{A}} \widetilde{\theta}_{t,m}^\top a$ until $A_{t,m,1} \neq A_{t,m,2}$. We pick $A_{t,m} = A_{t,m,1}$ with probability equal to $\beta \in (0, 1]$, and $A_{t,m} = A_{t,m,2}$ with probability equal to $1 - \beta$. In practice, we recommend $\beta = 1/2$, following Qin & Russo (2022). We also comment that, `SAMP` is a subroutine that is frequently used in the literature (see, e.g., Russo et al., 2018; Jourdan et al., 2022). It can be obtained exactly when a conjugate prior is used, or approximately through various methods such as MCMC and variational inference.

Algorithm 1 Sequential best-arm identification.

Input: The LLM ρ, the sampling oracle `SAMP`, the level of confidence δ, the sampling parameter β.
Output: A sequence of recommended actions $\{\psi_1, \ldots, \psi_M\}$.
for $m \in [M]$ **do**
  Construct the informative prior based on ρ and the history $\{\psi_1, \ldots, \psi_{m-1}\}$.
  Set $t = 0$.
  while TRUE **do**
    Set $t = t + 1$.
    Sample $\widetilde{\theta}_{t,m}$ from `SAMP`, and set $A_{t,m,1} = A_{t,m,2} = \text{argmax}_{a \in \mathcal{A}} \widetilde{\theta}_{t,m}^\top a$.
    while $A_{t,m,1} = A_{t,m,2}$ **do**
      Re-sample $\widetilde{\theta}_{t,m}$ from `SAMP`, and set $A_{t,m,2} = \text{argmax}_{a \in \mathcal{A}} \widetilde{\theta}_{t,m}^\top a$.
    end while
    Sample $C_{t,m} \sim \text{Bernoulli}(\beta)$.
    Pull arm $A_{t,m} = A_{t,m,1} C_{t,m} + A_{t,m,2}(1 - C_{t,m})$, and receive $R_{t,m}$.
    if $\max_{a_i \in \mathcal{A}} \min_{a_j \in \mathcal{A} \setminus \{a_i\}} Z_t(a_i, a_j) \geq \gamma_{t,\delta_M}$ **then**
      Identify the best arm ψ_m.
      STOP and break the while loop
    end if
    Update `SAMP`.
  end while
end for

3.2 Stopping rule and decision rule

For the fixed-confidence setting, we employ the Chernoff stopping rule introduced by Garivier & Kaufmann (2016); Shang et al. (2020). More specifically, let N_{t,a_i} denote the number of pulls of arm a_i before step t for each task, μ_{t,a_i} the posterior mean for arm a_i, and $\mu_{t,a_i,a_j} = (N_{t,a_i}\mu_{t,a_i} + N_{t,a_j}\mu_{t,a_j})/(N_{t,a_i} + N_{t,a_j})$. We further define $Z_{t,a_i,a_j} = N_{t,a_i}\text{KL}(\mu_{t,a_i}, \mu_{t,a_i,a_j})$ for any two arms a_i and a_j, such that $Z_t(a_i, a_j) = 0$ if $\mu_{t,a_j} \geq \mu_{t,a_i}$, and $Z_t(a_i, a_j) = Z_{t,a_i,a_j} + Z_{t,a_j,a_i}$ otherwise, where $\text{KL}(\mu_1, \mu_2)$ is the KL-divergence between two distributions with means μ_1 and μ_2. For each task m, the Chernoff stopping rule is,

$$\tau_m = \inf\left\{t \in \mathbb{N} : \max_{a_i \in \mathcal{A}} \min_{a_j \in \mathcal{A} \setminus \{a_i\}} Z_t(a_i, a_j) \geq \gamma_{t,\delta}\right\},$$

where $\gamma_{t,\delta}$ is the threshold parameter, and δ controls the level of confidence. As noted in Shang et al. (2020), $Z_t(a_i, a_j)$ can be interpreted as a generalized likelihood ratio statistic. As the agent interacts with M tasks sequentially, we employ the Bonferroni correction to handle multiple comparisons (Dunn, 1961). As such, the family-wise error rate $\delta_M = \delta/M$, and the level of confidence for each task becomes $1 - \delta/M$. For the final decision rule, we choose the Bayes optimal decision rule, $\psi_m = \text{argmax}_i \, \mu_{\tau_m,a_i}$. We summarize the full procedure for the fixed-confidence setting in Algorithm 1.

For the fixed-budget setting, the only change is we stop the algorithm when some pre-specified budget constraint is met.

4 Theoretical properties

We first consider the fixed-budget setting. When the recommended word at the end of a task is wrong, we say the agent makes a mistake. The next theorem derives the corresponding probability error bound.

Theorem 4.1. *Consider a sequence of M best-arm identification problems, and assume for each task m, all the sub-optimal arms have the same and known sub-optimality gap, $\Delta_m = \langle A_m^* - a, \theta_m \rangle$, for any $a \in \mathcal{A}$. If* `STTS` *is applied with $\beta \geq 1/2$ and with the Bayes optimal decision rule ψ_m, then for any positive integer-valued*

budget n,

$$\frac{1}{M}\sum_{m=1}^{M}\mathbb{P}\left(\psi_m \neq A_m^*\right) \leq \sum_{m=1}^{M}\frac{\sqrt{h_m}}{\Delta_m}\frac{6}{M}\sqrt{\frac{\log\{J(1+n)\}}{1+n}} + \left(1 - \frac{1}{M}\sum_{m=1}^{M}\prod_{j=1}^{m-1}\left[1 - \frac{6}{\Delta_m}\sqrt{\frac{\log\{J(1+n)\}h_m}{1+n}}\right]\right), \tag{4.1}$$

where $h_m = \mathbb{H}(A_m^*|A_{m-1}^*, \ldots, A_1^*)$, and $\mathbb{H}(\cdot|\cdot)$ is the conditional entropy.

We discuss the individual terms in the error bound in (4.1). The first term in the bound is the main term, where $h_m = \mathbb{H}(A_m^*|A_{m-1}^*, \ldots, A_1^*)$ specifically characterizes the effect of the prior. In the P300 speller, the prior distribution of A_m^* is informed by a language model such that $\mathbb{H}(A_m^*|A_{m-1}^*, \ldots, A_1^*)$ is much smaller than $\log(J)$. This can be validated through an experiment shown in Figure 2, left panel, where the entropy of the probability distribution over the next word outputted by GPT-2 is seen to be much smaller than an uniform distribution. The second term in the bound is the remainder term, which is the price the agent pays for making mistakes. Together, our bound in (4.1) explicitly characterizes the impact of the prior through the conditional entropy. When the prior is more informative, the conditional entropy decreases, and the error probability decreases too. In addition, the error probability converges to zero as the budget n goes to infinity.

In this theorem, we have assumed the gaps for all sub-optimal arms are the same and known. This assumption allows us to use a simple regret guarantee to bound the error probability. We argue this is a reasonable assumption, especially for the application such as the P300 speller, since all the non-target stimuli have the same level of EEG responses and the gap can usually be estimated though the offline data (see, e.g., Ma et al., 2021; Kalika et al., 2017; Mainsah et al., 2017). On the other hand, when the gaps are unknown, how to establish the theoretical guarantee for the top-two TS under a fixed budget is still an open question even for the single task setting, and we leave it for future research.

We next consider the fixed-confidence setting. Choosing the threshold $\gamma_{t,\delta} = 4\ln(4+\ln(t)) + 2C(\ln((J-1)/\delta)/2)$ for some constant C, and applying Theorem 1 of Shang et al. (2020) leads to an asymptotic sample complexity bound for `STTS`, coupled with the Bonferroni correction, as,

$$\limsup_{\delta\to 0}\frac{\sum_{m=1}^{M}\mathbb{E}[\tau_m]}{\log(1/\delta)} \leq \sum_{m=1}^{M}\int\frac{2J}{\Delta_m}\,\mathrm{d}\nu_m(\Delta_m)\,,$$

where the expectation is taken over the prior of θ_m on both sides. We remark that, this asymptotic result can not fully characterize the prior effect, as the asymptotic complexity measure only depends on the prior of the gap Δ_m, rather than the prior of the position of the optimal arm. Typically, the overall sample complexity consists of two parts: the cost due to the optimal allocation rule, and the cost due to finding the optimal allocation rule. The latter is a lower-order term with respect to $\log(1/\delta)$. We conjecture that a more informative prior could greatly reduce the cost of finding the optimal allocation rule in the finite time. We conduct a numerical experiment to verify our conjecture. The detailed simulation setting is described in Appendix B. We report in Figure 2, right panel, the KL-divergence between the asymptotically optimal allocation rule and the allocation rule induced by `STTS`. It is clearly seen that, when the prior is stronger, as reflected by a larger value of the parameter p that controls the strength of the prior, the allocation rule induced by `STTS` converges faster to the asymptotically optimal allocation rule. Meanwhile, a rigorous theoretical analysis requires a much more involved finite-time dependent analysis for top-two TS, and we leave it as future research.

5 Synthetic experiments

5.1 Simulation setup

We carry out simulations to investigate the empirical performance of our method and the effect of prior. We compare with the following baselines solutions.

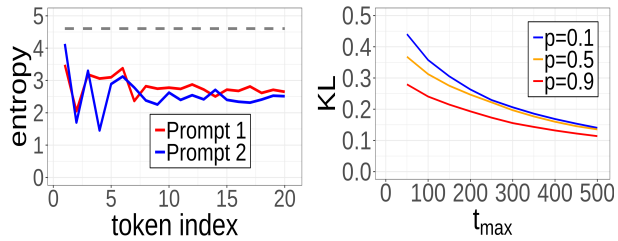

Figure 2: Left panel: the entropy of the next word under two different prompts (the red and blue curves) versus the entropy of a uniform distribution (the dashed line). Right panel: the KL divergence between the allocation rule of `STTS` and the optimal allocation rule versus the maximum number of steps when `STTS` stops. A larger p indicates a stronger prior.

- Vanilla top-two Thompson sampling (`VTTS`) (Russo, 2016; Qin & Russo, 2022): the top-two Thompson sampling that does not use any informative prior from the LLM.
- Random Policy (`Random`): the uniform sampling.
- Batch Racing (`BR`) (Jun et al., 2016): A frequentist algorithm that uses confidence interval to identify best arms in the fixed-confidence setting.

`VTTS` and `Random` use the same stopping rule and decision rule as described in Section 3.2, while `BR` is only for the fixed-confidence setting. There are several other popular best-arm identification algorithms, e.g., Jamieson & Nowak (2014); Garivier & Kaufmann (2016), in the literature, but none of them is designed for the sequential task setting.

We define the prior for the mean reward θ_m through (2.2), which requires the specification for the prior of the optimal arm, and the prior of the conditional mean reward. We assume the prior of the optimal arms satisfies the Markov property, in that the distribution of the mth optimal arm depends only on the $(m-1)$th optimal arm. More specifically, the optimal arm for mth task is sampled from $\mathbb{P}(A_m^* = j) = p$ if $j \in \{A_{m-1}^* + 1, A_{m-1}^* + 1 - J\} \cap [J]$, and $\mathbb{P}(A_m^* = j) = (1-p)/(J-1)$ if $j \in \{A_{m-1}^* + 1, A_{m-1}^* + 1 - J\}^c \cap [J]$, where S^c denotes the complement of a set S, and $p \in [0, 1]$ is a parameter that controls the strength of the prior.

Next, we set the prior of the mean reward of the jth arm to follow a Gaussian mixture distribution,

$$\mathbb{P}(\theta_{m,j} \in \cdot | A_1^*, \ldots, A_{m-1}^*) = p_{m,j}\mathcal{N}(\mu + \Delta, \sigma_0^2) + (1 - p_{m,j})\mathcal{N}(\mu, \sigma_0^2),$$

where we set the prior parameters as $\mu = 0, \sigma_0^2 = 0.2, \Delta = 2$. Moreover, when there is an oracle or external resource that reveals the identity of the optimal arm at the end of each task, `STTS` can start with an exact prior, which we call `STTS-Oracle`. We set $p_{m,j} \in [0, 1]$ for different algorithms as follows.

- For `STTS`, we set $p_{m,j} = \mathbb{P}(A_m^* = j | A_{m-1}^* = \psi_{m-1})$ for $j \in [J]$, where $\{\psi_1, \ldots, \psi_{m-1}\}$ are the recommended actions.
- For `STTS-Oracle`, we set $p_{m,j} = \mathbb{P}(A_m^* = j | A_{m-1}^* = a_{m-1}^*)$ for $j \in [J]$, where $\{a_1^*, \ldots, a_{m-1}^*\}$ denote the realized instances of the optimal arms.
- For `VTTS`, we set $p_{m,1} = \ldots = p_{m,J} = J^{-1}$.

Moreover, let $\mu_{m,t}$ denote the posterior mean, and $\sigma_{m,t,j}^2$ the posterior variance.

We take one sentence as one experiment, and one word in the sentence as one task, with each sentence consisting of $M = 20$ words. We replicate each experiment $B = 200$ times. We vary the number of arms $J \in \{10, 20\}$, and set the confidence level $\delta = 0.1$.

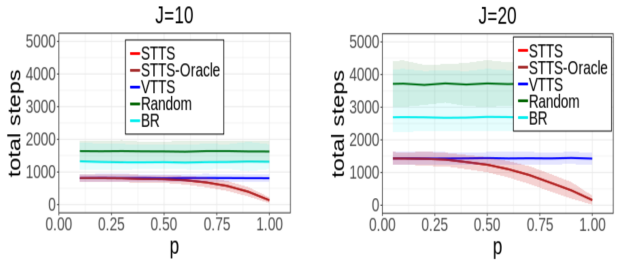

Figure 3: The total number of steps versus the prior strength p for the fixed-confidence setting. A smaller number of total steps indicates a better performance of the method, and a larger p indicates a stronger prior. The shaded area represents the 95% confidence interval.

We consider two accuracy measures for each experiment. One is the 0-1 accuracy, where as long as the agent makes at least one mistake among 20 words, we mark that experiment a failure. The other is the average accuracy, where we compute the percentage of correctly identified words among a sentence: $\sum_{m=1}^{M}\sum_{b=1}^{B}\mathbb{I}\{\psi_m^b = A_{m,b}^*\}/(BM)$, where $\psi_m^b, A_{m,b}^*$ denotes the recommended action and the optimal action for the mth task of the bth experiment, respectively.

We make a few comments. First, our proposed method does *not* require the Markov assumption. We have only adopted the Markov setting in our synthetic experiments for simplicity. Second, due to the Gaussian reward noise, the posterior distribution retains the form of a Gaussian mixture distribution. As such, the posterior sampling oracle `SAMP` can be easily derived. Finally, we note that the proposed `STTS` is a fairly general algorithm that can be coupled with any prior. As an illustration, we consider a different prior specification in Appendix C.

5.2 Fixed-confidence setting

We first consider the fixed-confidence setting. We vary the parameter in the prior of the optimal arm $p \in \{J^{-1}\} \cup \{0.1, 0.2, \ldots, 1.0\}$. A larger p indicates a stronger prior effect, and $p = J^{-1}$ means a non-informative prior. In our setting, we stop the algorithm when the Chernoff stopping rule is satisfied; i.e.,

$$\tau_m = \min\left\{ t : \min_{j \neq \psi_{t,m}} \frac{\mu_{m,t,j_1} - \mu_{m,t,j_2}}{\sqrt{\sigma^2_{m,t,j_1} + \sigma^2_{m,t,j_2}}} \geq \gamma_t \right\},$$

where $\psi_{t,m} = \text{argmax}_a\, a^\top \mu_{m,t}$, and $\gamma_t = [2\log\{\log(t)M/\delta\}]^{1/2}$. Here, we approximate the KL-divergence of two Gaussian mixture distributions by the KL-divergence of two Gaussian distributions with the same mean and variance (Hershey & Olsen, 2007). Moreover, as the theoretical stopping rule of `BR` is conservative, we multiply its range by a factor of 0.25.

We compare different methods via the total number of steps under the varying prior strength p, whereas all methods achieve at least $1 - \delta = 0.9$ for the 0-1 accuracy. Figure 3 reports the results. We see that, with the prior becoming more informative, `STTS` clearly outperforms the baseline solutions `VTTS`, `BR`, and `Random`. The improvement also increases with an increasing number of arms. When the prior of the optimal arms is uniformly distributed, `STTS` coincides with `VTTS` as expected. Furthermore, `STTS` and `STTS-Oracle` exhibit similar performances. This is because, as the accuracy of `STTS` is consistently close to one, $\mathbb{P}(A_m^* = \cdot | A_{m-1}^* = a_{m-1}^*)$ and $\mathbb{P}(A_m^* = \cdot | A_{m-1}^* = \psi_{m-1})$ are very close. We also briefly comment that, in real P300 speller experiments, it is impractical to run the number of steps into thousands. This step number can be adjusted by adjusting the accuracy measure. Here we mostly use this simulation example to illustrate the effect of the prior and the difference in performance of different solutions.

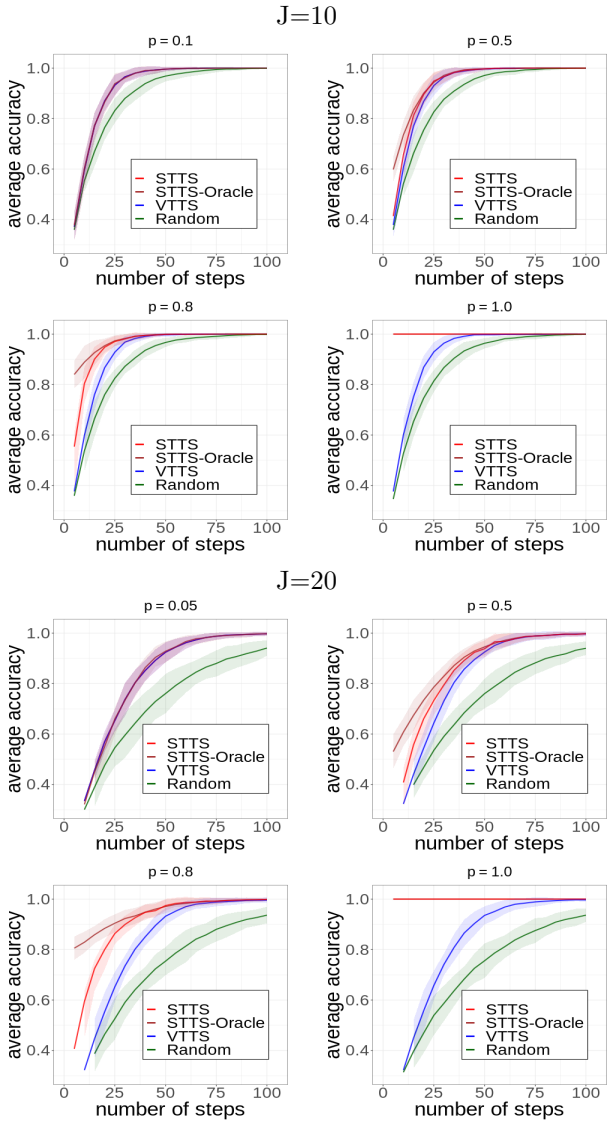

Figure 4: The average accuracy versus the maximum number of steps $t_{\max}$ under different prior strength p for the fixed-budget setting. A larger p indicates a stronger prior. The shaded area represents the 95% confidence interval.

5.3 Fixed-budget setting

We next consider the fixed-budget setting. The sampling and decision rules remain the same, while the stopping rule is determined by whether the algorithm reaches a pre-specified budget of maximum step $t_{\max}$ per task. We vary $t_{\max} \in \{5, 10, 15, \ldots, 100\}$, and $p \in \{J^{-1}, 0.5, 0.8, 1.0\}$.

We compare different methods via the average accuracy under the varying number of maximum step $t_{\max}$. Figure 4 reports the results. We see that, when the prior becomes more informative, `STTS` and `STTS-Oracle` reach a high accuracy with a smaller budget. When the prior is non-informative, `STTS` and `STTS-Oracle` perform similarly. We do not compare with `BR`, as it was designed for the fixed confidence setting only.

6 P300 speller simulator example

We carry out an experiment by generating the brain EEG signal data using a realistic P300 speller simulator developed by Ma et al. (2022). We do not use any public BCI data. This is because our method employs an adaptive policy that influences both data generation and collection, in contrast to the random and non-adaptive baseline policy that is typically associated with such type of datasets. Thus, offline data alone is insufficient for our needs, and we are not aware of any online benchmark data that is suitable for our purpose. On the other hand, the simulator of Ma et al. (2022) has been built based upon real BCI experiments, and can adeptly generate signals that capture and match the complex spatial-temporal EEG patterns observed in real experiments. Moreover, we generate the sentence of words that a participant wishes to type using GPT-3 (Brown et al., 2020), and obtain the prior distribution using GPT-2 (Radford et al., 2019). This mimics the potentially imperfect prior information that commonly arises in real-world scenarios. All computations were done using CPUs on Google Colab Cluster.

More specifically, the P300 speller simulator of Ma et al. (2022) is built on a Bayesian generative model, and was trained with the data from actual BCI experiments. It employs novel split-and-merge Gaussian process priors to ensure an accurate emulation of the EEG signal generation mechanism inherent in the P300 speller in response to both target and non-target stimuli. For instance, the generated EEG signals exhibit the target ERPs of the frontal and central channels with a negative drop around 100 ms and an ascend to the first peak approximately at 250 ms latency. This pattern aligns seamlessly with the N100 and P300 patterns outlined in Stemmer & Whitaker (2008), underscoring the simulator's capacity to mirror real experimental conditions effectively. In our experiment, we set the number of electrodes to 16, the noise variance $\sigma^2_{\text{EEG}} \in \{1, 2.5\}$, the noise spatial correlation based on a Gaussian kernel function, the noise temporal correlation from an $AR(1)$ model with an autocorrelation 0.9, and the mean magnitude of the target stimulus five times that of the non-target stimulus. In accordance with the current practice (Manyakov et al., 2011), we first train a binary classifier for the P300 offline data based on stepwise linear discriminant analysis (Donchin et al., 2000; Krusienski et al., 2008). This classifier converts the raw EEG signals into the classifier scores, and we take these scores as the rewards in our setting. A higher score indicates that the EEG signal is more likely to correspond to a target stimulus.

We use GPT-3 to generate $M = 20$ words, under a given prompt, as the words that the participant wishes to type, which form the optimal arms. We consider two prompts: Prompt 1 *"The most popular food in the United States is"*, and Prompt 2 *"My favorite sport is"*. We use GPT-2 to inform the prior distribution and to specify the prior of the optimal arms. That is, given an input, GPT-2 specifies a probability distribution of the next token over the full vocabulary space. We truncate the vocabulary size of GPT-2 from the original size of 50257 to 100, following the top-K sampling (Fan et al., 2018) and the nucleus sampling (Holtzman et al., 2019). This keeps the candidate words with the top-100 highest probabilities, and in effect substantially reduces the size of the action space to $J = 100$. We repeat each prompt $B = 100$ times, and set the confidence level at $1 - \delta = 0.9$.

We compare the algorithms studied in Section 5.1. When a wrong action is recommended, we let the experiment continue to run and reveal the identity of the optimal arm. As such, `STTS` and `STTS-Oracle` yield the same results. We also add the comparison with Beta-Bernoulli Thompson sampling (`BBTS`) algorithm of Ma et al. (2021).

For the fixed-confidence setting, we stop the algorithm when the Chernoff stopping rule is met. Table 1 reports the total number of steps, i.e., the stimulus flashes, whereas the 0-1 accuracy and the average accuracy for all methods are above 0.9. We see that, facilitated by the LLM-informed prior, `STTS` manages to reduce the total number of stimulus flashes by 25% to 50%, while maintaining about the same accuracy.

For the fixed-budget setting, we stop the algorithm when it reaches the pre-specified maximum number of flashes $t_{\max} \in \{5, 10, 15, \ldots, 150\}$. Figure 5 reports both the 0-1 accuracy and the average accuracy. We see that `STTS` achieves the highest accuracy compared to the alternative solutions. We also note that, the baseline random policy, which is currently used in most P300 experiments, requires about 150 steps to reach the 95% accuracy. This agrees with the practical observations that usually 100 to 150 steps are needed (Kindermans et al., 2012). In contrast, our method only requires 60 steps, which is a significant improvement.

Table 1: The total number of steps, i.e., the stimulus flashes, for the fixed-confidence setting using the P300 speller simulator and GPT. Two prompts are used. A larger σ^2_{EEG} indicates a larger noise level in the simulated EEG signals.

Prompt	σ^2_{EEG}	Method	total steps (std)
1	1	`STTS`	668.30 (136.5)
		`VTTS`	1445.0 (237.2)
		`Random`	6865.5 (826.6)
		`BR`	2611.8 (112.6)
		`BBTS`	1430.4 (203.5)
	2.5	`STTS`	1623.0 (240.6)
		`VTTS`	2493.2 (275.5)
		`Random`	13523.2 (829.9)
		`BR`	4396.4 (322.8)
		`BBTS`	1712.5 (248.1)
2	1	`STTS`	824.80 (171.3)
		`VTTS`	1441.1 (221.8)
		`Random`	7016.6 (956.8)
		`BR`	2662.0 (140.3)
		`BBTS`	1421.7 (197.2)
	2.5	`STTS`	1877.4 (258.2)
		`VTTS`	2445.8 (252.3)
		`Random`	13517.0 (952.6)
		`BR`	4441.7 (271.2)
		`BBTS`	1694.0 (266.8)

7 Conclusion

The BCI technology has the potential to revolutionize human communication by bypassing physical constraints and interacting directly with the brain. In this article, we propose a sequential best-arm identification formulation for the P300 speller BCI system that greatly enhances its sampling efficiency. We hope our work could open the door of using adaptive bandits algorithms for this important application.

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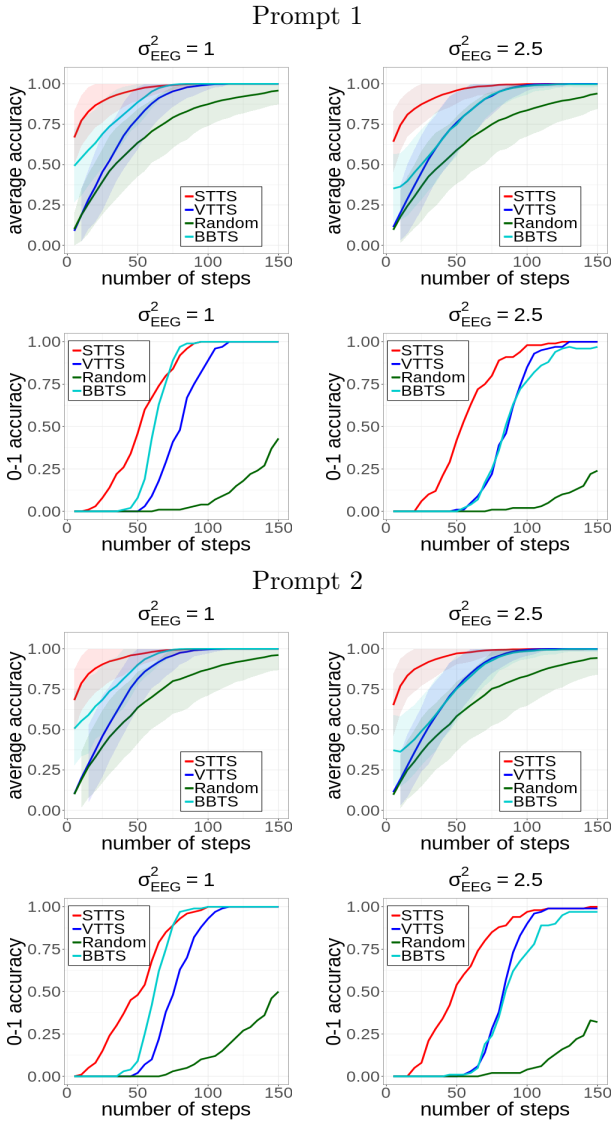

Figure 5: The average accuracy and the 0-1 accuracy for the fixed-budget setting using the P300 speller simulator and GPT. Two prompts are used. A larger σ^2_{EEG} indicates a larger noise level in the simulated EEG signals.

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A Proof of Theorem 4.1

We first consider the case when the recommended action is wrong, the subsequent task would use the wrong information. Suppose the prior is as specified in Section 2.2. Define the sub-optimality gap $\Delta_m = \langle A_m^*, \theta_m \rangle - \langle a, \theta_m \rangle$ for any $a \in \mathcal{A}, a \neq A_m^*$. Note that

$$\begin{aligned}\mathbb{E}\left[\langle A_m^*, \theta_m \rangle - \langle \psi_m, \theta_m \rangle\right] &= \mathbb{E}\left[\left(\langle A_m^*, \theta_m \rangle - \langle \psi_m, \theta_m \rangle\right) \mathbb{I}\left(\psi_m \neq A_m^*\right)\right] + \mathbb{E}\left[\left(\langle A_m^*, \theta_m \rangle - \langle \psi_m, \theta_m \rangle\right) \mathbb{I}\left(\psi_m = A_m^*\right)\right] \\ &= \mathbb{E}\left[\Delta_m \mathbb{I}\left(\psi_m \neq A_m^*\right)\right] ,\end{aligned}$$

where ψ_m is the recommeded action at the end of task m. Since we assume all the sub-optimal arms have the same mean reward, and the sub-optimality gap is known, the error probability can be bounded through the simple regret:

$$\mathbb{P}\left(\psi_m \neq A_m^*\right) = \frac{1}{\Delta_m} \mathbb{E}\left[\langle A_m^*, \theta_m \rangle - \langle \psi_m, \theta_m \rangle\right] ,$$

where the expectation in the left-hand-side is with respect to the prior distribution of θ_m. We then employ Proposition 1 of Qin & Russo (2022) for the top-two Thompson sampling for a single task. Define an event

$\mathcal{E}_m = \{\psi_1 = A_1^*, \ldots, \psi_{m-1} = A_{m-1}^*\}$. As Qin & Russo (2022) requires the top-two Thompson sampling starting with a correct prior, we decompose the simple regret term based on $\mathcal{E}_m$ as,

$$\mathbb{E}\left[\langle A_m^*, \theta_m\rangle - \langle \psi_m, \theta_m\rangle\right] = \underbrace{\mathbb{E}\left[(\langle A_m^*, \theta_m\rangle - \langle \psi_m, \theta_m\rangle)\,\mathbb{I}(\mathcal{E}_m)\right]}_{I_1} + \underbrace{\mathbb{E}\left[(\langle A_m^*, \theta_m\rangle - \langle \psi_m, \theta_m\rangle)\,\mathbb{I}(\mathcal{E}_m^c)\right]}_{I_2}. \tag{A.1}$$

Applying Proposition 1 of Qin & Russo (2022), we obtain that,

$$I_1 \leq 6\sqrt{\log(J(1+n))\mathbb{H}(A_m^*|A_{m-1}^*, \ldots, A_1^*)(1+n)^{-1}}, \tag{A.2}$$

where $\mathbb{H}(\cdot|\cdot)$ is the conditional entropy. To bound I_2, we have that,

$$\begin{aligned}
I_2 &= \mathbb{E}\left[(\langle A_m^*, \theta_m\rangle - \langle \psi_m, \theta_m\rangle)\,\mathbb{I}(\mathcal{E}_m^c)\right] \\
&\leq \Delta_m\left(1 - \mathbb{E}\left[\mathbb{P}\left(\psi_1 = A_1^*, \ldots, \psi_{m-1} = A_{m-1}^*\right)\right]\right) \\
&= \Delta_m\left(1 - \mathbb{E}\left[\prod_{j=1}^{m-1}\mathbb{P}\left(\psi_j = A_j^*|\psi_{j-1} = A_{j-1}^*, \ldots, \psi_1 = A_1^*\right)\right]\right) \\
&= \Delta_m\left(1 - \mathbb{E}\left[\prod_{j=1}^{m-1}\left(1 - \mathbb{P}\left(\psi_j \neq A_j^*|\psi_{j-1} = A_{j-1}^*, \ldots, \psi_1 = A_1^*\right)\right)\right]\right).
\end{aligned}$$

Applying Proposition 1 of Qin & Russo (2022) again, we have that,

$$\mathbb{P}\left(\psi_j \neq A_j^*|\psi_{j-1} = A_{j-1}^*, \ldots, \psi_1 = A_1^*\right) \leq \frac{6}{\Delta_m}\sqrt{\frac{\log(J(1+n))\mathbb{H}(A_m^*|A_{m-1}^*, \ldots, A_1^*)}{1+n}}. \tag{A.3}$$

Denote $p_m = 6\{\log(J(1+n))\mathbb{H}(A_m^*|A_{m-1}^*, \ldots, A_1^*)(1+n)^{-1}\}^{1/2}/\Delta_m$. Putting (A.1)-(A.3) together, we obtain that,

$$\mathbb{P}\left(\psi_m \neq A_m^*\right) \leq p_m + \left(1 - \prod_{j=1}^{m-1}(1-p_j)\right).$$

Summing over all the tasks, we obtain that,

$$\sum_{m=1}^{M}\mathbb{P}\left(\psi_m \neq A_m^*\right) \leq \sum_{m=1}^{M} p_m + \sum_{m=1}^{M}\left(1 - \prod_{j=1}^{m-1}(1-p_j)\right),$$

where the second term is the price the agent pays for the wrong prediction.

This completes the proof.

B Additional discussion on theory

First, we note that, for the fixed-budget setting, there is another practical scenario where at the end of task m, the agent is given the identity of the optimal arm A_m^*. If the recommended action is wrong, the agent pays an extra price c. For P300 speller, if the system outputs a wrong recommendation, the participant will gaze at the backspace and the system will repeat the process until the right word is recommended. In this case, the event $\mathcal{E}_m = \{\psi_1 = A_1^*, \ldots, \psi_{m-1} = A_{m-1}^*\}$ always holds. Based on (A.2), the error probability can be bounded by

$$\frac{1}{M}\sum_{m=1}^{M}\mathbb{P}\left(\psi_m \neq A_m^*\right) \leq \sum_{m=1}^{M}\frac{\sqrt{h_m}}{\Delta_m}\frac{6}{M}\sqrt{\frac{\log(J(1+n))}{1+n}},$$

Table 2: The accuracy and the number of total steps with a Gaussian prior specification. A larger μ_0 and a smaller σ_0 means a stronger prior information.

J	μ_0	σ_0	Method	$U^{(1)}$		$U^{(2)}$		$U^{(3)}$	
				accuracy	steps	accuracy	steps	accuracy	steps
5	4	0.5	STTS	100.0	60.0	100.0	50.8	98.5	307.5
			VTTS	100.0	253.1	100.0	251.4	99.5	715.9
			Random	100.0	405.1	100.0	406.1	98.0	1351.3
	5	0.5	STTS	100.0	47.1	100.0	47.3	99.5	76.6
			VTTS	100.0	184.1	100.0	183.3	99.5	370.6
			Random	100.0	295.7	100.0	296.6	99.5	708.5
	4	1	STTS	98.5	412.1	97.5	429.8	89.5	2008.0
			VTTS	98.5	644.3	98.5	655.8	89.0	2183.9
			Random	96.5	1052.1	98.5	1048.1	82.5	3332.3
	5	1	STTS	99.5	81.8	98.5	80.3	94.5	1115.3
			VTTS	99.5	267.6	99.5	273.2	93.5	1286.8
			Random	99.5	449.0	100.0	459.3	94.0	2088.8
10	4	0.5	STTS	100.0	62.6	98.5	62.2	98.5	455.5
			VTTS	100.0	480.9	100.0	482.1	99.5	1110.1
			Random	100.0	1030.0	99.5	1036.2	98.0	3173.1
	5	0.5	STTS	100.0	54.9	100.0	54.8	100.0	94.8
			VTTS	100.0	333.1	100.0	336.5	100.0	525.7
			Random	100.0	764.5	100.0	760.8	100.0	1517.3
	4	1	STTS	98.5	1588.8	96.0	1425.2	93.0	5362.4
			VTTS	99.0	2162.3	98.0	2145.4	94.5	5700.2
			Random	95.5	4803.8	95.0	4786.2	88.0	11594.4
	5	1	STTS	98.5	109.9	100.0	138.0	97.0	2341.1
			VTTS	100.0	597.4	100.0	616.5	97.5	2771.5
			Random	100.0	1375.8	98.5	1424.1	96.5	6288.7

and the number of mistakes is bounded by $c\sum_{m=1}^{M}\mathbb{P}\left(\psi_m \neq A_m^*\right)$.

Next, we verify the role of prior in the fixed-confidence setting in our conjecture in Section 4. Specifically, we study how fast the allocation rule of each arm using STTS converges to the optimal allocation rule and how the information of the prior affects the convergence. Following the prior specified in Section 5 and the computation in Shang et al. (2020), the optimal allocation rule p^* pulls the optimal arm with probability $\beta = 1/2$, and all the other arms with probability $1/(2J-2)$. We set the prior parameters $\mu = 0, \sigma_0^2 = 1, \Delta = 0.5$, the number of arms $J = 10$, and we vary $p \in \{0.1, 0.5, 0.9\}$. Define $p_{t,a_i} = N_{t,a_i}/t$ as the proportion of selections of arm a_i before step t. The algorithm stops when it reaches the pre-specified maximum number of steps $t_{\max} \in \{50, 100, 150, \ldots, 500\}$ per task. Figure 2, right panel, reports the KL divergence $\mathrm{KL}(p_t, p^*)$ that compares the STTS allocation rule and the optimal allocation rule. We see clearly that, when the prior becomes stronger, the allocation rule induced by STTS converges faster to the optimal allocation rule.

C Alternative prior specification

Our proposed algorithm STTS is general and can be coupled with any prior specification. As an illustration, we consider a Gaussian prior specification. Recall the language model ρ_m only requires the previous best arm A_{m-1}^* to define the prior distribution as $\nu_m = \mathcal{N}(\rho_m(A_{m-1}^*), \sigma_0^2 I)$, where σ_0 is the standard deviation parameter, and ρ_m is specified as follows. We first introduce three matrices $U^{(1)}$, $U^{(2)}$, and $U^{(3)}$.

- For $U^{(1)}$, we set
$$U^{(1)}_{j_1,j_2} = \begin{cases} 1, & (j_2 - j_1) \in \{1-J, 1\} \\ 0, & \text{otherwise} \end{cases},$$
which means that the optimal arm in the next task is more likely to be the one immediately following the current optimal arm.

- For $U^{(2)}$, we set
$$U^{(2)}_{j_1,j_2} = \begin{cases} 1, & (j_1, j_2) \in \{(1,2), (2,3), \ldots, (J-2,1), (J-1,J), (J,J-1)\} \\ 0, & \text{otherwise} \end{cases},$$
which consists of two groups where the first $(J-2)$ arms are in the first group, and the last two arms are in the second group.

- For $U^{(3)}$, we set
$$U^{(3)}_{j_1,j_2} = \begin{cases} 1, & (j_2 - j_1) \in \{1-J, 1\} \\ 0.5, & (j_2 - j_1) \in \{2-J, 2\} \\ 0, & \text{otherwise} \end{cases},$$
which possesses more uncertainty than $U^{(1)}$.

Let e_j be the unit vector with length J with the jth entry being 1. We define $\rho_m(A^*_{m-1}) = \mu_0 e^T_{A^*_{m-1}} U$, for $m \geq 2$, where U takes one of the forms of $U^{(1)}$, $U^{(2)}$, and $U^{(3)}$, μ_0 controls the magnitude of the best arm. For $m = 1$, we assign each arm to be the optimal arm with equal probability, and $\nu_1 = N(\mu_0 e_j, \sigma_0^2 I_J)$ if arm j is the optimal arm. The instance θ_m is sampled from the prior distribution ν_m. In this case, we set the posterior mean $\mu_{m,t,j} = \mathbb{E}(\theta_{m,j} \in \cdot | \mathcal{H}_{t,m}\mathcal{D}_m)$, and the posterior variance $\sigma^2_{m,t,j} = \text{Var}(\theta_{m,j} \in \cdot | \mathcal{H}_{t,m}\mathcal{D}_m)$. Moreover, we set the number of tasks $M = 20$, and the agent interacts with M bandit instances with the confidence level $1 - \delta = 0.9$, and the reward noise variance $\sigma^2 = 1$. For `VTTS`, we specify the non-informative Gaussian prior as $\mu_{0,m,j} = 0$, and $\Sigma_{0,m,j} = 10^2$ for $j \in [J]$.

We repeat the experiments 200 times for $J \in \{5, 10\}$, $\mu_0 \in \{4, 5\}$, and $\sigma_0 \in \{0.5, 1\}$. Table 2 reports the accuracy and the number of total steps. It is seen from the table that `STTS` achieves the smallest number of total steps, while maintaining the desired accuracy. When μ_0 is large and σ_0 is small, which indicates a strong prior information, the improvement of `STTS` over `VTTS` in terms of the total steps is more evident.

D Computational complexity evaluation

In the P300 speller system, a sequence of flashes is presented on a virtual screen to the user. The time interval between two consecutive flashes is a constant, denoted as c_0, which is also the time taken for each step. We follow Ma et al. (2021), and set this time interval $c_0 = 160$ ms. We observe that our proposed algorithm as well as the alternative algorithms require much shorter time than c_0 in each step. As such, instead of reporting the computational time alone, we employ the information transfer rate (ITR) (Wolpaw et al., 1998), and the BCI utility (Utility) (Dal Seno et al., 2009), as the metrics to evaluate the overall computational complexity of all the algorithms. Specifically, we compute the two measures as,

$$\begin{aligned} \text{ITR} &= \frac{1}{c_0}\left\{\log_2 J + \phi \log_2 \phi + (1-\phi)\log_2\left(\frac{1-\phi}{J-1}\right)\right\}, \\ \text{Utility} &= \frac{1}{c_0}\max\left\{0, (2\phi - 1)\log_2(J-1)\right\}, \end{aligned}$$

which take into account the time interval c_0, the average accuracy ϕ, and the total number of arms J.

Figure 6 reports the ITR and Utility measures for the P300 speller simulator example studied in Section 6. We see that `STTS` achieves the best performance in terms of the ITR and Utility measures.

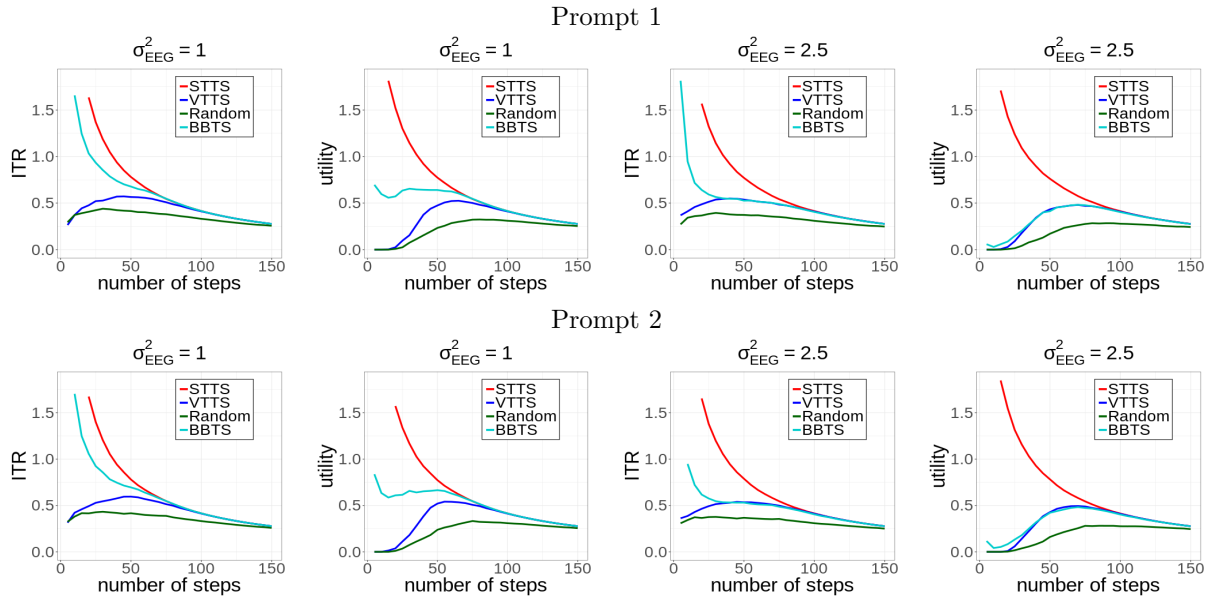

Figure 6: The information transfer rate (ITR) and the utility measure for the fixed-budget setting using the P300 speller simulator and GPT. Two prompts are used. A larger σ^2_{EEG} indicates a larger noise level in the simulated EEG signals.