NEARLY MINIMAX OPTIMAL OFFLINE REINFORCEMENT LEARNING WITH LINEAR FUNCTION APPROXIMATION: SINGLE-AGENT MDP AND MARKOV GAME

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ABSTRACT

Offline reinforcement learning (RL) aims at learning an optimal strategy using a pre-collected dataset without further interactions with the environment. While various algorithms have been proposed for offline RL in the previous literature, the minimax optimality has only been (nearly) established for tabular Markov decision processes (MDPs). In this paper, we focus on offline RL with linear function approximation and propose two new algorithms, LinPEVI-ADV+ and LinPMVI-ADV+, for single-agent MDPs and two-player zero-sum Markov games (MGs), respectively. The proposed algorithms establish pessimism in a variance-reduction manner via reference-advantage decomposition and variance-reweighted ridge regression. Theoretical analysis demonstrates that they can match the performance lower bounds up to logarithmic factors. We also establish new performance lower bounds for MDPs and MGs, which tighten the existing results, to demonstrate the nearly minimax optimality of the proposed algorithms. As a byproduct, equipped with the techniques developed in this paper, we can further improve the suboptimality bound when the feature vector set is finite. To the best of our knowledge, these are the first computationally efficient and nearly minimax optimal algorithms for offline single-agent MDPs and MGs with linear function approximation.

1 INTRODUCTION

Reinforcement learning (RL) has achieved tremendous empirical success in both single-agent (Kober et al., 2013) and multi-agent scenarios (Silver et al., 2016; 2017). Two components play a critical role – function approximations and efficient simulators. For RL problems with a large (or even infinite) number of states, storing a table as in classical Q-learning is generally infeasible. In these cases, practical algorithms (Mnih et al., 2015; Lillicrap et al., 2015; Schulman et al., 2015; 2017; Haarnoja et al., 2018) approximate the true value function or policy by a function class (e.g., neural networks). Meanwhile, an efficient simulator allows for learning a policy in an online trial-and-error fashion using millions or even billions of trajectories. However, due to the limited availability of data samples in many practical applications, e.g., healthcare (Wang et al., 2018) and autonomous driving (Pan et al., 2017), instead of collecting new trajectories, we may have to extrapolate knowledge only from past experiences, i.e., a pre-collected dataset. This type of RL problems is usually referred to as *offline RL* or *batch RL* (Lange et al., 2012; Levine et al., 2020).

An offline RL algorithm is usually measured by its sample complexity to achieve the desired statistical accuracy. A line of works (Xie et al., 2021b; Shi et al., 2022; Li et al., 2022) demonstrates that near-optimal sample complexity in *tabular* single-agent MDPs is attainable. However, these algorithms cannot solve the problem with large or infinite state spaces where function approximation is involved. To our best knowledge, existing algorithms cannot attain the statistical limit even for *linear function approximation*, which is arguably the simplest function approximation setting. Specifically, for linear function approximation, Jin et al. (2021c) proposes the first efficient algorithm for offline single-agent MDPs, but their upper bound is suboptimal compared with the existing lower bounds in Jin et al. (2021c); Zanette et al. (2021). Recently, Yin et al. (2022) claims that they can obtain the nearly minimax optimal bound for offline single-agent MDP with linear function approximation. However, a careful examination reveals a technical gap, and some additional assumptions may be needed to fix it (cf. Section 3 and Section 5). For the Markov games (MGs) with linear function approximation,

Zhong et al. (2022) provides the only provably efficient algorithm with a suboptimal result. Therefore, the following problem remains open:

Can we design computationally efficient offline RL algorithms for MDPs and MGs with linear function approximation that are nearly minimax optimal?

This paper answers this question affirmatively for offline linear MDPs (Jin et al., 2020) and two-player zero-sum linear MGs (Xie et al., 2020) by two new algorithms, LinPEVI-ADV+ and LinPMVI-ADV+, which attain the nearly minimax optimal rate when K exceeds certain thresholds. We first identify the complicated temporal dependency as the current bottleneck of algorithmic design and theoretical analysis (cf. Section 3). To overcome this issue, the proposed algorithms adapt the reference-advantage decomposition technique, which has been widely used in tabular settings (Azar et al., 2017; Zhang et al., 2020; Xie et al., 2021b), but is largely lacking under linear function approximation (cf. Section 4). We then further leverage the variance-reweighted ridge regression (Min et al., 2021; Yin et al., 2022) to obtain an instance-adaptive uncertainty estimation for offline RL with linear function approximation, where canceling the temporal dependency via reference-advantage decomposition is also critical (cf. Section 5). Finally, two nearly matching lower bounds are established to verify the minimax optimality of the proposed algorithms.

Related Work. Due to the space constraint, we defer the related work to Appendix B.

2 Offline Linear Markov Decision Process

Notations. Given a semi-definite matrix Λ and a vector u, we denote $\sqrt{u^\top \Lambda u}$ as $\|u\|_\Lambda$. The 2-norm of a vector w is $\|w\|_2$. We denote $\lambda_{\min}(A)$ as the smallest eigenvalue of the matrix A. The subscript of $f(x)_{[0,M]}$ means that we clip the value $f(x)$ to the range of $[0, M]$, i.e., $f(x)_{[0,M]} = \max\{0, \min\{M, f(x)\}\}$. Δ_X is the simplex on set X. We use the shorthand notations $\phi_h = \phi(x_h, a_h)$, $\phi_h^\tau = \phi(x_h^\tau, a_h^\tau)$, $r_h = r_h(x_h, a_h)$, and $r_h^\tau = r_h(x_h^\tau, a_h^\tau)$ (formally defined in the next subsection). With a slight abuse of notations, we also use similar notations (e.g. $\phi_h = \phi(x_h, a_h, b_h)$) for MGs, which shall be clear from the context. $Y \lesssim X$ means $Y \leq CX$ for some constant $C > 0$.

Markov Decision Process. We consider an episodic MDP, denoted as $\mathcal{M}(\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r)$, where $\mathcal{S}$ and $\mathcal{A}$ are the state space and action space, H is the episode length, $\mathbb{P} = \{\mathbb{P}_h\}_{h=1}^H$ and $r = \{r_h\}_{h=1}^H$ are the state transition kernels and reward functions, respectively. For each $h \in [H]$, $\mathbb{P}_h(\cdot|x, a)$ is the distribution of the next state given the state-action pair (x, a) at step h, $r_h(x, a) \in [0, 1]$ is the deterministic reward given the state-action pair (x, a) at step h. [1]

Policy and Value function. A policy $\pi = \{\pi_h\}_{h=1}^H$ is a collection of mappings from a state $x \in \mathcal{S}$ to a distribution of action space $\pi_h(\cdot|x) \in \Delta_{\mathcal{A}}$. For any policy π, we define the Q-value function $Q_h^\pi(x, a) = \mathbb{E}_\pi[\sum_{h'=h}^H r_{h'}(x_{h'}, a_{h'})|(x_h, a_h) = (x, a)]$ and the V-value function $V_h^\pi(x) = \mathbb{E}_\pi[\sum_{h'=h}^H r_{h'}(x_{h'}, a_{h'})|x_h = x]$, where $a_{h'} \sim \pi_{h'}(\cdot|x_{h'})$ and $x_{h'+1} \sim \mathbb{P}_{h'}(\cdot|x_{h'}, a_{h'})$. For any function $V : \mathcal{S} \to \mathbb{R}$, we denote the conditional mean as $(\mathbb{P}_h V)(x, a) := \sum_{x' \in \mathcal{S}} \mathbb{P}_h(x'|x, a)V(x')$ and conditional variance as $[\mathrm{Var}_h V](x, a) := [\mathbb{P}_h V^2](x, a) - ([\mathbb{P}_h V](x, a))^2$. The Bellman operator is defined as $(\mathcal{T}_h V)(x, a) := r_h(x, a) + (\mathbb{P}_h V)(x, a)$.

We consider the MDPs whose rewards and transitions possess a linear structure (Jin et al., 2020).

Definition 1 (Linear MDP)**.** *MDP$(\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r)$ is a linear MDP with a (known) feature map $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$, if for any $h \in [H]$, there exist d unknown signed measures $\mu_h = (\mu_h^{(1)}, \cdots, \mu_h^{(d)})$ over $\mathcal{S}$ and an unknown vector $\theta_h \in \mathbb{R}^d$, such that for any $(x, a) \in \mathcal{S} \times \mathcal{A}$, we have $\mathbb{P}_h(\cdot \mid x, a) = \langle \phi(x, a), \mu_h(\cdot) \rangle$, $r_h(x, a) = \langle \phi(x, a), \theta_h \rangle$. With loss of generality, we assume that $\|\phi(x, a)\| \leq 1$ for all $(x, a) \in \mathcal{S} \times \mathcal{A}$, and $\max\{\|\mu_h(\mathcal{S})\|, \|\theta_h\|\} \leq \sqrt{d}$ for all $h \in [H]$.*

For linear MDP, we have the following result, whose proof can be found in Jin et al. (2020).

Lemma 1. *For any function $V : \mathcal{S} \to [0, V_{\max} - 1]$ and $h \in [H]$, there exist vectors $\beta_h, w_h \in \mathbb{R}^d$ such that $\forall (x, a) \in \mathcal{S} \times \mathcal{A}$, we have $(\mathbb{P}_h V)(x, a) = \phi(x, a)^\top \beta_h$ and $(\mathcal{T}_h V)(x, a) = \phi(x, a)^\top w_h$, where $\|\beta_h\| \leq \|w_h\| \leq \sqrt{d} V_{\max}$.*

[1] The results readily generalize to the stochastic reward as the uncertainty of reward is non-dominating compared with that of state transition.

Offline RL. In offline RL, the algorithm needs to learn a near-optimal policy or to approximate Nash Equilibrium (NE) from a pre-collected dataset without further interacting with the environment. We suppose that we have access to a batch dataset $\mathcal{D} = \{x_h^\tau, a_h^\tau, r_h^\tau : h \in [H], \tau \in [K]\}$, where each trajectory is independently sampled by a behavior policy μ. The induced distribution of the state-action pair at step h is denoted as d_h^b. We make the following standard dataset coverage assumption for offline RL with linear function approximation (Wang et al., 2020a; Duan et al., 2020):

Assumption 1. *We assume* $\kappa = \min_{h\in[H]} \lambda_{\min}(\mathbb{E}_{d_h^b}[\phi(x,a)\phi(x,a)^\top]) > 0$ *for MDPs. With a slight abuse of notation, we also assume* $\kappa = \min_{h\in[H]} \lambda_{\min}(\mathbb{E}_{d_h^b}[\phi(x,a,b)\phi(x,a,b)^\top]) > 0$ *for MGs.*

This assumption requires the behavior policy to explore the state-action space well and is not information-theoretic as Jin et al. (2021c) does not necessarily require it. However, we make this assumption so that we can employ the variance information, which seems challenging otherwise. We remark the assumption is also made by the existing works that employ the variance information for the offline RL problems with linear function approximation (Min et al., 2021; Yin et al., 2022).

3 Preliminaries

Pessimistic Value Iteration (PEVI) is proposed in the seminal work of Jin et al. (2021c), which constructs the value function estimations $\{\widehat{V}_h\}_{h=1}^H$ and Q-value estimations $\{\widehat{Q}_h\}_{h=1}^H$ backward from $h = H$ to $h = 1$, with the initialization $\widehat{V}_{H+1} = 0$. Specifically, given $\widehat{V}_{h+1}$, PEVI approximates the Bellman equation by $\widehat{Q}_h(x,a) \leftarrow \widehat{\mathcal{T}}_h \widehat{V}_{h+1}(x,a) - \beta \|\phi(x,a)\|_{\Lambda_h^{-1}}$. Here the linear approximation $\widehat{\mathcal{T}}_h \widehat{V}_{h+1} = \phi(\cdot,\cdot)^\top \widehat{w}_h(\widehat{V}_{h+1})$ is the solution of the following value-target ridge regression:

$$\widehat{w}_h(\widehat{V}_{h+1}) = \operatorname*{arg\,min}_{w\in\mathbb{R}^d} \sum_{\tau\in\mathcal{D}} \left(r_h^\tau + \widehat{V}_{h+1}(x_{h+1}^\tau) - (\phi_h^\tau)^\top w\right)^2 + \lambda\|w\|_2^2 = \Lambda_h^{-1}\left(\sum_{\tau\in\mathcal{D}} \phi_h^\tau \left(r_h^\tau + \widehat{V}_{h+1}(x_{h+1}^\tau)\right)\right), \tag{1}$$

where $\Lambda_h = \sum_{\tau\in\mathcal{D}} \phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^\top + \lambda I_d$ and $\Gamma_h(x,a) = \beta\|\phi(x,a)\|_{\Lambda_h^{-1}}$ is the bonus function such that $|(\mathcal{T}_h\widehat{V}_{h+1} - \widehat{\mathcal{T}}_h\widehat{V}_{h+1})(x,a)| \le \Gamma_h(x,a)$ for all $(x,a) \in \mathcal{S}\times\mathcal{A}$ with high probability. In particular, pessimism means that we use lower confidence bound so that $\widehat{Q}_h(x,a) \le \mathcal{T}_h\widehat{V}_{h+1}(x,a)$. This leads to $V_1^*(x) - V_1^{\widehat{\pi}}(x) \le 2\sum_{h=1}^H \mathbb{E}_{\pi^*}[\Gamma_h(x_h,a_h)|x_1 = x]$ (formally presented in Lemma 2). Therefore, to establish a sharper suboptimality bound, it suffices to construct a smaller bonus function Γ_h that can ensure pessimism. There remains, however, a gap of $\tilde{O}(\sqrt{d}H)$ between the suboptimality bound of PEVI and the minimax lower bound in Zanette et al. (2021).

Self-normalized Process and Uniform Concentration. Given a function f_{h+1}, to construct the bonus function, we can bound $|\mathcal{T}_h f_{h+1} - \widehat{\mathcal{T}}_h f_{h+1}|$ as follows (formally presented in Lemma 3):

$$\left|(\mathcal{T}_h f_{h+1})(x,a) - \left(\widehat{\mathcal{T}}_h f_{h+1}\right)(x,a)\right| \lesssim \underbrace{\left\|\sum_{\tau\in\mathcal{D}} \phi(x_h^\tau, a_h^\tau)\cdot \xi_h^\tau(f_{h+1})\right\|_{\Lambda_h^{-1}}}_{(\mathtt{A})\le\beta} \|\phi(x,a)\|_{\Lambda_h^{-1}},$$

where $\xi_h^\tau(f_{h+1}) := r_h^\tau + f_{h+1}(x_{h+1}^\tau) - (\mathcal{T}_h f_{h+1})(x_h^\tau, a_h^\tau)$. Bounding $(\mathtt{A})$ is referred to as the concentration of a *self-normalized process* in the literature (Abbasi-Yadkori et al., 2011). For any *fixed* $\widehat{V}_{h+1}$, Lemma 9 ensures a high-probability upper bound $\tilde{O}\left(H\sqrt{d}\right)$ for $(\mathtt{A})$. However, in the backward iteration, $\widehat{V}_{h+1}$ is computed by ridge regression in later steps $[h+1, H]$ and thus inevitably depends on $\{x_{h+1}^\tau\}_{\tau\in\mathcal{D}}$, which is also used to estimate the Bellman equation at step h. Consequently, the concentration inequality cannot directly be applied. To resolve the above issue, the standard approach is a uniform concentration argument by ϵ-covering net, leading to an extra $\sqrt{d}$ from the logarithmic covering number $\log\mathcal{N}_{h+1}(\epsilon)$ (cf. Appendix A.2 for details). The prior work Yin et al. (2022) ignores the dependency between $\widehat{V}_{h+1}$ and $\{x_h^\tau, a_h^\tau, x_{h+1}^\tau\}_{\tau\in\mathcal{D}}$, and thus circumvents the uniform concentration, where additional assumption is required to fix the gap of their analysis.

4 REFERENCE-ADVANTAGE DECOMPOSITION UNDER LINEAR FUNCTION APPROXIMATION

Reference-advantage decomposition has been widely applied in tabular MDPs (Azar et al., 2017; Zanette and Brunskill, 2019; Zhang et al., 2020; Xie et al., 2021b) but is largely lacking under linear function approximation. For tabular MDP, we can estimate the transition matrix by counting each (x, a) pair separately and establish $\widehat{\mathbb{P}}_h$. To deal with the measurability issue, one may bound $\left(\widehat{\mathbb{P}}_h - \mathbb{P}_h\right) \widehat{V}_{h+1}(x, a)$ by $\left\|\widehat{\mathbb{P}}_h - \widehat{\mathbb{P}}_h\right\|_1 \left\|\widehat{V}_{h+1}\right\|_\infty$ as in (Auer et al., 2008) but this incurs an additional $\sqrt{|\mathcal{S}|}$. Instead, we can decompose the uncertainty into $\left(\widehat{\mathbb{P}}_h - \mathbb{P}_h\right) V^*_{h+1}(x, a) + \left(\widehat{\mathbb{P}}_h - \mathbb{P}_h\right) (\widehat{V}_{h+1} - V^*_{h+1})(x, a)$ where the first term can be handled directly because V^*_{h+1} is deterministic and the second term can be controlled by some more involved analysis.

While the high-level ideas are similar, the algorithmic designs and analysis techniques from tabular MDP cannot apply to the linear setting directly. Although we can establish $\widehat{\mathbb{P}}_h$ separately by counting each (x, a) pair and show that it is close to $\mathbb{P}_h$ in total variation distance in tabular MDP, one cannot expect this in the linear setting as $\mathbb{P}_h(\cdot|x, a) = \phi(x, a)^\top \mu_h(\cdot)$ has infinite degrees of freedom. Instead, we face a vector-valued martingale (the self-normalized process) in the linear setting where $\mathcal{T}_h f_{h+1}$ ($\mathbb{P}_h f_{h+1}$) is handled as a whole using the value-target ridge regression (Eqn. (1)) and the proof techniques are distinctly different. Luckily, it turns out that approximating $\mathbb{P}_h f_{h+1}$ as a whole is already sufficient for our goal. Moreover, both true Bellman equation and the least-square estimation admit closed forms and satisfy the following properties:

$$
\begin{aligned}
\left(\widehat{\mathcal{T}}_h(f_{h+1} + g_{h+1})\right)(\cdot, \cdot) &= \phi(\cdot, \cdot)^\top \Lambda_h^{-1} \left(\sum_{\tau \in \mathcal{D}} \phi_h^\tau \left(r_h^\tau + f_{h+1}(x_{h+1}^\tau)\right)\right) + \phi(\cdot, \cdot)^\top \Lambda_h^{-1} \left(\sum_{\tau \in \mathcal{D}} \phi_h^\tau g_{h+1}(x_{h+1}^\tau)\right) \\
&= \widehat{\mathcal{T}}_h f_{h+1}(\cdot, \cdot) + \widehat{\mathbb{P}}_h g_{h+1}(\cdot, \cdot), \\
\left(\mathcal{T}_h\left(f_{h+1} + g_{h+1}\right)\right)(\cdot, \cdot) &= \langle \phi(x, a), \theta_h \rangle + \int_{\mathcal{S}} f_{h+1}(x') \left\langle \phi(x, a), d\mu_h(x') \right\rangle + \int_{\mathcal{S}} g_{h+1}(x') \left\langle \phi(x, a), d\mu_h(x') \right\rangle \\
&= \mathcal{T}_h f_{h+1}(\cdot, \cdot) + \mathbb{P}_h g_{h+1}(\cdot, \cdot),
\end{aligned}
$$

where $\widehat{\mathbb{P}}_h g_{h+1}(\cdot, \cdot)$ is obtained by setting $r_h^\tau = 0$ in $\widehat{\mathcal{T}}_h g_{h+1}(\cdot, \cdot)$. These two observations ensure that we can decompose the uncertainty as follows:

$$
\begin{aligned}
&\widehat{\mathcal{T}}_h \widehat{V}_{h+1}(x, a) - \mathcal{T}_h \widehat{V}_{h+1}(x, a) = \left(\widehat{\mathcal{T}}_h(\widehat{V}_{h+1} + V^*_{h+1} - V^*_{h+1})\right)(x, a) - \left(\mathcal{T}_h(\widehat{V}_{h+1} + V^*_{h+1} - V^*_{h+1})\right)(x, a) \\
&= \underbrace{\widehat{\mathcal{T}}_h V^*_{h+1}(x, a) - \mathcal{T}_h V^*_{h+1}(x, a)}_{\text{Reference function} \le b_{0,h}(x,a)} + \underbrace{\widehat{\mathbb{P}}_h(\widehat{V}_{h+1} - V^*_{h+1})(x, a) - \mathbb{P}_h(\widehat{V}_{h+1} - V^*_{h+1})(x, a)}_{\text{Advantage function} \le b_{1,h}(x,a)}.
\end{aligned} \tag{2}
$$

Circumventing the Uniform Concentration for Reference Function. As V^*_{h+1} is deterministic, we can directly invoke Lemma 3 and Lemma 9 without uniform concentration to set $b_{0,h}(x, a) = \tilde{O}\left(\sqrt{d}H\right) \|\phi(x, a)\|_{\Lambda_h^{-1}}$, which saves a factor of $\sqrt{d}$.

High-order Error from Correlated Advantage Function. First, $\widehat{\mathbb{P}}_h f \approx \mathbb{P}_h f$ can be established similarly as it corresponds to a special case of $r_h \equiv 0$. Although we still need the standard uniform concentration argument to analyze the advantage function and obtain a sub-optimal d-dependency, under Assumption 1, we can show (by induction) that $\left\|\widehat{V}_{h+1} - V^*_{h+1}\right\|_\infty = \tilde{O}\left(\frac{\sqrt{d}H^2}{\sqrt{K\kappa}}\right)$. The much smaller range implies that we can invoke Lemma 9 to set $b_{1,h}(x, a) = \tilde{O}\left(\frac{d^{3/2}H^2}{\sqrt{K\kappa}}\right) \|\phi(x, a)\|_{\Lambda_h^{-1}}$, which is non-dominating when $K \ge \tilde{\Omega}\left(d^2 H^2/\kappa\right)$. The detailed proof can be found in Appendix D.

The Algorithm 1 is called LinPEVI-ADV and we have the following theoretical guarantee.

Theorem 1 (LinPEVI-ADV). *Under the Assumptions 1, and $K > \tilde{\Omega}(d^2 H^2/\kappa)$, if we set $\lambda = 1$ and $\beta_1 = \tilde{O}(\sqrt{d}H)$ in Algorithm 1, then, with probability at least $1 - \delta$, for any $x \in \mathcal{S}$, we have*

$$
V_1^*(x) - V_1^{\widehat{\pi}}(x) \le \tilde{O}\left(\sqrt{d}H\right) \sum_{h=1}^{H} \mathbb{E}_{\pi^*}\left[\|\phi(x, a)\|_{\Lambda_h^{-1}} \mid x_1 = x\right],
$$

Algorithm 1 LinPEVI-ADV

Initialize: Input dataset $\mathcal{D}$, β_1; $\widehat{V}_{H+1}(\cdot) = 0$.
for $h = H, \ldots, 1$ **do**
  $\Lambda_h \leftarrow \sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^\top + \lambda I_d$;
  $\widehat{w}_h \leftarrow \Lambda_h^{-1}(\sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau)(r_h^\tau + \widehat{V}_{h+1}(x_{h+1}^\tau))$;
  $\Gamma_h(\cdot,\cdot) \leftarrow \beta_1 \|\phi(\cdot,\cdot)\|_{\Lambda_h^{-1}}$;
  $\widehat{Q}_h(\cdot,\cdot) \leftarrow \{\phi(\cdot,\cdot)^\top \widehat{w}_h - \Gamma_h(\cdot,\cdot)\}_{[0,H-h+1]}$;
  $\widehat{\pi}_h(\cdot \mid \cdot) \leftarrow \arg\max_{\pi_h} \langle \widehat{Q}_h(\cdot,\cdot), \pi_h(\cdot \mid \cdot)\rangle_{\mathcal{A}}, \widehat{V}_h(\cdot) \leftarrow \langle \widehat{Q}_h(\cdot,\cdot), \widehat{\pi}_h(\cdot \mid \cdot)\rangle_{\mathcal{A}}$.
end for
Output: $\widehat{\pi} = \{\widehat{\pi}_h\}_{h=1}^H$ and $\widehat{V} = \{\widehat{V}_h\}_{h=1}^H$

where $\Lambda_h = \sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^\top + \lambda I_d$.

The Choice of Reference Function. Xie et al. (2021b) adapts the decomposition technique from online tabular MDP by explicitly constructing an independent reference $\widehat{V}_{h+1}^{\text{ref}}$. The main advantage is that $\widehat{V}_{h+1}^{\text{ref}}$ is available to the actually executed algorithm so we can further estimate the Bellman equation of the advantage function by another independent dataset. In this case, the temporal dependency in the advantage function is also cancelled so the analysis is straightforward. On the other hand, the algorithmic designs presented here are more similar to the original PEVI (Jin et al., 2021c). The tighter β_1 comes from the theoretical analysis and a carefully crafted induction-based analysis is employed to deal with the temporal dependency of the advantage function (cf. Appendix D).

Linear MDP with Finite Feature Set. Equipped with the reference-advantage decomposition technique for linear MDP, when $|\{\phi(x,a) \in \mathbb{R}^d : x \in \mathcal{S}, a \in \mathcal{A}\}| = M < \infty$, we can further improve the d-dependency by avoiding the analysis of the self-normalized process:

$$\left(\widehat{\mathcal{T}}_h V_{h+1}^* - \mathcal{T}_h V_{h+1}^*\right)(x,a) \overset{(a)}{\lesssim} \sum_{\tau \in \mathcal{D}} \left\langle \phi(x,a), \Lambda_h^{-1}\phi_h^\tau \right\rangle \xi_h^\tau(V_{h+1}^*) \overset{(b)}{\lesssim} \|\phi(x,a)\|_{\Lambda_h^{-1}} \left\| \sum_{\tau \in \mathcal{D}} \xi_h^\tau(V_{h+1}^*)\phi_h^\tau \right\|_{\Lambda_h^{-1}},$$

where the proofs of (a) and (b) are presented in Section I. For the previous analyses, we focus on (b), and obtain a bound simultaneously valid for all $(x,a) \in \mathcal{S} \times \mathcal{A}$ but also suffer a factor of $\sqrt{d}$ from bounding the self-normalized process by Lemma 9. Instead, for any fixed $\phi(x,a) \in \mathbb{R}^d$, we first fix $\{(x_h^\tau, a_h^\tau)\}_{\tau \in \mathcal{D}}$. This is possible because V_{h+1}^* is deterministic so $\{\xi_h^\tau(V_{h+1}^*)\}_{\tau \in \mathcal{D}}$ are still independent and bounded random variables conditioned on $\{(x_h^\tau, a_h^\tau)\}_{\tau \in \mathcal{D}}$. Therefore, we can apply the classical Hoeffding's inequality, and a union bound over all features $\phi(x,a)$, to obtain that with probability at least $1-\delta$, for all $(x,a) \in \mathcal{S} \times \mathcal{A}$, we have:

$$\left|\widehat{\mathcal{T}}_h V_{h+1}^* - \mathcal{T}_h V_{h+1}^*\right|(x,a) \lesssim \sum_{\tau \in \mathcal{D}} \underbrace{\left\langle \phi(x,a), \Lambda_h^{-1}\phi_h^\tau \right\rangle}_{\text{Constant}} \xi_h^\tau(V_{h+1}^*) \leq H\sqrt{\log\left(\frac{2M}{\delta}\right)} \|\phi(x,a)\|_{\Lambda_h^{-1}}, \tag{3}$$

where a $\sqrt{d}$ from Lemma 9 is further eliminated. This leads to the following theoretical guarantee.

Theorem 2 (LinPEVI-ADV with Finite Feature Set)**.** *Suppose that* $|\{\phi(x,a) \in \mathbb{R}^d : x \in \mathcal{S}, a \in \mathcal{A}\}| = M < \infty$ *and* $K \geq \tilde{\Omega}(d^2H^2/\kappa)$. *Under Assumptions 1 if we set* $\beta_1 = O\left(\sqrt{\log(2H^2M/\delta)}\right)$ *and* $\lambda = 1/d$ *in Algorithm 1, then, with probability at least* $1-\delta$, *for any* $x \in \mathcal{S}$, *with* $\Lambda_h = \sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^\top + \lambda I_d$, *we have*

$$V_1^*(x) - V_1^{\widehat{\pi}}(x) \leq O\left(H\sqrt{\log\left(\frac{2H^2M}{\delta}\right)}\right) \sum_{h=1}^H \mathbb{E}_{\pi^*}\left[\|\phi(x_h,a_h)\|_{\Lambda_h^{-1}} \middle| x_1 = x\right]. \tag{4}$$

5 Leverage the Variance Information

Variance-reweighted Regression. The reweighted ridge regression technique is introduced by Zhou et al. (2021) in the literature of RL and has been adapted by Yin et al. (2022); Min et al. (2021)

Algorithm 2 LinPEVI-ADV+

Initialize: Input datasets $\mathcal{D}$, $\mathcal{D}'$, and β_2; $\widehat{V}_{H+1}(\cdot) = 0$.
Construct variance estimator $\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)$ as in Section 5 with $\mathcal{D}'$.
for $h = H, \ldots, 1$ **do**
  $\Sigma_h = \sum_{\tau\in\mathcal{D}} \phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^\top / \widehat{\sigma}_h^2(x_h^\tau, a_h^\tau) + \lambda I_d$;
  $\widehat{w}_h = \Sigma_h^{-1}\left(\sum_{\tau\in\mathcal{D}} \phi(x_h^\tau, a_h^\tau)\frac{r_h^\tau + \widehat{V}_{h+1}(x_{h+1}^\tau)}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)}\right)$;
  $\Gamma_h(\cdot,\cdot) \leftarrow \beta_2 \left\|\phi(\cdot,\cdot)\right\|_{\Sigma_h^{-1}}$;
  $\widehat{Q}_h(\cdot,\cdot) \leftarrow \{\phi(\cdot,\cdot)^\top \widehat{w}_h - \Gamma_h(\cdot,\cdot)\}_{[0,H-h+1]}$;
  $\widehat{\pi}_h(\cdot \mid \cdot) \leftarrow \arg\max_{\pi_h} \langle \widehat{Q}_h(\cdot,\cdot), \pi_h(\cdot \mid \cdot)\rangle_{\mathcal{A}}, \widehat{V}_h(\cdot) \leftarrow \langle \widehat{Q}_h(\cdot,\cdot), \widehat{\pi}_h(\cdot \mid \cdot)\rangle_{\mathcal{A}}$.
end for
Output: $\widehat{\pi} = \{\widehat{\pi}_h\}_{h=1}^H$.

for offline linear MDP. Specifically, given a *fixed* function $f_{h+1} : \mathcal{S} \to [0, H-1]$, we perform the following variance-reweighted ridge regression to estimate $\widehat{w}_h(f_{h+1})$ as:

$$\operatorname*{argmin}_{w\in\mathbb{R}^d} \sum_{\tau\in\mathcal{D}} \frac{\left[(\phi_h^\tau)^\top w - r_h^\tau - f_{h+1}\left(x_{h+1}^\tau\right)\right]^2}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} + \lambda\|w\|_2^2 = \Sigma_h^{-1}\left(\sum_{\tau\in\mathcal{D}} \frac{\phi\left(x_h^\tau, a_h^\tau\right)\cdot\left(r_h^\tau + f_{h+1}\left(x_{h+1}^\tau\right)\right)}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)}\right), \tag{5}$$

where $\Sigma_h = \sum_{\tau\in\mathcal{D}} \frac{\phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^\top}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} + \lambda I_d$ and $\widehat{\sigma}_h^2(\cdot,\cdot) \in [1, H^2]$ is a variance estimator. In this case, we can bound the uncertainty by Lemma 3 as follows:

$$|\mathcal{T}_h f_{h+1} - \widehat{\mathcal{T}}_h f_{h+1}|(x,a) \lesssim \|\sum_{\tau\in\mathcal{D}} \phi\left(x_h^\tau, a_h^\tau\right)\cdot \xi_h^\tau(f_{h+1})\|_{\Sigma_h^{-1}} \|\phi(x,a)\|_{\Sigma_h^{-1}},$$

with $\xi_h^\tau(f_{h+1}) = \frac{r_h^\tau + f_{h+1}(x_{h+1}^\tau) - (\mathcal{T}_h f_{h+1})(x_h^\tau, a_h^\tau)}{\widehat{\sigma}_h(x_h^\tau, a_h^\tau)}$. The high-level idea is that the normalized $\xi_h^\tau(f_{h+1})$ is of a conditional variance $O(1)$ and the Bernstein-type inequality gives a bound of $\beta_2 := \tilde{O}(\sqrt{d})$ for the self-normalized process instead of $\tilde{O}(H\sqrt{d})$. Furthermore, as discussed in Yin et al. (2022), instead of depending on H explicitly, the new bound is in an instance-dependent manner that provides different convergence rates for different problems. In particular, as $\Sigma_h^{-1} \preccurlyeq H^2\Lambda_h^{-1}$, we know that $\beta_2 \|\phi(x,a)\|_{\Sigma_h^{-1}} \leq \beta_1 \|\phi(x,a)\|_{\Lambda_h^{-1}}$ always holds up to constant factors.

Limitation of the Existing Approach. Yin et al. (2022) adapts this technique with $f_{h+1} = \widehat{V}_{h+1}$ and the variance estimator $\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)$ is also constructed by $\widehat{V}_{h+1}$. However, two issues arise from the temporal dependency. First, in the analysis, they need to establish the concentration of the covariance matrix with Lemmas 12 (Lemma C.3 of Yin et al. (2022)) where $\{\frac{\phi(x_h^\tau, a_h^\tau)}{\widehat{\sigma}_h(x_h^\tau, a_h^\tau)}\}_{\tau=1}^K$ are required to be independent. This is not satisfied as the denominator depends on the state-action pairs at step h via $\widehat{V}_{h+1}$. Due to the same reason, the concentration inequality cannot be applied to $\{\xi_h^\tau(\widehat{V}_{h+1})\}$ even though we apply uniform concentration as the martingale is not well-defined. Second, the variance estimation of $\xi_h^\tau(\widehat{V}_{h+1})$ could be hard because one cannot take the denominator $\widehat{\sigma}_h(x_h^\tau, a_h^\tau)$ outside the (conditional) variance operator (cf. Lemma C.4 of Yin et al. (2022)) due to the dependency from $\widehat{V}_{h+1}$. To fix these issues, one may need to additionally assume independence across different h.

Variance Estimator. With the reference-advantage decomposition, if we can construct a $\widehat{\sigma}_h^2(\cdot,\cdot)$ that is independent of $\mathcal{D}$, then $\xi_h^\tau(V_{h+1}^*) = \frac{r_h^\tau + V_{h+1}^*(x_{h+1}^\tau) - (\mathcal{T}_h V_{h+1}^*)(x_h^\tau, a_h^\tau)}{\widehat{\sigma}_h(x_h^\tau, a_h^\tau)}$ would be independent with each other and easy to deal with. We first use an independent dataset $\mathcal{D}'$ to run Algorithm 1 and construct $\{\widehat{V}_h'\}_{h=1}^H$ (this only incurs a factor of 2 in the final sample complexity). Then, by Lemma 1, we know that there exist $\beta_{h,1}, \beta_{h,2} \in \mathbb{R}^d$ such that $[\mathbb{P}_h(\widehat{V}_{h+1}')^2](x,a) = \langle\phi(x,a), \beta_{h,2}\rangle$ and $([\mathbb{P}_h\widehat{V}_{h+1}'(x,a)])^2 = [\langle\phi(x,a), \beta_{h,1}\rangle]^2$. Similarly, we can approximate $\beta_{1,h}$ and $\beta_{2,h}$ via ridge

regression with $\mathcal{D}'$:

$$\begin{aligned}\widetilde{\beta}_{h,2} &= \operatorname*{argmin}_{\beta\in\mathbb{R}^d} \sum_{\tau\in\mathcal{D}'} \left[\langle \phi\left(x_h^\tau, a_h^\tau\right), \beta\rangle - (\widehat{V}'_{h+1})^2\left(x_{h+1}^\tau\right)\right]^2 + \lambda\|\beta\|_2^2,\\ \widetilde{\beta}_{h,1} &= \operatorname*{argmin}_{\beta\in\mathbb{R}^d} \sum_{\tau\in\mathcal{D}'} \left[\langle \phi\left(x_h^\tau, a_h^\tau\right), \beta\rangle - \widehat{V}'_{h+1}\left(x_{h+1}^\tau\right)\right]^2 + \lambda\|\beta\|_2^2.\end{aligned} \tag{6}$$

We then employ the following variance estimator:

$$\widehat{\sigma}_h^2(x,a) := \max\left\{1, \left[\phi(x,a)^\top \widetilde{\beta}_{h,2}\right]_{[0,H^2]} - \left[\phi(x,a)^\top, \widetilde{\beta}_{h,1}\right]_{[0,H]}^2 - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right)\right\}. \tag{7}$$

Then, we can show that $[\mathbb{V}_h V_{h+1}^*](x,a) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) \le \widehat{\sigma}_h^2(x,a) \le [\mathbb{V}_h V_{h+1}^*](x,a)$ with high probability (cf. Lemma 5), with $[\mathbb{V}_h V_{h+1}^*](x,a) = \max\{1, [\mathrm{Var}_h\, V_{h+1}^*](x,a)\}$. The monotonicity firstly implies $\|\phi(x,a)\|_{\Sigma_h^{-1}} \le \|\phi(x,a)\|_{\Sigma_h^{*-1}}$ with $\Sigma_h^* = \sum_{\tau\in\mathcal{D}} \frac{\phi(x_h^\tau,a_h^\tau)\phi(x_h^\tau,a_h^\tau)^\top}{[\mathbb{V}_h V_{h+1}^*](x_h^\tau,a_h^\tau)}$. Moreover, let $\mathbb{D}_{\tau-1,h}$ denote the variance operator conditioned on the sigma algebra generated by $\{(x_h^j, a_h^j)\}_{j=1}^\tau \cup \{(r_h^j, x_{h+1}^j)\}_{j=1}^{\tau-1}$. Then, the conditional variance of the reference function can be estimated by

$$\mathbb{D}_{\tau-1,h}[\xi_h^\tau(V_{h+1}^*)] = \frac{\mathbb{D}_{\tau-1,h}\left[V_{h+1}^*(x_{h+1}^\tau)\right]}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} \le \frac{[\mathbb{V}_h V_{h+1}^*](x_h^\tau, a_h^\tau)}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} = \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) = O(1),$$

where the first equality is because $\widehat{\sigma}_h^2(\cdot,\cdot)$ is independent of $\mathcal{D}$ and the last inequality holds after K exceeds a threshold. The new algorithm is called LinPEVI-ADV+ (Algorithm 2) and we have the following theoretical guarantee.

Theorem 3 (LinPEVI-ADV+). *Under Assumption 1, for $K \ge \tilde{\Omega}(d^2H^6/\kappa)$, if we set $\lambda = 1/H^2$ and $\beta_2 = \tilde{O}(\sqrt{d})$ in Algorithm 2, then, we have with probability at least $1-\delta$, for any $x \in \mathcal{S}$, we have*

$$V_1^*(x) - V_1^{\widehat{\pi}}(x) \le \tilde{O}(\sqrt{d}) \cdot \sum_{h=1}^H \mathbb{E}_{\pi^*}\left[\|\phi(x_h,a_h)\|_{\Sigma_h^{*-1}} | x_1 = x\right],$$

where $\Sigma_h^* = \sum_{\tau\in\mathcal{D}} \phi(x_h^\tau,a_h^\tau)\phi(x_h^\tau,a_h^\tau)^\top/[\mathbb{V}_h V_{h+1}^*](x_h^\tau,a_h^\tau) + \lambda I_d$.

We present the following matching lower bound.

Theorem 4 (Lower Bound for MDP). *For fixed episode length H, dimension d, probability δ, and sample size $K \ge \tilde{\Omega}(d^4)$. There exists a class $\mathcal{M}$ of linear MDPs and an offline dataset $\mathcal{D}$ with $|\mathcal{D}| = K$, such that for any policy $\widehat{\pi}$, it holds with probability at least $1-\delta$ that for some universal constant $c > 0$, we have*

$$\sup_{M\in\mathcal{M}} \mathbb{E}_M[V_1^*(x_1) - V_1^{\widehat{\pi}}(x_1)] \ge c\sqrt{d} \cdot \sum_{h=1}^H \mathbb{E}_{\pi^*}\|\phi(x_h,a_h)\|_{\Sigma_h^{*-1}}.$$

The lower bound matches the main term in Theorem 3, so LinPEVI-ADV+ is nearly minimax optimal after K exceeds the threshold. We also remark that Yin et al. (2022) establishes the lower bound $c\sqrt{d} \cdot \sum_{h=1}^H \|\mathbb{E}_{\pi^*}\phi(x_h,a_h)\|_{\Sigma_h^{*-1}}$, which is smaller than our lower bound by Jensen's inequality.

6 Results for Markov Game

We extend our techniques to the linear Markov games in this section.

Two-player Zero-sum Markov Game with Linear Function Approximation is defined by a tuple $(\mathcal{S}, \mathcal{A}, \mathcal{B}, H, \mathbb{P}, r)$, where $\mathcal{S}$ denotes the state space, $\mathcal{A}$ and $\mathcal{B}$ are the action spaces for the two players, H is the length of each episode, $\mathbb{P} = \{\mathbb{P}_h : \mathcal{S}\times\mathcal{A}\times\mathcal{B} \to \Delta_{\mathcal{S}}\}_{h=1}^H$ is the transition kernel, and $r = \{r_h : \mathcal{S}\times\mathcal{A}\times\mathcal{B} \to [0,1]\}_{h=1}^H$ is the reward function. The policy of the max-player and the policy for the min-player are defined by $\pi = \{\pi_h : \mathcal{S} \to \Delta_{\mathcal{A}}\}_{h=1}^H$ and $\nu = \{\nu_h : \mathcal{S} \to \Delta_{\mathcal{B}}\}$.

Value Function and Nash Equilibrium. For any fixed policy pair (π, ν), we define the value function $V_h^{\pi,\nu}$ and the Q-function $Q_h^{\pi,\nu}$ by $V_h^{\pi}(x) = \mathbb{E}_{\pi,\nu}[\sum_{h'=h}^{H} r_{h'}|x_h = x], Q_h^{\pi}(x,a,b) = \mathbb{E}_{\pi,\nu}[\sum_{h'=h}^{H} r_{h'}|(x_h, a_h, b_h) = (x,a,b)]$. For any function $V : \mathcal{S} \to \mathbb{R}$, we also define the shorthand notations for the conditional mean, conditional variance, and Bellman operator: $(\mathbb{P}_h V)(x,a,b) := \sum_{x' \in \mathcal{S}} \mathbb{P}_h(x'|x,a,b)V(x')$, $[\mathrm{Var}_h V](x,a,b) := [\mathbb{P}_h V^2](x,a,b) - ([\mathbb{P}_h V](x,a,b))^2, (\mathcal{T}_h V)(x,a,b) := r_h(x,a,b) + (\mathbb{P}_h V)(x,a,b)$. For any max-player's policy π, we define the best-response as $\mathrm{br}(\pi) = \mathrm{argmin}_\nu V_h^{\pi,\nu}(x)$ for all $(x,h) \in \mathcal{S} \times [H]$. Similarly, we can define $\mathrm{br}(\nu)$ by $\mathrm{br}(\nu) = \mathrm{argmax}_\pi V_h^{\pi,\nu}(x)$ for all $(x,h) \in \mathcal{S} \times [H]$. We say (π^*, ν^*) is a Nash equilibrium (NE) if π^* and ν^* are the best response to each other. For simplicity, we denote $V_h^{\pi,*} = V_h^{\pi,\mathrm{br}(\pi)}$, $V_h^{*,\nu} = V_h^{\mathrm{br}(\nu),\nu}$, and $V_h^*(x) = V_h^{\pi^*,\nu^*}(x)$. It is well known that (π^*, ν^*) is the solution to $\max_\pi \min_\nu V_h^{\pi,\nu}(x)$. Then we can measure the optimality of a policy pair (π, ν) by the duality gap, which is defined in Zhong et al. (2022); Xie et al. (2020) as follows:

$$\mathrm{Gap}((\pi,\nu),x) = V_1^{*,\nu}(x) - V_1^{\pi,*}(x). \tag{8}$$

We consider the linear MG (Xie et al., 2020), which generalizes the definition of linear MDP.

Definition 2 (Linear MG). *$MG(\mathcal{S}, \mathcal{A}, \mathcal{B}, H, \mathbb{P}, r)$ is a linear MG with a (known) feature map $\phi : \mathcal{S} \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}^d$, if for any $h \in [H]$, there exist d unknown signed measures $\mu_h = \left(\mu_h^{(1)}, \cdots, \mu_h^{(d)}\right)$ over $\mathcal{S}$ and an unknown vector $\theta_h \in \mathbb{R}^d$, such that for any $(x,a) \in \mathcal{S} \times \mathcal{A}$, we have $\mathbb{P}_h(\cdot \mid x,a,b) = \langle \phi(x,a,b), \mu_h(\cdot) \rangle$, $\quad r_h(x,a,b) = \langle \phi(x,a,b), \theta_h \rangle$. We assume that $\|\phi(x,a,b)\| \le 1$ for all $(x,a,b) \in \mathcal{S} \times \mathcal{A} \times \mathcal{B}$, and $\max\{\|\mu_h(\mathcal{S})\|, \|\theta_h\|\} \le \sqrt{d}$ for all $h \in [H]$.*

6.1 LinPMVI-ADV+

LinPMVI-ADV+ (Algorithm 3) is a variant of pessimistic minimax value iteration (PMVI) from (Zhong et al., 2022). At a high level, LinPMVI-ADV+ constructs *pessimistic* estimations of the Q-functions for both players and outputs a policy pair by solving two Nash Equilibrium (NE) based on these two estimated value functions. For linear MG, these can also be done by regressions. Suppose we have constructed value functions $(\underline{V}_{h+1}, \overline{V}_{h+1})$ at $(h+1)$-th step, and two *independent* variance estimators $\underline{\sigma}_h^2$ and $\overline{\sigma}_h^2$, which are constructed similarly to Section 5 and the details are deferred to Appendix A.1 due to space constraint. We can approximate the Bellman equations $\mathcal{T}_h \underline{V}_{h+1}$ and $\mathcal{T}_h \overline{V}_{h+1}$ by solving the following regression problems (cf. Line 5 of Algorithm 2):

$$\begin{aligned}
\underline{w}_h &\leftarrow \operatorname*{argmin}_{w \in \mathbb{R}^d} \sum_{\tau \in \mathcal{D}} \frac{[r_h^\tau + \underline{V}_{h+1}(x_{h+1}^\tau) - (\phi_h^\tau)^\top w]^2}{\underline{\sigma}_h^2(x_h^\tau, a_h^\tau, b_h^\tau)} + \lambda \|w\|_2^2, \quad \text{and} \quad \underline{\mathcal{T}}_h \underline{V}_{h+1}(o) := \phi(o)^\top \underline{w}_h, \\
\overline{w}_h &\leftarrow \operatorname*{argmin}_{w \in \mathbb{R}^d} \sum_{\tau \in \mathcal{D}} \frac{[r_h^\tau + \overline{V}_{h+1}(x_{h+1}^\tau) - (\phi_h^\tau)^\top w]^2}{\overline{\sigma}_h^2(x_h^\tau, a_h^\tau, b_h^\tau)} + \lambda \|w\|_2^2, \quad \text{and} \quad \overline{\mathcal{T}}_h \overline{V}_{h+1}(o) := \phi(o)^\top \overline{w}_h
\end{aligned} \tag{9}$$

where (o) is short for (x,a,b), and similarly $\overline{\mathbb{P}}_h g_{h+1}, \underline{\mathbb{P}}_h g_{h+1}$ can be obtained by setting $r_h^\tau = 0$ in $\overline{\mathcal{T}}_h g_{h+1}$ and $\underline{\mathcal{T}} g_{h+1}$. Then, we estimate the Q-functions by LCB (max-player) and UCB (min-player):

$$\underline{Q}_h(o) \leftarrow \phi(o)^\top \underline{w}_h - \beta_3 \|\phi(o)\|_{\underline{\Sigma}_h^{-1}}, \qquad \overline{Q}_h(o) \leftarrow \phi(o)^\top \overline{w}_h + \beta_3 \|\phi(o)\|_{\overline{\Sigma}_h^{-1}}. \tag{10}$$

where we remark that they are pessimistic for the max-player and the min-player, respectively. Next, we solve the matrix games with payoffs $\underline{Q}_h$ and $\overline{Q}_h$: $(\widehat{\pi}_h(\cdot \mid \cdot), \nu_h'(\cdot \mid \cdot)) \leftarrow \mathrm{NE}(\underline{Q}_h(\cdot,\cdot,\cdot))$, and $(\pi_h'(\cdot \mid \cdot), \widehat{\nu}_h(\cdot \mid \cdot)) \leftarrow \mathrm{NE}(\overline{Q}_h(\cdot,\cdot,\cdot))$. The V-functions $\underline{V}_h$ and $\overline{V}_h$ are then given by $\mathbb{E}_{a \sim \widehat{\pi}_h(\cdot|\cdot), b \sim \nu_h'(\cdot|\cdot)} \underline{Q}_h(\cdot,a,b)$ and $\mathbb{E}_{a \sim \pi_h'(\cdot|\cdot), b \sim \widehat{\nu}_h(\cdot|\cdot)} \overline{Q}_h(\cdot,a,b)$, respectively. After H steps, the algorithm outputs the policy pair $(\widehat{\pi} = \{\widehat{\pi}_h\}_{h=1}^H, \widehat{\nu} = \{\widehat{\nu}_h\}_{h=1}^H)$ and value functions $(\underline{V} = \{\underline{V}_h\}_{h=1}^H, \overline{V} = \{\overline{V}_h\}_{h=1}^H)$.

Similar to the linear MDP, a sharper bonus leads to a smaller suboptimality gap. The techniques developed for MDPs can be separately applied to the max-player and min-player. Specifically, given $(\underline{V}_{h+1}, \overline{V}_{h+1})$, if we denote the Nash Value as V_{h+1}^*, we can decompose the uncertainties as follows:

$$\begin{aligned}
\mathcal{T}_h \underline{V}_{h+1}(o) - \underline{\mathcal{T}}_h \underline{V}_{h+1}(o) &= \mathcal{T}_h V_{h+1}^*(o) - \underline{\mathcal{T}}_h V_{h+1}^*(o) + \mathbb{P}_h(\underline{V}_{h+1} - V_{h+1}^*)(o) - \underline{\mathbb{P}}_h(\underline{V}_{h+1} - V_{h+1}^*)(o), \\
\mathcal{T}_h \overline{V}_{h+1}(o) - \overline{\mathcal{T}}_h \overline{V}_{h+1}(o) &= \mathcal{T}_h V_{h+1}^*(o) - \overline{\mathcal{T}}_h V_{h+1}^*(o) + \mathbb{P}_h(\overline{V}_{h+1} - V_{h+1}^*)(o) - \overline{\mathbb{P}}_h(\overline{V}_{h+1} - V_{h+1}^*)(o).
\end{aligned}$$

Algorithm 3 LinPMVI-ADV+

Initialize: Input datasets $\mathcal{D}$, $\mathcal{D}'$, and β_3; $\overline{V}_{H+1}(\cdot) = \underline{V}_{H+1}(\cdot) = 0$.
Construct variance estimator $\overline{\sigma}_h^2$ and $\underline{\sigma}_h^2$ as in Appendix A.1 with $\mathcal{D}'$.
for $h = H, \ldots, 1$ **do**
  $\underline{\Sigma}_h = \sum_{\tau\in\mathcal{D}} \frac{\phi_h^\tau(\phi_h^\tau)^\top}{\underline{\sigma}_h^2(x_h^\tau, a_h^\tau, b_h^\tau)} + \lambda I_d$, $\overline{\Sigma}_h = \sum_{\tau\in\mathcal{D}} \frac{\phi_h^\tau(\phi_h^\tau)^\top}{\overline{\sigma}_h^2(x_h^\tau, a_h^\tau, b_h^\tau)} + \lambda I_d$;
  $\underline{w}_h = \underline{\Sigma}_h^{-1}(\sum_{\tau\in\mathcal{D}} \phi_h^\tau(r_h^\tau + \underline{V}_{h+1}(x_{h+1}^\tau)))$, $\overline{w}_h = \overline{\Sigma}_h^{-1}(\sum_{\tau\in\mathcal{D}} \phi_h^\tau(r_h^\tau + \overline{V}_{h+1}(x_{h+1}^\tau)))$;
  $\underline{\Gamma}_h(\cdot,\cdot,\cdot) \leftarrow \beta_3 \|\phi(\cdot,\cdot,\cdot)\|_{\underline{\Sigma}_h^{-1}}$, $\overline{\Gamma}_h(\cdot,\cdot,\cdot) \leftarrow \beta_3 \|\phi(\cdot,\cdot,\cdot)\|_{\overline{\Sigma}_h^{-1}}$
  $\underline{Q}_h(\cdot,\cdot,\cdot) \leftarrow \{\phi(\cdot,\cdot,\cdot)^\top \underline{w}_h - \underline{\Gamma}_h(\cdot,\cdot,\cdot)\}_{[0,H-h+1]}$.
  $\overline{Q}_h(\cdot,\cdot,\cdot) \leftarrow \{\phi(\cdot,\cdot,\cdot)^\top \overline{w}_h + \overline{\Gamma}_h(\cdot,\cdot,\cdot)\}_{[0,H-h+1]}$.
  $(\widehat{\pi}_h(\cdot \mid \cdot), \nu_h'(\cdot \mid \cdot)) \leftarrow \mathrm{NE}(\underline{Q}_h(\cdot,\cdot,\cdot))$; $\underline{V}_h(\cdot) \leftarrow \langle \underline{Q}_h(\cdot,\cdot,\cdot), \widehat{\pi}_h(\cdot \mid \cdot) \times \nu_h'(\cdot \mid \cdot)\rangle_{\mathcal{A}\times\mathcal{B}}$.
  $(\pi_h'(\cdot \mid \cdot), \widehat{\nu}_h(\cdot \mid \cdot)) \leftarrow \mathrm{NE}(\overline{Q}_h(\cdot,\cdot,\cdot))$; $\overline{V}_h(\cdot) \leftarrow \langle \overline{Q}_h(\cdot,\cdot,\cdot), \pi_h'(\cdot \mid \cdot) \times \widehat{\nu}_h(\cdot \mid \cdot)\rangle_{\mathcal{A}\times\mathcal{B}}$.
end for
Output: $(\widehat{\pi} = \{\widehat{\pi}_h\}_{h=1}^H, \widehat{\nu} = \{\widehat{\nu}_h\}_{h=1}^H)$.

Then, for sufficiently large K, the uncertainty is dominated by the reference function part with V_{h+1}^* and we can set $\overline{\Gamma}_h(x,a,b) = \tilde{O}\left(\sqrt{d}\right) \|\phi(x,a,b)\|_{\overline{\Sigma}_h^{-1}}$ and $\underline{\Gamma}_h(x,a,b) = \tilde{O}\left(\sqrt{d}\right) \|\phi(x,a,b)\|_{\underline{\Sigma}_h^{-1}}$. The full pseudo code is presented in Algorithm 3 and we have the following theoretical guarantee.

Theorem 5 (LinPMVI-ADV+). *Under Assumption 1, for $K \geq \tilde{\Omega}(d^2H^6/\kappa)$, if we set $0 < \lambda < \kappa$ and $\beta_3 = \tilde{O}(\sqrt{d})$ in Algorithm 3, then with probability at least $1-\delta$, we have*

$$V_1^{*,\widehat{\nu}}(x) - V_1^{\widehat{\pi},*}(x) \leq \tilde{O}(\sqrt{d}) \cdot \Big(\max_{\nu} \sum_{h=1}^H \mathbb{E}_{\pi^*,\nu} \|\phi(x_h,a_h,b_h)\|_{\Sigma_h^{*-1}} + \max_{\pi} \sum_{h=1}^H \mathbb{E}_{\pi,\nu^*} \|\phi(x_h,a_h,b_h)\|_{\Sigma_h^{*-1}} \Big),$$

where (π^*, ν^*) is an NE and $\Sigma_h^* = \sum_{\tau\in\mathcal{D}} \phi_h^\tau(\phi_h^\tau)^\top / [\mathbb{V}_h V_{h+1}^*](x_h^\tau, a_h^\tau, b_h^\tau) + \lambda I_d$.

We establish the following nearly matching lower bound which tightens that in Zhong et al. (2022).

Theorem 6 (Lower Bound for MG). *Fix horizon H, dimension d, probability $\delta > 0$, and sample size $K \geq \tilde{\Omega}(d^4)$. There exists a class $\mathcal{M}$ of linear MGs and an offline dataset $\mathcal{D}$ with $|\mathcal{D}| = K$, such that for any policy pair $(\widehat{\pi}, \widehat{\nu})$, it holds with probability at least $1-\delta$ that*

$$\sup_{M\in\mathcal{M}} \mathbb{E}_M[V_1^{*,\widehat{\nu}}(x_1) - V_1^{\widehat{\pi},*}(x_1)] \geq c\sqrt{d} \cdot \Big(\max_{\nu} \sum_{h=1}^H \mathbb{E}_{\pi^*,\nu} \|\phi_h\|_{\Sigma_h^{*-1}} + \max_{\pi} \sum_{h=1}^H \mathbb{E}_{\pi,\nu^*} \|\phi_h\|_{\Sigma_h^{*-1}} \Big),$$

where $\Sigma_h^* = \sum_{\tau\in\mathcal{D}} \phi_h^\tau(\phi_h^\tau)^\top / [\mathbb{V}_h V_{h+1}^*](x_h^\tau, a_h^\tau, b_h^\tau) + \lambda I_d$ and $c > 0$ is a universal constant.

As $[\mathbb{V}_h V_{h+1}^*](\cdot,\cdot,\cdot) \geq 1$, we know that $\|\phi(x_h,a_h,b_h)\|_{\Sigma_h^{*-1}} \geq \|\phi(x_h,a_h,b_h)\|_{\Lambda_h^{-1}}$. Therefore, it improves the lower bound in Zhong et al. (2022) at least by a factor of $\sqrt{d}$. Therefore, LinPMVI-ADV+ is nearly minimax optimal when K exceeds the threshold specified in the theorem.

7 Conclusion and Future Work

In this paper, we study the linear MDPs and two-player zero-sum MGs in the offline setting. To solve the issues from complicated temporal dependency across the time steps, we adapt the reference-advantage decomposition under linear function approximation, which further plays an important role in leveraging the variance information. Moreover, two nearly matching lower bounds are provided to verify the minimax optimality of the proposed algorithms. For future work, an interesting direction is to extend the techniques to the general function approximation scenario like that of Wang et al. (2020b) where similar issues arise from the construction of step-wise bonus functions. Moreover, an additional interesting problem for future research is how to leverage the variance information under general non-linear function approximation.

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Algorithm 4 LinPMVI-ADV

Initialize: Input dataset $\mathcal{D}$, β_3; $\overline{V}_{H+1}(\cdot) = \underline{V}_{H+1}(\cdot) = 0$.
for $h = H, \ldots, 1$ **do**
  $\Lambda_h \leftarrow \sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau, b_h^\tau)\phi(x_h^\tau, a_h^\tau, b_h^\tau)^\top + \lambda I$.
  $\underline{w}_h \leftarrow \Lambda_h^{-1}(\sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau, b_h^\tau)\left(r_h^\tau + \underline{V}_{h+1}(x_{h+1}^\tau)\right))$.
  $\overline{w}_h \leftarrow \Lambda_h^{-1}(\sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau, b_h^\tau)\left(r_h^\tau + \overline{V}_{h+1}(x_{h+1}^\tau)\right))$.
  $\Gamma_h(\cdot,\cdot,\cdot) \leftarrow \beta_3 \cdot (\phi(\cdot,\cdot,\cdot)^\top \Lambda_h^{-1} \phi(\cdot,\cdot,\cdot))^{1/2}$.
  $\underline{Q}_h(\cdot,\cdot,\cdot) \leftarrow \{\phi(\cdot,\cdot,\cdot)^\top \underline{w}_h - \Gamma_h(\cdot,\cdot,\cdot)\}_{[0,H-h+1]}$.
  $\overline{Q}_h(\cdot,\cdot,\cdot) \leftarrow \{\phi(\cdot,\cdot,\cdot)^\top \overline{w}_h + \Gamma_h(\cdot,\cdot,\cdot)\}_{[0,H-h+1]}$.
  $(\widehat{\pi}_h(\cdot \mid \cdot), \nu_h'(\cdot \mid \cdot)) \leftarrow \mathrm{NE}(\underline{Q}_h(\cdot,\cdot,\cdot))$.
  $(\pi_h'(\cdot \mid \cdot), \widehat{\nu}_h(\cdot \mid \cdot)) \leftarrow \mathrm{NE}(\overline{Q}_h(\cdot,\cdot,\cdot))$.
  $\underline{V}_h(\cdot) \leftarrow \langle \underline{Q}_h(\cdot,\cdot,\cdot), \widehat{\pi}_h(\cdot \mid \cdot) \times \nu_h'(\cdot \mid \cdot)\rangle_{\mathcal{A}\times\mathcal{B}}$.
  $\overline{V}_h(\cdot) \leftarrow \langle \overline{Q}_h(\cdot,\cdot,\cdot), \pi_h'(\cdot \mid \cdot) \times \widehat{\nu}_h(\cdot \mid \cdot)\rangle_{\mathcal{A}\times\mathcal{B}}$.
end for
Output: $(\widehat{\pi} = \{\widehat{\pi}_h\}_{h=1}^H, \widehat{\nu} = \{\widehat{\nu}_h\}_{h=1}^H)$.

A Missing Algorithms and Discussions

A.1 Missing Algorithms

In this subsection, we first present the full pseudo code for LinPMVI-ADV+ by filling in the details of variance estimators. To begin with, we present Algorithm 4, the LinPMVI-ADV, which employs regular ridge regression in the algorithmic design but employs reference-advantage decomposition in the analysis. In particular, it corresponds to the algorithm described in Section 6.1 but with the variance estimators set as the naive 1. LinPMVI-ADV admits the following theoretical guarantee:

Theorem 7 (LinPMVI-ADV)**.** *Under Assumption 1, for $K > \tilde{\Omega}(d^2H^2/\kappa)$, if we set $\lambda = 1$ and $\beta_4 = \tilde{O}(\sqrt{d}H)$ in Algorithm 1, then with probability at least $1-\delta$, we have*

$$V_1^{*,\widehat{\nu}}(x) - V_1^{\widehat{\pi},*}(x) \leq \tilde{O}(\sqrt{d}H) \cdot \Big(\max_{\nu} \sum_{h=1}^H \mathbb{E}_{\pi^*,\nu} \|\phi_h\|_{\Lambda_h^{-1}} + \max_{\pi} \sum_{h=1}^H \mathbb{E}_{\pi,\nu^*} \|\phi_h\|_{\Lambda_h^{-1}} \Big), \quad (11)$$

where (π^*, ν^*) is an NE and $\Lambda_h = \sum_{\tau \in \mathcal{D}} \phi(x_h^\tau, a_h^\tau, b_h^\tau)\phi(x_h^\tau, a_h^\tau, b_h^\tau)^\top + \lambda I_d$.

Construction of the Variance Estimators. To begin with, we run Algorithm 4 to construct $\{\overline{V}_h', \underline{V}_h'\}_{h=1}^H$ with an independent dataset $\mathcal{D}'$. This will only incur a factor of 2 in the final sample complexity. Similar to Lemma 1, we can show that there exist $\beta_{h,1}(\overline{V}_{h+1}')$ and $\beta_{h,2}(\overline{V}_{h+1}')$ such that $[\mathbb{P}_h(\overline{V}_{h+1}')^2](x,a) = \left\langle \phi(x,a), \beta_{h,2}(\overline{V}_{h+1}') \right\rangle$ and $[\mathbb{P}_h \overline{V}_{h+1}'](x,a) = \left\langle \phi(x,a), \beta_{h,1}(\overline{V}_{h+1}') \right\rangle$. We approximate them via ridge regression with $\mathcal{D}'$:

$$\widetilde{\beta}_{h,2} = \operatorname*{argmin}_{\beta \in \mathbb{R}^d} \sum_{\tau \in \mathcal{D}'} \left[\langle \phi_h^\tau, \beta \rangle - (\overline{V}_{h+1}')^2 \left(x_{h+1}^\tau\right) \right]^2 + \lambda \|\beta\|_2^2,$$

$$\widetilde{\beta}_{h,1} = \operatorname*{argmin}_{\beta \in \mathbb{R}^d} \sum_{\tau \in \mathcal{D}'} \left[\langle \phi_h^\tau, \beta \rangle - \overline{V}_{h+1}' \left(x_{h+1}^\tau\right) \right]^2 + \lambda \|\beta\|_2^2.$$

Then, the variance estimator is then constructed as

$$\overline{\sigma}_h^2(x,a) := \max\left\{1, \left[\phi(x,a)^\top \widetilde{\beta}_{h,2}\right]_{[0,H^2]} - \left[\phi(x,a)^\top, \widetilde{\beta}_{h,1}\right]_{[0,H]}^2 - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right)\right\}.$$

Similarly, we can construct the variance estimator $\underline{\sigma}_h^2(x,a)$ with $\underline{V}_{h+1}'$ and dataset $\mathcal{D}'$. In particular, as $\overline{\sigma}_h(\cdot,\cdot)$ and $\underline{\sigma}_h(\cdot,\cdot)$ only depend on the dataset $\mathcal{D}'$, they are independent of $\mathcal{D}$.

A.2 Discussions

Additional $\sqrt{d}$ Incurred by Uniform Concentration. To determine the value of β in PEVI, we need to establish the concentration of $\left\|\sum_{\tau\in\mathcal{D}}\phi\left(x_h^\tau,a_h^\tau\right)\cdot\xi_h^\tau(\widehat{V}_{h+1})\right\|_{\Lambda_h^{-1}}$ where $\xi_h^\tau(\widehat{V}_{h+1}) := r_h^\tau+\widehat{V}_{h+1}(x_{h+1}^\tau)-(\mathcal{T}_h\widehat{V}_{h+1})(x_h^\tau,a_h^\tau)$ and $\Lambda_h=\sum_{\tau\in\mathcal{D}}\phi\left(x_h^\tau,a_h^\tau\right)\phi\left(x_h^\tau,a_h^\tau\right)^\top+\lambda I_d$. For a fixed function V, Lemma 9 can provide us with a bound of $\tilde{O}\left(\sqrt{d}H\right)$. But it cannot be directly applied to $\widehat{V}_{h+1}$ because $\widehat{V}_{h+1}$ is computed by the ridge regression in the steps $[h+1,H]$ and thus depends on $\{(x_h^\tau,a_h^\tau,x_{h+1}^\tau)\}_{\tau\in\mathcal{D}}$ due to temporal dependency. Therefore, the martingale cannot be well defined as $\widehat{V}_{h+1}$ is not well measurable. To solve this issue, the standard approach is to construct a uniform concentration over all elements in an ϵ-covering of the following function class:

$$\mathcal{V}_{h+1}:=\max_a\{\phi(\cdot,a)^\top w+\beta\left\|\phi(\cdot,a)\right\|_{\Lambda_h^{-1}},\|w\|\le R,\beta\in[0,B],\Lambda\succeq\lambda\cdot I\}_{[0,H-h+1]}.$$

Then, self-normalized process is bounded by the uniform bound of the ϵ-covering, plus some approximation error. We can balance two parts by tuning $\epsilon>0$ and obtain that (Eqn. B.20 of Jin et al. (2021c)):

$$\begin{aligned}\left\|\sum_{\tau=1}^K\phi\left(x_h^\tau,a_h^\tau\right)\cdot\epsilon_h^\tau\left(V\right)\right\|_{\Lambda_h^{-1}}^2 &\le cH^2\cdot\left(\log\left(H\mathcal{N}_{h+1}(\epsilon)/\delta\right)+d\log(1+K)+d^2\right)\\ &\le cH^2\cdot\Big(\underbrace{d^2}_{\text{Uniform conver.}}+\underbrace{d}_{\text{Conver. + fixed }V}+\underbrace{d^2}_{\text{Approximation error}}\Big),\end{aligned}$$

where we use an upper bound of the covering number (Lemma 11), and omit all the logarithmic terms for a clear comparison. As we can see, the requirement of uniform concentration leads to two d^2 terms on the right-hand side. Therefore, the bottleneck of the analysis lies in the uniform concentration due to the temporal dependency and avoiding the uniform concentration leads to an improvement of $\sqrt{d}$.

On the Choice of Reference Function. Similar to the tabular MDP case Xie et al. (2021b), we can also explicitly construct an independent reference function and then apply the reference-advantage decomposition. Specifically, we can the dataset into 3 parts: $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3$ and constructs a reference function $\widehat{V}_{h+1}^{\text{ref}}$ explicitly via $\mathcal{D}_1$ by e.g. PEVI. Then, we further split $\mathcal{D}_3$ equally into H subsets $\mathcal{D}_{3,h}$. We set $\widehat{\mathcal{T}}_h\widehat{V}_{h+1}=\widehat{\mathcal{T}}_h^0\widehat{V}_{h+1}^{\text{ref}}+\widehat{\mathbb{P}}_h^1(\widehat{V}_{h+1}-\widehat{V}_{h+1}^{\text{ref}})$

$$\left|\left(\widehat{\mathcal{T}}_h-\mathcal{T}_h\right)\widehat{V}_{h+1}(x,a)\right|\le\left|\left(\widehat{\mathcal{T}}_h^0-\mathcal{T}_h\right)\widehat{V}_{h+1}^{\text{ref}}(x,a)\right|+\left|\left(\widehat{\mathbb{P}}_h^1-\mathbb{P}_h\right)(\widehat{V}_{h+1}-\widehat{V}_{h+1}^{\text{ref}})(x,a)\right|,\quad(12)$$

where $\widehat{\mathcal{T}}_h^0\widehat{V}_{h+1}^{\text{ref}}$ is estimated by $\mathcal{D}_2$ and $\widehat{\mathbb{P}}_h^1(\widehat{V}_{h+1}-\widehat{V}_{h+1}^{\text{ref}})$ is estimated by $\mathcal{D}_{3,h}$. The advantage is that as $\widehat{V}_{h+1}^{\text{ref}}$ is available to the executed algorithm, we can estimate $\mathbb{P}_h(\widehat{V}_{h+1}-\widehat{V}_{h+1}^{\text{ref}})$ explicitly by $\mathcal{D}_{3,h}$. As $\widehat{V}_{h+1}-\widehat{V}_{h+1}^{\text{ref}}$ depends on $\{\mathcal{D}_{3,h'}:h'\ge h+1\}$ and $\mathcal{D}_1$ so is independent of $\mathcal{D}_h$. In this case, the temporal dependency in the advantage function part is also cancelled, and we also do not need uniform concentration for the advantage function. The analysis for the advantage function would be rather easy and short and a tighter bonus for the advantage function can be achieved. The main drawback may lie in that the algorithm is rather complicated. On the other hand, the algorithmic designs presented here are more similar to the original PEVI (Jin et al., 2021c) and PMVI (Zhong et al., 2022).

On the Large-Sample Regime. The reference-advantage decomposition is of advantages only when K exceeds certain thresholds, which is common in the literature (Shi et al., 2022; Zhou et al., 2021; Xie et al., 2021b; Hu et al., 2022). Instead of decomposing the uncertainty into an optimal reference part and a high-order advantage part, Li et al. (2022) introduces a novel two-fold subsampling technique to directly cancel the temporal dependency among time steps h and thus circumvents the uniform concentration. It would be interesting to see whether the technique can be extended to the linear setting and more general scenarios of function approximation.

B Related Work

We review existing works that are closely related to our paper in this section.

Offline RL. The principle of pessimism is first used by Jin et al. (2021c) to empower efficient offline learning under only partial coverage. It shows that we can design efficient offline RL algorithm with only a sufficient coverage over the optimal policy, instead of the previous uniform one required by Precup (2000); Antos et al. (2008); Levine et al. (2020). After that, a line of work (Rashidinejad et al., 2021; Yin and Wang, 2021; Uehara et al., 2021; Zanette et al., 2021; Xie et al., 2021a; Uehara and Sun, 2021; Shi et al., 2022; Li et al., 2022) leverages the principle of pessimism, either in the tabular case or in the case with function approximation.

Offline tabular RL. For tabular MDP, a line of works have incorporated the principle of pessimism to design efficient offline RL algorithms (Rashidinejad et al., 2021; Yin and Wang, 2021; Xie et al., 2021b; Shi et al., 2022; Li et al., 2022). In particular, Xie et al. (2021b) proposes a variance-reduction offline RL algorithm for tabular MDP which is nearly optimal after the total sample size exceeds certain threshold. After that, Li et al. (2022) proposed an algorithm that is nearly optimal by introducing a novel subsampling trick to cancel the temporal dependency among time steps. Shi et al. (2022) proposed the first nearly optimal model-free offline RL algorithm.

Offline RL with function approximation. For RL problems with linear function approximation, Jin et al. (2021c) designs the first pessimism-based efficient offline algorithm for linear MDP. After that, Min et al. (2021) considers the linear mixture MDP and designs a novel offline algorithm which incorporates the variance information of the value function to improve the sample efficiency. This technique is later adopted by Yin et al. (2022). However, Min et al. (2021); Yin et al. (2022) depend (explicitly or implicitly) on an assumption that the data samples are independent across different time steps h so they do not need to handle the temporal dependency, which considerably complicates the analysis. Moreover, such an assumption is not very realistic when the dataset is collected by some behavior policy by interacting with the underlying MDP. Therefore, it remains open whether we can design computationally efficient algorithms that achieve minimax optimal sample efficiency for offline learning with linear MDP. Beyond the linear function approximation, Xie et al. (2021a); Uehara and Sun (2021) propose pessimistic offline RL algorithms with general function approximation. However, their works are only information-theoretic AS they require an optimization subroutine over the general function class which is computationally intractable in general.

Offline MGs. The existing works studying sample-efficient equilibrium finding in offline MARL include Zhong et al. (2021); Chen et al. (2021); Cui and Du (2022); Zhong et al. (2022). Among these works, Cui and Du (2022) and Zhong et al. (2022) are most closely related to our algorithm where they study the offline two-player zero-sum MGs in the tabular and linear case, respectively. In terms of the minimal dataset coverage assumption, both of them identify that the unilateral concentration, i.e., a good coverage on the $\{(\pi^*, \nu), (\pi, \nu^*) : (\pi^*, \nu^*)$ is an NE and (π, ν) is arbitrary.$\}$, is necessary and sufficient for efficient offline learning. In terms of sample complexity, similarly, while for tabular game, Cui and Du (2022) is nearly optimal in terms of the dependency on the number of state, there is a $\tilde{O}(\sqrt{d}H)$ gap between the upper and lower bounds for linear MG (Zhong et al., 2022).

Online RL with function approximation. Jin et al. (2020) and Xie et al. (2020) propose the first provably efficient algorithm for online linear MDP and linear MG, respectively. However, there is a gap between their regret bounds and existing lower bounds where we remark that similar issues of temporal dependency also exists in the analysis of online algorithms for linear MDP and linear MG. For instance, in Lemma C.3 of Jin et al. (2020), they also need to analyze the self-normalized process where the uniform concentration leads to an extra factor of $\sqrt{d}$. The recent work Hu et al. (2022) leverages similar ideas of reference-advantage decomposition, trying to improve the regret bound for the linear MDP but focus on the online setting, whose algorithmic design (optimism v.s. pessimism, choices of weights) and proof techniques (online v.s. offline) are different from ours. Chen et al. (2022) considers the linear mixture MG, which is different from the model considered in this paper. Beyond the linear function approximation, several works on MDP (Jiang et al., 2017; Jin et al., 2021a; Dann et al., 2021; Du et al., 2021; Foster et al., 2021) and MG (Jin et al., 2021b; Huang et al., 2021; Xiong et al., 2022) design algorithms in the online setting with general function approximation. When applied to the linear setting, though their regret bounds are sharper than that of Jin et al. (2020); Xie et al. (2020), their algorithms are only information-theoretic and are computationally inefficient.

C Auxiliary Lemmas

In this section, we provide several useful lemmas to facilitate the proof of linear MDP.

Lemma 2 (Regret Decomposition Lemma for MDP)**.** *Under the condition that with probability at least $1-\delta$, the functions $\Gamma_h : \mathcal{S}\times\mathcal{A}\to\mathbb{R}$ in Algorithms 1 and 2 ($\Gamma_h = b_{0,h} + b_{1,h}$) satisfying*

$$|\mathcal{T}_h\widehat{V}_{h+1}(x,a) - \widehat{\mathcal{T}}_h\widehat{V}_{h+1}(x,a)| \le \Gamma_h(x,a), \quad \forall (x,a,h)\in\mathcal{S}\times\mathcal{A}\times[H],$$

we have that with probability at least $1-\delta$, for Algorithm 1 and Algorithm 2 that for any $x\in\mathcal{S}$,

$$V_1^*(x) - V_1^{\widehat{\pi}}(x) \le V_1^*(x) - \widehat{V}_1(x) \le 2\sum_{h=1}^{H}\mathbb{E}_{\pi^*}[\Gamma_h(x_h,a_h)\mid x_1 = x].$$

Proof. See Lemma 3.1 and Theorem 4.2 in Jin et al. (2021c) for a detailed proof. □

Lemma 3 (Decomposition)**.** *For a function f_{h+1}, suppose that $\mathcal{T}_h f_{h+1}(\cdot,\cdot) = \phi(\cdot,\cdot)^\top w_h$. With $\widehat{w}_h = \Sigma_h^{-1}\left(\sum_{\tau\in\mathcal{D}}\frac{\phi(x_h^\tau,a_h^\tau)\cdot\left(r_h^\tau + f_{h+1}(x_{h+1}^\tau)\right)}{\widehat{\sigma}_h^2(x_h^\tau,a_h^\tau)}\right)$ where $\widehat{\sigma}_h^2(x_h^\tau,a_h^\tau)$ can be either 1 (for regular ridge regression) or the estimated variance (for variance-reweighted ridge regression) and $\Sigma_h = \sum_{\tau\in\mathcal{D}}\frac{\phi(x_h^\tau,a_h^\tau)\phi(x_h^\tau,a_h^\tau)^\top}{\widehat{\sigma}_h^2(x_h^\tau,a_h^\tau)} + \lambda I_d$. Then, we have the following decomposition:*

$$\begin{aligned}\left(\mathcal{T}_h f_{h+1} - \widehat{\mathcal{T}}_h f_{h+1}\right)(x,a) &= \phi(x,a)^\top w_h - \phi(x,a)^\top\widehat{w}_h \\ &\le \lambda\left\|w_h\right\|_{\Sigma_h^{-1}}\left\|\phi(x,a)\right\|_{\Sigma_h^{-1}} + \left\|\sum_{\tau\in\mathcal{D}}\frac{\phi\left(x_h^\tau,a_h^\tau\right)}{\widehat{\sigma}_h(x_h^\tau,a_h^\tau)}\xi_h^\tau(f_{h+1})\right\|_{\Sigma_h^{-1}}\left\|\phi(x,a)\right\|_{\Sigma_h^{-1}},\end{aligned} \tag{13}$$

where $\xi_h^\tau(f_{h+1}) = \frac{r_h^\tau + f_{h+1}(x_{h+1}^\tau) - \mathcal{T}_h f_{h+1}(x_h^\tau,a_h^\tau)}{\widehat{\sigma}_h(x_h^\tau,a_h^\tau)}$. In particular, if $|f_{h+1}|$ is bounded by $H-1$ and $\left\|\sum_{\tau\in\mathcal{D}}\frac{\phi(x_h^\tau,a_h^\tau)}{\widehat{\sigma}_h(x_h^\tau,a_h^\tau)}\cdot\xi_h^\tau(f_{h+1})\right\|_{\Sigma_h^{-1}} \le \beta$, then we can set λ sufficiently small so that $\sqrt{\lambda d}H \le \beta$. In this case, the second term is dominating. $\mathbb{P}_h f_{h+1}(\cdot,\cdot)$ admits similar results by setting $r_h \equiv 0$.

Proof. See Appendix I for a detailed proof. □

D Proofs of LinPEVI-ADV

D.1 Proof of Theorem 1

The proof requires a more refined induction analysis to deal with the temporal dependency. For instance, when analyzing the h-step, we cannot take the condition on that $\left\|\widehat{V}_{h+1} - V_{h+1}^*\right\|$ is small, which is required to make sure that the uncertainty of the advantage function is non-dominating. This is because due to the temporal dependency, taking condition on $\widehat{V}_{h+1}$ may influence the distribution at step h. A carefully crafted induction analysis is employed to solve the challenge.

Proof of Theorem 1. We will prove the theorem by induction. For $h = H$, for a function $\|g_{h+1}\|_\infty \le R-1$, we invoke Lemma 3:

$$\left|\mathcal{T}_h g_{h+1} - \widehat{\mathcal{T}}_h g_{h+1}\right|(x,a) \le \underbrace{\sqrt{\lambda d}R\left\|\phi(x,a)\right\|_{\Sigma_h^{-1}}}_{\text{(a)}} + \underbrace{\left\|\sum_{\tau\in\mathcal{D}}\phi\left(x_h^\tau,a_h^\tau\right)\cdot\xi_h^\tau(g_{h+1})\right\|_{\Sigma_h^{-1}}\left\|\phi(x,a)\right\|_{\Sigma_h^{-1}}}_{\text{(b)}}.$$

For the reference part with $g_{h+1} = V_{H+1}^*$, since V_{H+1}^* is independent of $\mathcal{D}$, we can directly apply Lemma 9 with $\lambda = 1$ to obtain that with probability at least $1-\delta_H = 1-\delta/H^2$,

$$\left|\mathcal{T}_H V_{H+1}^* - \widehat{\mathcal{T}}_H V_{H+1}^*\right|(x,a) \le 2\sqrt{d}H\sqrt{\iota}\left\|\phi(x,a)\right\|_{\Lambda_H^{-1}},$$

where $\iota = \log(2H^2K/\delta) \ge 1$. To simplify the proof, we set $b_{0,H}(x,a) = 3\sqrt{d}H\sqrt{\iota}\left\|\phi(x,a)\right\|_{\Lambda_H^{-1}}$ to further capture the uncertainty of (a) from the advantage function. Then, we can focus

on the analysis of (b) for the advantage function. By construction, we have $\mathcal{E}_{H+1} = \{0 \leq V^*_{H+1} - \widehat{V}_{H+1} \leq \frac{8\sqrt{d}H\cdot 0\sqrt{\iota}}{\sqrt{K\kappa}}\}$ holds with probability 1. By Lemma 9, it suffices to set $b_{1,H}(x,a) = \frac{8d^{3/2}H^2\iota}{\sqrt{K\kappa}}\|\phi(x,a)\|_{\Lambda_H^{-1}}$. It follows that we can set

$$\Gamma_H(\cdot,\cdot) = b_{0,H}(\cdot,\cdot)+b_{1,H}(\cdot,\cdot) = \left(3\sqrt{d}H\sqrt{\iota} + \frac{8d^{3/2}H^2\iota}{\sqrt{K\kappa}}\right)\|\phi(\cdot,\cdot)\|_{\Lambda_H^{-1}} \leq 4\sqrt{d}H\sqrt{\iota}\,\|\phi(\cdot,\cdot)\|_{\Lambda_H^{-1}},$$

where we use $K \geq \tilde{\Omega}(d^2H^2/\kappa)$ to obtain the last inequality. If pessimism at step H is achieved, we know that

$$Q^*_H(x,a) = \mathcal{T}_H V^*_{H+1}(x,a) \geq \mathcal{T}_H \widehat{V}_{H+1}(x,a) \geq \widehat{Q}_H(x,a), \forall (x,a) \in \mathcal{S}\times\mathcal{A},$$

where the last step is due to $|\mathcal{T}_h\widehat{V}_{H+1}(x,a) - \widehat{\mathcal{T}}_h\widehat{V}_{H+1}(x,a)| \leq \Gamma_H(x,a)$. This implies that $V^*_H(x) \geq \widehat{V}_H(x)$ for all $x \in \mathcal{S}$ and we proceed to bound the error as follows:

$$\begin{aligned}
&V^*_H(x) - \widehat{V}_H(x) = \left\langle Q^*_H(x,\cdot) - \widehat{Q}_H(x,\cdot), \pi^*_H(\cdot|x)\right\rangle + \left\langle \widehat{Q}_H(x,\cdot), \pi^*_H(\cdot|x) - \widehat{\pi}_H(\cdot|x)\right\rangle \\
&\leq \left\langle \mathcal{T}_H\widehat{V}_{H+1}(x,\cdot) - \widehat{\mathcal{T}}_H\widehat{V}_{H+1}(x,\cdot) + \Gamma_H(x,\cdot), \pi^*_H(\cdot|x)\right\rangle + \left\langle \mathcal{T}_H(V^*_{h+1} - \widehat{V}_{H+1})(x,\cdot), \pi^*_H(\cdot|x)\right\rangle \\
&\leq 2\mathbb{E}_{\pi^*}[\Gamma_H(\cdot,\cdot)|x_H = x] + 0, \\
&\leq \frac{8\sqrt{d}H\cdot 1\sqrt{\iota}}{\sqrt{K\kappa}} := R_H, \forall x \in \mathcal{S},
\end{aligned}$$

where the last inequality uses Lemma 13. To summarize, the event $\mathcal{E}_H = \{0 \leq V^*_H(x) - \widehat{V}_H(x) \leq R_H, \forall x \in \mathcal{S}\}$ holds with probability at least $1-\delta_H = 1 - \frac{\delta}{H^2}$. This is the base case. Now suppose that the event $\mathcal{E}_{h+1} = \{0 \leq V^*_{h+1}(x) - \widehat{V}_{h+1}(x) \leq R_{h+1} := \frac{8\sqrt{d}H(H-h)\sqrt{\iota}}{\sqrt{K\kappa}}, \forall x \in \mathcal{S}\}$ holds with probability at least $1-\delta_{h+1}$. We are going to establish the result for step h. Clearly, we can still set $b_{0,h}(\cdot,\cdot) = 3\sqrt{d}H\sqrt{\iota}\,\|\phi(\cdot,\cdot)\|_{\Lambda_h^{-1}}$. It remains to determine $b_{1,h}(\cdot,\cdot)$ for (b) of the advantage function and to ensure that it is non-dominating. We need to deal with the temporal dependency, which requires a more involved analysis. We first state the following lemma.

Lemma 4 (Lemma B.2 of Jin et al. (2021c))**.** *Let $f : \mathcal{S} \to [0, R-1]$ be any fixed function. For any $\delta \in (0,1)$, we have*

$$\mathbb{P}\left(\left\|\sum_{\tau\in\mathcal{D}} \phi^\tau_h \cdot \xi^\tau_h(f)\right\|^2_{\Lambda_h^{-1}} \geq R^2(2\log(\frac{1}{\delta}) + d\log(1+\frac{K}{\lambda}))\right) \leq \delta.$$

However, as $(\widehat{V}_{h+1} - V^*_{h+1})$ is correlated to $\{(x^\tau_h, a^\tau_h, x^\tau_{h+1})\}_{\tau\in\mathcal{D}}$, we need a uniform concentration argument. In particular, we remark that we cannot take condition that $V^*_{h+1} - \widehat{V}_{h+1} \leq R_{h+1}$ directly. We consider the function class

$$\begin{aligned}
&\mathcal{V}_h(D,B,\lambda) = \{V_h(x;\theta,\beta,\Sigma) : \mathcal{S} \to [0,H] \text{ with } \|\theta\| \leq D, \beta \in [0,B], \Sigma \succeq \lambda\cdot I\}, \\
&\text{where } V_h(x;\theta,\beta,\Sigma) = \max_{a\in\mathcal{A}}\left\{\phi(x,a)^\top\theta - \beta\cdot\sqrt{\phi(x,a)^\top\Sigma^{-1}\phi(x,a)}\right\}_{[0,H-h+1]}.
\end{aligned} \tag{14}$$

With $V^*_{h+1} - \widehat{V}_{h+1}$ denoted as f, we can estimate D as follows:

$$\begin{aligned}
\|\widehat{w}_h(f)\| &= \left\|\Lambda_h^{-1}\left(\sum_{\tau\in\mathcal{D}}\phi(x^\tau_h,a^\tau_h)\cdot(r^\tau_h + f(x^\tau_{h+1}))\right)\right\| \\
&\leq H\sum_{\tau\in\mathcal{D}}\sqrt{\phi(x^\tau_h,a^\tau_h)^\top\Lambda_h^{-1/2}\Lambda_h^{-1}\Lambda_h^{-1/2}\phi(x^\tau_h,a^\tau_h)} \\
&\leq H\sqrt{\frac{K}{\lambda}}\sqrt{\sum_{\tau\in\mathcal{D}}\phi(x^\tau_h,a^\tau_h)^\top\Lambda_h^{-1}\phi(x^\tau_h,a^\tau_h)} \\
&= H\sqrt{\frac{K}{\lambda}}\sqrt{\mathrm{Tr}\left(\Lambda_h^{-1}(\Lambda_h - \lambda I_d)\right)} \leq H\sqrt{\frac{Kd}{\lambda}}.
\end{aligned}$$

It follows that

$$f = V_{h+1}^* - \widehat{V}_{h+1} \in \mathcal{F}_{h+1} := \{V_{h+1}^* - V_{h+1} : V_{h+1} \in \mathcal{V}_{h+1}(D_0, B_0, \lambda)\}$$

where $D_0 = H\sqrt{\frac{Kd}{\lambda}}, \lambda = 1$, and $B_0 = 8\sqrt{d}H\sqrt{\iota}$. For any $\epsilon > 0$, we denote the ϵ-cover of $\mathcal{F}_{h+1}$ with respect to the supremum norm as $\mathcal{N}_{h+1}(\epsilon)$ (short for $\mathcal{N}_{h+1}(\epsilon; D, B, \lambda)$) and its ϵ-covering number as $|\mathcal{N}_{h+1}(\epsilon)|$. For each $f \in \mathcal{F}_{h+1}$, we can find $f_\epsilon \in \mathcal{N}_{h+1}(\epsilon)$, such that $\sup_{x\in\mathcal{S}} |f(x) - f_\epsilon(x)| \le \epsilon$. It follows that

$$\begin{aligned}
&\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot \xi_h^\tau(f)\right\|_{\Lambda_h^{-1}}^2 \mathbf{1}\left\{\|f\|_\infty \le R_{h+1}\right\} \\
&\le 2\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot \xi_h^\tau(f_\epsilon)\right\|_{\Lambda_h^{-1}}^2 \mathbf{1}\left\{\|f_\epsilon\|_\infty \le (R_{h+1}+\epsilon)\right\} + 2\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot (\xi_h^\tau(f) - \xi_h^\tau(f_\epsilon))\right\|_{\Lambda_h^{-1}}^2 \\
&\le 2\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot \xi_h^\tau(f_\epsilon)\right\|_{\Lambda_h^{-1}}^2 \mathbf{1}\left\{\|f_\epsilon\|_\infty \le (R_{h+1}+\epsilon)\right\} + 2\epsilon^2 K^2/\lambda,
\end{aligned}$$

where the first inequality uses $\{\|f\|_\infty \le R_{h+1}\}$ implies $\{\|f_\epsilon\|_\infty \le (R_{h+1}+\epsilon)\}$ and the second inequality is because the following estimation of the second term:

$$\begin{aligned}
2\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot (\xi_h^\tau(f) - \xi_h^\tau(f_\epsilon))\right\|_{\Lambda_h^{-1}}^2 &\le 2\epsilon^2 \sum_{\tau,\tau'=1}^{K} |\phi_h^\tau \Lambda_h^{-1} \phi_h^{\tau'}| \le 2\epsilon^2 \|\phi_h^\tau\| \|\phi_h^{\tau'}\| \left\|\Lambda_h^{-1}\right\|_{\mathrm{op}} \\
&\le 2\epsilon^2 K^2/\lambda,
\end{aligned}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm and $\left\|\Lambda_h^{-1}\right\|_{\mathrm{op}} \le \lambda^{-1}$. With a union bound over $\mathcal{N}_{h+1}(\epsilon)$ and Lemma 4, we obtain that

$$\begin{aligned}
\mathbb{P}\Bigg(\sup_{f_\epsilon\in\mathcal{N}_{h+1}(\epsilon)} &\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot \xi_h^\tau(f_\epsilon)\right\|_{\Lambda_h^{-1}}^2 \mathbf{1}\left\{\|f_\epsilon\|_\infty \le R_{h+1}+\epsilon\right\} > \\
&(R_{h+1}+\epsilon)^2 (2\log(\frac{H^2 \cdot |\mathcal{N}_{h+1}(\epsilon)|}{\delta}) + d\log(1+\frac{K}{\lambda}))\Bigg) \le \delta/H^2.
\end{aligned}$$

With probability at least $1 - \delta/H^2$, for all $f \in \mathcal{F}_{h+1}$, we have

$$\begin{aligned}
&\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \cdot \xi_h^\tau(f)\right\|_{\Lambda_h^{-1}}^2 \mathbf{1}\left\{\|f\|_\infty \le R_{h+1}\right\} \\
&\le \inf_{\epsilon>0}\left\{\left(\frac{8\sqrt{d}H^2\sqrt{\iota}}{\sqrt{K\kappa}} + \epsilon\right)^2 \left(2\log(\frac{H^2 \cdot |\mathcal{N}_{h+1}(\epsilon)|}{\delta}) + d\log(1+\frac{K}{\lambda})\right) + 2\epsilon^2 K^2/\lambda\right\} \qquad (15) \\
&\le \left(\frac{128dH^4\iota}{K\kappa}\right)\left(2\log(H^2/\delta) + 4d^2\log(\frac{512K^3\iota}{d^{3/2}H^2})\right) + \frac{2d^3H^4}{K\kappa} \\
&\le \left(\frac{256dH^4\iota}{K\kappa}\right)\left(2\log(H^2/\delta) + 4d^2\log(\frac{512K^3\iota}{d^{3/2}H^2})\right),
\end{aligned}$$

where the second inequality is because the covering number of $\mathcal{F}_{h+1}$ is bounded by that of $\mathcal{V}_{h+1}(D, B, \lambda)$ and by Lemma 11, we can take $\epsilon = d^{3/2}H^2/(K^{3/2}\sqrt{\kappa})$ to obtain

$$\log|\mathcal{N}_{h+1}(\epsilon)| \le d\log(1+\frac{4K^2\sqrt{\kappa}}{dH}) + d^2\log(1+\frac{512K^3\kappa\iota}{d^{3/2}H^2}) \le 2d^2\log(\frac{512K^3\iota}{d^{3/2}H^2}),$$

where the second inequality holds when $K > \sqrt{d}H/(128\sqrt{\kappa}\iota)$. As $\iota > 1$ and $\log\iota \le \iota$, it further holds that

$$\begin{aligned}
\left(\frac{256dH^4\iota}{K\kappa}\right)\left(2\log(H^2/\delta) + 4d^2\log(\frac{512K^3\iota}{d^{3/2}H^2})\right) &\le \left(\frac{256dH^4\iota}{K\kappa}\right)\left(2\iota + 4d^2\left(\iota + \log(512)\iota + 3\iota\right)\right) \\
&\le \frac{8704d^3H^4\iota^2}{K\kappa},
\end{aligned}$$

To summarize, it suffices to set $b_{1,h} = \frac{94d^{3/2}H^2\iota}{\sqrt{K\kappa}} \|\phi(x,a)\|_{\Lambda_h^{-1}}$ and we have

$$\begin{aligned}
&\mathbb{P}\left(\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \xi_h^\tau (V_{h+1}^* - \widehat{V}_{h+1})\right\| > \frac{94d^{3/2}H^2\iota}{\sqrt{K\kappa}}\right) \\
&\leq \mathbb{P}\left(\left\|\sum_{\tau\in\mathcal{D}} \phi_h^\tau \xi_h^\tau (V_{h+1}^* - \widehat{V}_{h+1})\right\| \mathbf{1}\left\{\left\|V_{h+1}^* - \widehat{V}_{h+1}\right\|_\infty \leq R_{h+1}\right\} > \frac{94d^{3/2}H^2\iota}{\sqrt{K\kappa}}\right) \\
&\quad + \mathbb{P}\left(\mathbf{1}\left\{\left\|V_{h+1}^* - \widehat{V}_{h+1}\right\|_\infty > R_{h+1}\right\}\right) \\
&\leq \delta/H^2 + \delta_{h+1} := \delta_h,
\end{aligned}$$

where δ_{h+1} is the failure probability at step $h+1$. As $K > \tilde{\Omega}\left(d^2H^2/\kappa\right)$, we can set

$$\Gamma_h(\cdot,\cdot) \leq \left(3\sqrt{d}H\sqrt{\iota} + \frac{94d^{3/2}H^2\iota}{\sqrt{K\kappa}}\right) \|\phi(\cdot,\cdot)\|_{\Lambda_h^{-1}} \leq 4\sqrt{d}H\sqrt{\iota}\, \|\phi(\cdot,\cdot)\|_{\Lambda_h^{-1}},$$

With $R_h := \frac{8\sqrt{d}H(H-h+1)\sqrt{\iota}}{\sqrt{K\kappa}}$, we proceed to analyze the failure probability of $\mathcal{E}_h = \{0 \leq V_{h+1}^*(x) - \widehat{V}_{h+1}(x) \leq R_h, \forall x \in \mathcal{S}\}$. First of all, if $|\mathcal{T}_h\widehat{V}_{h+1} - \widehat{\mathcal{T}}_h\widehat{V}_{h+1}| \leq \Gamma_h$ and event $\mathcal{E}_{h+1}$ holds, we know that

$$Q_h^*(x,a) = \mathcal{T}_h V_{h+1}^*(x,a) \geq \mathcal{T}_h\widehat{V}_{h+1}(x,a) \geq \widehat{Q}_h(x,a), \forall (x,a) \in \mathcal{S}\times\mathcal{A},$$

and thus $V_h^*(x) \geq \widehat{V}_h(x)$ for all $x \in \mathcal{S}$. We also have

$$\begin{aligned}
&V_h^*(x) - \widehat{V}_h(x) = \left\langle Q_h^*(x,\cdot) - \widehat{Q}_h(x,\cdot), \pi_h^*(\cdot|x)\right\rangle + \left\langle \widehat{Q}_h(x,\cdot), \pi_h^*(\cdot|x) - \widehat{\pi}_h(\cdot|x)\right\rangle \\
&\leq \left\langle \mathcal{T}_h\widehat{V}_{h+1}(x,\cdot) - \widehat{\mathcal{T}}_h\widehat{V}_{h+1}(x,\cdot) + \Gamma_h(x,\cdot), \pi_h^*(\cdot|x)\right\rangle + \left\langle \mathcal{T}_h(V_{h+1}^* - \widehat{V}_{h+1})(x,\cdot), \pi_h^*(\cdot|x)\right\rangle \\
&\leq 2\mathbb{E}_{\pi^*}[\Gamma_h(\cdot,\cdot)|x_h = x] + R_{h+1}, \\
&\leq \frac{8\sqrt{d}H\cdot(H-h)\sqrt{\iota}}{\sqrt{K\kappa}} := R_h, \forall x \in \mathcal{S}.
\end{aligned}$$

Therefore, the failure probability at step h can be upper bounded as

$$\mathbb{P}\left(\mathcal{E}_h^c\right) \leq \mathbb{P}\left(\mathcal{E}_{h+1}^c \cup \mathcal{E}_{h+1} \cap \{\Gamma_h(\cdot,\cdot) \text{ does not ensures pessimism}\}\right) \leq \delta_{h+1} + \frac{\delta}{H^2} = \delta_h.$$

Therefore, we have shown that with probability at least $1-\delta_h = 1-\delta_{h+1}-\frac{\delta}{H^2}$, pessimism is achieved at step h and $\mathcal{E}_h$ holds with probability at least $1-\delta_h$. By induction, and a union bound over $h \in [H]$, we know that if we set $\Gamma_h = 4\sqrt{d}H\sqrt{\iota}\,\|\phi(\cdot,\cdot)\|_{\Lambda_h^{-1}}$, then with probability at least $1-(\delta_H+\cdots+\delta_1) = 1-\frac{\delta H(H+1)}{2H^2} > 1-\delta$, $|\mathcal{T}_h\widehat{V}_{h+1} - \widehat{\mathcal{T}}_h\widehat{V}_{h+1}|(x,a) \leq \Gamma_h(x,a)$ holds for all $(h,x,a) \in [H]\times\mathcal{S}\times\mathcal{A}$. The theorem then follows from Lemma 2. □

D.2 Proof of Theorem 2

Proof of Theorem 2. The proof basically follows the same arguments of that of Theorem 1 except that we can leverage the finite feature condition to derive a different bonus term for the dominating reference function. We focus on deriving the $\Gamma_h(\cdot,\cdot)$ and omit other details for simplicity.

We first elaborate Eqn. (4) via the proof of Lemma 3 (see Section I for details):

$$\left|\mathcal{T}_h V_{h+1}^* - \widehat{\mathcal{T}}_h V_{h+1}^*\right|(x,a) \leq \underbrace{\sqrt{\lambda d}H\,\|\phi(x,a)\|_{\Sigma_h^{-1}}}_{\text{(a)}} + \underbrace{\sum_{\tau\in\mathcal{D}} \left\langle \phi(x,a), \Lambda_h^{-1}\phi_h^\tau\right\rangle \xi_h^\tau(V_{h+1}^*)}_{\text{(b)}},$$

where $\xi_h^\tau(V_{h+1}^*) = r_h^\tau + V_{h+1}^*(x_{h+1}^\tau) - \mathcal{T}_h V_{h+1}^*(x_h^\tau, a_h^\tau)$. We will bound (b) by $\beta\,\|\phi(x,a)\|_{\Lambda_h^{-1}}$ with some $\beta > 0$ and we can set λ sufficiently small so that $\sqrt{\lambda d}H \leq \beta$. Therefore, we can focus on the analysis of (b).

We denote the state-action pairs at step h as $\mathcal{D}_h = \{(x_h^\tau, a_h^\tau)\}_{\tau\in\mathcal{D}}$. The key is that because V_{h+1}^* is deterministic, conditioned on $\mathcal{D}_h$, the only randomness is from x_{h+1}^τ and $\{\xi_h^\tau(V_{h+1}^*)\}_{\tau\in\mathcal{D}}$ are still independent and bounded random variables. In particular, for any fixed $\mathcal{D}_h = D_h$ and fixed $\phi(x,a)$, by the Hoeffding's inequality, with probability at least $1-\frac{\delta}{HM}$, we have

$$
\begin{aligned}
\text{(b)} &\le \sqrt{\sum_{\tau\in\mathcal{D}} H^2 \left\langle \phi(x,a), \Lambda_h^{-1}\phi(x_h^\tau, a_h^\tau)\right\rangle^2 \log\left(\frac{2H^2M}{\delta}\right)} \\
&= H\sqrt{\left(\|\phi(x,a)\|_{\Lambda_h^{-1}}^2 - \lambda\|\phi(x,a)\|_{\Lambda_h^{-2}}^2\right)\log\left(\frac{2H^2M}{\delta}\right)} \\
&\le H\sqrt{\log\left(\frac{2H^2M}{\delta}\right)}\|\phi(x,a)\|_{\Lambda_h^{-1}}, \qquad \forall (x,a,h)\in\mathcal{S}\times\mathcal{A}\times[H],
\end{aligned}
$$

where in the equality, we use

$$
\begin{aligned}
\sum_{\tau\in\mathcal{D}} \left\langle \phi(x,a), \Lambda_h^{-1}\phi(x_h^\tau, a_h^\tau)\right\rangle^2 &= \sum_{\tau\in\mathcal{D}} \phi(x,a)^\top\Lambda_h^{-1}\phi(x_h^\tau,a_h^\tau)\phi(x_h^\tau,a_h^\tau)^\top\Lambda_h^{-1}\phi(x,a) \\
&= \|\phi(x,a)\|_{\Lambda_h^{-1}}^2 - \lambda\phi(x,a)^\top(\Lambda_h^{-1})^2\phi(x,a).
\end{aligned}
$$

Then, if we denote the event $\mathcal{E}_h(x,a) := \{\sum_{\tau\in\mathcal{D}}\left\langle\phi(x,a),\Lambda_h^{-1}\phi_h^\tau\right\rangle\xi_h^\tau(V_{h+1}^*) > H\sqrt{\log\left(\frac{2H^2M}{\delta}\right)}\}$

$$
\mathbb{P}(\mathcal{E}_h(x,a)) = \int \mathbb{P}\left(\mathcal{E}_h(x,a)|\mathcal{D}_h = D_h\right) d\mu(D_h) \le \frac{\delta}{H^2M},
$$

where the last inequality holds for any fixed D_h. By a union bound over all $\phi(x,a)$, we know that with probability at least $1-\delta/H^2$, for all $(x,a)\in\mathcal{S}\times\mathcal{A}$, we have

$$
\sum_{\tau\in\mathcal{D}}\left\langle\phi(x,a),\Lambda_h^{-1}\phi_h^\tau\right\rangle\xi_h^\tau(V_{h+1}^*) \le H\sqrt{\log\left(\frac{2H^2M}{\delta}\right)}.
$$

By similar induction procedure, we can set $\Gamma_h(\cdot,\cdot) = O\left(H\sqrt{\log\left(\frac{2H^2M}{\delta}\right)}\right)\|\phi(x,a)\|_{\Lambda_h^{-1}}$, where we require $K \ge \tilde{\Omega}\left(d^2H^2/\kappa\right)$ and $\lambda = 1/d$ to make the advantage function and (a) non-dominating. Then, the theorem follows from Lemma 2. □

E PROOF OF LINPEVI-ADV+

In this section, we present the proof of Theorem 3.

E.1 ANALYSIS OF THE VARIANCE ESTIMATION ERROR

First, we analyze the estimation error of the conditional variance estimator. We recall that we estimate $[\mathbb{V}_h V_{h+1}^*](x,a)$ based on dataset $\mathcal{D}'$ as

$$
\widehat{\sigma}_h^2(x,a) = \max\left\{1, \underbrace{\left[\left\langle\phi^\top(x,a),\widetilde{\beta}_{h,2}\right\rangle\right]_{[0,H^2]} - \left[\left\langle\phi^\top(x,a),\widetilde{\beta}_{h,1}\right\rangle\right]_{[0,H]}^2}_{\mathbb{B}_h(x,a)} - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right)\right\}.
$$

Lemma 5 (Variance Estimator)**.** *Under Assumption 1, if $K \ge \tilde{\Omega}(d^2H^2/\kappa)$, then with probability at least $1-\delta$, for all $(x,a,h)\in\mathcal{S}\times\mathcal{A}\times[H]$, we have*

$$
[\mathbb{V}_h V_{h+1}^*](x,a) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) \le \widehat{\sigma}_h^2(x,a) \le [\mathbb{V}_h V_{h+1}^*](x,a) \tag{16}
$$

Proof. We first bound the difference between $\mathbb{B}_h(x,a)$ and $[\mathrm{Var}_h \widehat{V}'_{h+1}](x,a)$:

$$\begin{aligned}
&\left|\left\langle \phi^\top(x_h,a_h), \widetilde{\beta}_{h,2}\right\rangle_{[0,H^2]} - \left[\left\langle \phi^\top(x_h,a_h), \widetilde{\beta}_{h,1}\right\rangle\right]^2_{[0,H]} - [\mathbb{P}_h(\widehat{V}'_{h+1})^2](x_h,a_h) - ([\mathbb{P}_h \widehat{V}'_{h+1}(x_h,a_h)])^2\right| \\
\leq &\left|\left\langle \phi^\top(x_h,a_h), \widetilde{\beta}_{h,2}\right\rangle_{[0,H^2]} - [\mathbb{P}_h(\widehat{V}'_{h+1})^2](x_h,a_h)\right| + \left[\left\langle \phi^\top(x_h,a_h), \widetilde{\beta}_{h,1}\right\rangle\right]^2_{[0,H]} - ([\mathbb{P}_h \widehat{V}'_{h+1}(x_h,a_h)])^2 \\
\leq &\underbrace{\left|\left\langle \phi^\top(x_h,a_h), \widetilde{\beta}_{h,2}\right\rangle - [\mathbb{P}_h(\widehat{V}'_{h+1})^2](x_h,a_h)\right|}_{\text{(a)}} + 2H \underbrace{\left|\left\langle \phi^\top(x_h,a_h), \widetilde{\beta}_{h,1}\right\rangle - [\mathbb{P}_h \widehat{V}'_{h+1}(x_h,a_h)]\right|}_{\text{(b)}}.
\end{aligned}$$

We note that both (a) and (b) are analysis of the regular value-target ridge regression, with target $(\widehat{V}'_{h+1})^2$ and $(\widehat{V}'_{h+1})$, respectively. The analysis thus follows the same line of those presented in Section D when we deal with the correlated advantage part, except that we invoke Lemma 9 with range H for $\widehat{V}'_{h+1}$ and H^2 for $(\widehat{V}'_{h+1})^2$. We omit the details here for simplicity and present the results directly: with probability at least $1-\delta/2$, for all $(x,a,h) \in \mathcal{S}\times\mathcal{A}\times[H]$,

$$\left|\mathbb{B}_h(x,a) - [\mathrm{Var}_h \widehat{V}'_{h+1}](x,a)\right| \leq \text{(a)} + 2H\text{(b)} \leq \tilde{O}\left(\frac{dH^2}{\sqrt{K\kappa}}\right). \tag{17}$$

We then use Theorem 1 and Lemma 13 to show that with probability at least $1-\delta/2$, for all $(x,a,h) \in \mathcal{S}\times\mathcal{A}\times[H]$,

$$\begin{aligned}
&\left\|\mathrm{Var}_h \widehat{V}'_{h+1} - \mathrm{Var}_h V^*_{h+1}\right\|(x,a) \\
\leq &\left\|\mathbb{P}_h\left((\widehat{V}'_{h+1})^2 - (V^*_{h+1})^2\right)\right\|(x,a) + \left\|\left(\mathbb{P}_h \widehat{V}'_{h+1}\right)^2 - \left(\mathbb{P}_h V^*_{h+1}\right)^2\right\|(x,a) \\
\leq &2H\left\|\widehat{V}'_{h+1} - V^*_{h+1}\right\|(x,a) + 2H\left\|\mathbb{P}_h \widehat{V}'_{h+1} - \mathbb{P}_h V^*_{h+1}\right\|(x,a) \leq \tilde{O}\left(\frac{\sqrt{d}H^3}{\sqrt{K\kappa}}\right).
\end{aligned} \tag{18}$$

With a union bound over the estimations in Eqn. (17) and Eqn. (18), we know that with probability at least $1-\delta$, the following derivations hold. First, by triangle inequality, we have

$$\begin{aligned}
\left\|\mathbb{B}_h - \mathrm{Var}_h V^*_{h+1}\right\|(x,a) &\leq \left\|\mathbb{B}_h - \mathrm{Var}_h \widehat{V}'_{h+1}\right\|(x,a) + \left\|\mathrm{Var}_h \widehat{V}'_{h+1} - \mathrm{Var}_h V^*_{h+1}\right\|(x,a) \\
&\leq \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right),
\end{aligned}$$

where we use Eqn. (17) and Eqn. (18) in the last inequality. This shows that $\mathbb{B}_h(x,a) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) \leq [\mathrm{Var}_h V^*_{h+1}](x,a)$ and the second inequality of the lemma follows from the fact that $\max\{1,\cdot\}$ preserves the order of two numbers. On the other hand, we note that $\max\{1,\cdot\}$ is non-expansive, meaning that $|\max\{1,a\} - \max\{1,b\}| \leq |a-b|$. Then, we have

$$\begin{aligned}
\left\|\widehat{\sigma}^2_h - \mathbb{V}_h V^*_{h+1}\right\|(x,a) &\leq \left\|\widehat{\sigma}^2_h - \mathbb{V}_h \widehat{V}'_{h+1}\right\|(x,a) + \left\|[\mathbb{V}_h \widehat{V}'_{h+1}] - [\mathbb{V}_h V^*_{h+1}]\right\|(x,a) \\
&\leq \left\|\mathbb{B}_h - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) - \mathrm{Var}_h \widehat{V}'_{h+1}\right\|(x,a) + \left\|\mathrm{Var}_h \widehat{V}'_{h+1} - \mathrm{Var}_h V^*_{h+1}\right\|(x,a) \\
&\leq \tilde{O}\left(\frac{dH^2}{\sqrt{K\kappa}}\right) + \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) + \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) \\
&= \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right),
\end{aligned}$$

where the third inequality follows from Eqn. (17) and Eqn. (18). □

E.2 Proof of Theorem 3

The proof idea is similar to that of LinPEVI-ADV, except that we need to additionally estimate the conditional variance and apply the Bernstein-type Lemma 10 for the reference function. To simplify the presentation, we will focus on the uncertainty of the reference function as it is dominating, and focus on determining the threshold. To this end, we will omit the constants and logarithmic terms by $\tilde{O}(\cdot)$ throughout the proof.

Proof of Theorem 3. We still start with the following decomposition: for a function $\|g_{h+1}\|_\infty \leq R-1$, we invoke Lemma 3:

$$\begin{aligned}
&\left|\mathcal{T}_h g_{h+1} - \widehat{\mathcal{T}}_h g_{h+1}\right|(x,a) \\
&\leq \underbrace{\sqrt{\lambda d} R \|\phi(x,a)\|_{\Sigma_h^{-1}}}_{\text{(a)}} + \underbrace{\left\| \sum_{\tau\in\mathcal{D}} \frac{\phi\left(x_h^\tau, a_h^\tau\right)}{\sqrt{[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau)}} \cdot \xi_h^\tau(g_{h+1}) \right\|_{\Sigma_h^{-1}} \|\phi(x,a)\|_{\Sigma_h^{-1}}}_{\text{(b)}}.
\end{aligned}$$

Similarly, we can set $\lambda = 1/H^2$ to ensure that $\text{(a)} \leq \beta_2 \|\phi(x,a)\|_{\Sigma_h^{-1}} = \tilde{O}\left(\sqrt{d}\right) \|\phi(x,a)\|_{\Sigma_h^{-1}}$, so we focus on the analysis of (b). For the reference function, as $[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau) \geq 1$, we know that $\xi_h^\tau(V_{h+1}^*) \leq H$. Then we consider the filtration $\mathcal{F}_{\tau-1,h} = \sigma\left(\{(x_h^j, a_h^j)\}_{j=1,j\in\mathcal{D}}^{\tau} \cup \{(r_h^j, x_{h+1}^j)\}_{j=1,j\in\mathcal{D}}^{\tau-1}\right)$ where $\sigma(\cdot)$ denotes the σ-algebra generated by the random variables. As $[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau)$ is independent of $\mathcal{D}$, $\xi_h^\tau(V_{h+1}^*)$ is mean-zero conditioned on $\mathcal{F}_{\tau-1,h}$. We proceed to estimate the conditional variance:

$$\begin{aligned}
\mathbb{D}_{\tau-1,h}[\xi_h^\tau(V_{h+1}^*)] &= \frac{\mathbb{D}_{\tau-1,h}[V_{h+1}^*(x_{h+1}^\tau)]}{[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau)} \leq \frac{[\mathbb{V}_h V_{h+1}^*(x_{h+1}^\tau)]}{[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau)} \\
&\leq 1 + \frac{\tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right)}{[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right)} \\
&\leq 1 + 2\tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) = O(1),
\end{aligned}$$

where in the first equality, we use the fact that $[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](\cdot,\cdot)$ is independent of $\mathcal{D}$, and in the last inequality, we use $K \geq \tilde{\Omega}\left(d^2H^6/\kappa\right)$ to ensure that $[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) \geq \frac{1}{2}$. Then, we can directly invoke Lemma 10 to obtain that:

$$\left\| \sum_{\tau\in\mathcal{D}} \phi\left(x_h^\tau, a_h^\tau\right) \cdot \xi_h^\tau(V_{h+1}^*) \right\|_{\Sigma_h^{-1}} \leq \tilde{O}\left(\sqrt{d}\right).$$

Similar to the proof of LinPEVI-ADV, the uncertainty of the advantage function is non-dominating for sufficiently large K (determined later) so we can set

$$\Gamma_h(\cdot,\cdot) = \tilde{O}\left(\sqrt{d}\right) \|\phi(x,a)\|_{\Sigma_h^{-1}}.$$

Moreover, by Lemma 5, we have $[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau) \leq [\mathbb{V}_h V_{h+1}^*](x_h^\tau, a_h^\tau) \leq H^2$, which implies that

$$\left(\sum_{\tau\in\mathcal{D}} \frac{\phi_h^\tau(\phi_h^\tau)^\top}{[\bar{\mathbb{V}}_h \widehat{V}'_{h+1}](x_h^\tau, a_h^\tau)} + \lambda I_d\right)^{-1} \preceq \left(\sum_{\tau\in\mathcal{D}} \frac{\phi_h^\tau(\phi_h^\tau)^\top}{[\mathbb{V}_h V_{h+1}^*](x_h^\tau, a_h^\tau)} + \lambda I_d\right)^{-1} \preceq H^2 \left(\sum_{\tau\in\mathcal{D}} \phi_h^\tau(\phi_h^\tau)^\top + \lambda I_d\right)^{-1}.$$

This implies that

$$\|\phi(x,a)\|_{\Sigma_h^{-1}} \leq \|\phi(x,a)\|_{\Sigma_h^{*-1}} \leq H \|\phi(x,a)\|_{\Lambda_h^{-1}}, \qquad \forall (x,a).$$

Following the same induction analysis procedure of the proof of Theorem 1, we know that $\left\|\widehat{V}_{h+1} - V_{h+1}^*\right\|_\infty \leq \tilde{O}\left(\frac{\sqrt{d}H^2}{\sqrt{K\kappa}}\right)$. Using the standard ϵ-covering argument and Lemma 9, we know that we can set

$$b_{1,h}(x,a) = \tilde{O}\left(\frac{d^{3/2}H^2}{\sqrt{K\kappa}}\right) \|\phi(x,a)\|_{\Sigma_h^{-1}}.$$

To make it non-dominating, we require that $K \geq \tilde{\Omega}\left(d^2H^4/\kappa\right)$. Moreover, to make $\text{(a)} = \sqrt{\lambda d} R \|\phi(x,a)\|_{\Sigma_h^{-1}}$ non-dominating, we set $\lambda = 1/H^2$. Then, the theorem follows from Lemma 2. □

F PROOF OF MARKOV GAME

Proof of Theorem 5. The techniques developed for MDPs are readily extended to the MGs by decoupling the estimations into the max-player part and min-player part so we start with the following lemma, which is a counterpart of Lemma 2.

Lemma 6 (Decomposition Lemma for MG)**.** *Under the condition that the functions* $\Gamma_h : \mathcal{S}\times\mathcal{A}\times\mathcal{B} \to \mathbb{R}$ *in Algorithms 3 and 4 satisfying*

$$|\mathcal{T}_h\underline{V}_{h+1}(x,a,b) - \widehat{\mathcal{T}}_h\underline{V}_{h+1}(x,a,b)| \le \underline{\Gamma}_h(x,a,b),$$
$$|\mathcal{T}_h\overline{V}_{h+1}(x,a,b) - \widehat{\mathcal{T}}_h\overline{V}_{h+1}(x,a,b)| \le \overline{\Gamma}_h(x,a,b),$$

for any $(x,a,b,h) \in \mathcal{S}\times\mathcal{A}\times\mathcal{B}\times[H]$*, then for Algorithms 4 and 3, we have*

$$V_1^{*,\widehat{\nu}}(x) - V_1^*(x) \le \overline{V}_1(x) - V_1^*(x) \le 2\sup_{\nu}\sum_{h=1}^{H}\mathbb{E}_{\pi^*,\nu}[\overline{\Gamma}_h(x,a,b) \mid x_1 = x],$$

$$V_1^*(x) - V_1^{\widehat{\pi},*}(x) \le V_1^*(x) - \underline{V}_1(x) \le 2\sup_{\pi}\sum_{h=1}^{H}\mathbb{E}_{\pi,\nu^*}[\underline{\Gamma}_h(x,a,b) \mid x_1 = x].$$

Furthermore, we can obtain that for any $x \in \mathcal{S}$*,*

$$\begin{aligned} V_1^{*,\widehat{\nu}}(x) - V_1^{\widehat{\pi},*}(x) &= V_1^{*,\widehat{\nu}}(x) - V_1^*(x) + V_1^*(x) - V_1^{\widehat{\pi},*}(x) \\ &\le 2\sup_{\nu}\sum_{h=1}^{H}\mathbb{E}_{\pi^*,\nu}[\overline{\Gamma}_h(x,a,b) \mid x_1 = x] + 2\sup_{\pi}\sum_{h=1}^{H}\mathbb{E}_{\pi,\nu^*}[\underline{\Gamma}_h(x,a,b) \mid x_1 = x]. \end{aligned}$$

Proof. See Appendix A in Zhong et al. (2022) for a detailed proof. □

Therefore, it suffices to determine the $\Gamma_h(\cdot,\cdot,\cdot)$ that establishes pessimism. Before continuing, we first prove Theorem 7, which is required in our subsequent analysis.

Proof of Theorem 7. We first note that Lemma 3 is constructed for (weighted) ridge regression and can be applied to linear MG by replacing $\phi(x,a) \in \mathbb{R}^d$ with $\phi(x,a,b)$ accordingly. Therefore, for a function $g : \mathcal{S}\times\mathcal{A}\times\mathcal{B} \to [0, H-1]$, we have

$$\begin{aligned} \left|\mathcal{T}_h g_{h+1} - \overline{\mathcal{T}}_h g_{h+1}\right|(x,a,b) &\lesssim \left\|\sum_{\tau\in\mathcal{D}}\frac{\phi\left(x_h^\tau,a_h^\tau,b_h^\tau\right)}{\overline{\sigma}_h(x_h^\tau,a_h^\tau,b_h^\tau)}\cdot\overline{\xi}_h^\tau(g_{h+1})\right\|_{\overline{\Sigma}_h^{-1}}\|\phi(x,a)\|_{\overline{\Sigma}_h^{-1}} \\ |\mathcal{T}_h g_{h+1} - \underline{\mathcal{T}}_h g_{h+1}|(x,a,b) &\lesssim \left\|\sum_{\tau\in\mathcal{D}}\frac{\phi\left(x_h^\tau,a_h^\tau,b_h^\tau\right)}{\underline{\sigma}_h(x_h^\tau,a_h^\tau,b_h^\tau)}\cdot\underline{\xi}_h^\tau(g_{h+1})\right\|_{\underline{\Sigma}_h^{-1}}\|\phi(x,a)\|_{\underline{\Sigma}_h^{-1}}. \end{aligned} \tag{19}$$

where $\overline{\Sigma}_h = \sum_{\tau\in\mathcal{D}}\frac{\phi(x_h^\tau,a_h^\tau,b_h^\tau)\phi(x_h^\tau,a_h^\tau,b_h^\tau)^\top}{\overline{\sigma}_h^2(x_h^\tau,a_h^\tau,b_h^\tau)} + \lambda I_d$, $\overline{\xi}_h^\tau(g_{h+1}) = \frac{r_h^\tau + g_{h+1}(x_{h+1}^\tau) - \mathcal{T}_h g_{h+1}(x_h^\tau,a_h^\tau,b_h^\tau)}{\overline{\sigma}_h(x_h^\tau,a_h^\tau,b_h^\tau)}$, and $\underline{\Sigma}_h, \underline{\xi}_h^\tau(g_{h+1})$ are defined similarly for $\underline{\sigma}_h(\cdot,\cdot,\cdot)$. Moreover, we again omit the $\sqrt{\lambda d}H$ as we can set λ sufficiently small (determined later).

For LinPMVI-ADV, we set $\overline{\sigma}_h = \underline{\sigma}_h = 1$ so $\overline{\Sigma}_h = \underline{\Sigma}_h = \Lambda_h$. By the reference-advantage decomposition for MG (Eqn. (6.1)), it suffices to focus on the reference function with the Nash value V_{h+1}^* where Lemma 9 can be applied directly:

$$\left|\left(\mathcal{T}_h\overline{V}_{h+1} - \overline{\mathcal{T}}_h\overline{V}_{h+1}\right)(x,a,b)\right| \lesssim \tilde{O}\left(\sqrt{d}H\right)\|\phi(x,a,b)\|_{\Lambda_h^{-1}} = \Gamma_h(x,a,b),$$
$$\left|\left(\mathcal{T}_h\underline{V}_{h+1} - \underline{\mathcal{T}}_h\underline{V}_{h+1}\right)(x,a,b)\right| \lesssim \tilde{O}\left(\sqrt{d}H\right)\|\phi(x,a,b)\|_{\Lambda_h^{-1}} = \Gamma_h(x,a,b).$$

We follow the same induction analysis procedure of Theorem 1 to obtain that $\left\|\overline{V}_{h+1} - V_{h+1}^*\right\|_\infty \le \tilde{O}\left(\frac{H(H-h)}{\sqrt{K\kappa}}\right)$ and $\left\|\underline{V}_{h+1} - V_{h+1}^*\right\|_\infty \le \tilde{O}\left(\frac{H(H-h)}{\sqrt{K\kappa}}\right)$. By standard ϵ-covering argument and Lemma 9, we can set

$$b_{1,h}(x,a,b) = O\left(\frac{d^{3/2}H^2\iota}{\sqrt{K\kappa}}\right)\|\phi(x,a,b)\|_{\Lambda_h^{-1}}.$$

To make it non-dominating, we require that $K \geq \tilde{\Omega}(d^2H^2/\kappa)$. Also, to make $\sqrt{\lambda d}H \leq \tilde{O}\left(\sqrt{d}H\right)$, it suffices to set $\lambda = 1$. Now we invoke Lemma 6 with $\Gamma_h = \overline{\Gamma}_h = \underline{\Gamma}_h$ to obtain that for any $x \in \mathcal{S}$,

$$\begin{aligned}
&V_1^{*,\widehat{\nu}}(x) - V_1^{\widehat{\pi},*}(x) \\
&\leq 2\sup_{\nu}\sum_{h=1}^{H}\mathbb{E}_{\pi^*,\nu}[\Gamma_h(x,a,b) \mid x_1 = x] + 2\sup_{\pi}\sum_{h=1}^{H}\mathbb{E}_{\pi,\nu^*}[\Gamma_h(x,a,b) \mid x_1 = x] \\
&\leq \tilde{O}(\sqrt{d}H)\cdot\left(\max_{\nu}\sum_{h=1}^{H}\mathbb{E}_{\pi^*,\nu}\|\phi(x_h,a_h,b_h)\|_{\Lambda_h^{-1}} + \max_{\pi}\sum_{h=1}^{H}\mathbb{E}_{\pi,\nu^*}\|\phi(x_h,a_h,b_h)\|_{\Lambda_h^{-1}}\right).
\end{aligned}$$

□

With Theorem 7 in hand, similar to Lemma 5, we have the following lemma to control the variance estimation error and we omit the proof for simplicity.

Lemma 7 (Variance Estimation Error for MGs)**.** *Under Assumption 1, in Algorithm 3, if $K \geq \tilde{\Omega}(d^2H^2/\kappa)$, then with probability at least $1-\delta$, for all $(x,a,b,h) \in \mathcal{S}\times\mathcal{A}\times\mathcal{B}\times[H]$, we have*

$$\begin{aligned}
[\mathbb{V}_h V_{h+1}^*](x,a,b) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) &\leq \overline{\sigma}_h(x,a,b) \leq [\mathbb{V}_h V_{h+1}^*](x,a,b) \\
[\mathbb{V}_h V_{h+1}^*](x,a,b) - \tilde{O}\left(\frac{dH^3}{\sqrt{K\kappa}}\right) &\leq \underline{\sigma}_h(x,a,b) \leq [\mathbb{V}_h V_{h+1}^*](x,a,b).
\end{aligned}$$

Similar to the proof of Theorem 3, if $K \geq \tilde{\Omega}\left(d^2H^6/\kappa\right)$, the conditional variances of $\overline{\xi}_h^{\tau}(V_{h+1}^*)$ and $\underline{\xi}_h^{\tau}(V_{h+1}^*)$ are $O(1)$ so it suffices to set $\overline{\Gamma}_h(\cdot,\cdot,\cdot) = \tilde{O}\left(\sqrt{d}\right)\|\phi(\cdot,\cdot,\cdot)\|_{\overline{\Sigma}_h^{-1}}$ and $\underline{\Gamma}_h(\cdot,\cdot,\cdot) = \tilde{O}\left(\sqrt{d}\right)\|\phi(\cdot,\cdot,\cdot)\|_{\underline{\Sigma}_h^{-1}}$. Moreover, because $[\mathbb{V}_h V_{h+1}^*](x,a,b) \leq H^2$, we have

$$\overline{\Sigma}_h^{-1} \preceq \Sigma_h^{*-1} \preceq H^2\Lambda_h^{-1}, \qquad \underline{\Sigma}_h^{-1} \preceq \Sigma_h^{*-1} \preceq H^2\Lambda_h^{-1}.$$

We can similarly establish $\left\|\overline{V}_{h+1} - V_{h+1}^*\right\|_\infty \leq \tilde{O}\left(\frac{\sqrt{d}H^2}{\sqrt{K\kappa}}\right)$ and $\left\|\underline{V}_{h+1} - V_{h+1}^*\right\|_\infty \leq \tilde{O}\left(\frac{\sqrt{d}H^2}{\sqrt{K\kappa}}\right)$, respectively. It suffices to set $\overline{b}_{1,h}(\cdot,\cdot,\cdot) = \tilde{O}\left(d^{3/2}H^2/\sqrt{K\kappa}\right)\|\phi(\cdot,\cdot,\cdot)\|_{\overline{\Sigma}_h^{-1}}$ and $\underline{b}_{1,h}(\cdot,\cdot,\cdot) = \tilde{O}\left(d^{3/2}H^2/\sqrt{K\kappa}\right)\|\phi(\cdot,\cdot,\cdot)\|_{\underline{\Sigma}_h^{-1}}$ where $K \geq \tilde{\Omega}(d^2H^4/\kappa)$ is sufficient to make them non-dominating. Moreover, we need to set $\lambda = 1/H^2$ to make $\sqrt{\lambda d H} \leq \tilde{O}\left(\sqrt{d}\right)$. The theorem then follows from Lemma 6. □

G Proof of Lower Bounds

We only provide the proof of the lower bound for MGs, and the lower bound for MDP follows from the similar argument (see Remark 1 for details). In particular, we remark that the $\sqrt{d}$-dependency does not contradict Theorem 2 as the feature size of the constructed instances is exponentially in d.

Proof of Theorem 6. Our proof largely follows Zanette et al. (2021); Yin et al. (2022). We construct a family of MGs $\mathcal{M} = \{\mathcal{M}_u\}_{u\in\mathcal{U}}$, where $\mathcal{U} = \{u = (u_1,\ldots,u_H) \mid u_h \in \{-1,1\}^{d-2}, \forall h \in [H]\}$. For any fixed $u \in \mathcal{U}$, the associated MDP $\mathcal{M}_u$ is defined by

State space. The state space $\mathcal{S} = \{-1,+1\}$.

Action space. The action space $\mathcal{A} = \mathcal{B} = \{-1,0,1\}^{d-2}$.

Feature map. The feature map $\phi : \mathcal{S}\times\mathcal{A}\times\mathcal{B} \to \mathbb{R}^d$ defined by

$$\phi(1,a,b) = \begin{pmatrix} \frac{a}{\sqrt{2d}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \in \mathbb{R}^d, \quad \phi(-1,a,b) = \begin{pmatrix} \frac{a}{\sqrt{2d}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathbb{R}^d.$$

Transition kernel. Let

$$\mu_h(1) = \mu_h(-1) = \begin{pmatrix} \mathbf{0}_{d-2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathbb{R}^d.$$

By the assumption that $\mathbb{P}_h(s' \mid s, a, b) = \langle \phi(s, a, b), \mu_h(s') \rangle$, we know the MDP reduces to a homogeneous Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Reward observation. Let

$$\theta_{u,h} = \begin{pmatrix} \zeta u_h \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \in \mathbb{R}^d,$$

where $\zeta \in [0, \frac{1}{\sqrt{3d}}]$. By the assumption that $r_h(s, a, b) = \langle \phi(s, a, b), \theta_h \rangle$, we have

$$r_{u,h}(s, a, b) = \frac{s}{\sqrt{6}} + \frac{\zeta}{\sqrt{2d}} \langle a, u_h \rangle$$

We further assume the reward observation follows a Gaussian distribution:

$$R_{u,h}(s, a, b) \sim \mathcal{N}\left(\frac{s}{\sqrt{6}} + \frac{\zeta}{\sqrt{2d}} \langle a, u_h \rangle, 1 \right).$$

Date collection process. Let $\{e_1, e_2, \cdots, e_{d-2}\}$ be the canonical bases of $\mathbb{R}^{d-2}$ and $\mathbf{0}_{d-2} \in \mathbb{R}^{d-2}$ be the zero vector. The behavior policy $\mu : \mathcal{S} \to \Delta_{\mathcal{A}\times\mathcal{B}}$ is defined as

$$\mu(e_j, \mathbf{0}_{d-2} \mid s) = \frac{1}{d}, \quad \forall j \in [d-2], \quad \mu(\mathbf{0}_{d-2}, \mathbf{0}_{d-2} \mid s) = \frac{2}{d}.$$

By construction, a Nash equilibrium of $\mathcal{M}_u$ is (π^*, ν^*) satisfying $\pi_h^*(\cdot) = u_h$ and $\nu_h^*(\cdot) = u_h$. Since the reward and transition are irrelevant with the min-player's policy, we use the notation $u^\pi = (u_1^\pi, \ldots, u_H^\pi) = (\mathrm{sign}(\mathbb{E}_\pi[a_1]), \ldots, \mathrm{sign}(\mathbb{E}_\pi[a_H]))$. Moreover, for any vector v, we denote by its i-th element $v[i]$. By the proof of Lemma 9 in Zanette et al. (2021) we know

$$\begin{aligned} V_u^* - V_u^{\pi,\nu} &\geq \frac{\zeta}{\sqrt{2d}} \sum_{h=1}^{H} \sum_{i=1}^{d} \mathbf{1}\{u_h^\pi[i] - u_h[i]\} \\ &:= \frac{\zeta}{\sqrt{2d}} D_H(u^\pi, u). \end{aligned} \tag{20}$$

By Assouad's method (cf. Lemma 2.12 in Tsybakov (2009)), we have

$$\sup_{u\in\mathcal{U}} \mathbb{E}_u[D_H(u^\pi, u)] \geq \frac{(d-2)H}{2} \min_{u,u' \mid D_H(u,u')=1} \inf_{\psi} [\mathbb{P}_u(\psi \neq u) + \mathbb{P}_{u'}(\psi \neq u')],$$

where ψ is the test function mapping from observations to $\{u, u'\}$. Furthermore, by Theorem 2.12 in Tsybakov (2009), we have

$$\min_{u,u' \mid D_H(u,u')=1} \inf_{\psi} [\mathbb{P}_u(\psi \neq u) + \mathbb{P}_{u'}(\psi \neq u')] \geq 1 - \left(\frac{1}{2} \max_{u,u' \mid D_H(u,u')=1} \mathrm{KL}(\mathcal{Q}_u \| \mathcal{Q}_{u'}) \right)^{1/2}, \tag{21}$$

where $\mathcal{Q}_u$ takes form

$$\mathcal{Q}_u = \prod_{k=1}^{K} \xi_1\left(s_1^k\right) \prod_{h=1}^{H} \mu\left(a_h^k, b_h^k \mid s_h^k\right) \left[R_{u,h}\left(s_h^k, a_h^k, b_h^k\right)\right] \left(r_h^k\right) \mathbb{P}_h\left(s_{h+1}^k \mid s_h^k, a_h^k, b_h^k\right),$$

where $\xi = [\frac{1}{2}, \frac{1}{2}]$ is the initial distribution.

$$\begin{aligned}
\mathrm{KL}(\mathcal{Q}_u \| \mathcal{Q}_{u'}) &= K \cdot \sum_{h=1}^{H} \mathbb{E}_u[\log([R_{u,h}(s_h^1, a_h^1, b_h^1)](r_h^1)/[R_{u',h}(s_h^1, a_h^1, b_h^1)](r_h^1))] \\
&= \frac{K}{d} \sum_{j=1}^{d-2} \mathrm{KL}\left(\mathcal{N}\Big(\frac{\zeta}{\sqrt{2d}}\langle e_j, u_h\rangle, 1\Big) \Big\| \mathcal{N}\Big(\frac{\zeta}{\sqrt{2d}}\langle e_j, u_h'\rangle, 1\Big)\right) \\
&= \frac{K}{d} \cdot \mathrm{KL}\left(\mathcal{N}\Big(\frac{\zeta}{\sqrt{2d}}, 1\Big) \Big\| \mathcal{N}\Big(\frac{-\zeta}{\sqrt{2d}}, 1\Big)\right) = \frac{2K\zeta^2}{d^2}, \qquad (22)
\end{aligned}$$

where the third equality uses the fact that $D_H(u, u') = 1$. Choosing $\zeta = \Theta(d/\sqrt{K})$, (20), (21), and (22) imply that

$$\inf_{\pi,\nu} \max_{u \in \mathcal{U}} \mathbb{E}_u[V_u^* - V_u^{\pi,\nu}] \gtrsim \frac{d\sqrt{d}H}{\sqrt{K}}. \qquad (23)$$

Here $K \geq \Omega(d^3)$ ensures that $\zeta \leq \frac{1}{\sqrt{3d}}$. On the other hand, we have

$$\begin{aligned}
&\mathrm{Var}[V_{h+1}^*](s, a, b) \\
&= \frac{1}{2}\left[V_{h+1}^*(-1) - \frac{1}{2}\left(V_{h+1}^*(+1) + V_{h+1}^*(-1)\right)\right]^2 + \frac{1}{2}\left[V_{h+1}^*(+1) - \frac{1}{2}\left(V_{h+1}^*(+1) + V_{h+1}^*(-1)\right)\right]^2 \\
&= \left[\frac{1}{2}\left(r_{h+1}^*(+1, u_{h+1}, u_{h+1}) - r_{h+1}^*(-1, u_{h+1}, u_{h+1})\right)\right]^2 \\
&= \frac{1}{6},
\end{aligned}$$

where the second equality follows from Bellman equation, and the last inequality uses the facts that $r_{u,h+1}(1, a, b) - r_{u,h+1}(-1, a, b) = \frac{2}{\sqrt{6}}$. By calculation, we have

$$\mathbb{E}\Sigma_h^* = \frac{3K}{2}\begin{pmatrix} \frac{2}{d^2}\mathbf{I}_{d-2} & \frac{1}{d\sqrt{d}}\mathbf{1}_{(d-2)\times 2} \\ \frac{1}{d\sqrt{d}}\mathbf{1}_{2\times(d-2)} & \mathbf{I}_2 \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

By Gaussian elimination, we know $\|(\mathbb{E}\Sigma_h^*/K)^{-1}\| \leq d^2$. For all $h \in [H]$, (s, a, b) and $K \geq \Omega(d^4 \log(2Hd/\delta))$, it holds with probability $1 - \delta$ that

$$\begin{aligned}
&\|\phi(s, a, b)\|_{(\Sigma_h^*)^{-1}}^2 \\
&\quad \leq 2 \cdot \|\phi(s, a, b)\|_{(\mathbb{E}\Sigma_h^*)^{-1}}^2 \\
&\quad = \frac{4}{3K}\left\{\frac{d}{4}a^\top\left\{\mathbf{I}_{d-2} + \frac{1}{d-2}\mathbf{1}_{(d-2)\times(d-2)}\right\}a - \frac{d}{2(d-2)}\mathbf{1}_{d-2}^\top a + \frac{d-1}{2(d-2)}\right\} \\
&\quad = \frac{4}{3K}\left\{\frac{d}{4} \cdot a^\top a + \frac{d}{4(d-2)}\left(1 - \mathbf{1}_{d-2}^\top a\right)^2 + \frac{1}{4}\right\} \\
&\quad \leq \frac{4}{3K}\left\{\frac{d(d-2)}{4} + \frac{d}{4(d-2)}\left(1 - \mathbf{1}_{d-2}^\top a\right)^2 + \frac{1}{4}\right\} \\
&\quad \leq \frac{4}{3K}\left\{\frac{d(d-2)}{4} + \frac{d(d-1)^2}{4(d-2)} + \frac{1}{4}\right\} \lesssim \frac{d^2}{K},
\end{aligned}$$

where the first inequality uses Lemma 13. Hence, we have

$$\max_{\nu} \sum_{h=1}^{H} \mathbb{E}_{\pi^*,\nu}\|\phi(s_h, a_h, b_h)\|_{(\Sigma_h^*)^{-1}} + \max_{\pi} \sum_{h=1}^{H} \mathbb{E}_{\pi,\nu^*}\|\phi(s_h, a_h, b_h)\|_{(\Sigma_h^*)^{-1}} \lesssim \frac{dH}{\sqrt{K}}. \qquad (24)$$

for any $u \in \mathcal{U}$. Combining (23), (24), and the fact that $|V_u^* - V_u^{\pi,\nu}| \leq |V_u^{*,\nu} - V_u^{\pi,*}|$ for any $u \in \mathcal{U}$, we have with probability at least $1 - \delta$ that

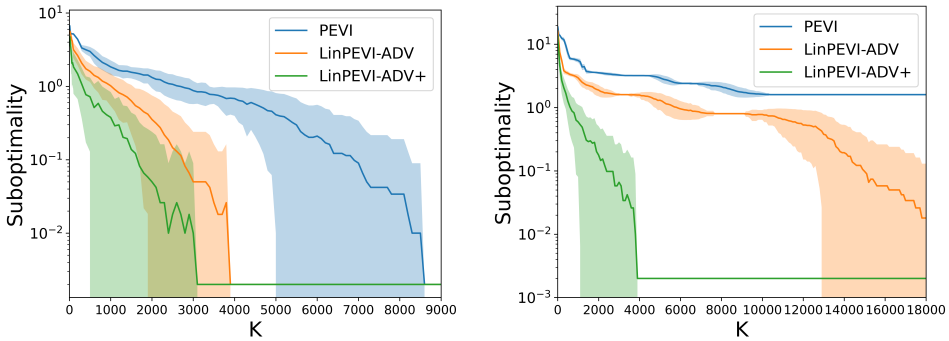
$$\begin{aligned}
&\inf_{\pi,\nu} \max_{u \in \mathcal{U}} \mathbb{E}_u[V_u^{*,\nu} - V_u^{\pi,*}] \\
&\quad \geq c\sqrt{d} \cdot \Big(\max_{\nu} \sum_{h=1}^{H} \mathbb{E}_{\pi^*,\nu}\|\phi(s_h, a_h, b_h)\|_{(\Sigma_h^*)^{-1}} + \max_{\pi} \sum_{h=1}^{H} \mathbb{E}_{\pi,\nu^*}\|\phi(s_h, a_h, b_h)\|_{(\Sigma_h^*)^{-1}}\Big),
\end{aligned}$$

which concludes our proof. □

Remark 1. *In our construction, the min-player will not affect the rewards and transitions. So by the similar derivations of* (23) *and* (24) *, we can obtain* $\inf_\pi \max_{u\in\mathcal{U}} \mathbb{E}_u[V_u^* - V_u^\pi] \gtrsim \frac{d\sqrt{d}H}{\sqrt{K}}$ *and* $\sum_{h=1}^H \mathbb{E}_{\pi^*}\|\phi(s_h,a_h)\|_{(\Sigma_h^*)^{-1}} \lesssim \frac{dH}{\sqrt{K}}$. *Hence, we can establish the lower bound for MDP as desired.*

H Numerical Simulations

For completeness, we adopt a similar synthetic linear MDP instance that is also used in Min et al. (2021) and Yin et al. (2022), and redo the experiments to verify the theoretical findings. The adopted MDP has $\mathcal{S} = \{1, 2\}$, $\mathcal{A} = \{0, \cdots, 99\}$, and $d = 10$. For the feature, we apply binary encoding to represent $a \in \mathcal{A}$ by $\mathbf{a} \in \mathbb{R}^8$. For the last two bits of the feature, we define $\delta(x, a) = \mathbf{1}(x = 0, a = 0)$ where $\mathbf{1}$ is the indicator function. Then, the MDP is characterized as follows.



(a) Horizon $H = 20$.

(b) Horizon $H = 40$.

Figure 1: Suboptimality v.s. The number of trajectories K. The results are averaged aver 100 independent trails and the mean result is plotted as solid lines. The error bar area corresponds to the standard deviation.

- Feature mapping:
$$\phi(x, a) = (\mathbf{a}^\top, \delta(x, a), 1 - \delta(x, a))^\top \in \mathbb{R}^d.$$
- True measure ν_h:
$$\nu_h(x) = (0, \cdots, 0, (1 - s) \oplus \alpha_h, x \oplus \alpha_h),$$
where $\{\alpha_h\}_{h\in[H]}$ is a sequence of integers taking values in 0 or 1 generated randomly and fixed, and $\oplus$ is the XOR operator. The tranision is given by $\mathbb{P}_h(x'|x, a) = \langle\phi(x, a), \nu_h(x')\rangle$.
- Reward function: we define
$$\theta_h = (0, \cdots, 0, r, 1 - r) \in \mathbb{R}^{10}$$
with $r = 0.9$ to obtain the mean reward $r_h(x, a) = \langle\phi(x, a), \theta_h\rangle$. Thr reward is then generated as a Bernoulli random variable.
- Behavior policy: always choose $a = 0$ with probability p, and other actions uniformly with $(1 - p)/99$. We choose $p = 0.5$.
- The initial state is chosen uniformly from $\mathcal{S}$.
- The regularization parameter λ is set to be 0.1 as suggested by Yin et al. (2022) and we estimate the value of the learned policy by 1000 i.i.d. trajectories where we also set the reward to be its mean during the this evaluation process.

Figure 1(a) and Figure 1(b) match our theoretical findings that a sharper bonus function leads to a smaller suboptimality. Therefore, LinPEVI-ADV+ achieves the best sample complexity, and PEVI performs worst. In particular, as H increases, we can see that both LinPEVI-ADV and PEVI perform worse significantly, while the convergence rate of LinPEVI-ADV+ is rather stable. This demonstrates the power of variance information, which has been observed by the previous work on offline linear MDP (Min et al., 2021; Yin et al., 2022).

I PROOF OF AUXILIARY LEMMAS

Proof of Lemma 3. We add and subtract $\phi(x,a)^\top \Sigma_h^{-1}(\Sigma_h - \lambda I_d) w_h$ in the second equality to obtain that

$$
\begin{aligned}
&\left(\mathcal{T}_h f_{h+1}\right)(x,a) - \left(\widehat{\mathcal{T}}_h f_{h+1}\right)(x,a) = \phi(x,a)^\top \left(w_h - \widehat{w}_h\right) \\
=&\phi(x,a)^\top w_h - \phi(x,a)^\top \Sigma_h^{-1} \left(\sum_{\tau \in \mathcal{D}} \phi_h^\tau \cdot \frac{\left(r_h^\tau + f_{h+1}\left(x_{h+1}^\tau\right)\right)}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} \right) \\
=&\phi(x,a)^\top w_h - \phi(x,a)^\top \Sigma_h^{-1} \left(\Sigma_h - \lambda I_d\right) w_h \\
&+ \phi(x,a)^\top \Sigma_h^{-1} \left(\sum_{\tau \in \mathcal{D}} \frac{\phi_h^\tau (\phi_h^\tau)^\top}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} w_h \right) - \phi(x,a)^\top \Sigma_h^{-1} \left(\sum_{\tau \in \mathcal{D}} \phi_h^\tau \cdot \frac{\left(r_h^\tau + f_{h+1}\left(x_{h+1}^\tau\right)\right)}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} \right) \\
=&\lambda \phi(x,a)^\top \Sigma_h^{-1} w_h + \phi(x,a)^\top \Sigma_h^{-1} \left(\sum_{\tau \in \mathcal{D}} \phi_h^\tau \cdot \frac{\left(\mathcal{T}_h f_{h+1}(x_h^\tau, a_h^\tau) - r_h^\tau - f_{h+1}(x_{h+1}^\tau)\right)}{\widehat{\sigma}_h^2(x_h^\tau, a_h^\tau)} \right) \\
\leq& \underbrace{\lambda \left\|w_h\right\|_{\Sigma_h^{-1}} \left\|\phi(x,a)\right\|_{\Sigma_h^{-1}}}_{\text{(i)}} + \underbrace{\left\| \sum_{\tau \in \mathcal{D}} \frac{\phi_h^\tau}{\widehat{\sigma}_h(x_h^\tau, a_h^\tau)} \cdot \xi_h^\tau(f_{h+1}) \right\|_{\Sigma_h^{-1}}}_{\text{(ii)}} \left\|\phi(x,a)\right\|_{\Sigma_h^{-1}}.
\end{aligned}
$$

This proves the first part of the lemma. Now suppose that $|f_{h+1}|$ is bounded by $H-1$. We have,

$$
\left\|w_h\right\| = \left\| \theta_h + \int_{\mathcal{S}} f_{h+1}(x') \mu_h(x') dx' \right\| \leq (1 + \max_{x'} f_{h+1}(x')) \sqrt{d} \leq H\sqrt{d},
$$

and

$$
\lambda \sqrt{w_h^\top \Sigma_h^{-1} w_h} \leq \left\|w_h\right\| \sqrt{\left\|\Sigma_h^{-1}\right\|} \leq \lambda \left\|w_h\right\| \sqrt{\lambda^{-1}} \leq \sqrt{\lambda d} H,
$$

where we use $\lambda_{\min}(\Sigma_h) \leq \lambda^{-1}$. Therefore, by setting λ sufficiently small such that $\sqrt{\lambda d} H \leq \beta$, (i) is non-dominating and we can focus on bounding (ii). □

J TECHNICAL LEMMAS

Lemma 8 (Hoeffding's inequality (Wainwright, 2019))**.** *Let $X_1, \cdots, X_n$ be mean-zero independent random variables such that $|X_i| \leq \xi_i$ almost surely. Then, for any $t > 0$, we have*

$$
\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n X_i \geq t \right) \leq \exp\left(-\frac{2n^2 t^2}{\sum_{i=1}^n \xi_i^2} \right).
$$

Lemma 9 (Hoeffding-type inequality for self-normalized process Abbasi-Yadkori et al. (2011))**.** *Let $\{\eta_t\}_{t=1}^\infty$ be a real-valued stochastic process and let $\{\mathcal{F}_t\}_{t=0}^\infty$ be a filtration such that η_t is $\mathcal{F}_t$-measurable. Let $\{x_t\}_{t=1}^\infty$ be an $\mathbb{R}^d$-valued stochastic process where x_t is $\mathcal{F}_{t-1}$ measurable and $\|x_t\| \leq L$. Let $\Lambda_t = \lambda I_d + \sum_{s=1}^t x_s x_s^\top$. Assume that conditioned on $\mathcal{F}_{t-1}$, η_t is mean-zero and R-subGaussian. Then for any $\delta > 0$, with probability at least $1-\delta$, for all $t > 0$, we have*

$$
\left\| \sum_{s=1}^t x_s \eta_s \right\|_{\Lambda_t^{-1}} \leq R\sqrt{d \log\left(1 + (tL)/\lambda\right) + 2\log(1/\delta)} \leq R\sqrt{d \log\left(\frac{\lambda + tL}{\lambda\delta}\right)} = \tilde{O}\left(R\sqrt{d}\right).
$$

Lemma 10 (Bernstein-type inequality for self-normalized process Zhou et al. (2021))**.** *Let $\{\eta_t\}_{t=1}^\infty$ be a real-valued stochastic process and let $\{\mathcal{F}_t\}_{t=0}^\infty$ be a filtration such that η_t is $\mathcal{F}_t$-measurable. Let $\{x_t\}_{t=1}^\infty$ be an $\mathbb{R}^d$-valued stochastic process where x_t is $\mathcal{F}_{t-1}$ measurable and $\|x_t\| \leq L$. Let $\Lambda_t = \lambda I_d + \sum_{s=1}^t x_s x_s^\top$. Assume that*

$$
|\eta_t| \leq R, \mathbb{E}[\eta_t | \mathcal{F}_{t-1}] = 0, \mathbb{E}[\eta_t^2 | \mathcal{F}_{t-1}] \leq \sigma^2.
$$

Then for any $\delta > 0$, with probability at least $1-\delta$, for all $t > 0$, we have

$$
\left\| \sum_{s=1}^t \mathbf{x}_s \eta_s \right\|_{\Lambda_t^{-1}} \leq 8\sigma \sqrt{d \log\left(1 + \frac{tL^2}{\lambda d}\right) \cdot \log\left(\frac{4t^2}{\delta}\right)} + 4R \log\left(\frac{4t^2}{\delta}\right) = \tilde{O}\left(\sigma\sqrt{d}\right).
$$

Lemma 11 (ϵ-Covering Number (Jin et al., 2021c)). *For all $h \in [H]$ and all $\epsilon > 0$, we have*

$$\log |\mathcal{N}(\epsilon; R, B, \lambda)| \leq d \cdot \log(1 + 4R/\epsilon) + d^2 \cdot \log(1 + 8d^{1/2}B^2/(\epsilon^2\lambda)).$$

Lemma 12 (Lemma H.4 of Min et al. (2021)). *Let $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ satisfying $\|\phi(x, a)\| \leq C$ for all $(x, a) \in \mathcal{S} \times \mathcal{A}$. For any $K > 0$ and $\lambda > 0$, define $\bar{\mathbb{G}}_K = \sum_{k=1}^{K} \phi(x_k, a_k)\phi(x_k, a_k)^\top + \lambda I_d$ where (x_k, a_k)'s are i.i.d. samples from some distribution ν over $\mathcal{S} \times \mathcal{A}$. Let $\mathbb{G} = \mathbb{E}_v[\phi(x, a)\phi(x, a)^\top]$. Then, for any $\delta \in (0, 1)$, with probability at least $1 - \delta$, it holds that*

$$\left\| \frac{\bar{\mathbb{G}}_K^{-1}}{K} - \mathbb{E}_\nu[\frac{\bar{\mathbb{G}}_K^{-1}}{K}] \right\| \leq \frac{4\sqrt{2}C^2}{\sqrt{K}} \left(\log \frac{2d}{\delta} \right)^{1/2}.$$

Lemma 13 (Lemma H.5 of Min et al. (2021)). *Let $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ satisfying $\|\phi(x, a)\| \leq C$ for all $(x, a) \in \mathcal{S} \times \mathcal{A}$. For any $K > 0$ and $\lambda > 0$, define $\bar{\mathbb{G}}_K = \sum_{k=1}^{K} \phi(x_k, a_k)\phi(x_k, a_k)^\top + \lambda I_d$ where (x_k, a_k)'s are i.i.d. samples from some distribution ν over $\mathcal{S} \times \mathcal{A}$. Let $\mathbb{G} = \mathbb{E}_v[\phi(x, a)\phi(x, a)^\top]$. Then, for any $\delta \in (0, 1)$, if K satisfies that*

$$K \geq \max \left\{ 512C^4 \left\| \mathbb{G}^{-1} \right\|^2 \log \left(\frac{2d}{\delta} \right), 4\lambda \left\| \mathbb{G}^{-1} \right\| \right\}. \tag{25}$$

Then with probability at least $1 - \delta$, it holds simultaneously for all $u \in \mathbb{R}^d$ that

$$\|u\|_{\bar{\mathbb{G}}_K^{-1}} \leq \frac{2}{\sqrt{K}} \|u\|_{\mathbb{G}^{-1}}.$$