Adaptive Extra-Gradient Methods for Min-Max Optimization and Games

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Abstract

We present a new family of min-max optimization algorithms that automatically exploit the geometry of the gradient data observed at earlier iterations to perform more informative extra-gradient steps in later ones. Thanks to this adaptation mechanism, the proposed methods automatically detect the problem's smoothness properties, without requiring any prior tuning by the optimizer. As a result, the algorithm simultaneously achieves the optimal smooth/non-smooth min-max convergence rates, i.e., it converges to an ε-optimal solution within $\mathcal{O}(1/\varepsilon)$ iterations in smooth problems, and within $\mathcal{O}(1/\varepsilon^2)$ iterations in non-smooth ones. Importantly, these guarantees do not require any of the standard boundedness or Lipschitz continuity conditions that are typically assumed in the literature; in particular, they apply even to problems with divergent loss functions (such as log-likelihood learning and the like). This "off-the-shelf" adaptation is achieved through the use of a geometric apparatus based on Finsler metrics and a suitably chosen mirror-prox template that allows us to derive sharp convergence rates for the family of methods at hand.

1 Introduction

The surge of recent breakthroughs in generative adversarial networks (GANs) [24], robust reinforcement learning [48], and other adversarial learning models [32] has sparked renewed interest in the theory of min-max optimization problems and games. In this broad setting, it has become empirically clear that, ceteris paribus, the simultaneous training of two (or more) antagonistic models faces drastically new challenges relative to the training of a single one. Perhaps the most prominent of these challenges is the appearance of cycles and recurrent (or even chaotic) behavior in min-max games. This has been studied extensively in the context of bilinear games [12, 15, 19, 22, 23, 36, 37, 45, 47], and the methods proposed to overcome cycling and recurrent behaviors typically focus on mitigating the rotational component of min-max games that is at the root of these phenomena.

The method with the richest history in this context is the *extra-gradient* (EG) algorithm of Korpelevich [29] and its variants – for an appetizer, see [15, 33, 34, 40, 42, 50, 51] and references therein. Originally applied to variational inequalities, the EG algorithm exploits the Lipschitz smoothness of the problem and, if coupled with an iterate averaging scheme – the *mirror-prox* (MP) template [40] – it achieves an $\mathcal{O}(1/T)$ rate of convergence in smooth, convex-concave min-max problems. This rate is known to be tight [39, 44] but, in order to achieve it, the original method requires the problem's Lipschitz constant to be known in advance. If the problem is not Lipschitz (or the algorithm is run with a vanishing step-size schedule), the method's rate of convergence drops to $\mathcal{O}(1/\sqrt{T})$ [40].

Our contributions. Our aim in this paper is to provide an algorithm that automatically adapts to smooth / non-smooth min-max problems and games, and achieves optimal rates in both classes without requiring any tuning by the optimizer. In this regard, we propose a flexible algorithmic scheme, which we call UniProx, and which exploits gradient data observed at earlier iterations to perform more informative extra-gradient steps in later ones. Thanks to this mechanism, and to the best of our knowledge, UniProx is the first algorithm that simultaneously achieves the following:

1. An $\mathcal{O}(1/\sqrt{T})$ convergence rate in non-smooth problems and $\mathcal{O}(1/T)$ in smooth ones.
2. Convergence in min-max problems and games where the standard boundedness / Lipschitz continuity conditions required in the literature do not hold.
3. "Off-the-shelf" applicability without requiring prior knowledge of the problem's parameters.

	EG [28, 29, 40]	GRAAL [34]	AMP [1, 21]	BL [2]	UniProx [ours]
Parameter-Agnostic	✗	✓	✓	Partial	✓
Rate Interpolation	✗	✗	✗	✓	✓
Non-Lips. / Unbounded	✗/ ✗	Partial / –	✓/ –	✗/ ✗	✓/ ✓

Table 1: Overview of related work. For the purposes of this table, "parameter-agnostic" means that the method does not require prior knowledge of any of the parameters of the problem it was designed to solve (Lipschitz constant, domain diameter, etc.). "Rate interpolation" means that the algortihm's convergence rate is $\mathcal{O}(1/T)$ or $\mathcal{O}(1/\sqrt{T})$ in smooth / non-smooth problems respectively. Finally, "non-Lipschitz / unbounded" refers to smooth / non-smooth problems with a non-Lipschitz / unbounded vector field respectively, and means that the method can be applied to problems with singularities (D-design, Poisson inverse problems, etc.).

The method we propose achieves the above by fusing the following ingredients: *a*) a family of local norms – a *Finsler metric* – capturing any singularities in the problem at hand; *b*) a suitable mirror-prox template for the algorithm; and *c*) an adaptive step-size policy in the spirit of Rakhlin & Sridharan [51]. As an extra feature of our analysis, we also show that the sequence of iterates generated by the algorithm converges, thus providing an appealing alternative to iterate averaging in cases where the method's "last iterate" is more appropriate.

Related works. There have been several works improving on the guarantees of the original extra-gradient/mirror-prox template. We review the most relevant of these works below; for convenience, we also tabulate these contributions in Table 1 above. Because many of these works appear in the literature on variational inequalities [18], we also use this language in the sequel.

In unconstrained problems with an operator that is locally Lipschitz continuous (but not necessarily globally so), the *golden ratio algorithm* (GRAAL) [34] achieves convergence without requiring prior knowledge of the problem's Lipschitz parameter. However, GRAAL provides no rate guarantees for non-smooth problems and hence, a fortiori, no interpolation guarantees either. Moreover, nothing is known about the convergence of GRAAL in constrained problems with divergent operators.

Convergence guarantees for problems with divergent operators have been provided in a recent series of papers [1, 21, 55] exploiting a "Bregman continuity" or "Lipschitz-like" condition. These methods achieve optimal rates in the smooth case, without requiring any knowledge of the problem's parameters. However, like GRAAL, they do not provide "off-the-shelf" adaptivity to the non-smooth regime or any rate interpolation guarantees.

To the best of our knowledge, the only method that simultaneously achieves an $\mathcal{O}(1/\sqrt{T})$ rate in non-smooth problems and an $\mathcal{O}(1/T)$ rate in smooth ones is the recent algorithm of Bach & Levy [2]. The BL algorithm employs an adaptive, AdaGrad-like step-size policy which allows the method to interpolate between the two regimes – and this, even with noisy gradient feedback. On the negative side, the BL algorithm requires a bounded domain with a (Bregman) diameter that is known in advance; as a result, its theoretical guarantees do not apply to unbounded problems. In addition, the BL algorithm makes crucial use of boundedness and Lipschitz continuity requirements; extending the BL method beyond this standard framework is a highly non-trivial endeavor which formed a big part of this paper's motivation.

2 Problem Setup and Blanket Assumptions

We begin in this section by reviewing some basics for min-max optimization problems and games.

2.1. Min-max / Saddle-point problems. A *min-max game* is a saddle-point problem of the form

$$\min_{\theta\in\Theta} \max_{\phi\in\Phi} L(\theta,\phi) \tag{SP}$$

where Θ, Φ are convex subsets of some ambient real space and $L\colon \Theta\times\Phi\to\mathbb{R}$ is the problem's *loss function*. In the game-theoretic interpretation of (SP), the player controlling θ seeks to minimize $L(\theta,\phi)$ for any value of the maximization variable ϕ, while the player controlling ϕ seeks to maximize $L(\theta,\phi)$ for any value of the minimization variable θ. Accordingly, solving (SP) consists of finding a *Nash equilibrium* (NE), i.e., an action profile $(\theta^*,\phi^*)\in\Theta\times\Phi$ such that

$$L(\theta^*,\phi) \le L(\theta^*,\phi^*) \le L(\theta,\phi^*) \quad \text{for all } \theta\in\Theta,\ \phi\in\Phi. \tag{1}$$

By the minimax theorem of von Neumann [56], Nash equilibria are guaranteed to exist when Θ, Φ are compact and L is convex-concave (i.e., convex in θ and concave in ϕ). Much of our paper is motivated by the question of calculating a Nash equilibrium (θ^*, ϕ^*) of (SP) in the context of von Neumann's theorem; we expand on this below.

2.2. Games. Going beyond the min-max setting, a *continuous game in normal form* is defined as follows: First, consider a finite set of players $\mathcal{N} = \{1, \dots, N\}$, each with their own action space $\mathcal{K}_i \in \mathbb{R}^{d_i}$ (assumed convex). During play, each player selects an action x_i from $\mathcal{K}_i$ with the aim of minimizing a loss determined by the ensemble $x := (x_i; x_{-i}) := (x_1, \dots, x_N)$ of all players' actions. In more detail, writing $\mathcal{K} := \prod_i \mathcal{K}_i$ for the game's total action space, we assume that the loss incurred by the i-th player is $\ell_i(x_i; x_{-i})$, where $\ell_i \colon \mathcal{K} \to \mathbb{R}$ is the player's *loss function*.

In this context, a Nash equilibrium is any action profile $x^* \in \mathcal{K}$ that is *unilaterally stable*, i.e.,

$$\ell_i(x_i^*; x_{-i}^*) \leq \ell_i(x_i; x_{-i}^*) \quad \text{for all } x_i \in \mathcal{K}_i \text{ and all } i \in \mathcal{N}. \tag{NE}$$

If each $\mathcal{K}_i$ is compact and ℓ_i is convex in x_i, existence of Nash equilibria is guaranteed by the theorem of Debreu [16]. Given that a min-max problem can be seen as a two-player zero-sum game with $\ell_1 = L$, $\ell_2 = -L$, von Neumann's theorem may in turn be seen as a special case of Debreu's; in the sequel, we describe a first-order characterization of Nash equilibria that encapsulates both.

2.3. The vector field formulation and variational inequalities. In most cases of interest, the players' loss functions are *individually subdifferentiable* on a subset $\mathcal{X}$ of $\mathcal{K}$ with $\operatorname{ri} \mathcal{K} \subseteq \mathcal{X} \subseteq \mathcal{K}$ [25, 52]. This means that there exists a (possibly discontinuous) vector field $V_i \colon \mathcal{X} \to \mathbb{R}^{d_i}$ such that

$$\ell_i(x_i'; x_{-i}) \geq \ell_i(x_i; x_{-i}) + \langle V_i(x), x_i' - x_i \rangle \tag{2}$$

for all $x \in \mathcal{X}$, $x' \in \mathcal{K}$ [25]. In the simplest case, if ℓ_i is differentiable at x, then $V_i(x)$ can be interpreted as the gradient of ℓ_i with respect to x_i. The *raison d'être* of the more general definition (2) is that it allows us to treat non-smooth loss functions that are common in machine learning (such as L^1-regularized losses). We make this distinction precise below:

1. If there is no continuous vector field $V_i(x)$ satisfying (2), the problem is called *non-smooth*.
2. If there is a continuous vector field $V_i(x)$ satisfying (2), the problem is called *smooth*.

Remark. We stress here that the adjective "smooth" refers to the game itself: for instance, if $\ell(x) = |x|$ for $x \in \mathbb{R}$, the game is not smooth and any V satisfying (2) is discontinuous at 0. In this regard, the above boils down to whether the (individual) subdifferential of each ℓ_i admits a continuous selection.

In both cases (smooth and non-smooth), we have the following characterization of Nash equilibria:

Proposition 1. *$x^* \in \mathcal{X}$ is a Nash equilibrium if and only if it satisfies the variational inequality*

$$\langle V(x^*), x - x^* \rangle \geq 0 \quad \textit{for all } x \in \mathcal{X}, \tag{VI}$$

where, in obvious notation, we set $V(x) = (V_1(x), \dots, V_N(x))$.

This equivalent formulation is quite common in the literature on min-max / equilibrium problems [17, 18, 30, 35], and it is often referred to as the "vector field formulation" [3, 9, 27]. Its usefulness lies in that it allows us to abstract away from the underlying game-theoretic complications (multiple indices, individual subdifferentials, etc.) and provides a unifying framework for a wide range of problems in machine learning, signal processing, operations research, and many other fields [18, 53]. For this reason, our analysis will focus almost exclusively on solving (VI), and we will treat V and $\mathcal{X} \subseteq \mathbb{R}^d$, $d = \sum_i d_i$, as the problem's primitive data.

2.4. Merit functions and monotonicity. A widely used assumption in the literature on equilibrium problems and variational inequalities is the *monotonicity condition*

$$\langle V(x) - V(x'), x - x' \rangle \geq 0 \quad \text{for all } x, x' \in \mathcal{X}. \tag{MC}$$

In single-player games, monotonicity is equivalent to convexity of the optimizer's loss function; in min-max games, it is equivalent to L being convex-concave [30]; etc. In the absence of monotonicity, approximating an equilibrium is PPAD-hard [14], so (MC) will be our blanket assumption.

Now, to assess the quality of a candidate solution $\hat{x} \in \mathcal{X}$, we will employ the *restricted merit function*

$$\operatorname{Gap}_{\mathcal{C}}(\hat{x}) = \sup_{x \in \mathcal{C}} \langle V(x), \hat{x} - x \rangle, \tag{3}$$

where the "*test domain*" $\mathcal{C}$ is a nonempty convex subset of $\mathcal{X}$ [18, 28, 42]. The motivation for this is provided by the following proposition:

Proposition 2. *Let $\mathcal{C}$ be a nonempty convex subset of $\mathcal{X}$. Then: a) $\operatorname{Gap}_{\mathcal{C}}(\hat{x}) \geq 0$ whenever $\hat{x} \in \mathcal{C}$; and b) if $\operatorname{Gap}_{\mathcal{C}}(\hat{x}) = 0$ and $\mathcal{C}$ contains a neighborhood of $\hat{x}$, then $\hat{x}$ is a solution of* (VI).

Proposition 2 generalizes an earlier characterization by Nesterov [42] and justifies the use of $\operatorname{Gap}_{\mathcal{C}}(x)$ as a merit function for (VI); to streamline our presentation, we defer the proof to the paper's supplement. Moreover, to avoid trivialities, we will also assume that the solution set $\mathcal{X}^*$ of (VI) is nonempty and we will reserve the notation x^* for solutions of (VI). Together with monotonicity, this will be our only blanket assumption.

3 The Extra-Gradient Algorithm and its Limits

Perhaps the most widely used solution method for games and variational inequalities (VIs) is the *extra-gradient* (EG) algorithm of Korpelevich [29] and its variants [33, 50, 51]. This algorithm has a rich history in optimization, and it has recently attracted considerable interest in the fields of machine learning and AI; for an overview, see [9, 15, 22, 26, 27, 37, 38] and references therein.

In its simplest form, the algorithm proceeds recursively as

$$X_{t+1/2} = \Pi(X_t - \gamma_t V_t), \qquad X_{t+1} = \Pi(X_t - \gamma_t V_{t+1/2}), \tag{EG}$$

where $\Pi(x) = \arg\min_{x' \in \mathcal{X}} \|x' - x\|$ is the Euclidean projection on $\mathcal{X}$, $V_t := V(X_t)$ for $t = 1, 3/2, \dots$, and $\gamma_t > 0$, is the method's step-size. Then, running (EG) for T iterations, the algorithm returns the "ergodic average"

$$\bar{X}_T = \frac{\sum_{t=1}^{T} \gamma_t X_{t+1/2}}{\sum_{t=1}^{T} \gamma_t}. \tag{4}$$

In this setting, the main guarantees for (EG) date back to [40] and can be summarized as follows:

1. *For non-smooth problems* (discontinuous V): Assume V is *bounded*, i.e., there exists some $M > 0$ such that
$$\|V(x)\| \leq M \quad \text{for all } x \in \mathcal{X}. \tag{BD}$$
Then, if (EG) is run with a step-size of the form $\gamma_t \propto 1/\sqrt{t}$, we have
$$\operatorname{Gap}_{\mathcal{C}}(\bar{X}_T) = \mathcal{O}(1/\sqrt{T}). \tag{5}$$
2. *For smooth problems* (continuous V): Assume V is *L-Lipschitz continuous*, i.e.,
$$\|V(x) - V(x')\| \leq L\|x - x'\| \quad \text{for all } x, x' \in \mathcal{X}. \tag{LC}$$
Then, if (EG) is run with a constant step-size $\gamma < 1/L$, we have
$$\operatorname{Gap}_{\mathcal{C}}(\bar{X}_T) = \mathcal{O}(1/T). \tag{6}$$

Remark. In the above, $\|\cdot\|$ is tacitly assumed to be the standard Euclidean norm. Non-Euclidean considerations will play a crucial role in the sequel, but they are not necessary for the moment.

Importantly, the distinction between smooth and non-smooth problems cannot be lifted: the bounds (5) and (6) are tight in their respective problem classes and they cannot be improved without further assumptions [39, 44]. Moreover, we should also note the following:

1. The algorithm changes drastically from the non-smooth to the smooth case: non-smoothness requires a vanishing γ_t, but such a step-size cannot achieve an accelerated $\mathcal{O}(1/T)$ rate.
2. If (EG) is run with a constant step-size, L must be known in advance; otherwise, running (EG) with an ill-adapted step-size ($\gamma > 1/L$) could lead to non-convergence.

We illustrate this convergence failure of (EG) in Fig. 1. As we discussed in the introduction, our aim in the sequel will be to provide a single, *universal* algorithm that simultaneously achieves the following: *a*) an $\mathcal{O}(1/\sqrt{T})$ convergence rate in non-smooth problems and $\mathcal{O}(1/T)$ in smooth ones; *b*) convergence in problems where the boundedness / Lipschitz continuity conditions (BD) / (LC) no longer hold; and *c*) achieves all this without prior knowledge of any of the problem's parameters.

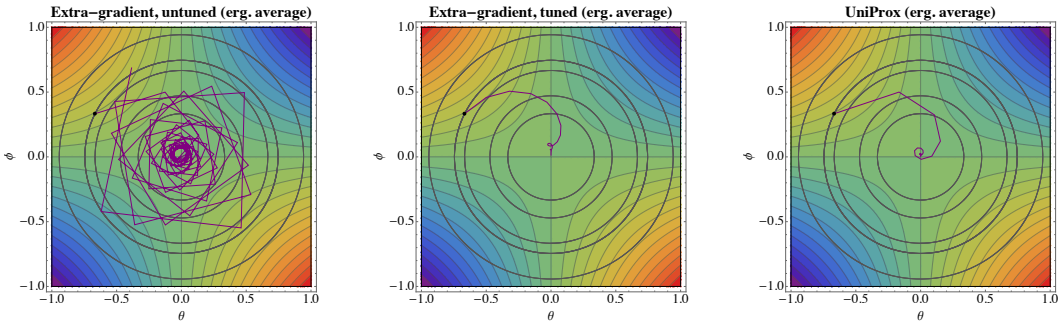

Figure 1: The behavior of (EG) in the bilinear min-max problem $L(\theta, \phi) = \theta\phi$ with $\theta, \phi \in [-1, 1]$. Running the algorithm with a step-size only slightly above the $1/L$ bound ($L = 1$, $\gamma = 1.04$) results in a dramatic convergence failure (left plot). Running (EG) with a finely tuned step-size resolves this problem (center), but the use of a constant step-size makes the algorithm unnecessarily conservative towards the end. The proposed UniProx algorithm automatically exploits the geometry of the gradient data observed at early iterations to perform more informative extra-gradient steps in later ones, thus achieving faster convergence without any tuning.

4 Rate Interpolation: the Euclidean Case

As a prelude to our main result, we provide in this section an adaptive version of (EG) that achieves the "best of both worlds" in the Euclidean setting of Section 3, i.e., an $\mathcal{O}(1/\sqrt{T})$ convergence rate in problems satisfying (BD), and an $\mathcal{O}(1/T)$ rate in problems satisfying (LC). Our starting point is the observation that, if the sequence X_t produced by (EG) converges to a solution of (VI), the difference

$$\delta_t \coloneqq \|V_{t+1/2} - V_t\| = \|V(X_{t+1/2}) - V(X_t)\| \tag{7}$$

must itself become vanishingly small if V is (Lipschitz) continuous. On the contrary, if V is *discontinuous*, this difference may remain bounded away from zero (consider for example the L^1 loss $\ell(x) = |x|$ near 0). Based on this observation, we consider the adaptive step-size policy:

$$\gamma_t = 1 \Big/ \sqrt{1 + \textstyle\sum_{s=1}^{t-1} \delta_s^2}. \tag{8}$$

The intuition behind (8) is as follows: If V is not smooth and $\liminf_{t\to\infty} \gamma_t > 0$, then γ_t will vanish at a $\Theta(1/\sqrt{t})$ rate, which is the optimal step-size schedule for problems satisfying (BD) but not (LC). Instead, if V satisfies (LC) and X_t converges to a solution x^* of (VI), it is plausible to expect that the infinite series $\sum_t \delta_t^2$ is summable, in which case the step-size γ_t will not vanish as $t \to \infty$. Furthermore, since δ_t is defined in terms of successive gradient differences, it automatically exploits the variation of the gradient data observed up to time t, so it can be expected to adjust to the "local" Lipschitz constant of V around a solution x^* of (VI).

Our step-size policy and motivation are similar in spirit to the "predictable sequence" approach of [51]. However, making our reasoning precise (especially the summability of $\sum_t \delta_t^2$ in the smooth case) involves considerable conceptual and technical difficulties that we present in detail in the supplement. For now, we only state (without proof) our main result for problems satisfying (BD) or (LC).

Theorem 1. *Let $\mathcal{C}$ be a compact neighborhood of a solution of* (VI) *and let $H = \sup_{x\in\mathcal{C}} \|X_1 - x\|^2$ Then, if* (EG) *is run with the adaptive step-size policy* (8)*, we have:*

$$\text{a) If } V \text{ satisfies (BD):} \quad \operatorname{Gap}_{\mathcal{C}}(\bar{X}_T) = \mathcal{O}\left(\frac{H + 4M^2 + \log(1 + 4M^2 T)}{\sqrt{T}}\right). \tag{9a}$$

$$\text{b) If } V \text{ satisfies (LC):} \quad \operatorname{Gap}_{\mathcal{C}}(\bar{X}_T) = \mathcal{O}(H/T). \tag{9b}$$

We stress here that Theorem 1 holds even for problems with an unbounded domain ($\operatorname{diam}(\mathcal{X}) = \infty$). In fact, even when $\mathcal{X}$ is bounded, the adaptive policy (8) does not require *any* prior knowledge of its diameter (which might be difficult to estimate). These limitations are both present in the work of Bach & Levy [2], so Theorem 1 provides a significant improvement in this regard. Going beyond this improvement, in the next sections we drop even the Euclidean regularity requirements (BD)/(LC), and we provide a corresponding rate interpolation result that does not require either condition.

5 Finsler Regularity

To motivate our analysis outside the setting of (BD)/(LC), consider the generalized *Kullback–Leibler* (KL) loss

$$\ell(x) = \mathrm{KL}(p, x) = \sum_{i=1}^{d} [p_i \log(p_i/x_i) + x_i - p_i], \tag{10}$$

where $p \in [0,1]^d$ is a fixed target point (typically a ground truth profile). This loss function is widely used in Poisson inverse problems [6], logistic regression and log-likelihood models [20], experiment design [46], and many other fields of statistical learning. The KL loss is well-defined for all $x \in [0,1]^d$ such that $x_i = 0$ when $p_i = 0$ and its gradient is $V(x) = (1 - p_i/x_i)_{i=1,\dots,d}$ for all $x \in \mathcal{X} = (0,1]^d$. Consequently, (BD) and (LC) both fail for the KL loss because $\|V(x)\| \to \infty$ whenever $x_i \to 0^+$ (note here that $\mathcal{X}$ is convex but neither open nor closed).

On the other hand, if we consider the "local" norm $\|v\|_{x,*} = \sum_{i=1}^{d} x_i|v_i|$, we have $\|V(x)\|_{x,*} \leq 2d$, so V *is bounded relative to* $\|\cdot\|_{x,*}$. This observation motivates the use of a *local* – as opposed to *global* – norm on $\mathcal{X}$; formally, this amounts to equipping $\mathcal{X}$ with a *Finsler metric* [4, 11].

Definition 1. A *Finsler metric* on a convex subset $\mathcal{X}$ of $\mathbb{R}^d$ is a continuous function $F\colon \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}_+$ which satisfies the following properties for all $x \in \mathcal{X}$ and all $z, z' \in \mathbb{R}^d$:

1. *Subadditivity:* $F(x; z + z') \leq F(x; z) + F(x; z')$.
2. *Absolute homogeneity:* $F(x; \lambda z) = |\lambda| F(x; z)$ for all $\lambda \in \mathbb{R}$.
3. *Positive-definiteness:* $F(x; z) \geq 0$ with equality if and only if $z = 0$.

Given a Finsler metric on $\mathcal{X}$, the induced *primal/dual local norms* on $\mathcal{X}$ are respectively defined as

$$\|z\|_x = F(x; z) \quad \text{and} \quad \|v\|_{x,*} = \max\{\langle v, z\rangle : F(x; z) = 1\} \tag{11}$$

for all $x \in \mathcal{X}$ and all $z, v \in \mathbb{R}^d$. A Finsler metric on $\mathcal{X}$ is further called *regular* when $\|v\|_{x',*}/\|v\|_{x,*} = 1 + \mathcal{O}(\|x' - x\|_x)$ for all $x, x' \in \mathcal{X}$, $v \in \mathbb{R}^d$. Finally, for simplicity, we will also assume in the sequel that $\|\cdot\|_x \geq \mu\|\cdot\|$ for some $\mu > 0$ and all $x \in \mathcal{X}$ (this last assumption is for convenience only, as the norm could be redefined to $\|\cdot\|_x \leftarrow \|\cdot\|_x + \mu\|\cdot\|$ without affecting our results).

When $\mathcal{X}$ is equipped with a regular Finsler metric as above, we will say that it is a *Finsler space*.

► **Example 5.1.** Let $F(x; z) = \|z\|$ where $\|\cdot\|$ denotes the reference norm of $\mathcal{X} = \mathbb{R}^d$. Then the axioms of a Finsler space are satisfied trivially.

► **Example 5.2.** A more interesting example of a Finsler structure on $\mathcal{X} = (0,1]^d$ is provided by the metric $\|z\|_x = \max_i |z_i|/x_i$, $z \in \mathbb{R}^d$, $x \in \mathcal{X}$. In this case $\|v\|_{x,*} = \sum_{i=1}^{d} x_i|v_i|$ for all $v \in \mathbb{R}^d$, and the only axiom left to verify is that of regularity. To that end, we have

$$\|v\|_{x',*} - \|v\|_{x,*} \leq \textstyle\sum_{i=1}^{d} |v_i| \cdot |x'_i - x_i| = \sum_{i=1}^{d} x_i|v_i| \cdot |x'_i - x_i|/x_i \leq \|v\|_{x,*} \cdot \|x' - x\|_x. \tag{12}$$

Hence, by dividing by $\|v\|_{x,*}$, we readily get $\|v\|_{x',*}/\|v\|_{x,*} \leq 1 + \|x - x'\|_x$ i.e., $\|\cdot\|_x$ is regular in the sense of Definition 1. As we discuss in the sequel, this metric plays an important role for problems with loss functions of the generalized KL form (10). ◄

With all this in hand, we will say that a vector field $V\colon \mathcal{X} \to \mathbb{R}^d$ is

1. *Metrically bounded* if there exists some $M > 0$ such that
$$\|V(x)\|_{x,*} \leq M \quad \text{for all } x \in \mathcal{X}. \tag{MB}$$
2. *Metrically smooth* if there exists some $L > 0$ such that
$$\|V(x') - V(x)\|_{x,*} \leq L\|x' - x\|_{x'} \quad \text{for all } x', x \in \mathcal{X}. \tag{MS}$$

The notion of metric boundedness/smoothness extends that of ordinary boundedness/Lipschitz continuity to a Finsler context. Our next example, based on the KL loss function (10), shows that this extension is *proper*, i.e., (BD)/(LC) may both fail while (MB)/(MS) both hold:

► **Example 5.3.** Let $V(x) = (1/x_1, \dots, 1/x_d)$ for $x \in \mathcal{X} = (0,1]^d$. Obviously, since the components of $V(x)$ diverge to infinity as $x_i \to 0$, both (BD) and (LC) fail to hold for *any* global norm on $\mathbb{R}^d$. On the other hand, under the local norm $\|z\|_x = \max_i |z|_i/x_i$ of Example 5.2, we have:

1. V is metrically bounded with $M = d$: $\|V(x)\|_{x,*} = \sum_{i=1}^{d} x_i \cdot (1/x_i) = d$.
2. V is metrically smooth with $L = d$: indeed, for all $x, x' \in \mathcal{X}$, we have
$$\|V(x') - V(x)\|_{x,*} = \textstyle\sum_{i=1}^{d} x_i \left|\frac{1}{x'_i} - \frac{1}{x_i}\right| = \sum_{i=1}^{d} \frac{|x'_i - x_i|}{x'_i} \leq d \max_i \frac{|x'_i - x_i|}{x'_i} = d\|x' - x\|_{x'}. \tag{13}$$

6 The UniProx Algorithm and its Guarantees

The method. We are now in a position to define a family of algorithms that is capable of interpolating between the optimal smooth/non-smooth convergence rates for solving (VI) without requiring either (BD) or (LC). To do so, the key steps in our approach will be to (i) equip $\mathcal{X}$ with a suitable Finsler structure (as in Section 5); and (ii) replace the Euclidean projection in (EG) with a suitable "Bregman proximal" step that is compatible with the chosen Finsler structure on $\mathcal{X}$.

We begin with the latter (assuming that $\mathcal{X}$ is equipped with an arbitrary Finsler structure):

Definition 2. We say that $h : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is a *Bregman-Finsler function* on $\mathcal{X}$ if:

1. h is convex, lower semi-continuous (l.s.c.), and $\operatorname{cl}(\operatorname{dom} h) = \operatorname{cl}(\mathcal{X})$.
2. The subdifferential of h admits a *continuous selection* $\nabla h(x) \in \partial h(x)$ for all $x \in \operatorname{dom} \partial h$.
3. h is *strongly convex*, i.e., there exists some $K > 0$ such that
$$h(x') \geq h(x) + \langle \nabla h(x), x' - x \rangle + \tfrac{K}{2}\|x' - x\|_x^2 \tag{14}$$
for all $x \in \operatorname{dom} \partial h$ and all $x' \in \operatorname{dom} h$.

The *Bregman divergence* induced by h is defined for all $x \in \operatorname{dom} \partial h$, $x' \in \operatorname{dom} h$ as
$$D(x', x) = h(x') - h(x) - \langle \nabla h(x), x' - x \rangle \tag{15}$$
and the associated *prox-mapping* is defined for all $x \in \operatorname{dom} \partial h$ and $y \in \mathbb{R}^d$ as
$$P_x(y) = \arg\min_{x' \in \mathcal{X}} \{\langle y, x - x' \rangle + D(x', x)\}. \tag{16}$$

Definition 2 is fairly technical, so some clarifications are in order. First, to connect this definition with the Euclidean setup of Section 4, the prox-mapping (16) should be seen as the Bregman equivalent of a Euclidean projection step, i.e., $\Pi(x + y) \leftrightsquigarrow P_x(y)$. Second, the main difference between Definition 2 and other definitions of Bregman proximal mappings in the literature [5, 7, 8, 10, 28, 41, 42, 54] is that h is assumed strongly convex relative to a *local* norm – not a global norm. This "locality" will play a crucial role in allowing the proposed methods to adapt to the geometry of the problem. For concreteness, we provide below an example that expands further on Examples 5.2 and 5.3:

► **Example 6.1.** Consider the local norm $\|z\|_x = \max_i |z_i|/x_i$ on $\mathcal{X} = (0, 1]^d$ and let $h(x) = \sum_{i=1}^d 1/x_i$ on $(0, 1]^d$. We then have
$$D(x', x) = \sum_{i=1}^d \left[\tfrac{1}{x_i'} - \tfrac{1}{x_i} + \tfrac{x_i' - x_i}{x_i^2}\right] = \sum_{i=1}^d \tfrac{(x_i' - x_i)^2}{x_i^2 x_i'} \geq \sum_{i=1}^d (1 - x_i'/x_i)^2 \geq \|x' - x\|_x^2 \tag{17}$$
i.e., h is 1-strongly convex relative to $\|\cdot\|_x$ on $\mathcal{X}$. ◄

With all this is in place, the extra-gradient method can be adapted to our present setting as follows:
$$\begin{aligned} X_{t+1/2} &= P_{X_t}(-\gamma_t V_t) & \delta_t &= \|V_{t+1/2} - V_t\|_{X_{t+1/2},*} \\ X_{t+1} &= P_{X_t}(-\gamma_t V_{t+1/2}) & \gamma_{t+1} &= 1\Big/\sqrt{1 + \textstyle\sum_{s=1}^t \delta_s^2} \end{aligned} \tag{UniProx}$$
with $V_t = V(X_t)$, $t = 1, 3/2, \dots$, as in Section 3. In words, this method builds on the template of (EG) by (i) replacing the Euclidean projection with a mirror step; (ii) replacing the global norm in (8) with a dual Finsler norm evaluated at the algorithm's leading state $X_{t+1/2}$. The first of these two steps is also the defining ingredient of the *mirror-prox* (MP) algorithm of Nemirovski [40]; the name "UniProx" has beeen chosen precisely because the proposed method can be seen as a mirror-prox method with universal rate interpolation and convergence guarantees. We expand on this below.

Trajectory convergence. We begin our analysis with a trajectory convergence result that governs the *actual* iterates of the method:

Theorem 2. *Suppose that V satisfies* (MB) *or* (MS). *If* $\operatorname{Gap}_{\mathcal{C}}(x^*) < 0$ *whenever $\mathcal{C}$ is relatively closed in $\mathcal{X}$ and disjoint from $\mathcal{X}^*$, then the iterates X_t of UniProx converge to a solution of* (VI).

The importance of this result is twofold: First, it ratifies our initial bet that X_t converges, so the difference sequence $\delta_t = \|V_{t+1/2} - V_t\|_{X_{t+1/2},*}^2$ must vanish when (MS) holds (we provide an explicit

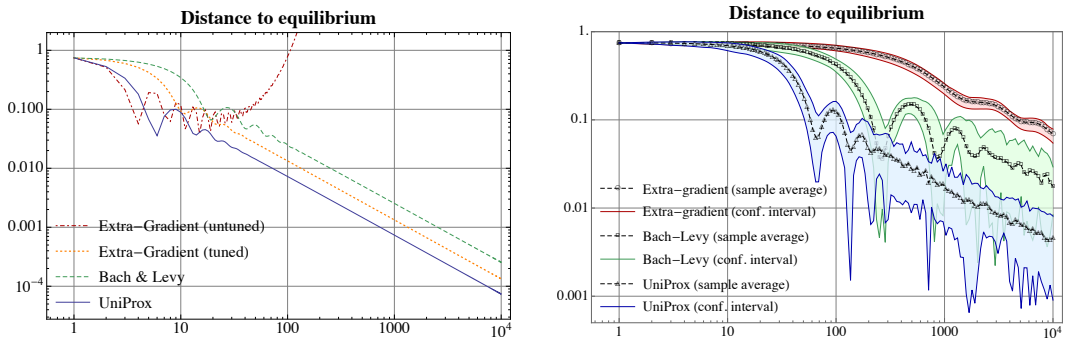

Figure 2: Numerical comparison between the extra-gradient algorithm, the BL algorithm [2] and UniProx. The experimental model is an unconstrained min-max game of the form $L(\theta, \phi) = (\theta - \theta^*)^\top A(\phi - \phi^*)$ where A is a 1000×1000 matrix. On the left, all algorithms were run with perfect gradient information on L. In particular, (EG) was run with constant γ, both tuned (with a grid search) and untuned; the BL algorithm was also tuned with a hyper-parameter search; by contrast, UniProx has no parameters, so it was run "off-the-shelf". On the right, the gradients of L were subject to zero-mean Gaussian errors with covariance matrix $A^\top A$ and the experiment was repeated $S = 1000$ times to obtain the average performance of each algorithm and a 95% confidence interval (shaded areas in the plot). In all cases, UniProx exhibits a consistent gain in performance.

statement of this below). Second, in many practical applications, it is more common to harvest the "last iterate" X_t of an algorithm rather than $\bar{X}_T$; Theorem 2 justifies this implementation choice.

The proof of Theorem 2 relies on non-standard arguments, so we provide a sketch below while relegating details to the supplement. The first step is to show that X_t visits any neighborhood of a solution point $x^* \in \mathcal{X}^*$ infinitely often (this is where the assumption on $\operatorname{Gap}_{\mathcal{C}}(x^*)$ is used; note however that this assumption is *not* used in the sequel). The second is to use this trapping property in conjunction with a suitable "energy inequality" to establish convergence via the use of a quasi-Fejér technique as in [13]; this part is detailed in a separate appendix.

Convergence rate analysis. We are finally in a position to state our main result for UniProx:

Theorem 3. *Let $\mathcal{C}$ be a compact neighborhood of a solution of* (VI) *and set $H = \sup_{x \in \mathcal{C}} D(x, X_1)$ Then, the UniProx algorithm enjoys the universal guarantees:*

$$\text{a) If } V \text{ satisfies (MB):} \quad \operatorname{Gap}_{\mathcal{C}}(\bar{X}_T) = \mathcal{O}\left(\frac{H + M^2(1 + 1/K)^2 + \log(1 + 4M^2(1 + 2/K)^2 T)}{\sqrt{T}}\right). \tag{18a}$$

$$\text{b) If } V \text{ satisfies (MS):} \quad \operatorname{Gap}_{\mathcal{C}}(\bar{X}_T) = \mathcal{O}(H/T). \tag{18b}$$

The proof of Theorem 3 is quite intricate, so we defer it to the paper's supplement. The key element throughout is the determination of the asymptotic behavior of the adaptive step-size policy γ_t in the non-smooth and smooth regimes, i.e., under (MB) and (MS) respectively. At a very high level, (MB) guarantees that the difference sequence δ_t is bounded, which implies in turn that $\sum_{t=1}^T \gamma_t = \Omega(\sqrt{T})$ and eventually yields the bound (18a) for the algorithm's ergodic average $\bar{X}_T$. On the other hand, if (MS) kicks in, we can show the much finer result that δ_t is actually square summable:

Lemma 1. *Assume V satisfies* (MS)*. Then, a) γ_t decreases monotonically to a strictly positive limit $\gamma_\infty = \lim_{t\to\infty} \gamma_t > 0$; and b) the sequence δ_t is square summable; in particular, $\sum_{t=1}^\infty \delta_t^2 = 1/\gamma_\infty^2 - 1$.*

By means of this lemma (which we prove in the paper's supplement), it follows that $\sum_{t=1}^T \gamma_t \geq \gamma_\infty T = \Omega(T)$. Because the algorithm's rate of convergence is controlled by this quantity, it ultimately follows that UniProx enjoys an accelerated $\mathcal{O}(1/T)$ rate of convergence under (MS). However, the details of the ensuing calculations are quite complicated, so we defer them to the supplement.

Conclusions. The appealing properties of UniProx are illustrated in a series of numerical experiments in Fig. 2. Together with our theoretical analysis, Fig. 2 confirms that UniProx can provide a fruitful template for adaptive min-max optimization methods that attain optimal rates in smooth/non-smooth problems, without requiring any tuning by the optimizer. This opens the door to several future research directions, from adaptation to other regimes (strong monotonicity, noise, etc.) to applications to non-convex/non-concave games and reinforcement learning problems.

7 Ethical Concerns

Our paper is primarily theoretical and, as such, we do not feel that it presents any ethical concerns to the community. Our main objective was to design a universal algorithm for min-max optimization problems and games that simultaneously achieves the following: *a*) optimal rates in smooth / non-smooth problems; *b*) convergence beyond the standard boundedness / Lipschitz continuity conditions; and *c*) "off-the-shelf" applicability without requiring prior tuning by the optimizer. The proposed method provides a flexible, freely available toolkit that can prove beneficial to researchers and practitioners in the field (from GAN training to variational inequalities), and we cannot see any possible ethical concerns arising from its use.

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