DEEP DECLARATIVE DYNAMIC TIME WARPING FOR END-TO-END LEARNING OF ALIGNMENT PATHS

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ABSTRACT

This paper addresses end-to-end learnable models for time series data that include a temporal alignment step via dynamic time warping (DTW). Existing approaches to differentiable DTW either differentiate through a fixed warping path or apply a continuous relaxation to the min operator found in the recursive steps used to solve the DTW problem. We instead propose a DTW layer based around deep declarative networks. By formulating the DTW problem as a continuous, inequality constrained optimisation problem, we can compute exact gradients for the solution of the optimal alignment (with respect to the underlying time series) using implicit differentiation. Our formulation yields a major improvement over existing approaches; our DTW layer outputs the entire warping path between two time series as opposed to only the DTW discrepancy value. This enables the specification of downstream loss functions on the alignment path itself, useful for instance, to improve the precision of predicted alignments when ground-truth alignments are available. We evaluate our method on two such applications, namely the audio-to-score alignment task in music information retrieval and the visual place recognition task in robotics, demonstrating state-of-the-art results in both.

1 INTRODUCTION

The dynamic time warping (DTW) algorithm computes a similarity measure between two temporal sequences, which is invariant to shifting and non-linear scaling in time. Because of this desirable invariance, DTW is ubiquitous in fields which analyze temporal sequences such as speech recognition, motion capture, time series classification and bioinformatics (Kovar & Gleicher, 2003; Zhu & Shasha, 2003; Sakoe & Chiba, 1978; Bagnall et al., 2017; Berndt & Clifford, 1994; Petitjean et al., 2014; Needleman & Wunsch, 1970). The original formulation of DTW computes the minimum cost matching between elements of the two sequences, called an alignment (or warping) path, subject to temporal constraints imposed on the matches. For two sequences of length m and n, this can be computed by first constructing an $m \times n$ pairwise cost matrix between sequence elements and subsequently solving a dynamic program (DP) using Bellman's recursion in $O(mn)$ time. Figure 2 illustrates the mechanics of the DTW algorithm.

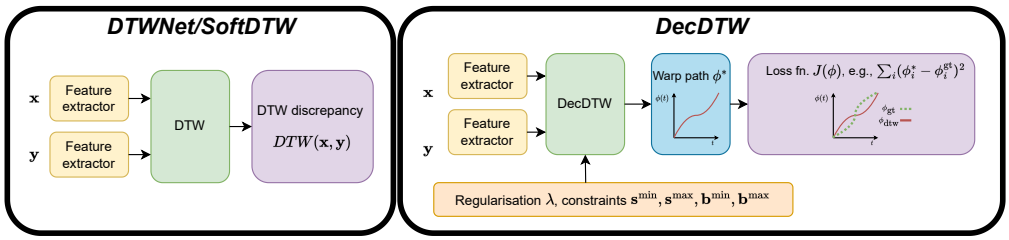


Figure 1: **Left:** DTWNet and SoftDTW only allow the DTW discrepancy between two time series to be differentiable w.r.t. its inputs. **Right:** DecDTW outputs the full warping path $\boldsymbol{\phi}$, which can be used in downstream losses on the warping path (e.g., MSE to a ground truth alignment). DecDTW also allows the regularisation and constraint values to be learnable parameters in an end-to-end system.

There has been interest in recent years around embedding DTW within deep learning models (Cuturi & Blondel, 2017; Cai et al., 2019; Lohit et al., 2019; Chang et al., 2019; 2021), with applications spread across a variety of learning tasks utilising audio and video data (Garreau et al., 2014; Dvornik et al., 2021; Haresh et al., 2021; Chang et al., 2019; 2021). However, the classic DTW algorithm solves a discrete optimisation problem; thus the DTW similarity is not differentiable w.r.t. the cost matrix. This is caused by the use of $\min$ functions found in the DP recursion, preventing gradients from being evaluated by unrolling the computations. Several approaches in the literature address this limitation in distinct ways: Cuturi & Blondel (2017) and Chang et al. (2019) use a continuous relaxation of $\min$ in the DP recursion, Cai et al. (2019) and Chang et al. (2021) observe that differentiation is possible after fixing the warping path, and a body of work predicts warping paths directly from sequence elements without the need for an optimisation step (Lohit et al., 2019; Shapira Weber et al., 2019; Grabocka & Schmidt-Thieme, 2018; Kazlauskaite et al., 2019; Abid & Zou, 2018).

We propose a novel approach to differentiable DTW based on deep implicit layers that moves away from discrete optimisation entirely. By adapting a continuous time formulation of the DTW problem proposed by Deriso & Boyd (2019) as an inequality constrained non-linear program (NLP), we can use the deep declarative networks (DDN) framework (Gould et al., 2021) to define the forward and backward passes in our DTW layer. The forward pass involves solving for the optimal (continuous) warping path; we achieve this using a custom dynamic programming approach similar to the original DTW algorithm. The backward pass uses the identities in Gould et al. (2021) to derive gradients implicitly through the solution computed in the forward pass.

We will show that our declarative approach, named DecDTW, has multiple benefits compared to existing approaches; the most important of which is that our DTW layer enables the alignment *path* output by DTW to be differentiable w.r.t. its inputs, rather than only the DTW similarity value. This enables for the first time the use of the alignment path itself in a downstream loss. This is particularly useful for many applications where ground truth alignments between time series pairs are available, allowing us to directly minimise the estimated alignment error using gradient descent. An overview of our proposed DTW layer and contributions are illustrated in Figure 1.

Our Contributions First, we propose a novel, inequality constrained NLP formulation of the DTW problem, building on the approach in Deriso & Boyd (2019). Second, we use this NLP formulation to specify a novel declarative DTW layer, where gradients in the backward pass are computed implicitly as in the DDN framework (Gould et al., 2021). Third, we show how discrepancies to ground truth alignments can be minimised using our DTW layer. Last, we use our method to attain state-of-the-art performance on challenging real-world alignment tasks.

2 Related Works

Differentiable DTW Layers Early approaches to learning with temporal alignment (Zhou & De la Torre, 2009; Su & Wu, 2019; Zhou & De la Torre, 2012) involve learning a linear transformation of features placed before a DTW alignment step. The objective of learning is to minimise the DTW discrepancy and learning proceeds by alternating between first estimating the temporal alignment using the classic DTW algorithm and then analytically solving for the optimal feature transform. This framework was extended to using non-linear feature extractors in Trigeorgis et al. (2016). DTW-Net (Cai et al., 2019) and DP-DTW (Chang et al., 2021) adapt the aforementioned approaches to the mini-batch gradient descent setting, where at each descent step the alignment path is estimated using DTW for each training observation and is then fixed. Along this fixed path, gradients of the DTW discrepancy w.r.t. the associated entries in the cost matrix can be computed.

Another class of differentiable DTW methods involves applying a continuous relaxation to the $\min$ operation in the DP recursion. Soft-DTW (Cuturi & Blondel, 2017) proposes replacing the $\min$ step with a *soft-min*, allowing gradients to be backpropagated through the recursion. D^3TW (Chang et al., 2019) combined Soft-DTW with a contrastive loss and OTAM (Cao et al., 2020) extends Soft-DTW to allow for subsequence alignment. Variations on soft-min have been proposed (Hadji et al., 2021) and there has been recent work on outlier rejection (Dvornik et al., 2021), aligning across different domains (Cohen et al., 2021) and spatio-temporal optimal transport (Janati et al., 2020). Our proposed method allows the full alignment path to be differentiable w.r.t. DTW layer inputs, in contrast to the aforementioned approaches which only differentiate through the DTW discrepancy. This property

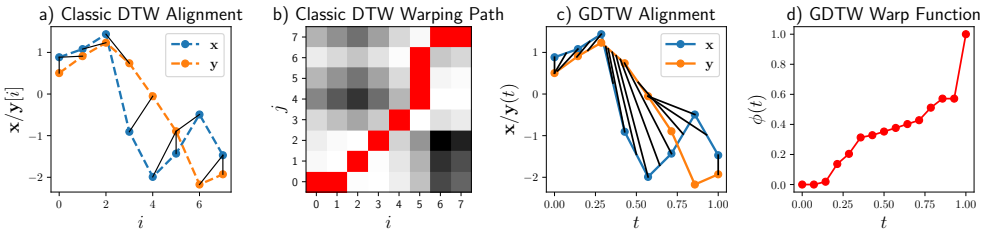


Figure 2: Classic DTW (a) is a discrete optimisation problem which finds the minimum cost warping path through a pairwise cost matrix (b). DecDTW uses a continuous variant of classic DTW (GDTW) (c) and finds an optimal time warp *function* between two continuous time signals (d).

allows us now to specify loss functions over the alignment path itself, e.g., mean squared error against a ground truth alignment. We will present applications where this can be used in Section 7.

Generalised DTW Deriso & Boyd (2019) proposed a generalised framework for dynamic time warping (GDTW) which transforms the classic discrete DTW optimisation problem over sequences to a variational optimisation problem over continuous time signals. This is achieved by first converting a time series from a discrete temporal sequence to a continuous time signal, e.g., by using some form of piecewise smooth interpolation. Second, the warping path is extended from a finite set of indices in the pairwise cost matrix to a continuous *warp function*. We will show that this can be reduced to a continuous optimisation problem, i.e., an inequality constrained NLP in Sections 3 and 4. We use this NLP formulation of DTW as the basis for our declarative layer.

Deep Declarative Networks The declarative framework that allows us to incorporate differentiable optimisation algorithms into a deep network is described in Gould et al. (2021). They present theoretical results and analyses on how to differentiate constrained optimization problems via implicit differentiation. Differentiable convex problems have also been studied recently, including quadratic programs (Amos & Kolter, 2017) and cone programs (Agrawal et al., 2019a;b). This technique has been applied to various application domains including optimisation-based control (Amos et al., 2018), video classification (Fernando & Gould, 2016; 2017), action recognition (Cherian et al., 2017), visual attribute ranking (Santa Cruz et al., 2019), few-shot learning for visual recognition (Lee et al., 2019), and camera pose estimation (Campbell et al., 2020; Chen et al., 2020). To the best of our knowledge, this is the first embedding of an inequality constrained NLP within a declarative layer.

3 Continuous Time Formulation for DTW

In this section, we introduce a continuous time version of the DTW problem which is heavily inspired by GDTW (Deriso & Boyd, 2019). This more general description of the DTW problem is used to derive the NLP formulation for our declarative layer. We will describe how the results in this section can be used to derive a corresponding NLP in Section 4.

3.1 Preliminaries

Signal A time-varying signal $\mathbf{x} : [0, 1] \to \mathbb{R}^d$ is a vector-valued function of time. Signals are assumed to be differentiable (at least piecewise smooth) and can be constructed from a time series comprised of observation times $\mathbf{t} \in [0, 1]^N$ where $0 = t_1 < t_2 < \cdots < t_N \leq 1$ and associated features $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \in \mathbb{R}^{N \times d}$ by continuously interpolating between observations (e.g., linear, cubic splines). Without loss of generality, we rescale time for all signals to $[0, 1]$.

Time warp function A time warp function $\phi : [0, 1] \to [0, 1]$ defines correspondences from times in one signal to another, and is similarly assumed to be piecewise smooth. Warp functions typically come with constraints; a common constraint requires that warp functions are nondecreasing, i.e., $\phi'(t) \geq 0$ for all t. We interpret $\mathbf{x} \circ \phi$ to be the *time warped* version of $\mathbf{x}$.

Dynamic time warping The dynamic time warping problem can be formulated as follows. Given two signals $\mathbf{x}$ and $\mathbf{y}$, find a warp function ϕ such that $\mathbf{y} \circ \phi \approx \mathbf{x}$ in some sense over the time series

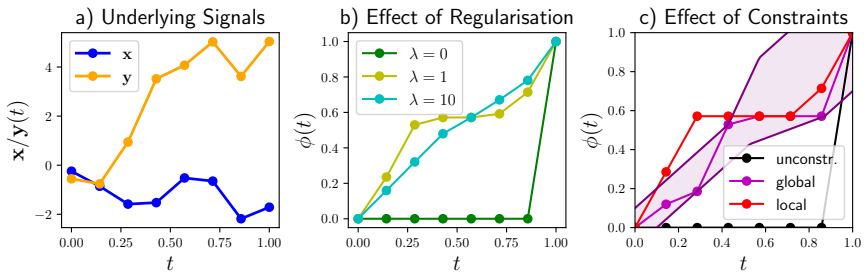


Figure 3: Regularisation (b) and constraints (c) can applied to prevent pathological warps, e.g., $\lambda = 0$ in (b). Global and local constraints bound warp function values and derivatives, respectively.

features. In other words, we wish to bring signals $\mathbf{x}$ and $\mathbf{y}$ together by warping time in $\mathbf{y}$. Figure 2 illustrates how GDTW differs from classic DTW for an example sequence pair.

3.2 Optimisation problem formulation

DTW can be specified as a constrained optimisation problem where the decision variable is the time warp function ϕ and input parameters are signals $\mathbf{x}$ and $\mathbf{y}$. Concretely, the problem is given by

$$\begin{aligned} \text{minimise} \quad & f(\mathbf{x}, \mathbf{y}, \lambda, \phi) \triangleq \mathcal{L}(\mathbf{x}, \mathbf{y}, \phi) + \lambda \mathcal{R}(\phi) \\ \text{subject to} \quad & \underbrace{s^{\min}(t) \leq \phi'(t) \leq s^{\max}(t)}_{\textbf{local constraints}}, \quad \underbrace{b(t)^{\min} \leq \phi(t) \leq b(t)^{\max}}_{\textbf{global constraints}} \quad \forall t. \end{aligned} \tag{1}$$

The objective function f can be decomposed into the *signal loss* $\mathcal{L}$ relating to feature error and a warp regularisation term $\mathcal{R}$, where $\lambda \geq 0$ is a hyperparameter controlling the strength of regularisation. Since ϕ, $\mathbf{x}$ and $\mathbf{y}$ are functions, our objective function f is a functional, revealing the variational nature of the optimisation problem. We will see in Section 4 that restricting ϕ to the space of piecewise linear functions will allow us to easily solve Equation 1 while still allowing for expressive warps.

Furthermore, a number of potentially time-varying constraints are usually imposed on ϕ. We partition these constraints into *local* and *global* constraints as in Morel et al. (2018). Local constraints bound the time derivatives of the warp function and can be used to enforce the typical nondecreasing warp assumption. Global constraints bound the warp function values directly, with prominent examples being the R-K (Ratanamahatana & Keogh, 2004) and commonly used Sakoe-Chiba (Sakoe & Chiba, 1978) bands. Furthermore, global constraints are used to enforce boundary constraints, i.e., $\phi(t) \in [0, 1]$ for all t and endpoint constraints, i.e., $\phi(0) = 0, \phi(1) = 1$. Not including endpoint constraints allows for *subsequence* GDTW, where a subsequence $\mathbf{y}$ is aligned to only a portion of $\mathbf{x}$. Figure 3 illustrates how constraints can be used to change the resultant warps returned by GDTW.

Signal Loss The signal loss functional $\mathcal{L}$ in Equation 1 is defined as

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \phi) = \int_0^1 L\big(\mathbf{y}(\phi(t)) - \mathbf{x}(t)\big) \mathrm{d}t, \tag{2}$$

where $L : \mathbb{R}^d \to \mathbb{R}$ is a twice-differentiable penalty function defined over features. In our experiments we use $L(\cdot) \triangleq \| \cdot \|_2^2$, however other penalty functions (e.g., the 1-norm, Huber loss) can be used. The signal loss (Equation 2) measures the separation of features between $\mathbf{x}$ and time-warped $\mathbf{y}$ and is analogous to the classic DTW discrepancy given by the cost along the optimal warping path.

Warp Regularisation The warp regularisation functional $\mathcal{R}$ in Equation 1 is defined as

$$\mathcal{R}(\phi) = \int_0^1 R\big(\phi'(t) - 1\big) \mathrm{d}t, \tag{3}$$

where $R : \mathbb{R} \to \mathbb{R}$ is a penalty function on deviations from the identity warp, i.e., $\phi'(t) = 1$. We use a quadratic penalty $R(u) = u^2$ in this work, consistent with Deriso & Boyd (2019). This regularisation term penalises warp functions with large jumps and sufficiently high λ brings ϕ close to the identity. Regularisation is crucial for preventing pathological warps (Zhang et al., 2017) from being produced from GDTW and greatly increases stability when training bi-level problems using declarative DTW.

4 SIMPLIFIED NLP FORMULATION FOR DTW

We follow the approach in Deriso & Boyd (2019) to reduce the infinite dimensional variational problem in Equation 1 to a finite dimensional NLP. This is achieved by first assuming that ϕ is piecewise linear, allowing it to be fully defined by its value at m knot points $0 = t_1 < t_2 < \cdots < t_m = 1$. Knot points can be uniformly spaced or just be the observation times used to parameterise $\mathbf{y}$. Formally, piecewise linearity allows us to represent ϕ as $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^n$ where $\phi_i = \phi(t_i)$. The other crucial assumption involved in reducing Equation 1 to an NLP is replacing the continuous integrals in the signal loss and warp regularisation in Section 3 with numerical approximations given by the trapezoidal rule. These assumptions yield an alternative optimisation problem given by

$$\begin{array}{ll} \text{minimise} & \hat{f}(\mathbf{x}, \mathbf{y}, \lambda, \boldsymbol{\phi}) \triangleq \hat{\mathcal{L}}(\mathbf{x}, \mathbf{y}, \boldsymbol{\phi}) + \lambda \hat{\mathcal{R}}(\boldsymbol{\phi}) \\ \text{subject to} & s_i^{\min} \leq \frac{\phi_{i+1} - \phi_i}{\Delta t_i} \leq s_i^{\max}, \quad b_i^{\min} \leq \phi_i \leq b_i^{\max} \quad \forall i, \end{array} \tag{4}$$

where $\Delta t_i = t_{i+1} - t_i$. The new signal loss $\hat{\mathcal{L}}$ is given by

$$\hat{\mathcal{L}}(\mathbf{x}, \mathbf{y}, \boldsymbol{\phi}) = \sum_{i=1}^{m-1} \frac{L_{i+1} + L_i}{2} \Delta t_i, \quad L_i \coloneqq L(\mathbf{x}(\phi_i) - \mathbf{y}(t_i)), \tag{5}$$

which follows from the definition of $\boldsymbol{\phi}$ and the trapezoidal approximation of the integral in Equation 2. Note, the non-convexity of objective $\hat{f}$ assuming convex L and R is caused by the $\mathbf{x}(\phi_i)$ terms for arbitrary signal $\mathbf{x}$. The new regularisation term $\hat{\mathcal{R}}$ is given by

$$\hat{\mathcal{R}}(\boldsymbol{\phi}) = \sum_{i=1}^{m-1} R\left(\frac{\phi_{i+1} - \phi_i}{\Delta t_i} - 1\right) \Delta t_i, \tag{6}$$

noting that since ϕ' is assumed to be piecewise constant, we can simply use Riemann integration to evaluate the continuous integral in $\mathcal{R}$ exactly. It is easy to see that the simplified problem in Equation 4 is a continuous non-linear program with decision variable $\boldsymbol{\phi} \in \mathbb{R}^n$ and linear constraints. However, the objective function is non-convex (Deriso & Boyd, 2019) and we will now describe the method used to find good solutions to Equation 4 using approximate dynamic programming.

5 DECLARATIVE DTW FORWARD PASS

Our DecDTW layer encodes an implicit function $\boldsymbol{\phi}^\star = \text{DecDTW}(\mathbf{x}, \mathbf{y}, \lambda, \mathbf{s}^{\min}, \mathbf{s}^{\max}, \mathbf{b}^{\min}, \mathbf{b}^{\max})$, which yields the optimal warp function given input signals $\mathbf{x}, \mathbf{y}$, regularisation λ and global/local constraints $\mathbf{s}^{\min} = \{s_i^{\min}\}_{i=1}^m$, $\mathbf{s}^{\max} = \{s_i^{\max}\}_{i=1}^m$, $\mathbf{b}^{\min} = \{b_i^{\min}\}_{i=1}^m$, $\mathbf{b}^{\max} = \{b_i^{\max}\}_{i=1}^m$. The forward pass yields the solution to Equation 4 given the input parameters. We solve Equation 4 using a dynamic programming (DP) approach as proposed in Deriso & Boyd (2019) instead of calling a general-purpose NLP solver; this is to minimise computation time. The DP approach uses the fact that $\boldsymbol{\phi}$ lies on a compact subset of $\mathbb{R}^m$ defined by the global bounds, allowing for efficient discretisation. Finding the globally optimal solution to the discretised version of Equation 4 can be done using DP. The resultant solution $\boldsymbol{\phi}^\star$ is an approximation to the solution of the continuous NLP, with the approximation accuracy dependent on the resolution of discretisation. We can reduce the approximation error given a fixed resolution M using multiple rounds of refinement. In each round, a tighter discretisation is generated around the previous solution and the new discrete problem solved. A detailed description of the solver can be found in Deriso & Boyd (2019) and the appendix.

A key distinction between our DecDTW and existing methods like DTWNet (Cai et al., 2019) and SoftDTW (Cuturi & Blondel, 2017) is that our DTW layer provides a differentiable *alignment path* $\boldsymbol{\phi}^\star$ instead of just the DTW discrepancy value. This allows us to specify a larger class downstream loss functions $J(\boldsymbol{\phi}^\star)$, e.g., the discrepancy between the predicted warp $\boldsymbol{\phi}^\star$ and a desired ground truth warp $\boldsymbol{\phi}^{\text{gt}}$. Note, we can trivially recover the DTW discrepancy by setting $J = \hat{f}$.

6 DECLARATIVE DTW BACKWARD PASS

We now derive the analytical solution of the gradients of $\boldsymbol{\phi}^\star$ recovered in the forward pass w.r.t. inputs $\mathbf{z} = \{\mathbf{x}, \mathbf{y}, \lambda, \mathbf{s}^{\min}, \mathbf{s}^{\max}, \mathbf{b}^{\min}, \mathbf{b}^{\max}\} \in \mathbb{R}^n$ using Proposition 4.6 in Gould et al. (2021) (note that

gradients w.r.t. signals $\mathbf{x}, \mathbf{y}$ are w.r.t. the underlying features $\mathbf{X}, \mathbf{Y}$). Unlike existing approaches for differentiable DTW, DecDTW allows the regularisation weight λ and constraints to be learnable parameters in a deep network. Let $\tilde{h} = [h_1, \ldots, h_p]$, where each $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ corresponds to an *active* constraint from the full set of inequality constraints described in Equation 4, rearranged in standard form, i.e., $h_i(\cdot) \leq 0$. For instance, an active local constraint relating to time t_j can be expressed as $h_i(\mathbf{z}, \boldsymbol{\phi}) = \phi_{j+1} - \phi_j - \Delta t_j s_j^{\text{max}} \leq 0$. Assuming non-singular H (note, this always holds for norm-squared L and R) and $\text{rank}(D_{\boldsymbol{\phi}} \tilde{h}(\mathbf{z}, \boldsymbol{\phi})) = p$, the backward pass gradient is given by

$$D\boldsymbol{\phi}(\mathbf{z}) = H^{-1}A^\top(AH^{-1}A^\top)^{-1}(AH^{-1}B - C) - H^{-1}B, \tag{7}$$

where $D\phi(\mathbf{z}) \in \mathbb{R}^{m \times n}$ is the Jacobian of estimated warp $\boldsymbol{\phi}$ w.r.t. inputs $\mathbf{z}$ and

$$\begin{aligned} A &= D_{\boldsymbol{\phi}} \tilde{h}(\mathbf{z}, \boldsymbol{\phi}) \in \mathbb{R}^{p \times m} \quad & B &= D^2_{\mathbf{z}\boldsymbol{\phi}} \hat{f}(\mathbf{z}, \boldsymbol{\phi}) \in \mathbb{R}^{m \times n} \\ C &= D_{\mathbf{z}} \tilde{h}(\mathbf{z}, \boldsymbol{\phi}) \in \mathbb{R}^{p \times n} \quad & H &= D^2_{\boldsymbol{\phi}\boldsymbol{\phi}} \hat{f}(\mathbf{z}, \boldsymbol{\phi}) \in \mathbb{R}^{m \times m}. \end{aligned} \tag{8}$$

Observe that the objective $\hat{f}$ as defined in Equation 4 only depends on a subset of $\mathbf{z}$ corresponding to $\mathbf{x}, \mathbf{y}, \lambda$ and similarly, constraints $\tilde{h}$ only depend on $\mathbf{s}^{\text{min}}, \mathbf{s}^{\text{max}}, \mathbf{b}^{\text{min}}, \mathbf{b}^{\text{max}}$. While Proposition 4.6 in Gould et al. (2021) has additional terms in B and H involving Lagrange multipliers, we note that since $\tilde{h}$ is at most first order w.r.t. $\{\mathbf{z}, \boldsymbol{\phi}\}$, these terms evaluate to be zero and can be ignored. We discuss how to evaluate Equation 7 using vector-Jacobian products efficiently in the appendix.

7 Experiments

Our experiments involve two distinct application areas and utilise real-world datasets. For both applications, the goal is to use learning to improve the accuracy of predicted temporal alignments using a training set of labelled ground truth alignments. We show that this is easily achieved using our declarative DTW layer since we can specify downstream loss functions on the predicted warping path itself. We have implemented our declarative DTW layer in PyTorch (Paszke et al., 2019) with open source code available to reproduce all experiments at https://anonymised.github.repo[1].

7.1 Learning Features for Audio-to-Score Alignment

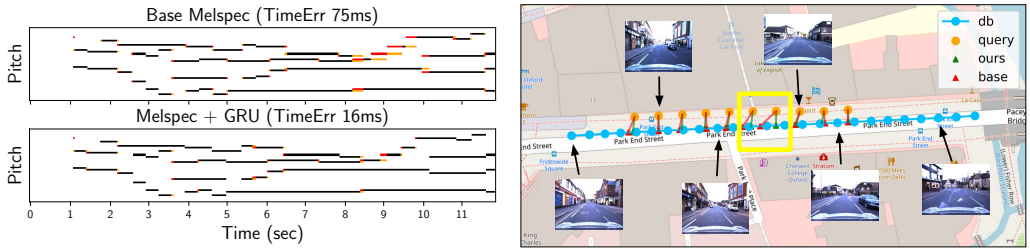
Problem Formulation Our first experiment relates to audio-to-score alignment, which is a fundamental problem in music information retrieval (Thickstun et al., 2020; Ewert et al., 2009; Orio et al., 2001; Shalev-Shwartz et al., 2004), with applications ranging from score following to music transcription (Thickstun et al., 2020). The goal of this task is to align an audio recording of a music performance to its corresponding musical score/sheet music. Thickstun et al. (2020) provide a formal mathematical framework for evaluating predicted audio-to-score alignments against a ground truth alignment, which we will now summarise. An alignment as a monotonic function $\phi : [0, S) \to [0, T)$ which maps positions in a score (measured in beats) to a corresponding position in the performance recording (measured in seconds). Two evaluation metrics between a predicted alignment ϕ and a ground truth alignment ϕ^{gt} are proposed, namely the temporal average error (TimeErr) and temporal standard deviation (TimeDev), given formally by

$$\text{TimeErr}(\phi, \phi^{\text{gt}}) = \frac{1}{S}\int_0^S |\phi(s) - \phi^{\text{gt}}(s)| \mathrm{d}s, \quad \text{TimeDev}(\phi, \phi^{\text{gt}}) = \sqrt{\frac{1}{S}\int_0^S (\phi(s) - \phi^{\text{gt}}(s))^2 \mathrm{d}s}. \tag{9}$$

Thickstun et al. (2020) assumes alignments are piecewise linear in between changepoints (where the set of notes being played changes) within the score, allowing Equation 9 to be analytically tractable.

Methodology Thickstun et al. (2020) provide a method for generating candidate alignments between a performance recording and a musical score as follows: First, synthesize the musical score (using a MIDI file) to an audio recording. Second, extract audio features from the raw audio for both the synthesised score and performance. Third, use DTW to align the two sets of features. Last, use Equation 9 to evaluate the predicted alignment. In these experiments, we learn a feature extractor on top of the base audio features with the goal of improving the alignment accuracy from DTW.

[1] reviewers see attached zip



(a) (b)

Figure 4: **(a)** Example audio-to-score alignment before and after training. Red (resp. yellow) identifies notes indicated by audio feature (resp. ground truth) alignments, but not by the ground truth (resp. audio features). DecDTW training yields dramatic improvements for melspec features, even surpassing the highly optimised CQT features (30ms TimeErr). **(b)** Example VPR alignment. Query images are aligned to a database sequence using learned image embeddings. Red (green) illustrates where the pre-trained (resp. fine-tuned) system estimates the query poses using database geotags. The maximum error is reduced from 5.2m to 1.2m after fine-tuning (yellow region).

Experimental Setup We use the dataset in Thickstun et al. (2020), which is comprised of 193 piano performances and scores with ground truth alignments taken from a subset of the MAESTRO (Hawthorne et al., 2019) and Kernscores (Sapp, 2005) datasets. We use 50% of performances for training and 25% each for validation and testing. We extract three types of base frequency-domain features from the raw audio; constant-Q transform (CQT), (log-)chromagram (chroma) and (log-)mel spectrogram (melspec). The CQT and chroma features are highly optimised for pitch, making them already performant for this task (Thickstun et al., 2020). In contrast, melspec features are less optimised for pitch and perform worse as a base feature, despite retaining more information from the raw audio compared to the CQT and chroma features. We uniformly slice all audio recordings to length 256 (approx. 11s) under our hyperparameter settings. Separate feature extractors are trained for the synthesised score and performance recordings. The network architecture for both is a single layer bi-directional Gated Recurrent Unit (GRU) (Cho et al., 2014) with total output dimensionality of 128. The full set of hyperparameters for this experiment are provided in the appendix. We evaluated features produced by five different methods using this dataset: *1)* base features (no learning) *2)* DTWNet (Cai et al., 2019) training *3)* SoftDTW (Cuturi & Blondel, 2017) training *4)* loss based on matching cost along ground truth alignment path (Along GT) *5)* TimeErr loss training (DecDTW only). Regularisation λ is selected for GDTW using the validation set with the base features.

Table 1: Summary of results for the audio-to-score alignment experiments

Reported metrics are TimeErr / TimeDev (ms)								
Method		CQT	Chroma	Melspec		CQT	Chroma	Melspec
Base		**42 / 98**	**60 / 131**	170 / 300		25 / 39	30 / 48	49 / 71
DTWNet	Classic DTW	56 / 138	61 / 147	54 / 135		29 / 43	30 / 46	29 / 42
SoftDTW	Classic DTW	56 / 138	61 / 147	54 / 135	GDTW	29 / 43	30 / 46	29 / 42
Along GT	Classic DTW	56 / 138	61 / 150	**50 / 132**		29 / 43	30 / 46	26 / 38
DecDTW		61 / 144	83 / 196	78 / 178		**23 / 36**	**24 / 38**	**20 / 34**

Results A summary of results is presented in Table 1 (colors indicate method used to produce alignments during *training*), including results where both classic DTW and GDTW are used for generating alignments[2]. Training using the DTW discrepancy only (i.e., DTWNet and SoftDTW) yields degraded alignment performance because the ground truth alignment is not used during learning. Similarly, minimising the feature matching cost along the ground truth alignment path also fails to improve alignment accuracy, since the predicted alignment is not used during learning.

[2]Note that the randomly initialised GRU feature extractor degrades alignment accuracy compared to base features; this explains why on occasion DTWNet and SoftDTW can perform worse than base features.

In contrast, training using the $\mathrm{TimeErr}$ loss with DecDTW improves the alignment accuracy since both the predicted and ground truth alignments are used for learning. DecDTW is able to slightly improve the alignment accuracy of CQT and chroma features and dramatically improves the accuracy of melspec feature alignments. In fact, DecDTW is able to utilise the additional information in the melspec features to surpass the performance of CQT and chroma. Finally, note that training with $\mathrm{TimeErr}$ specialises our feature extractor to improve alignment accuracy for GDTW with a given λ; performance degrades when using classic DTW to generate alignments. Figure 4a illustrates an example test alignment, with more examples provided in the appendix.

7.2 Transfer Learning for Sequence-based Visual Place Recognition

Our second experiment relates to the Visual Place Recognition (VPR) problem, which is an active area of research in computer vision (Berton et al., 2022; Arandjelovic et al., 2016) and robotics (Garg et al., 2021; Lowry et al., 2015; Cummins & Newman, 2011) and is often an important component of the navigation systems found in mobile robots and autonomous cars. The VPR task involves recognising the approximate geographic location where an image is captured, typically to a tolerance of a few meters (Zaffar et al., 2021; Berton et al., 2022) and is commonly formulated as image retrieval, where a query image is compared to a database of geo-tagged reference images. VPR using image sequences has been shown to improve retrieval performance over single image methods (Garg & Milford, 2021; Xu et al., 2020; Stenborg et al., 2020). In this experiment we first formulate sequence-based VPR as a temporal alignment problem and then fine-tune a deep feature extractor for this alignment task.

Problem formulation Let $\{I_i^r\}_{i=1}^N$, $\mathbf{t}^r \in \mathbb{R}^N$ and $\{\mathbf{x}_i^r\}_{i=1}^N$ be a reference image sequence, associated (sorted) timestamps and geotags, respectively. Similarly, let $\{I_i^q\}_{i=1}^M$, $\mathbf{t}^q \in \mathbb{R}^M$ and $\mathbf{X}^q = \{\mathbf{x}_i^q\}_{i=1}^N$ be the equivalent for a query sequence, noting that query geotags are used only for training and evaluation. Furthermore, when using deep networks for VPR, we have a feature extractor f_θ with weights θ that is used to extract embeddings $\mathbf{Z}^r = \{f_\theta(I_i^r)\}_{i=1}^N \in \mathbb{R}^{N\times d}$ and $\mathbf{Z}^q = \{f_\theta(I_i^q)\}_{i=1}^M \in \mathbb{R}^{M\times d}$ from reference and query images, respectively. Finally, let $\mathbf{x}^r$, $\mathbf{x}^q$, $\mathbf{z}^r$, $\mathbf{z}^q$ be continuous time signals built from geotags and image embeddings using associated timestamps.

We can formulate sequence-based VPR as a temporal alignment problem, where the objective is to minimise the discrepancy between the predicted alignment derived from the query and reference image embeddings and the optimal alignment derived from the geotags. Concretely, the goal of our learning problem is to minimise w.r.t. network weights θ, a measure of localisation error $G(\mathbf{x}^r(\boldsymbol{\phi}), \mathbf{X}^q)$, where $\boldsymbol{\phi} = \mathrm{DTW}(\mathbf{z}^r, \mathbf{z}^q; \theta)$ defined as $\phi_i = \phi(t_i^q)$ is the estimated alignment which maps query images to positions in the reference sequence and $\mathbf{x}^r(\boldsymbol{\phi}) = \{\mathbf{x}^r(\phi_i)\}_{i=1}^M$. We select G as the (squared) maximum error over the queries, given by $G(\mathbf{X}^1, \mathbf{X}^2) = \max\{\|\mathbf{x}_i^1 - \mathbf{x}_i^2\|_2^2 : i = 1, \ldots, M\}$.

Experimental setup We source images and geotags from the Oxford RobotCar dataset (Maddern et al., 2017), commonly used as a benchmark to evaluate VPR methods. This dataset is comprised of autonomous vehicle traverses of the same route captured over two years across varying times of day and environmental conditions. We use a subset of ∼1M images captured across 50+ traverses provided by Thoma et al. (2020) with geotags taken from RTK GPS (Maddern et al., 2020) where available, accurate to 15cm. These images are split into training, validation and test sets, which are used to train a base network for single image retrieval as well as our sequence-based methods. The train and validation sets are captured in the same geographic areas on distinct dates whereas the test set is captured in an area geographically disjoint from validation and training.

Paired sequences are generated using the GPS ground truth; reference sequences have 25 images spaced ∼5m apart and query sequences have 10 images sampled at ∼1Hz. This setup is close to a deployment scenario where geotags are available for the reference image and query images exhibit large changes in velocity compared to the reference. Subsequence GDTW is used to align queries to the reference sequence. In total, we use ∼22k, ∼4k, ∼1.7k sequence pairs for training, validation and testing, respectively. Our test set contains a diverse set of sequences over a large geographic area, and exhibits lighting, seasonal and structural change between reference and query. We provide our paired sequence dataset in the supplementary material. Finally, for feature extraction, we use a VGG backbone with NetVLAD aggregation layer (Arandjelovic et al., 2016) (yields 32k-dim embeddings) and train the single image network using triplet margin loss. We fine-tune the conv5 and NetVLAD layers for sequence training. See the appendix for a detailed description of the training setup.

Table 2: VPR experiment results (maximum error): Test Accuracy @2/3/5/10m

Method	Overcast	Sunny	Snow
Independent (Base)	0.0/11.6/47.9/77.0	0.0/7.5/40.4/81.1	1.6/10.0/50.1/87.1
DTW (Base)	10.2/31.7/65.6/**90.3**	9.0/33.2/75.1/**99.3**	6.2/25.7/73.2/**100.0**
Fine-tune Along GT	10.2/31.7/65.6/**90.3**	9.0/33.2/75.1/**99.3**	6.2/25.7/73.2/**100.0**
Fine-tune OTAM	10.2/31.7/65.6/**90.3**	9.0/33.2/75.1/**99.3**	6.2/25.7/73.2/**100.0**
Fine-tune DecDTW	**21.5/41.6/67.6**/89.1	**24.0/50.6/81.6**/98.3	**14.4/42.6/86.0**/99.1
Independent (DecDTW)	0.0/9.9/36.6/63.7	0.2/9.9/35.8/56.7	0.4/6.0/39.7/77.2

Results We evaluate all methods by measuring the proportion of test sequences where the maximum error is below predefined distance thresholds; these metrics are commonly used for single image methods (Garg et al., 2021; Berton et al., 2022). Results are split into three different environmental conditions including overcast, sunny and snow. We present results for two methods using the base single image network (1-2) and three methods which perform fine-tuning of the base network for sequence VPR (3-5): *1)* independent matching (no DTW); each query image inherits the geotag of the most similar reference image *2)* DTW sequence alignment (no fine-tuning) *3)* Feature cost along the ATE minimising alignment (Along GT) *4)* OTAM (SoftDTW w/subsequence alignment) loss and *5)* max error loss (DecDTW). We additionally evaluate the weights after our sequence fine-tuning in 5) on the independent retrieval method in 1). Table 2 summarises the results.

We found that fine-tuning with the OTAM and Along GT losses does not improve performance over the base single image network. This is because the OTAM loss does not use GPS ground-truth and the Along GT loss does not use the predicted alignment. However, training with DecDTW to directly reduce the maximum GPS error between predicted and ground truth yields significant improvements across tighter (< 5m) error tolerances. Interestingly, fine-tuning using DecDTW dramatically reduces independent retrieval performance. Figure 4b illustrates an example of a test sequence before and after fine-tuning, with more examples provided in the appendix.

7.3 Scalability

Figure 5 illustrates how DecDTW scales with time series length n compared to SoftDTW (Cuturi & Blondel, 2017) and DTWNet (Cai et al., 2019). Timings are evaluated on an Nvidia GeForce GTX 1080Ti 11Gb. DecDTW is between 4-20 (resp. 10-50) times slower than SoftDTW (resp. DTWNet). The additional computation is largely attributed to the forward pass; SoftDTW and DTWNet solves a DP with $O(mn)$ nodes and $O(mn)$ edges, whereas DecDTW solves a DP with $O(mM)$ edges and $O(mM^2)$ edges (we set $M = n$). This added computation is necessary for the fine-grained warp estimation required for GDTW. We provide a detailed discussion of the DecDTW solver in the appendix. Future work will look at developing methods to set solver resolution M adaptively.

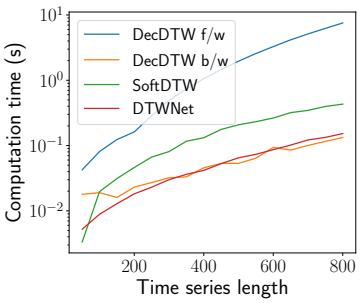


Figure 5: DecDTW scalability

8 Conclusion and Future Works

We present a novel formulation for a differentiable DTW layer that can be embedded into a deep network, based on continuous time warping and implicit deep layers. Unlike existing approaches, DecDTW is able to yield the entire alignment path as a differentiable function of layer inputs, allowing it to be used in a downstream loss function. We demonstrate how we can use this added capability to yield considerable performance gains in challenging, real-world alignment tasks. Future work will focus on improving the scalability of our method. In addition, we presented methodology for allowing regularisation and constraints to be learnable parameters in a deep network but did not explore this in our experiments. Future work will also explore this capability in more detail.

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A APPENDIX

We provide additional implementation details for DecDTW, as well as additional qualitative examples for both experiments in this section.

A.1 DYNAMIC PROGRAMMING SOLVER FOR GDTW

The mechanics of the solver are described as follows. For each i we can discretise ϕ_i into M values $\{\phi_{i,j}\}_{j=1}^{M}$ uniformly between global bounds $b_i^{\text{min}}, b_i^{\text{max}}$. This forms a graph where each of the mM values correspond to nodes and furthermore, temporally adjacent nodes, i.e., $\phi_{i-1,j}, \phi_{i,k}$ for all j, k are connected with edges, for a total of $(m-1) \times M^2$ edges. Node costs correspond to signal loss values in Equation 5 and edge costs correspond to warp regularisation values in Equation 6. Edges which violate the local constraints $s_i^{\text{min}}, s_i^{\text{max}}$ are given a cost of ∞. The global minima of this new discrete optimisation problem corresponds to the minimum cost path through the graph and is solved using dynamic programming in $O(mM^2)$ time complexity. Iterative refinement of the discretisation and solution is performed as described in Deriso & Boyd (2019). We use three rounds of discretisation with discretisation factor $\eta = 0.125$ in all of our experiments. Figure 6 illustrates the mechanics of the DP solver for a subsequence alignment problem and shows that the solution after refinement is comparable to calling an SLSQP solver (Virtanen et al., 2020) directly.

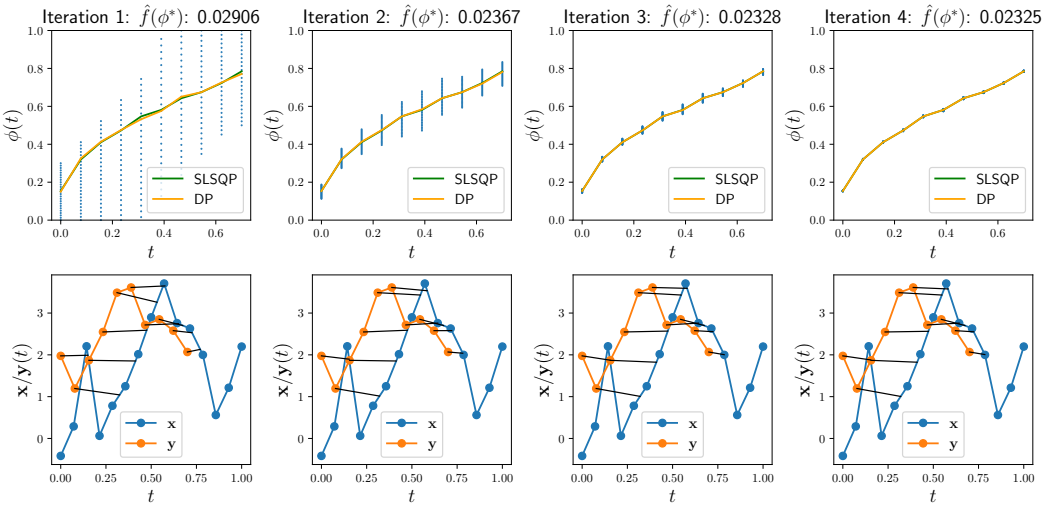


Figure 6: Illustration of the DP solver mechanics on a subsequence alignment problem, where ϕ maps times in the shorter sequence $\mathbf{y}$ to longer sequence $\mathbf{x}$. A uniform discretisation is generated between global constraints in the first iteration. Subsequent discretisations are made between tighter bounds around the previous solution. The objective function value decreases at each iteration until convergence after 3 iterations. The objective value $\hat{f}$ from the SLSQP solver (Virtanen et al., 2020) is 0.02325, close to the DP solver at the final iteration and furthermore, we can see qualitatively that the solved warps are indistinguishable between the DP and SLSQP approach.

A.2 Efficient Computation of Backward Pass

We do not construct the full Jacobian $D\phi(\mathbf{z})$ from Equation 7 explicitly, instead directly computing the vector-Jacobian products required for automatic differentiation libraries. Let $J(\boldsymbol{\phi})$ be a downstream loss function defined over the estimated alignment (e.g., MSE to a ground truth alignment $\boldsymbol{\phi}^{\text{gt}}$). Our goal for the end-to-end model is to compute $DJ(\mathbf{z}) = v^\top D\boldsymbol{\phi}(\mathbf{z})$, where $v^\top = J(\boldsymbol{\phi}) \in \mathbb{R}^{1\times m}$. To do this, we evaluate $v^\top D\boldsymbol{\phi}(\mathbf{z})$ from left to right, pre-computing and caching $H^{-1}A^\top$ beforehand. In addition, note that H is tridiagonal from the definition of $\hat{f}$, where off-diagonal elements correspond only to the warp regularisation component $\hat{\mathcal{R}}$. This allows us to solve $v^\top H^{-1}$ in $O(m)$ time. Finally, observe that B and C are in a sense complementary to each other. Specifically, columns of B relating to $\mathbf{s}^{\text{min}}, \mathbf{s}^{\text{max}}, \mathbf{b}^{\text{min}}, \mathbf{b}^{\text{max}}$ are zero since the objective function $\hat{f}$ only depends on $\mathbf{x}, \mathbf{y}, \lambda$. Conversely, columns of C relating to $\mathbf{x}, \mathbf{y}, \lambda$ are zero since the constraint functions only depend on $\boldsymbol{\phi}$ and $\mathbf{s}^{\text{min}}, \mathbf{s}^{\text{max}}, \mathbf{b}^{\text{min}}, \mathbf{b}^{\text{max}}$. This allows efficient evaluation of Equation 7 w.r.t the two blocks $\{\mathbf{x}, \mathbf{y}, \lambda\}$ and $\{\mathbf{s}^{\text{min}}, \mathbf{s}^{\text{max}}, \mathbf{b}^{\text{min}}, \mathbf{b}^{\text{max}}\}$ separately by setting C (resp. B) to zero.

A.3 Experimental Setup for Audio-to-Score Alignment

Ground truth alignments were generated for the provided 193 piano performances in Thickstun et al. (2020) using the methodology proposed in Thickstun et al. (2020) using the authors' default hyperparameters. These 193 performances are divided into train (93), validation (48) and test (49) splits and sliced uniformly, yielding 1010, 580 and 508 slices, respectively. Audio is sampled at 22.1kHz with a hop size of 1024, yielding ∼11s of audio per slice. Otherwise, the hyperparameters used to generate constant-Q transform (48-dim) and chromagram (12-dim) features are identical to the ones used in Thickstun et al. (2020). For the mel spectrogram features, we used 128 mel bands, yielding a 128-dim feature. The full dataset generation procedure is provided in our code.

We use the Adam (Kingma & Ba, 2015) optimiser with a learning rate of 0.0005 and a batch size of 5 for all methods. We trained for 20 epochs and selected the model which yielded minimum TimeErr over the validation set for all methods. For declarative DTW training and GDTW eval we used $\lambda = 0.08, 0.6, 1.7$ for CQT, chroma and melspec features, repsectively. These regularisation weight values were selected using a grid search over the average TimeErr across the validation set

for the base features. For the SoftDTW (Cuturi & Blondel, 2017) comparison method we used $\gamma = 1$, noting that results do not meaningfully change for different γ.

A.4 Experimental Setup for Visual Place Recognition

Single Image VPR We train single image NetVLAD representations with VGG-16 backbone pretrained on ImageNet. NetVLAD layer uses 64 clusters, leading to a 32,768 dimensional time series embedding. Images are resized to 240×180 (width $\times$ height), consistent with Thoma et al. (2020). Triplet loss margin is set to 0.1 and 10 negatives per anchor-positive pair are used. The model is trained using the Adam optimizer with a learning rate of 0.00001 and a batch size of 4. Our training setup is based on the recently released visual geolocalization benchmark (Berton et al., 2022) and the codebase[3]. We provide training and dataset setup code in the supplementary materials.

Sequence-based VPR For DTW sequence fine-tuning, we use a learning rate of 0.0001 and a batch size of 8 for a maximum of 30 epochs. We checkpoint models during training based on the average maximum GPS error over the full validation set. We selected a fixed regularisation weight of $\lambda = 0.1$ for DecDTW using the validation set and the pre-trained single-image model.

A.5 Additional Qualitative Examples for Audio-to-Score Alignment Experiments

In Figure 7 we present additional visualisations of example alignments for the audio to score alignment task. All features types (CQT, chroma, melspec) and before/after learning are presented in the figures. Example alignments are randomly sampled from the test set. The visualisations are a way proposed by Thickstun et al. (2020) to visually compare two alignments (e.g., an estimated and ground truth alignment) by plotting *performance aligned scores*, where a performance aligned score is generated by mapping the score through a warping function. We generate a performance aligned score using both the ground truth alignment and the estimated alignments using audio features. Red identifies notes indicated by audio feature alignments, but not by the ground-truth, and yellow identifies notes indicated by the ground-truth, but not by the audio features. Note, even if a predicted alignment is close to the ground truth (in a $\mathrm{TimeErr}$ sense), mistakes made by the performer may yield significant looking errors (orange and red) in the visualisations.

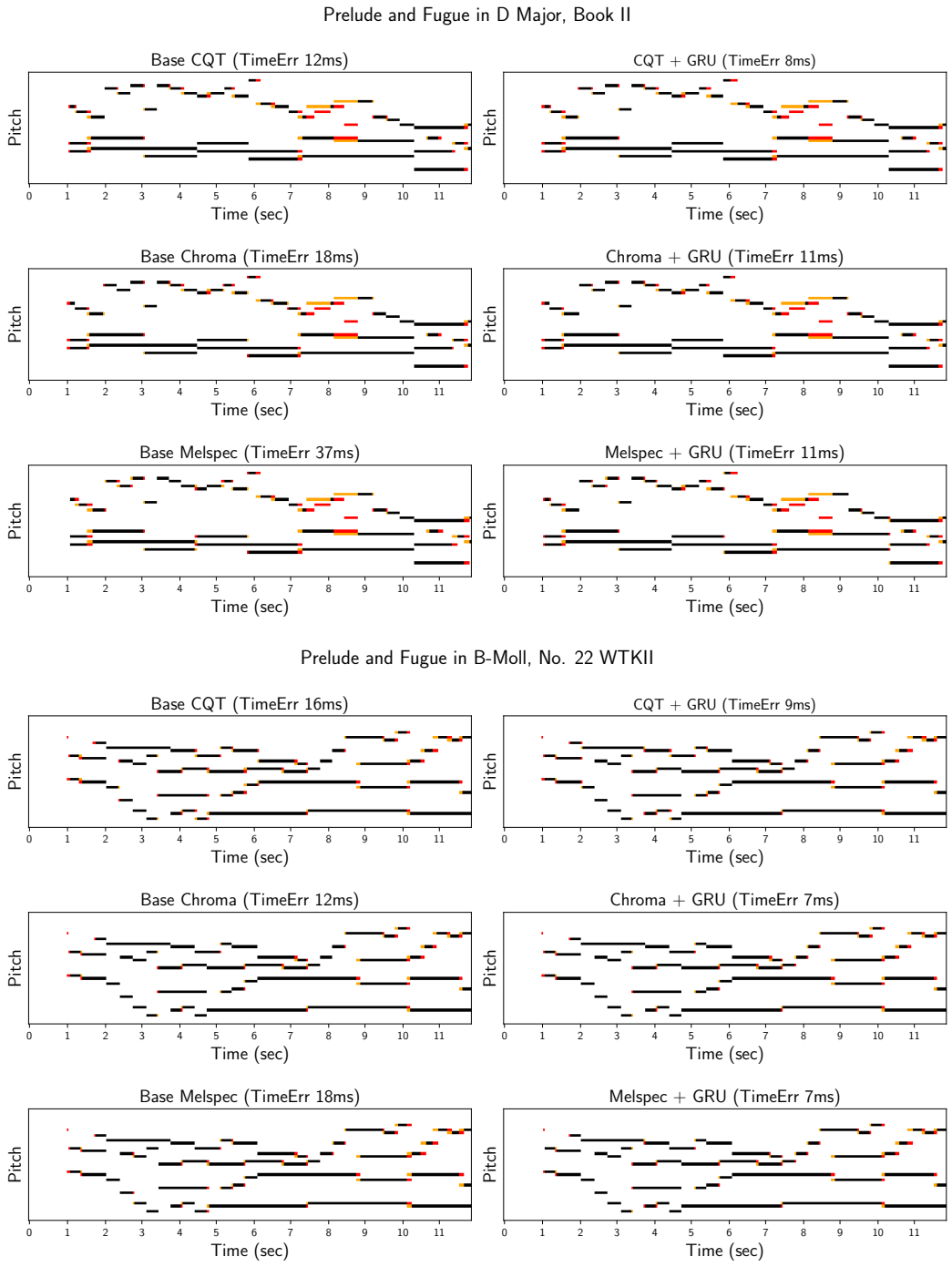
A.6 Additional Results for Visual Place Recognition Experiments

We additionally present results for the *mean* pose error across the sequence (as opposed to the maximum in Table 2) in Table 3. This measure of positioning error is commonly referred to as the absolute trajectory error (ATE). The conclusions are equivalent to the maximum error case; fine-tuning on the GPS error directly using our declarative DTW layer yields considerable performance improvements at sub 5m tolerances over the comparison methods.

Table 3: Additional VPR experiment results (mean error): Test Accuracy @2/3/5/10m

Method	Overcast	Sunny	Snow
Independent (Base)	18.6/55.7/80.1/91.5	16.9/50.8/85.4/96.8	17.0/58.5/90.9/98.4
DTW (Base)	44.8/70.2/90.1/**98.1**	47.0/85.2/**99.5/99.8**	43.5/79.2/**100.0/100.0**
Fine-tune Along GT	44.8/70.2/90.1/**98.1**	47.0/85.2/**99.5/99.8**	43.5/79.2/**100.0/100.0**
Fine-tune OTAM	44.8/70.2/90.1/**98.1**	47.0/85.2/**99.5/99.8**	43.5/79.2/**100.0/100.0**
Fine-tune Ours	**54.0/73.4/91.0**/96.1	**63.4/88.4**/98.5/**99.8**	**53.7/88.9**/98.5/**100.0**
Independent (DecDTW)	15.5/43.6/66.8/80.9	17.4/43.8/59.3/72.6	10.2/49.7/78.9/91.6

[3]https://github.com/gmberton/deep-visual-geo-localization-benchmark



A.7 Additional Qualitative Examples for Visual Place Recognition Experiments

We provide additional qualitative examples for the VPR experiments in Figure 8. These figures illustrate the effect of DecDTW fine-tuning on resultant positioning accuracy for a randomly selected set of test sequence pairs. In addition, Figure 8 provides example test images to illustrate the amount of appearance change observed between reference and query images.

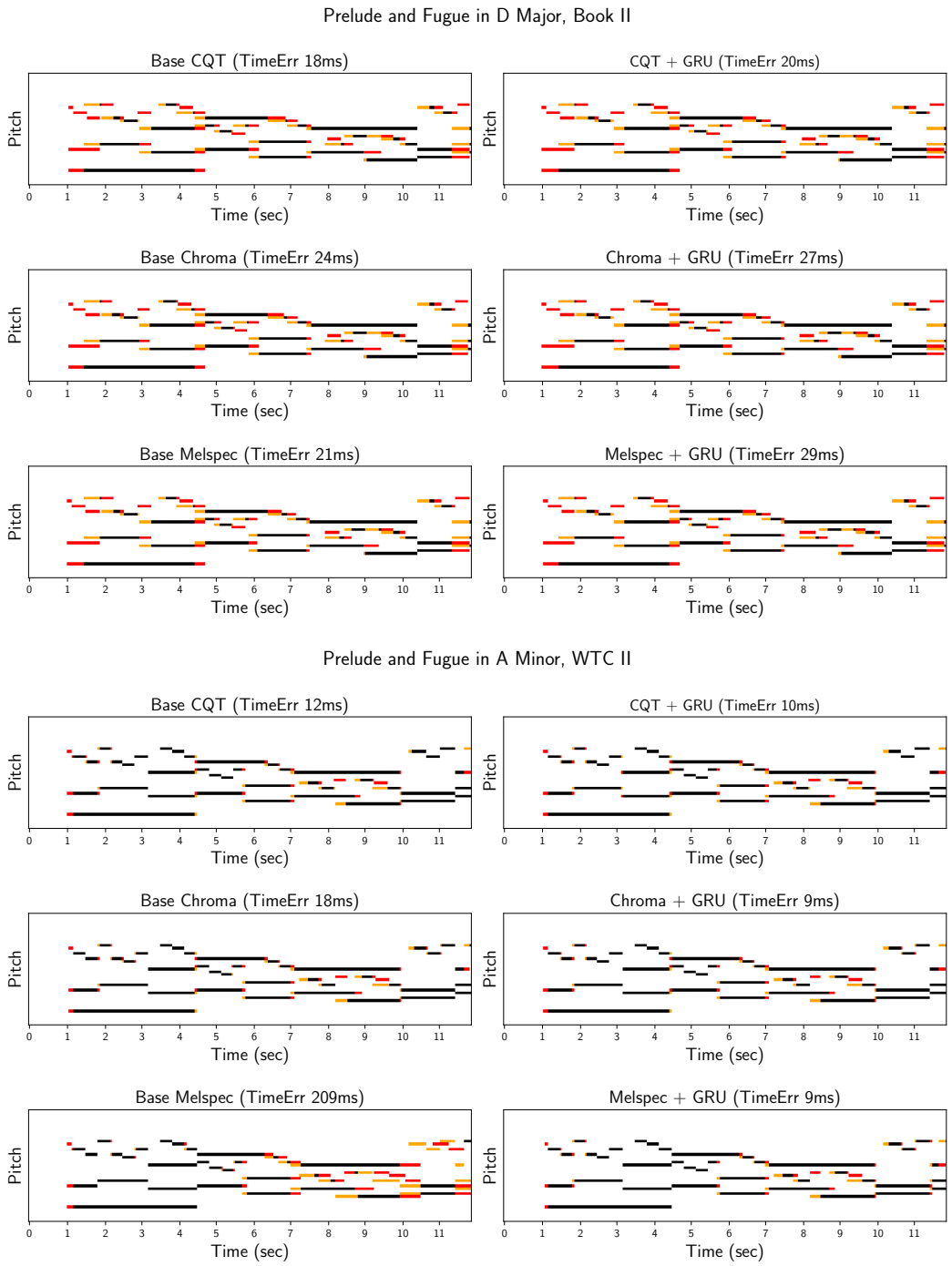
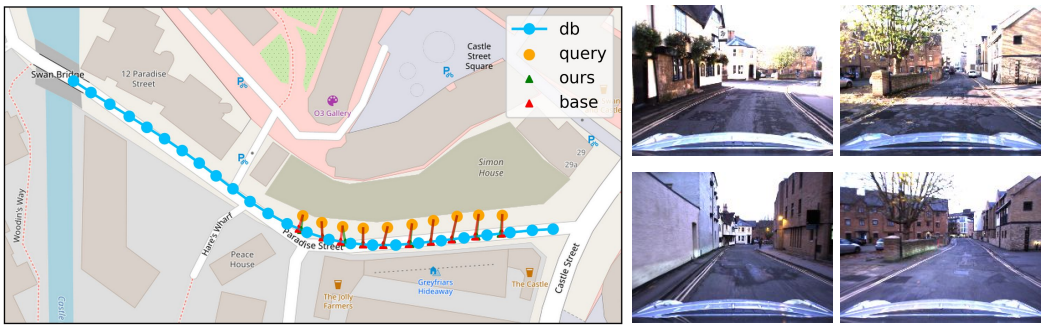
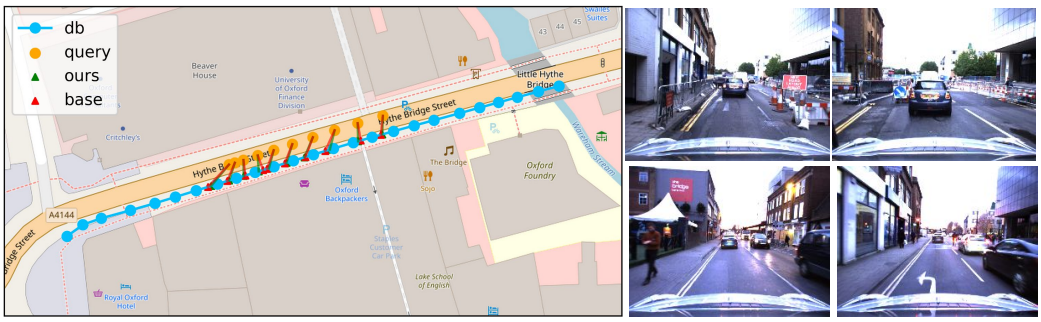


Figure 7: Example test set alignments for the audio-to-score alignment experiments. **Left:** Alignments produced using base features. **Right:** Alignments produced using DecDTW learned features.

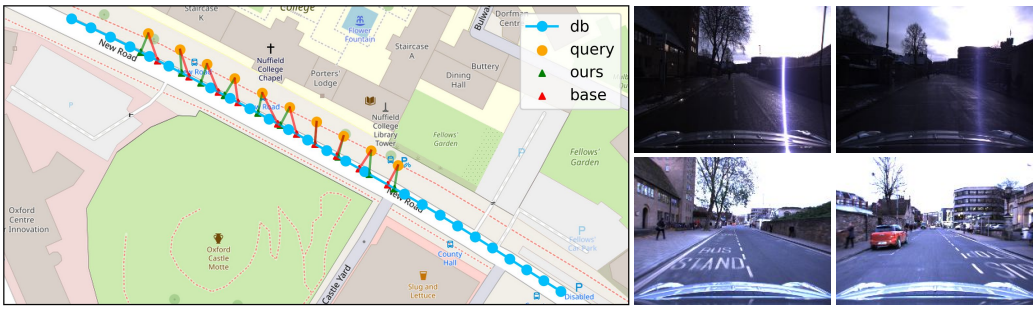
Sunny Query Max Error: Ours 1.93m, Baseline 2.01m



Overcast Query Max Error: Ours 4.71m, Baseline 5.56m



Snow Query Max Error: Ours 2.52m, Baseline 5.51m



Sunny Query Max Error: Ours 5.04m, Baseline 4.18m

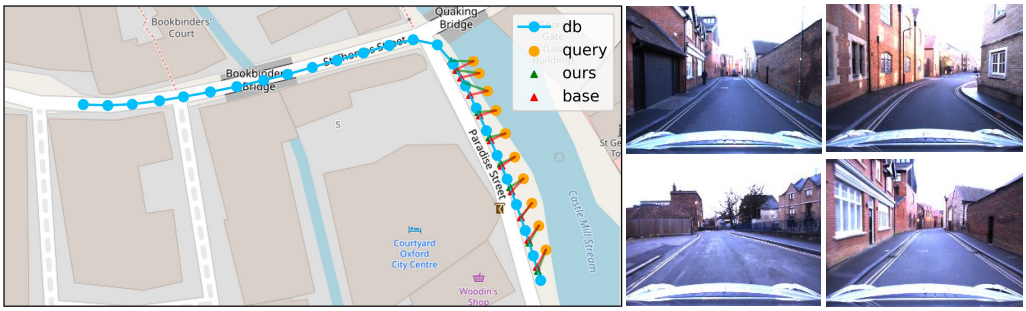


Figure 8: Additional example localisation output for the VPR experiments randomly selected from the test set. Example query images from the sequence are provided in the top row and example database images are provided in the bottom row. Query poses are offset from the database for visualisation purposes only.