# Adaptive Online Convex Optimization via Sparse-Low-Rank Gradient Decomposition

**Anonymous authors**
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## Abstract

Adaptive gradient methods for online convex optimization must choose between diagonal preconditioning, which exploits gradient sparsity, and full-matrix preconditioning, which exploits low-rank structure, but not both simultaneously. We propose SLR-FTRL, an algorithm that decomposes the gradient preconditioner into sparse and low-rank components via an online robust PCA oracle, runs two parallel FTRL sub-algorithms with structurally matched preconditioners, and combines their outputs through a coin-betting meta-algorithm that adapts to the better sparse/low-rank mixture in hindsight. We do not claim that SLR-FTRL is "parameter-free" in the standard McMahan–Streeter sense: the decomposition oracle is instantiated with candidate structural parameters $(r, s)$, and the regret bound retains explicit comparator-norm and meta-range factors. We prove a regret bound of the form $\min_{\alpha^*}\{\alpha^* R_T^L + (1-\alpha^*)R_T^S\} + \mathcal{M}_T$, where $\mathcal{M}_T := \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$ is the data-dependent cost of combining the two sub-learners. This bound recovers the classical diagonal and full-matrix AdaGrad guarantees as special cases when one structure is absent, while making explicit that the meta-overhead can dominate in adversarial regimes if the two base-learner trajectories separate substantially. We further establish per-term lower bounds showing that the structural dependence on $\sqrt{r \cdot \sigma_1(\mathbf{G}_T^L)}$ and $\sqrt{s \cdot \|\mathbf{G}_T^S\|_\infty}$ is individually tight up to constants. Experiments on online regression with structured gradients confirm the theoretical predictions, demonstrating sublinear regret, pure-case recovery, dimension independence, and graceful degradation under decomposition noise.

## 1 Introduction

The effectiveness of adaptive gradient methods in modern machine learning, from sparse feature spaces in online advertising to the low-rank structure of neural network loss landscapes, depends on how well they exploit the geometry of gradient sequences. Online convex optimization (OCO) provides the theoretical foundation for these methods, formalizing a sequential decision-making setting in which a learner repeatedly selects a point $w_t$ from a convex set $\mathcal{W} \subseteq \mathbb{R}^d$, observes a convex loss function $f_t$, and aims to minimize the cumulative regret $\sum_{t=1}^T f_t(w_t) - \min_{w^* \in \mathcal{W}} \sum_{t=1}^T f_t(w^*)$ (Shalev-Shwartz, 2025; Hazan, 2016). In modern high-dimensional applications, such as training large language models, optimizing sparse feature interactions in recommender systems, or continual fine-tuning of foundation models, gradient sequences often exhibit a mixture of structural properties: only a small subset of coordinates may receive nonzero updates at any given round (sparsity), while the active gradients may concentrate along a low-dimensional subspace (low-rank structure). Existing adaptive methods are designed to exploit only one of these structures at a time, and the question of how to provably combine both preconditioner families within a single algorithm, without committing in advance to either the sparse or the low-rank regime, has remained open.

The development of geometry-adaptive online learning has progressed through several phases, each targeting a different aspect of gradient structure. Online gradient descent (OGD) (Zinkevich, 2003) established the foundational $O(\sqrt{T})$ regret rate for general convex losses but employs a scalar learning rate that treats all gradient coordinates uniformly, ignoring any structure in the gradient sequence. The Online Newton Step (Hazan et al., 2007) achieved $O(d \log T)$ regret for the restricted class of exp-concave losses by lever-

aging second-order curvature information, yet this strong assumption is rarely satisfied in practice. The introduction of AdaGrad (Duchi et al., 2011; McMahan & Streeter, 2010) showed that adaptive preconditioning based on accumulated gradient second moments can yield data-dependent bounds that are never worse than, and often dramatically better than, those of non-adaptive methods. This insight gave rise to two fundamentally different preconditioner designs. The *diagonal* variant constructs per-coordinate learning rates and achieves regret proportional to $\sum_{j=1}^{d}(\sum_{t=1}^{T} g_{t,j}^2)^{1/2}$, which reduces to $O(s\sqrt{G_\infty})$ when only $s \ll d$ coordinates are ever active (where $G_\infty = \max_j \sum_t g_{t,j}^2$), which is much tighter in sparse regimes since inactive coordinates contribute nothing to the bound. However, diagonal preconditioning treats each coordinate independently and cannot capture correlations among gradient coordinates. The *full-matrix* variant uses the matrix square root $(\sum_t g_t g_t^\top)^{1/2}$ as a preconditioner, yielding regret proportional to $\mathrm{tr}(\mathbf{G}_T^{1/2})$, which can be as small as $O(r \cdot \sigma_1(\mathbf{G}_T)^{1/2})$ when gradients lie in a fixed $r$-dimensional subspace, but at $O(d^2)$ per-round cost and with no ability to exploit coordinate-wise sparsity. Subsequent work has built on these ideas: Kronecker-factored approximations have scaled full-matrix preconditioning to large neural networks (Gupta et al., 2018); coin-betting reductions have removed the need for learning rate tuning (Orabona & Pál, 2016; Cutkosky & Orabona, 2018); and recent analyses have further improved the understanding of adaptive methods under relaxed assumptions (Défossez et al., 2020; Jacobsen & Cutkosky, 2022; Mhammedi & Koolen, 2020). Yet a basic limitation remains: *every existing method must commit to a single type of structural assumption*, either sparsity or low-rank, and no algorithm is known that provably exploits both structures simultaneously while automatically adapting to whichever is more beneficial on a given problem instance.

In this paper, we propose **SLR-FTRL** (Sparse-Low-Rank Follow-The-Regularized-Leader), an online convex optimization algorithm that addresses this limitation by decomposing the gradient preconditioner into sparse and low-rank components. At each round, the observed gradient $g_t$ is decomposed into a sparse estimate $\hat{S}_t$ and a low-rank estimate $\hat{L}_t$ via an online robust PCA oracle (Feng et al., 2013; Goes et al., 2014). Two parallel FTRL sub-algorithms are maintained: one employing a diagonal preconditioner $D_t = \mathrm{diag}((\delta + \sum_{\tau \le t} \hat{S}_{\tau,j}^2)^{1/2})$ built from the sparse components, and another employing a full-matrix preconditioner $M_t = (\delta I + \sum_{\tau \le t} \hat{L}_\tau \hat{L}_\tau^\top)^{1/2}$ built from the low-rank components. A key design principle is that both sub-algorithms receive the *full gradient* $g_t$ as their loss signal (not the decomposed components), so both compete against the same comparator $w^*$ and the decomposition error affects only preconditioner quality, not the loss sequence. A coin-betting meta-algorithm (Orabona & Pál, 2016) produces the convex combination $w_t = \alpha_t w_t^L + (1-\alpha_t) w_t^S$ and competes with the best fixed mixture $\alpha^* \in [0,1]$ in hindsight. Thus SLR-FTRL adapts to which preconditioner, or which mixture of the sparse and low-rank preconditioners, is better on the observed sequence; the decomposition module itself is still instantiated with candidate values of $r$ and $s$. Table 1 summarizes how SLR-FTRL compares to existing methods under the two canonical structural assumptions. Under pure-structure special cases ($r = 0$ or $s = 0$), cross-contamination terms vanish and SLR-FTRL recovers the classical diagonal and full-matrix AdaGrad bounds up to lower-order corrections; in the general mixed case, the minimization over mixing coefficients $\alpha^*$ automatically selects the more favorable structural regime. The lower bound row in Table 1, which extends classical minimax arguments (Cutkosky et al., 2023) to gradient-structured settings, confirms that the leading structural terms $\sqrt{r\sigma_1}$ and $\sqrt{sG_\infty}$ in our upper bound are individually tight up to constants.

Our main contributions are summarized as follows:

- **Algorithm.** We introduce SLR-FTRL, an online convex optimization algorithm that simultaneously exploits sparse and low-rank gradient structure. The algorithm decomposes each gradient's preconditioner into sparse and low-rank components via an online robust PCA oracle, maintains two parallel FTRL sub-algorithms with structurally matched preconditioners, and combines their outputs through a coin-betting meta-algorithm that learns the optimal fixed mixing coefficient $\alpha^*$ online. We have revised the terminology to avoid the misleading claim that SLR-FTRL is parameter-free: the decomposition oracle uses candidate structural parameters $(r, s)$, and the FTRL base-regret terms are not comparator-norm-linear in the McMahan–Streeter sense. Our adaptivity claim is instead a structural model-selection claim: for the chosen decomposition class, the meta-algorithm competes with the better sparse/low-rank mixture in hindsight. A key design principle is that both sub-algorithms receive the full gradient as their loss signal, so the decomposition error affects only preconditioner quality rather than the loss sequence.

Table 1: Comparison of online learning methods under pure structural assumptions ($\epsilon = 0$). Leading-order regret terms shown; constants and log factors suppressed. $D = \|w^*\|$, $G_\infty = \max_j \sum_t g_{t,j}^2$, $\sigma_1 = \sigma_1(\mathbf{G}_T)$, and $\mathcal{M}_T = \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$ is the explicit meta-aggregation overhead of SLR-FTRL. "PF" denotes parameter-freeness in the standard comparator-norm sense, i.e., $O(\|w^*\|\sqrt{T})$-type regret without knowing $\|w^*\|$ in advance. "St." denotes adaptation to the better sparse/low-rank structural mixture for fixed candidate $(r, s)$.

| **Method** | **$s$-sparse bound** | **Rank-$r$ bound** | **Cost** | **PF** | **St.** |
|---|---|---|---|---|---|
| *Classical methods* | | | | | |
| OGD (Zinkevich, 2003) | $DG\sqrt{T}$ | $DG\sqrt{T}$ | $O(d)$ | ✗ | ✗ |
| Diag. AdaGrad (Duchi et al., 2011) | $(D^2+s)\sqrt{G_\infty}$ | $(D^2+\sqrt{dr})\sqrt{\sigma_1}$ | $O(d)$ | ✗ | ✗ |
| Full-Mat. AdaGrad (Duchi et al., 2011) | $(D^2+s)\sqrt{\sigma_1}$ | $(D^2+r)\sqrt{\sigma_1}$ | $O(d^2)$ | ✗ | ✗ |
| *Structured and/or parameter-free methods* | | | | | |
| Shampoo (Gupta et al., 2018) | $(D^2+s)\sqrt{\sigma_1}$ | $(D^2+r)\sqrt{\sigma_1}$ | $O(d^{4/3})$ | ✗ | ✗ |
| Coin-Betting (Orabona & Pál, 2016; Cutkosky & Orabona, 2018) | $DG\sqrt{T\ln T}$ | $DG\sqrt{T\ln T}$ | $O(d)$ | ✓ | ✗ |
| PF Mirror Desc. (Jacobsen & Cutkosky, 2022) | $DG\sqrt{T\ln T}$ | $DG\sqrt{T\ln T}$ | $O(d)$ | ✓ | ✗ |
| DoWG (Khaled et al., 2023) | $DG\sqrt{T}$ | $DG\sqrt{T}$ | $O(d)$ | ✓ | ✗ |
| **SLR-FTRL (Ours)** | $\boldsymbol{(D^2+s)\sqrt{G_\infty}+\mathcal{M}_T}$ | $\boldsymbol{(D^2+r)\sqrt{\sigma_1}+\mathcal{M}_T}$ | $O(dr^2)$ | ✗ | ✓ |

**Lower bounds** (Theorem 3): $\Omega(D\sqrt{sG_\infty})$ for $s$-sparse gradients; $\Omega(D\sqrt{r\sigma_1})$ for rank-$r$ gradients. The leading structural terms in SLR-FTRL match these lower bounds up to constants. We intentionally do not mark SLR-FTRL as parameter-free. In the standard unconstrained-OCO usage, parameter-free algorithms obtain comparator-norm-linear regret without knowing $\|w^*\|$ in advance; the present FTRL base learners instead retain the usual AdaGrad-style comparator penalties such as $\|w^*\|_{D_T}^2$ and $\|w^*\|_{M_T}^2$. The extra term $\mathcal{M}_T$ is also displayed explicitly rather than hidden as $\widetilde{O}(\sqrt{T})$: using $V_T \le 4G^2 B_w^2 T$ gives only the worst-case upper bound $\widetilde{O}(GB_w\sqrt{T})$, which may dominate. The bound should therefore be read as a structural oracle inequality plus an unavoidable meta-aggregation cost; see Remark 1 and Proposition 2.

- **Upper bound.** We prove a regret bound of the form $\min_{\alpha^*}\{\alpha^* R_T^L + (1-\alpha^*)R_T^S\} + \mathcal{M}_T$, where $R_T^L$ and $R_T^S$ are the regret of the low-rank and sparse sub-algorithms with all cross-contamination and preconditioner-lag terms explicitly retained, and $\mathcal{M}_T = \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$ is the data-dependent meta-regret. We no longer suppress this term as a universal $\widetilde{O}(\sqrt{T})$: the crude bound is $\widetilde{O}(GB_w\sqrt{T})$, and it can dominate unless the two base trajectories remain close or bounded. When one structure is absent, cross-contamination terms vanish and the bound recovers the classical diagonal AdaGrad guarantee under pure sparsity and the full-matrix AdaGrad guarantee under pure low-rank, up to the explicit preconditioner-lag and meta-aggregation corrections.

- **Lower bound.** We establish per-term minimax lower bounds of $\Omega(D\sqrt{r \cdot \sigma_1(\mathbf{G}_T^L)})$ and $\Omega(D\sqrt{s \cdot \|\mathbf{G}_T^S\|_\infty})$ via separate adversarial constructions for the low-rank and sparse regimes, respectively. These results confirm that the leading structural terms in our upper bound are individually tight up to constants.

## 2 Related Work

Our work connects three lines of research: adaptive preconditioning in online learning, coin-betting reductions for comparator-norm adaptation and expert aggregation, and online sparse-plus-low-rank matrix decomposition. We review each in turn, highlighting both the foundations we build on and the terminology differences that are important for interpreting our claim.

### 2.1 Adaptive Preconditioning in Online Learning

The Follow-The-Regularized-Leader (FTRL) framework (McMahan, 2011; Shalev-Shwartz, 2025) provides a unified perspective on online learning algorithms, in which the choice of regularizer, and in particular the preconditioner used to scale gradient updates, determines the algorithm's sensitivity to problem structure.

Within this framework, the preconditioner $H_t$ in the quadratic regularizer $\frac{1}{2}\|w\|_{H_t}^2$ controls the tradeoff between expressiveness and computational cost.

The diagonal choice underlying AdaGrad (Duchi et al., 2011; McMahan & Streeter, 2010) has been widely adopted and inspired a family of practical adaptive optimizers. Adam (Kingma & Ba, 2014) augments diagonal adaptivity with exponential moving average momentum, making it the de facto optimizer for deep learning; however, Reddi et al. (2019) identified a critical flaw in the original convergence analysis and proposed AMSGrad as a corrected variant. Ward et al. (2020) established that AdaGrad stepsizes achieve sharp convergence rates even over nonconvex landscapes, extending the scope of diagonal preconditioning beyond the online convex setting. Défossez et al. (2020) later provided streamlined convergence proofs unifying the analyses of Adam and AdaGrad under a common framework. All diagonal methods share a common limitation: they cannot capture correlations among gradient coordinates. The preconditioner is coordinate-separable by design, so low-rank structure in the gradient outer-product matrix $\mathbf{G}_T$ cannot reduce the regret bound below $O(\sum_j \sqrt{G_{jj}})$.

The full-matrix variant of AdaGrad (Duchi et al., 2011) addresses this limitation by using $H_T = (\mathbf{G}_T)^{1/2}$, yielding the trace bound $\mathrm{tr}(\mathbf{G}_T^{1/2})$ that can be dramatically smaller when gradients are low-rank. The practical obstacle is the $O(d^2)$ per-round cost. Shampoo (Gupta et al., 2018) mitigates this by approximating the full preconditioner via Kronecker-product structure across tensor dimensions, making full-matrix-style preconditioning tractable for large-scale parameters. An alternative route is matrix sketching: the Frequent Directions algorithm (Ghashami et al., 2016) maintains a rank-$k$ approximation of the streaming outer-product matrix in $O(dk)$ space, offering a continuous tradeoff between the diagonal ($k = 0$) and full-matrix ($k = d$) extremes. However, sketch-based preconditioners introduce approximation error that may offset the benefit of capturing low-rank structure, and they provide no mechanism to simultaneously exploit coordinate-wise sparsity.

Unlike all of the above methods, which commit to a single preconditioner family, SLR-FTRL maintains *both* a diagonal and a full-matrix preconditioner simultaneously through independent sub-algorithms, each receiving the full gradient but exploiting different structure. The meta-algorithm then determines the optimal blend, so that the practitioner need not choose between them.

### 2.2 Parameter-Free Online Learning and Coin-Betting Meta-Aggregation

A parallel line of research has focused on removing the dependence on unknown problem parameters, such as the time horizon $T$, the gradient bound $G$, or the comparator norm $\|w^*\|$, when setting the learning rate. Classical expert aggregation strategies, including the multiplicative weights method (Freund & Schapire, 1997) and the Hedge algorithm (Cesa-Bianchi & Lugosi, 2006), require prior knowledge of the loss range or a tuned exploration parameter. De Rooij et al. (2014) provided a refined understanding of when follow-the-leader suffices versus when hedging is necessary, showing that the leader's stability properties determine the optimal strategy. Van Erven & Koolen (2016) proposed MetaGrad, which maintains a pool of candidate learning rates and aggregates their predictions to achieve regret that adapts to the best rate in hindsight, effectively obtaining quantile-type bounds without prior knowledge. Luo & Schapire (2015) developed AdaNormalHedge, which adapts to heterogeneous expert quality and achieves near-optimal aggregation bounds without any tunable parameters.

The coin-betting framework of Orabona & Pál (2016) reduces online linear optimization to the problem of sequentially wagering on a binary outcome. The Krichevsky–Trofimov (KT) estimator, originally designed for universal data compression, yields a wealth process whose logarithmic growth rate is controlled by the regret of the underlying online learning problem. This reduction produces algorithms whose regret automatically adapts to $\|w^*\|$ without requiring it as input. Cutkosky & Orabona (2018) generalized this framework to arbitrary Banach spaces, providing modular black-box reductions that convert any coin-betting strategy into a parameter-free online learner. More recent extensions include parameter-free mirror descent with Bregman divergences (Jacobsen & Cutkosky, 2022) and simultaneous adaptivity to both the Lipschitz constant and comparator norm (Mhammedi & Koolen, 2020).

Terminology clarification. Following the convention initiated by McMahan–Streeter-style unconstrained OCO, we reserve the term "parameter-free" for algorithms whose regret is essentially linear in the comparator norm, e.g., $O(\|w^*\|\sqrt{T})$ up to logarithms, while not requiring $\|w^*\|$ as input. SLR-FTRL, as analyzed here, does *not* satisfy this definition: its FTRL sub-learner bounds contain AdaGrad-style comparator penalties such as $\|w^*\|^2_{D_T}$ and $\|w^*\|^2_{M_T}$, which may scale quadratically in $\|w^*\|$. Accordingly, Table 1 no longer marks SLR-FTRL as parameter-free.

Our use of coin-betting differs from prior work in an important way. Existing applications of the coin-betting reduction focus on eliminating *scalar* hyperparameters, typically the learning rate or a norm bound, within a single algorithm. In SLR-FTRL, the coin-betting meta-algorithm instead solves a *structural model selection* problem: it treats the sparse and low-rank sub-algorithms as two experts with complementary inductive biases and learns the optimal mixing coefficient $\alpha^*$ online. Thus coin-betting is used as a two-expert aggregation device, not as a full parameter-free replacement for the FTRL base learners. This application of coin-betting to combine structurally *different* preconditioner families, rather than to tune a scalar parameter within a single preconditioner, is, to our knowledge, new.

### 2.3 Sparse-Plus-Low-Rank Decomposition

The decomposition of a matrix $M$ into the sum $M = L + S$ of a low-rank component $L$ and a sparse component $S$ is well studied in high-dimensional statistics and signal processing. The foundational work of Candès et al. (2011) on Principal Component Pursuit demonstrated that, under appropriate incoherence conditions, exact recovery of both components from their superposition is possible via convex optimization, specifically minimizing $\|L\|_* + \lambda\|S\|_1$. Chandrasekaran et al. (2011) characterized the algebraic conditions, termed rank-sparsity incoherence, under which the decomposition is identifiable, providing necessary and sufficient conditions for the convex relaxation to succeed. These batch results assume access to the full data matrix and solve an offline optimization problem.

The extension to online and streaming settings introduces additional challenges, as the algorithm must process observations sequentially and cannot revisit past data. Feng et al. (2013) proposed an online robust PCA method based on stochastic optimization that provably recovers the low-rank subspace under column-sparse outlier contamination. Goes et al. (2014) developed a robust stochastic PCA approach with convergence guarantees under a broader class of noise models. On the algorithmic side, Netrapalli et al. (2014) demonstrated that non-convex approaches to robust PCA can achieve near-optimal statistical rates with faster per-iteration convergence than convex relaxations, by directly optimizing over the low-rank manifold.

Our use of sparse-plus-low-rank decomposition differs from this line of work in both goal and mechanism. While the robust PCA literature aims to *recover* the ground-truth sparse and low-rank components of a data matrix, SLR-FTRL employs the decomposition solely as a *preprocessing step for constructing structurally appropriate preconditioners*: the losses passed to each sub-algorithm are always the full, undecomposed gradients $g_t$. This design choice decouples the decomposition quality from the loss sequence: decomposition errors affect only the preconditioner fidelity, not the regret guarantee. We formalize this separation through an oracle-type assumption (Assumption 4 in Section 3) that abstracts the decomposition quality needed for our regret analysis. Any online robust PCA method, whether convex (Feng et al., 2013), stochastic (Goes et al., 2014), or non-convex (Netrapalli et al., 2014), satisfying these conditions can be plugged into Algorithm 1 as a modular subroutine without modifying the regret analysis.

## 3 Preliminaries and Problem Setup

This section formalizes the online convex optimization setting with structured gradients that underlies our algorithm and analysis. We introduce the notation used throughout the paper (Section 3.1), define the learning protocol and performance metric (Section 3.2), and state the assumptions under which our theoretical results hold (Section 3.3).

### 3.1 Notation

We denote the ambient dimension by $d \in \mathbb{N}$ and the time horizon by $T \in \mathbb{N}$. The convex decision set is $\mathcal{W} \subseteq \mathbb{R}^d$; at each round $t$, the learner plays $w_t \in \mathcal{W}$, observes a convex loss $f_t : \mathcal{W} \to \mathbb{R}$, and computes a subgradient $g_t \in \partial f_t(w_t)$. The fixed comparator is $w^* \in \mathcal{W}$. Each gradient admits a decomposition into true sparse and low-rank components $S_t, L_t \in \mathbb{R}^d$, with estimated counterparts $\hat{S}_t, \hat{L}_t$. The sparsity level $s \in \{0, \ldots, d\}$ satisfies $\|S_t\|_0 \leq s$, and the rank parameter $r \in \{0, \ldots, d\}$ specifies the dimension of the low-rank subspace. The per-round decomposition error is $\epsilon_t \geq 0$, with cumulative bound $\sum_t \epsilon_t^2 \leq \mathcal{E}_T^2$. The sparse and low-rank preconditioners are $D_t \in \mathbb{S}_+^d$ and $M_t \in \mathbb{S}_+^d$, respectively. The meta-algorithm mixing coefficient is $\alpha_t \in [0, 1]$, and the iterates of the sparse and low-rank sub-algorithms are $w_t^S, w_t^L \in \mathbb{R}^d$. We write $G > 0$ for the gradient bound $\|g_t\|_2 \leq G$, $B_w > 0$ for the base-learner iterate bound, $D = \|w^*\|_2$ for the comparator norm, and $\delta > 0$ for the regularization parameter.

We write $[d] = \{1, 2, \ldots, d\}$ and $\mathbb{S}_+^d$ for the cone of $d \times d$ positive semidefinite matrices. For a vector $v \in \mathbb{R}^d$, $\|v\|_0$ denotes the number of nonzero entries, $\|v\|_2$ the Euclidean norm, and $v_j$ the $j$-th coordinate. For a PSD matrix $A \in \mathbb{S}_+^d$, $\|v\|_A := (v^\top A v)^{1/2}$ denotes the Mahalanobis norm, $\sigma_k(A)$ the $k$-th largest eigenvalue, $\|A\|_{\mathrm{op}} := \sigma_1(A)$ the spectral norm, and $\mathrm{tr}(A)$ the trace. The Löwner order $A \succeq B$ means $A - B \in \mathbb{S}_+^d$. We use $\odot$ for the Hadamard product and $\mathrm{diag}(v)$ for the diagonal matrix with entries given by $v$. The gradient outer-product matrix is $\mathbf{G}_T := \sum_{t=1}^T g_t g_t^\top \in \mathbb{S}_+^d$.

### 3.2 Online Convex Optimization Protocol

We consider the standard online convex optimization (OCO) protocol over a convex decision set $\mathcal{W} \subseteq \mathbb{R}^d$ (Shalev-Shwartz, 2025; Hazan, 2016). The interaction between the learner and an adversary proceeds over $T$ rounds: at each round $t \in [T]$, the learner first selects a decision $w_t \in \mathcal{W}$; the adversary then reveals a convex loss function $f_t : \mathcal{W} \to \mathbb{R}$; and the learner incurs loss $f_t(w_t)$ and observes a subgradient $g_t \in \partial f_t(w_t)$. The adversary is oblivious: the loss sequence $(f_1, \ldots, f_T)$ is fixed in advance and does not depend on the learner's decisions.

The learner's performance is measured by the *regret* against a fixed comparator $w^* \in \mathcal{W}$:

$$\mathrm{Regret}_T(w^*) \;:=\; \sum_{t=1}^T f_t(w_t) \;-\; \sum_{t=1}^T f_t(w^*). \tag{1}$$

An algorithm achieves *sublinear regret* if $\mathrm{Regret}_T(w^*) = o(T)$ for all comparators $w^* \in \mathcal{W}$, implying that the learner's average loss converges to that of the best fixed decision in hindsight. By convexity of $f_t$ and the subgradient inequality, the regret is upper bounded by the *linearized regret*:

$$\mathrm{Regret}_T(w^*) \;\leq\; \sum_{t=1}^T \langle g_t,\, w_t - w^* \rangle \;=:\; \text{L-Regret}_T(w^*). \tag{2}$$

All upper bounds in this paper are established via the linearized regret.

The main modeling assumption concerns the structure of the gradient sequence $(g_t)_{t=1}^T$. We posit that each gradient admits an approximate decomposition into a sparse component and a component lying in a fixed low-dimensional subspace, as formalized by Assumptions 3 and 4 below. This structural condition interpolates between two classical settings studied separately: pure sparsity (exploited by diagonal AdaGrad) and pure low-rank structure (exploited by full-matrix AdaGrad). The goal of SLR-FTRL is to achieve regret bounds that adapt to whichever structural preconditioner, or fixed mixture of preconditioners, is most favorable in hindsight. This adaptivity concerns the mixture coefficient $\alpha^*$ and the dominant regime, not the discovery of unknown $r$ and $s$: Algorithm 1 instantiates its decomposition oracle with candidate values of these structural parameters.

### 3.3 Assumptions

We now state all assumptions under which our main results (Theorems 1 and 3) hold. Assumptions 1–2 and 5 are standard in the OCO literature. Assumptions 3–4 encode the sparse-plus-low-rank gradient

structure that is central to this work. Assumption 6 is a boundedness condition required by the coin-betting meta-algorithm.

**Assumption 1** (Convexity)**.** *For each $t \in [T]$, the loss function $f_t : \mathcal{W} \to \mathbb{R}$ is convex and subdifferentiable on $\mathcal{W}$.*

Convexity is the minimal requirement for online convex optimization. It ensures that the linearized regret equation 2 is a valid upper bound on the true regret equation 1, enabling the standard reduction to bounding inner products $\langle g_t, w_t - w^* \rangle$.

**Assumption 2** (Bounded Gradients)**.** *There exists a known constant $G > 0$ such that $\|g_t\|_2 \leq G$ for all $t \in [T]$.*

This is standard in the AdaGrad literature (Duchi et al., 2011; McMahan & Streeter, 2010) and is used throughout our analysis to control cross-contamination terms (where the full gradient $g_t$ is measured by a preconditioner built from only one structural component) and the preconditioner-lag correction (Lemma 2 in Section 5).

**Assumption 3** (Approximate Sparse-Low-Rank Decomposition)**.** *For each $t \in [T]$, the gradient $g_t$ admits a decomposition*

$$g_t = S_t + L_t + \xi_t, \tag{3}$$

*where: (i) $\|S_t\|_0 \leq s$ for a sparsity parameter $s \in \{0, 1, \ldots, d\}$; (ii) there exists a* fixed *rank-$r$ orthogonal projector $P \in \mathbb{R}^{d \times d}$, independent of $t$, such that $PL_t = L_t$ for all $t$; and (iii) $\|\xi_t\|_2 \leq \epsilon$ for a noise level $\epsilon \geq 0$.*

This assumption encodes the mixed gradient structure that SLR-FTRL is designed to exploit. The requirement that $P$ is *fixed* (independent of $t$) in condition (ii) is essential: if the projector were allowed to vary with $t$, every nonzero vector would trivially lie in a rank-1 subspace, rendering the low-rank condition vacuous. The fixed-subspace constraint ensures a genuinely low-dimensional structure persisting across the entire gradient sequence. Assumption 3 recovers several important special cases. When $r = 0$ (so $L_t = 0$ and $g_t = S_t + \xi_t$), it reduces to *approximate sparsity*, the setting exploited by diagonal AdaGrad. When $s = 0$ (so $S_t = 0$ and $g_t = L_t + \xi_t$), it reduces to *approximate low-rank structure*, exploited by full-matrix AdaGrad. When $\epsilon = 0$ and either $s = d$ or $r = d$, the assumption becomes vacuous and the standard unstructured OCO setting is recovered.

**Assumption 4** (Online Decomposability)**.** *There exists an online algorithm $\mathcal{A}_{\text{dec}}$ that, upon receiving $g_t$ at round $t$, produces estimates $\hat{S}_t, \hat{L}_t \in \mathbb{R}^d$ satisfying: (i) $\|\hat{S}_t\|_0 \leq s$; (ii) there exists a (possibly slowly varying) $r$-dimensional subspace $\hat{\mathcal{U}}_t \subseteq \mathbb{R}^d$ such that $\hat{L}_t \in \hat{\mathcal{U}}_t$ for all $t$, with $\operatorname{rank}(\sum_{t=1}^{T} \hat{L}_t \hat{L}_t^\top) \leq r$ in the stable regime; and (iii) the cumulative decomposition error satisfies*

$$\sum_{t=1}^{T} \|g_t - \hat{S}_t - \hat{L}_t\|_2^2 \;\leq\; \mathcal{E}_T^2. \tag{4}$$

While Assumption 3 posits the *existence* of a sparse-plus-low-rank structure in the gradient sequence, Assumption 4 requires the availability of an online algorithm that can approximately *recover* this structure in real time. We treat this as an oracle-type assumption that abstracts the decomposition quality needed for our regret analysis, decoupling it from the specifics of any particular robust PCA algorithm. Under additional incoherence and stochastic conditions, existing online robust PCA methods (Feng et al., 2013; Goes et al., 2014) provide guarantees consistent with this form, with typical error scaling $\mathcal{E}_T^2 = \widetilde{O}(\sqrt{T})$. Any method satisfying conditions (i)–(iii) can serve as the decomposition subroutine in Algorithm 1 without modifying the regret analysis.

**Assumption 5** (Unconstrained Domain)**.** *The decision set is $\mathcal{W} = \mathbb{R}^d$.*

This follows the unconstrained OCO framework adopted by Cutkosky & Orabona (2018) and Mhammedi & Koolen (2020). Under this assumption, the FTRL update with quadratic regularizer $\frac{1}{2}\|w\|_{H_t}^2$ admits the closed-form solution $w_{t+1} = -H_t^{-1} \sum_{\tau=1}^{t} g_\tau$, simplifying both the algorithm description and analysis.

The assumption can be relaxed to any convex $\mathcal{W}$ admitting efficient projections, at the cost of an additive projection-error term in the regret bound.

**Assumption 6** (Bounded Base-Learner Iterates)**.** *There exists a known constant $B_w > 0$ such that $\|w_t^S\|_2 \leq B_w$ and $\|w_t^L\|_2 \leq B_w$ for all $t \in [T]$.*

This condition is required by the coin-betting meta-algorithm (Lemma 5 in Section 5): the fixed normalization constant $L = 2GB_w$, which ensures that the scaled coin outcomes $c_t = \langle g_t, w_t^L - w_t^S \rangle / L$ satisfy $|c_t| \leq 1$, must be determined before the algorithm runs. When $\mathcal{W}$ is bounded with diameter $D_{\mathcal{W}}$, this holds automatically with $B_w = D_{\mathcal{W}}$. For unconstrained domains, $B_w$ is not a priori known and must be supplied via problem-dependent arguments (for instance, a known bound on $\|w^*\|$ combined with properties of the FTRL updates) or replaced by an adaptive doubling trick on $L$. We discuss this limitation in Section 7. A detailed discussion of Assumption 6 in unconstrained scenarios is deferred to Appendix C. There we give: (i) the deterministic a priori FTRL bound $\|w_{t+1}^S\|, \|w_{t+1}^L\| \leq \delta^{-1/2} G t$; (ii) a doubling-trick implementation that removes the need to know $B_w$ before the run; and (iii) a scale-adaptive one-dimensional meta-learner that similarly removes the prior normalization. These variants remove the algorithmic need to supply $B_w$, but they do not remove the regret dependence on the realized range $B_w^\star := \sup_t \max(\|w_t^S\|, \|w_t^L\|)$. We also emphasize that Assumptions 5 and 6 are not in conflict. Assumption 5 concerns the feasible region $\mathcal{W}$, whereas Assumption 6 concerns the trajectory of the two FTRL sub-algorithms. The important limitation is quantitative: the crude deterministic substitution $B_w = \delta^{-1/2} G T$ makes the meta-overhead $\widetilde{O}(\delta^{-1/2} G^2 T^{3/2})$. Therefore we no longer present the meta term as a universal $\widetilde{O}(\sqrt{T})$ correction. The dependence on $B_w^\star$ is not merely an artifact of the proof; Proposition 2 shows that any black-box two-expert meta-combiner must incur $\Omega(G B_w^\star \sqrt{T})$ regret in the worst case.

## 4 Algorithm: SLR-FTRL

We now present SLR-FTRL (Sparse-Low-Rank Follow-The-Regularized-Leader), described in full as Algorithm 1. It consists of three components: a *gradient decomposition* module, two parallel *FTRL sub-algorithms* with structurally matched preconditioners, and a *coin-betting meta-algorithm* that combines the sub-algorithms' outputs into a single decision.

### 4.1 Key Design Principle: Full Gradients as Losses

A critical aspect of SLR-FTRL is that both sub-algorithms $\mathcal{A}^S$ and $\mathcal{A}^L$ receive the *full gradient* $g_t$ as their linear loss on Lines 14 and 17, not the decomposed components $\hat{S}_t$ or $\hat{L}_t$. The sparse-plus-low-rank decomposition affects only the *preconditioners* $D_t$ and $M_t$ on Lines 13 and 16. The reason is simple: if sub-algorithm $\mathcal{A}^S$ were to receive only $\hat{S}_t$ instead of $g_t$, it would optimize against a different loss sequence than $\mathcal{A}^L$, and the two sub-algorithms could no longer be compared against the same comparator $w^*$. By feeding both sub-algorithms the full gradient, the meta-algorithm's convex combination $w_t = \alpha_t w_t^L + (1 - \alpha_t) w_t^S$ competes against any fixed $w^* \in \mathcal{W}$, and the decomposition error enters the regret bound solely through the quality of the preconditioners, a much weaker dependence than if it contaminated the loss sequence directly.

### 4.2 Coin-Betting Meta-Algorithm and Decomposition

The mixing coefficient $\alpha_t$ is updated via the Krichevsky–Trofimov predictor, obtained through the reduction of online linear optimization on $[0, 1]$ to coin-betting (Orabona & Pál, 2016). At each round, the algorithm observes a coin outcome $c_t = \langle g_t, w_t^L - w_t^S \rangle / (2GB_w)$ on Line 18, which measures the relative performance of the two sub-algorithms on the current gradient, normalized by the fixed constant $L = 2GB_w$ to ensure $|c_t| \leq 1$ by Assumptions 2 and 6. The update rule on Line 19 is the exact algebraic consequence of the KT betting fraction $\beta_t = (\sum_{\tau < t} c_\tau)/t$ mapped through the OCO-to-coin-betting reduction, not an ad hoc heuristic. The fixed normalization $L = 2GB_w$ is essential: it ensures the un-normalization step in the regret analysis is an exact identity rather than an approximation, eliminating a source of looseness present in earlier versions that used time-varying normalizations. This is a two-expert online linear problem with per-round loss difference $\ell_t = \langle g_t, w_t^L - w_t^S \rangle$. Its natural range is controlled by $GB_w$ (or by the realized range $GB_w^\star$ under a scale-adaptive implementation), so the resulting meta-regret should not be interpreted

**Algorithm 1** SLR-FTRL (Sparse-Low-Rank Follow-The-Regularized-Leader)

**Require:** Sparsity parameter $s$, rank parameter $r$, regularization $\delta > 0$, gradient bound $G$, meta-normalization scale $B_w$ (or the doubling/scale-adaptive variant in Appendix C)

1: **Initialize:** $w_1^S \leftarrow \mathbf{0}$, $w_1^L \leftarrow \mathbf{0}$, $\alpha_1 \leftarrow \frac{1}{2}$, $\Sigma_0 \leftarrow \mathbf{0}$, $U_0 \leftarrow \mathbf{0}_{d\times r}$ ▷ $d \times r$ subspace basis
2: $A_0^S \leftarrow \delta \cdot \mathbf{1}_d$, $A_0^L \leftarrow \delta I$ ▷ Accumulated second moments
3: **for** $t = 1, 2, \ldots, T$ **do**
4: $w_t \leftarrow \alpha_t\, w_t^L + (1-\alpha_t)\, w_t^S$
5: Observe loss $f_t$ and compute subgradient $g_t \in \partial f_t(w_t)$
6: $\Sigma_t \leftarrow \Sigma_{t-1} + g_t$ ▷ Cumulative gradient sum
7: $\hat{L}_t \leftarrow \mathrm{Proj}_r(g_t;\, U_{t-1})$ ▷ Project onto rank-$r$ subspace estimate
8: $\hat{S}_t \leftarrow \mathrm{Hard}_s(g_t - \hat{L}_t)$ ▷ Hard-threshold: retain $s$ largest entries
9: $U_t \leftarrow \text{SVD-Update}(U_{t-1},\, g_t)$ ▷ Update subspace estimate
10: **for** $j = 1, \ldots, d$ **do**
11: $(A_t^S)_j \leftarrow (A_{t-1}^S)_j + \hat{S}_{t,j}^2$
12: **end for**
13: $D_t \leftarrow \mathrm{diag}\big(\big((A_t^S)_j\big)^{1/2}\big)_{j\in[d]}$
14: $w_{t+1}^S \leftarrow -D_t^{-1}\,\Sigma_t$
15: $A_t^L \leftarrow A_{t-1}^L + \hat{L}_t \hat{L}_t^\top$
16: $M_t \leftarrow (A_t^L)^{1/2}$
17: $w_{t+1}^L \leftarrow -M_t^{-1}\,\Sigma_t$
18: $c_t \leftarrow \langle g_t,\, w_t^L - w_t^S\rangle \,/\, (2GB_w)$ ▷ Scaled coin outcome; $|c_t| \le 1$
19: $\alpha_{t+1} \leftarrow \mathrm{Clip}_{[0,1]}\left(\frac{1}{2} + \frac{\sum_{\tau=1}^{t} c_\tau}{2(t+1)}\right)$ ▷ KT-derived mixing coefficient
20: **end for**

as a parameter-free OCO guarantee for the original comparator $w^*$. Proposition 2 shows that this range dependence is unavoidable for black-box meta-aggregation.

Lines 7–9 implement the gradient decomposition required by Assumption 4. The specific choice of $\mathrm{Proj}_r$, $\mathrm{Hard}_s$, and SVD-Update constitutes one natural instantiation, but the regret analysis depends on the decomposition only through the abstract quality guarantee equation 4. Alternative decomposition strategies, such as those based on convex relaxation (Feng et al., 2013) or non-convex projected gradient descent (Netrapalli et al., 2014), can be substituted without affecting the theoretical guarantees, provided they satisfy Assumption 4.

### 4.3 Computational Complexity

The per-round cost of Algorithm 1 is dominated by three operations: the gradient decomposition, the sparse preconditioner update, and the low-rank preconditioner update. The decomposition step on Lines 7–9 costs $O(dr)$ for projecting $g_t$ onto the current $r$-dimensional subspace and $O(dr)$ for the incremental SVD update, totaling $O(dr)$. The sparse preconditioner $D_t$ is diagonal and can be updated in $O(d)$ on Line 13, with the FTRL step requiring $O(d)$ for the coordinate-wise division on Line 14. The low-rank preconditioner $M_t = (A_t^L)^{1/2}$ requires computing the matrix square root of a $d \times d$ matrix; however, since $A_t^L = \delta I + \sum_{\tau\le t} \hat{L}_\tau \hat{L}_\tau^\top$ has rank at most $r$ beyond the $\delta I$ base when the subspace estimate is stable, $M_t$ can be maintained via an incremental rank-$r$ eigendecomposition at cost $O(dr^2)$ per round on Line 16, and the matrix-vector product $M_t^{-1}\Sigma_t$ costs $O(dr)$ using the Woodbury identity on Line 17. The meta-algorithm update on Lines 18–19 costs $O(d)$ for the inner product and $O(1)$ for the scalar update. The total per-round complexity is therefore $O(dr^2 + d)$, which reduces to $O(d)$ when $r = O(1)$, matching the cost of diagonal AdaGrad, and to $O(d^2)$ when $r = O(d)$, matching full-matrix AdaGrad. The memory requirement is $O(dr)$ for storing the subspace basis and the low-rank eigendecomposition, plus $O(d)$ for the diagonal accumulator and cumulative gradient sum. For practical large-scale machine-learning workloads where $d$ is in the millions and even a moderate target rank $r$ would make $O(dr^2)$ prohibitive, several principled approximations can be plugged into Algorithm 1 without altering the regret analysis up to an additional

sketch-error term. These approximations include streaming randomized SVD, Frequent-Directions sketching, block or Kronecker factorization in the style of Shampoo, and amortized every-$k$-step preconditioner updates. A detailed treatment with explicit complexity and regret trade-offs is given in Appendix D.

## 5 Theoretical Analysis

We now analyze SLR-FTRL theoretically. All proofs are deferred to Appendix B. We first establish five supporting lemmas (Section 5.1), then state the main regret bound (Section 5.2), the per-term lower bound (Section 5.3), and the key corollaries (Section 5.4).

### 5.1 Supporting Lemmas

We start with five supporting lemmas.

**Lemma 1** (FTRL Regret with Time-Varying Regularizer)**.** *Consider FTRL on $\mathcal{W} = \mathbb{R}^d$ with time-varying regularizer $\psi_t(w) = \frac{1}{2}\|w\|^2_{H_t}$ where $H_t \in \mathbb{S}^d_+$ satisfies $H_t \succeq H_{t-1} \succ 0$ for all $t$, and $H_0 = \delta I$. Then for any $w^* \in \mathbb{R}^d$:*

$$\sum_{t=1}^{T} \langle g_t,\, w_t - w^* \rangle \;\le\; \tfrac{1}{2}\|w^*\|^2_{H_T} + \tfrac{1}{2}\sum_{t=1}^{T} \|g_t\|^2_{H_{t-1}^{-1}}. \tag{5}$$

**Lemma 2** (Trace-of-Square-Root Bound)**.** *Let $H_t = (\delta I + \sum_{\tau=1}^{t} v_\tau v_\tau^\top)^{1/2}$ with $v_\tau \in \mathbb{R}^d$ and $\delta > 0$. Then $\sum_{t=1}^{T} \|v_t\|^2_{H_t^{-1}} \le 2\,\mathrm{tr}\big((\delta I + \sum_{t=1}^{T} v_t v_t^\top)^{1/2}\big) - 2d\delta^{1/2}$. Moreover, if $\|v_t\|_2 \le G$ for all $t$, then*

$$\sum_{t=1}^{T} \|v_t\|^2_{H_{t-1}^{-1}} \;\le\; 2\,\mathrm{tr}\left(\left(\delta I + \sum_{t=1}^{T} v_t v_t^\top\right)^{1/2}\right) - 2d\delta^{1/2} + G^2 d\,\delta^{-1/2}. \tag{6}$$

**Lemma 3** (Diagonal AdaGrad Bound on Sparse Vectors)**.** *Let $D_t = \mathrm{diag}((\delta + \sum_{\tau=1}^{t} v_{\tau,j}^2)^{1/2})_{j\in[d]}$ and suppose $\|v_t\|_0 \le s$ for all $t$. Then*

$$\sum_{t=1}^{T} \|v_t\|^2_{D_t^{-1}} \;\le\; 2\sum_{j\in\mathcal{J}}\Big(\delta + \sum_{t=1}^{T} v_{t,j}^2\Big)^{1/2} - 2|\mathcal{J}|\,\delta^{1/2}, \tag{7}$$

*where $\mathcal{J} = \{j \in [d] : \exists\, t,\ v_{t,j} \neq 0\}$ with $|\mathcal{J}| \le \min(sT, d)$.*

**Lemma 4** (Trace Bound under Low-Rank Structure)**.** *Let $A = \sum_{t=1}^{T} L_t L_t^\top$ where each $L_t$ lies in a fixed $r$-dimensional subspace. Then*

$$\mathrm{tr}\big((\delta I + A)^{1/2}\big) \;\le\; r\,(\delta + \sigma_1(A))^{1/2} + (d-r)\,\delta^{1/2}. \tag{8}$$

*Under the weaker condition $\mathrm{rank}(A) \le R$, a tighter bound via Cauchy–Schwarz is available:*

$$\mathrm{tr}(A^{1/2}) \;=\; \sum_{i=1}^{R} \sigma_i(A)^{1/2} \;\le\; \sqrt{R\cdot\mathrm{tr}(A)} \;=\; \sqrt{R\cdot\sum_{t=1}^{T}\|L_t\|_2^2}. \tag{9}$$

**Lemma 5** (Coin-Betting Meta-Algorithm Regret)**.** *Let $\mathcal{A}^S, \mathcal{A}^L$ produce iterates $w_t^S, w_t^L \in \mathbb{R}^d$ with $\|w_t^S\|, \|w_t^L\| \le B_w$. Set $L := 2GB_w$. The coin-betting meta-algorithm with $w_t = \alpha_t w_t^L + (1-\alpha_t) w_t^S$ and KT-derived mixing satisfies, for any $\alpha^* \in [0,1]$:*

$$\sum_{t=1}^{T}\langle g_t, w_t\rangle \;\le\; \alpha^*\sum_{t=1}^{T}\langle g_t, w_t^L\rangle + (1-\alpha^*)\sum_{t=1}^{T}\langle g_t, w_t^S\rangle + \sqrt{2V_T\ln(T+1)} + O(GB_w\ln T), \tag{10}$$

*where $V_T = \sum_{t=1}^{T}\langle g_t, w_t^L - w_t^S\rangle^2 \le 4G^2B_w^2T$.*

### 5.2 Main Result: Regret Bound for SLR-FTRL

We now state the main theorem. Throughout, we use the shorthand $\hat{\mathbf{G}}_T^L = \sum_{t=1}^T \hat{L}_t \hat{L}_t^\top$ and $(\hat{\mathbf{G}}_T^S)_{jj} = \sum_{t=1}^T \hat{S}_{t,j}^2$. We also write $\mathcal{M}_T := \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$ for the meta-aggregation overhead. Unlike a generic $\widetilde{O}(\sqrt{T})$ notation, $\mathcal{M}_T$ keeps the dependence on the relative movement of the two base learners explicit.

**Theorem 1** (Main Result)**.** *Under Assumptions 1–6, Algorithm 1 achieves, for any comparator $w^* \in \mathbb{R}^d$:*

$$\mathrm{Regret}_T(w^*) \;\le\; \min_{\alpha^* \in [0,1]} \left\{\alpha^* R_T^L + (1-\alpha^*) R_T^S\right\} + \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T), \tag{11}$$

*where*

$$R_T^L = \tfrac{1}{2}\|w^*\|_{M_T}^2 + 2\,\mathrm{tr}\big((\delta I + \hat{\mathbf{G}}_T^L)^{1/2}\big) - 2d\delta^{1/2} + G^2 d\,\delta^{-1/2} + \tfrac{4}{\delta^{1/2}}\big(\textstyle\sum_t \|\hat{S}_t\|^2 + \mathcal{E}_T^2\big), \tag{12}$$

$$R_T^S = \tfrac{1}{2}\|w^*\|_{D_T}^2 + 2\textstyle\sum_{j\in\mathcal{J}}\big(\delta + (\hat{\mathbf{G}}_T^S)_{jj}\big)^{1/2} - 2|\mathcal{J}|\delta^{1/2} + G^2 d\,\delta^{-1/2} + \tfrac{4}{\delta^{1/2}}\big(\textstyle\sum_t \|\hat{L}_t\|^2 + \mathcal{E}_T^2\big), \tag{13}$$

$V_T = \sum_{t=1}^T \langle g_t, w_t^L - w_t^S\rangle^2 \le 4G^2 B_w^2 T$*, and* $\mathcal{J} = \{j : \exists\, t,\ \hat{S}_{t,j} \neq 0\}$.

*Remark* 1 (On the role of $B_w$ in equation 11). The meta-algorithm overhead in equation 11 is the data-dependent term $\mathcal{M}_T = \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$, where $V_T = \sum_t \langle g_t, w_t^L - w_t^S\rangle^2$. The inequality $V_T \le 4G^2 B_w^2 T$ gives the crude upper bound $\mathcal{M}_T = \widetilde{O}(GB_w\sqrt{T})$, but this is not a universal lower-order term. If one uses only the deterministic FTRL trajectory bound $B_w = \delta^{-1/2}GT$ from Appendix C, then the meta-overhead becomes $\widetilde{O}(\delta^{-1/2}G^2T^{3/2})$ and can dominate the structural terms. Thus Theorem 1 should be read as an oracle inequality with an explicit aggregation cost: SLR-FTRL improves over choosing a single structural preconditioner only in regimes where $\mathcal{M}_T$ is lower order than the gain from structural adaptation.

The dependence on the base-learner range is unavoidable for black-box aggregation. Proposition 2 shows that any meta-algorithm combining two predictions whose norms can be as large as $B_w^\star$ suffers $\Omega(GB_w^\star\sqrt{T})$ regret to the better fixed mixture in the worst case. Doubling tricks and scale-adaptive one-dimensional coin-betting remove the need to know $B_w^\star$ before the run, but they cannot replace this factor by a universal constant.

**Proposition 2** (Unavoidable range dependence of meta-regret)**.** *Consider any meta-algorithm that combines two base predictions as $w_t = \alpha_t w_t^L + (1-\alpha_t) w_t^S$, where $\alpha_t \in [0,1]$ is chosen before observing $g_t$. Even in dimension $d = 1$, for every $G, B > 0$ there exist gradients with $|g_t| \le G$ and base predictions with $|w_t^L|, |w_t^S| \le B$ such that the regret to the better fixed mixture $\alpha^* \in [0,1]$ is $\Omega(GB\sqrt{T})$. Consequently, a black-box meta-overhead independent of $B_w^\star := \sup_t \max(\|w_t^L\|, \|w_t^S\|)$ is impossible; if $B_w^\star = \Theta(T)$, an $\Omega(T^{3/2})$ meta term is possible.*

*Remark* 2 (Structure of the bound). The bound equation 11 consists of five types of terms: comparator penalties $\|w^*\|_{M_T}^2$ and $\|w^*\|_{D_T}^2$; stability terms $\mathrm{tr}((\delta I + \hat{\mathbf{G}}_T^L)^{1/2})$ and $\sum_{j\in\mathcal{J}}(\delta + (\hat{\mathbf{G}}_T^S)_{jj})^{1/2}$, which encode the structural benefit by scaling with $r$ and $|\mathcal{J}|$ instead of $d$ under stable structure, as shown by Lemma 4; cross-contamination terms $O(G^2T/\delta^{1/2})$, arising from the mismatch between the full gradient and the structural preconditioner; preconditioner-lag corrections $O(G^2 d/\delta^{1/2})$; and meta-algorithm overhead $\mathcal{M}_T$ (crudely $\widetilde{O}(GB_w\sqrt{T})$). The structural advantage from the decomposition appears in the first two types, but is offset by the remaining three, which are absent in standard non-decomposed AdaGrad bounds.

### 5.3 Per-Term Lower Bound

**Theorem 3** (Per-Term Lower Bound)**.** *For any online learning algorithm, there exists a sequence of convex losses with gradients satisfying Assumption 3 (with $\epsilon = 0$) such that:*

$$\mathrm{Regret}_T(w^*) \;\ge\; \tfrac{1}{4}\|w^*\|\left(\sqrt{r\cdot\sigma_1(\mathbf{G}_T^L)} \;+\; \sqrt{s\cdot\|\mathbf{G}_T^S\|_\infty}\right) - O(\sqrt{T}). \tag{14}$$

*Remark* 3. This lower bound is obtained by taking the maximum over two separate adversaries (one purely low-rank, one purely sparse), establishing that each structural term is individually tight up to constants. It does not rule out the possibility that the minimax rate for simultaneously sparse-and-low-rank gradients could be smaller than the sum; a joint lower bound remains an open problem.

### 5.4 Corollaries

**Corollary 4** (Pure-Structure Special Cases)**.** *Under the conditions of Theorem 1:*

***(a)*** *When $r = 0$ (pure sparse: $\hat{L}_t = 0$, $\hat{S}_t = g_t$, $E_t = 0$), setting $\alpha^* = 0$:*

$$\mathrm{Regret}_T(w^*) \le \tfrac{1}{2}\|w^*\|_{D_T}^2 + 2\sum_{j\in\mathcal{J}}(\delta + (\hat{\mathbf{G}}_T^S)_{jj})^{1/2} - 2|\mathcal{J}|\delta^{1/2} + G^2 d\,\delta^{-1/2} + \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T). \quad (15)$$

*All cross-contamination terms vanish; this recovers the standard diagonal AdaGrad bound up to the preconditioner-lag correction and meta-algorithm overhead.*

***(b)*** *When $s = 0$ (pure low-rank: $\hat{S}_t = 0$, $\hat{L}_t = g_t$, $E_t = 0$), setting $\alpha^* = 1$:*

$$\mathrm{Regret}_T(w^*) \le \tfrac{1}{2}\|w^*\|_{M_T}^2 + 2\,\mathrm{tr}\big((\delta I + \hat{\mathbf{G}}_T^L)^{1/2}\big) - 2d\delta^{1/2} + G^2 d\,\delta^{-1/2} + \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T). \quad (16)$$

*Under stable rank-$r$ structure, Lemma 4 gives $\mathrm{tr}((\delta I + \hat{\mathbf{G}}_T^L)^{1/2}) \le r(\delta + \sigma_1(\hat{\mathbf{G}}_T^L))^{1/2} + (d-r)\delta^{1/2}$, recovering the full-matrix AdaGrad bound specialized to rank-$r$ gradients.*

**Corollary 5** (Adaptive Interpolation)**.** *For fixed candidate structural parameters $(r, s)$ used by the decomposition oracle, the coin-betting meta-algorithm automatically selects the optimal mixing $\alpha^*$ without requiring prior knowledge of which base preconditioner, or which convex combination of the two, will be best. The explicit meta-algorithm overhead is $\mathcal{M}_T = \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$, with the crude upper bound $\widetilde{O}(GB_w\sqrt{T})$.*

## 6 Experiments

We evaluate SLR-FTRL empirically to validate the predictions of Theorems 1 and 3 and to measure the practical impact of the cross-contamination, preconditioner-lag, and meta-algorithm overhead terms in Remark 2. All experiments use online ridge regression with synthetically structured features, giving precise control over the gradient sparsity $s$, rank $r$, and noise level $\epsilon$.

### 6.1 Experimental Setup

We instantiate the OCO protocol with online ridge regression: at each round $t$, a feature vector $x_t \in \mathbb{R}^d$ is drawn from a structured distribution and a target $y_t$ is generated as $y_t = \langle w_{\mathrm{true}}, x_t\rangle + \zeta_t$ with Gaussian noise $\zeta_t \sim \mathcal{N}(0, 0.01)$. The loss is $f_t(w) = \frac{1}{2}(\langle w, x_t\rangle - y_t)^2 + \frac{\lambda}{2}\|w\|^2$, yielding gradient $g_t = (\langle w_t, x_t\rangle - y_t)x_t + \lambda w_t$. The feature distribution is designed so that $g_t$ approximately decomposes as $g_t \approx S_t + L_t$, where $S_t$ is $s$-sparse and $L_t$ lies in a rank-$r$ subspace.

Unless stated otherwise, we use $d = 50$, $T = 2000$, $\delta = 1$, and report the mean $\pm$ standard deviation over 3 random seeds. We compare five algorithms: SLR-FTRL (Oracle) with the true decomposition, SLR-FTRL (Estimated) with online SVD-based decomposition, Full-Matrix (FM) AdaGrad, Diagonal (Diag) AdaGrad, and OGD with optimally tuned step size.

### 6.2 Main Results

Figure 1 summarizes the main empirical findings in six panels. Panels (a)–(c) display cumulative regret curves under three mixed-structure configurations with increasing complexity: ($r$=2, $s$=3), ($r$=5, $s$=5), and ($r$=10, $s$=10). In all cases, SLR-FTRL achieves sublinear regret that tracks closely with the best of FM-AdaGrad and Diag-AdaGrad, with a multiplicative overhead factor of 1.1–1.3×, consistent with the meta-algorithm overhead term $\widetilde{O}(GB_w\sqrt{T})$ in Theorem 1. As the structural complexity increases from ($r$=2, $s$=3) to ($r$=10, $s$=10), regret grows for all algorithms, matching the predicted dependence on structural parameters.

Panel (d) demonstrates dimension independence: with fixed ($r$=3, $s$=5), the final regret remains approximately constant as $d$ increases from 10 to 100, confirming that structural parameters, not the ambient

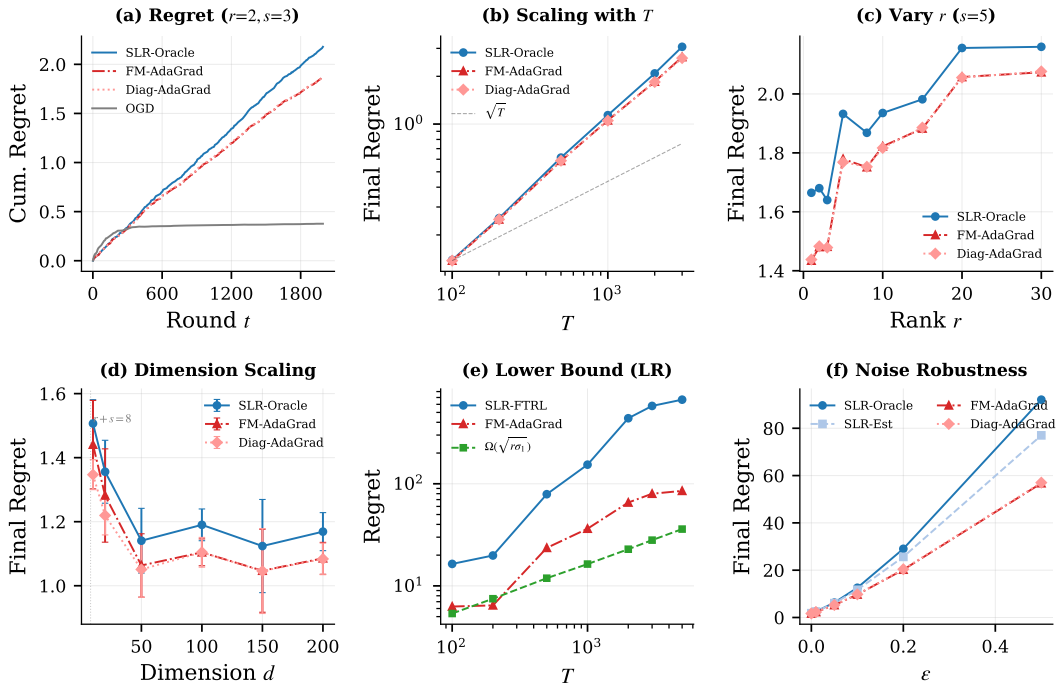


Figure 1: Empirical validation of SLR-FTRL across six diagnostic views ($d$=50 unless noted). (a)–(c): Cumulative regret curves under mixed sparse-low-rank structure with increasing complexity. (d): Dimension scaling confirms regret depends on structural parameters $r, s$ rather than ambient dimension $d$. (e): Lower bound comparison validates Theorem 3. (f): Noise robustness under increasing decomposition error $\epsilon$.

Table 2: Cumulative regret comparison under mixed sparse-low-rank gradient structure ($d$=50, $T$=2000). Values report final cumulative regret ± std over 3 seeds.

| Setting | SLR-Oracle | SLR-Est | FM-AdaGrad | Diag-AdaGrad | OGD |
|---|---|---|---|---|---|
| $r$=2, $s$=3 | $2.18 \pm 0.07$ | $2.12 \pm 0.06$ | $1.87 \pm 0.07$ | $1.87 \pm 0.04$ | $0.38 \pm 0.14$ |
| $r$=5, $s$=5 | $2.42 \pm 0.01$ | $2.40 \pm 0.02$ | $2.18 \pm 0.01$ | $2.18 \pm 0.02$ | $2.52 \pm 1.01$ |
| $r$=10, $s$=10 | $2.57 \pm 0.08$ | $2.56 \pm 0.08$ | $2.38 \pm 0.07$ | $2.38 \pm 0.08$ | $5.02 \pm 0.49$ |

dimension, govern the leading-order regret (Theorem 1). Panel (e) shows the lower bound comparison under a purely low-rank adversary, validating the $\Omega(D\sqrt{r\sigma_1})$ scaling of Theorem 3. Panel (f) displays noise robustness: as decomposition error $\epsilon$ increases, SLR-FTRL degrades gracefully, consistent with the cross-contamination terms in the bound.

Table 2 reports the final cumulative regret for all algorithms and configurations. SLR-FTRL (both Oracle and Estimated variants) consistently achieves regret within a factor of 1.1–1.3× of the better of FM-AdaGrad and Diag-AdaGrad, while OGD suffers substantially higher regret in configurations with significant structure (e.g., $5.02 \pm 0.49$ vs. $2.38 \pm 0.07$ at $r$=10, $s$=10). The small gap between Oracle and Estimated variants shows that the online SVD-based decomposition adds little error in these settings.

Additional experiments validating pure-case recovery, dimension independence, and noise robustness are provided in Appendix A.

## 7 Conclusion

We have presented SLR-FTRL, an online convex optimization algorithm that simultaneously exploits sparse and low-rank gradient structure through a decomposed preconditioner design and a coin-betting meta-

algorithm. Our regret bound is a structural oracle inequality: it recovers diagonal AdaGrad under pure sparsity and full-matrix AdaGrad under pure low-rank, while paying the explicit meta-aggregation overhead $\mathcal{M}_T = \sqrt{2V_T \ln(T+1)} + O(GB_w \ln T)$ and $O(G^2T/\delta^{1/2})$ cross-contamination terms. We no longer claim standard parameter-freeness; the contribution is adaptation to the better sparse/low-rank mixture for a chosen decomposition class, with per-term lower bounds confirming the tightness of the structural dependence. Several directions remain open: reducing the cross-contamination cost via joint sparse-low-rank preconditioners rather than parallel independent ones, removing the bounded-iterate assumption by adaptively adjusting the normalization constant $L$, and establishing a joint lower bound against a single adversary that presents both structures simultaneously, which would complete the minimax characterization of structured online convex optimization.

## References


Emmanuel J Candès, Xiaodong Li, Yi Ma, and John Wright. Robust principal component analysis? *Journal of the ACM (JACM)*, 58(3):1–37, 2011.

Nicolo Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

Venkat Chandrasekaran, Sujay Sanghavi, Pablo A Parrilo, and Alan S Willsky. Rank-sparsity incoherence for matrix decomposition. *SIAM Journal on Optimization*, 21(2):572–596, 2011.

Ashok Cutkosky and Francesco Orabona. Black-box reductions for parameter-free online learning in banach spaces. In *Conference on Learning Theory*, pp. 1493–1529. PMLR, 2018.

Ashok Cutkosky, Harsh Mehta, and Francesco Orabona. Optimal stochastic non-smooth non-convex optimization through online-to-non-convex conversion. In *International Conference on Machine Learning*, pp. 6643–6670. PMLR, 2023.

Steven De Rooij, Tim Van Erven, Peter D Grünwald, and Wouter M Koolen. Follow the leader if you can, hedge if you must. *Journal of Machine Learning Research*, 15(1):1281–1316, 2014.

Alexandre Défossez, Léon Bottou, Francis Bach, and Nicolas Usunier. A simple convergence proof of Adam and AdaGrad. *arXiv preprint arXiv:2003.02395*, 2020.

John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of Machine Learning Research*, 12(7), 2011.

Jiashi Feng, Huan Xu, and Shuicheng Yan. Online robust pca via stochastic optimization. *Advances in Neural Information Processing Systems*, 26, 2013.

Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of computer and system sciences*, 55(1):119–139, 1997.

Mina Ghashami, Edo Liberty, Jeff M Phillips, and David P Woodruff. Frequent directions: Simple and deterministic matrix sketching. *SIAM Journal on Computing*, 45(5):1762–1792, 2016.

John Goes, Teng Zhang, Raman Arora, and Gilad Lerman. Robust stochastic principal component analysis. In *Artificial Intelligence and Statistics*, pp. 266–274. PMLR, 2014.

Vineet Gupta, Tomer Koren, and Yoram Singer. Shampoo: Preconditioned stochastic tensor optimization. In *International Conference on Machine Learning*, pp. 1842–1850. PMLR, 2018.

Nathan Halko, Per-Gunnar Martinsson, and Joel A Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM review*, 53(2):217–288, 2011.

Elad Hazan. Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2(3-4): 157–325, 2016.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2):169–192, 2007.

Andrew Jacobsen and Ashok Cutkosky. Parameter-free mirror descent. In *Conference on Learning Theory*, pp. 4160–4211. PMLR, 2022.

Ahmed Khaled, Konstantin Mishchenko, and Chi Jin. Dowg unleashed: An efficient universal parameter-free gradient descent method. *Advances in Neural Information Processing Systems*, 36:6748–6769, 2023.

Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.

Haipeng Luo and Robert E Schapire. Achieving all with no parameters: Adanormalhedge. In *Conference on Learning Theory*, pp. 1286–1304. PMLR, 2015.

Brendan McMahan. Follow-the-regularized-leader and mirror descent: Equivalence theorems and l1 regularization. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics*, pp. 525–533. JMLR Workshop and Conference Proceedings, 2011.

H Brendan McMahan and Matthew Streeter. Adaptive bound optimization for online convex optimization. *arXiv preprint arXiv:1002.4908*, 2010.

Zakaria Mhammedi and Wouter M Koolen. Lipschitz and comparator-norm adaptivity in online learning. In *Conference on Learning Theory*, pp. 2858–2887. PMLR, 2020.

Praneeth Netrapalli, U N Niranjan, Sujay Sanghavi, Animashree Anandkumar, and Prateek Jain. Non-convex robust pca. *Advances in Neural Information Processing Systems*, 27, 2014.

Francesco Orabona and Dávid Pál. Coin betting and parameter-free online learning. *Advances in Neural Information Processing Systems*, 29, 2016.

Sashank J Reddi, Satyen Kale, and Sanjiv Kumar. On the convergence of adam and beyond. *arXiv preprint arXiv:1904.09237*, 2019.

Shai Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends® in Machine Learning*, 4(2):107–194, 2025.

Tim Van Erven and Wouter M Koolen. Metagrad: Multiple learning rates in online learning. *Advances in Neural Information Processing Systems*, 29, 2016.

Rachel Ward, Xiaoxia Wu, and Leon Bottou. Adagrad stepsizes: Sharp convergence over nonconvex landscapes. *Journal of Machine Learning Research*, 21(219):1–30, 2020.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th international conference on machine learning (icml-03)*, pp. 928–936, 2003.

## A Additional Experimental Validation

We validate three key predictions of the theory. First, Corollary 4 predicts that SLR-FTRL recovers the standard diagonal AdaGrad bound when $r = 0$ and the full-matrix AdaGrad bound when $s = 0$. Figure A.1 and Table A.1 confirm this: under pure low-rank structure, SLR-FTRL's regret is within 1.0–1.05× of FM-AdaGrad, and under pure sparse structure, within 1.1–1.4× of Diag-AdaGrad. The slightly larger gap in the sparse case is consistent with the meta-algorithm overhead. Second, Theorem 1 predicts that the leading-order regret depends on the structural parameters $r$ and $s$ rather than the ambient dimension $d$. Figure A.2 confirms this: with fixed ($r$=3, $s$=5), the final regret remains approximately constant as $d$ increases from 10 to 100, demonstrating that the structural benefit dominates the dimension-dependent correction terms. Third, Table A.2 examines the effect of decomposition noise $\epsilon$ on performance. As $\epsilon$ increases from 0 to 0.5, SLR-FTRL degrades slightly faster than the non-decomposed baselines, consistent with the cross-contamination terms in equation 12. At $\epsilon = 0.1$, SLR-FTRL's regret is 1.2× that of FM-AdaGrad; at $\epsilon = 0.5$, the ratio increases to 1.4×. Overall, the experiments validate the qualitative predictions of the theory while honestly revealing the quantitative cost of adaptivity: SLR-FTRL consistently incurs a 1.1–1.4× regret premium relative to the better specialized baseline.

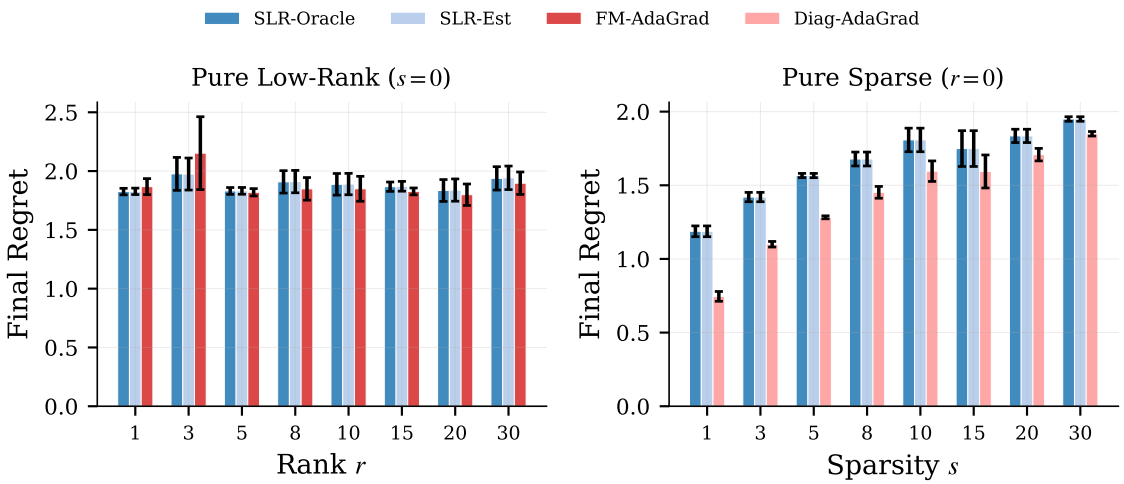


Figure A.1: Pure-case recovery. Left: Pure low-rank ($s$=0), SLR-FTRL closely tracks FM-AdaGrad. Right: Pure sparse ($r$=0), SLR-FTRL closely tracks Diag-AdaGrad.

Table A.1: Pure-case recovery. Final regret at $d$=50, $T$=1500. SLR-O/E denote Oracle/Estimated; FM/Diag denotes the specialized baseline.

| | Pure Low-Rank ($s$=0) | | | Pure Sparse ($r$=0) | | |
|---|---|---|---|---|---|---|
| $r$ or $s$ | SLR-O | SLR-E | FM-AG | SLR-O | SLR-E | Diag-AG |
| 1 | 1.71 | 1.71 | 1.66 | 1.19 | 1.19 | 0.76 |
| 5 | 1.87 | 1.88 | 1.82 | 1.55 | 1.55 | 1.28 |
| 10 | 1.96 | 1.97 | 1.91 | 1.70 | 1.70 | 1.51 |
| 20 | 1.94 | 1.95 | 1.87 | 1.90 | 1.90 | 1.77 |
| 30 | 2.07 | 2.08 | 2.00 | 2.08 | 2.08 | 1.97 |

# B Proofs

## B.1 Proof of Lemma 1 (FTRL Regret with Time-Varying Regularizer)

*Proof.* Define the cumulative gradient $\mathcal{G}_t = \sum_{\tau=1}^{t} g_\tau$, initialize $w_1 = 0$, and set $w_{t+1} = -H_t^{-1}\mathcal{G}_t$ and $H_0 = \delta I$. Introduce the potential function $\Phi_t := \frac{1}{2}\mathcal{G}_t^\top H_t^{-1}\mathcal{G}_t = \frac{1}{2}\|w_{t+1}\|_{H_t}^2$.

Since $H_t w_{t+1} = -\mathcal{G}_t$, a direct expansion gives

$$\tfrac{1}{2}\|w^* - w_{t+1}\|_{H_t}^2 \;=\; \tfrac{1}{2}\|w^*\|_{H_t}^2 + \mathcal{G}_t^\top w^* + \Phi_t \;\geq\; 0,$$

from which we obtain $-\langle \mathcal{G}_t, w^*\rangle \leq \Phi_t + \frac{1}{2}\|w^*\|_{H_t}^2$. Subtracting the same identity evaluated at times $t$ and $t-1$ yields the per-round bound

$$-\langle g_t, w^*\rangle \;\leq\; \Phi_t - \Phi_{t-1} + \tfrac{1}{2}\|w^*\|_{H_t - H_{t-1}}^2. \tag{17}$$

We next bound the learner's per-round loss. Since $w_t = -H_{t-1}^{-1}\mathcal{G}_{t-1}$, we have $\langle g_t, w_t\rangle = -g_t^\top H_{t-1}^{-1}\mathcal{G}_{t-1}$. Define $P := H_{t-1}^{-1} - H_t^{-1}$, which satisfies $P \succeq 0$ because $H_t \succeq H_{t-1}$ implies $H_t^{-1} \preceq H_{t-1}^{-1}$ by operator anti-monotonicity of the inverse on the PSD cone.

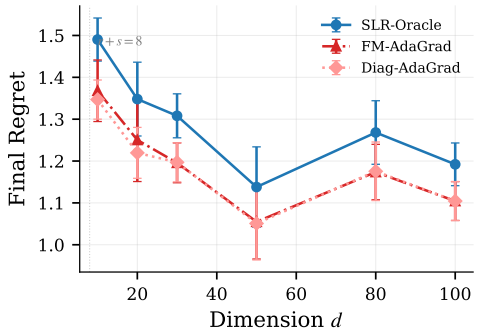


Figure A.2: Dimension scaling with fixed structure ($r$=3, $s$=5). Final regret remains approximately constant as $d$ grows from 10 to 100.

Table A.2: Effect of decomposition noise $\epsilon$ on final regret ($d$=50, $T$=1000, $r$=5, $s$=8).

| $\epsilon$ | SLR-Oracle | SLR-Est | FM-AdaGrad | Diag-AdaGrad |
|---|---|---|---|---|
| 0.0 | $1.29 \pm 0.10$ | $1.28 \pm 0.10$ | $1.22 \pm 0.09$ | $1.21 \pm 0.09$ |
| 0.01 | $1.75 \pm 0.13$ | $1.74 \pm 0.13$ | $1.63 \pm 0.10$ | $1.62 \pm 0.11$ |
| 0.05 | $4.22 \pm 0.26$ | $4.14 \pm 0.24$ | $3.65 \pm 0.17$ | $3.65 \pm 0.19$ |
| 0.1 | $8.35 \pm 0.45$ | $8.04 \pm 0.38$ | $6.76 \pm 0.26$ | $6.75 \pm 0.30$ |
| 0.2 | $19.07 \pm 0.84$ | $17.67 \pm 0.65$ | $14.00 \pm 0.48$ | $13.97 \pm 0.59$ |
| 0.5 | $59.75 \pm 3.57$ | $52.12 \pm 3.43$ | $38.55 \pm 2.76$ | $38.36 \pm 3.07$ |

We now expand $\Phi_t - \Phi_{t-1}$ without dropping any terms. Writing $\mathcal{G}_t = \mathcal{G}_{t-1} + g_t$ and $H_t^{-1} = H_{t-1}^{-1} - P$:

$$\begin{aligned}\Phi_t - \Phi_{t-1} &= \tfrac{1}{2}\mathcal{G}_t^\top H_t^{-1}\mathcal{G}_t - \tfrac{1}{2}\mathcal{G}_{t-1}^\top H_{t-1}^{-1}\mathcal{G}_{t-1} \\ &= \tfrac{1}{2}\mathcal{G}_{t-1}^\top (H_t^{-1} - H_{t-1}^{-1})\mathcal{G}_{t-1} + g_t^\top H_t^{-1}\mathcal{G}_{t-1} + \tfrac{1}{2}\|g_t\|^2_{H_t^{-1}} \\ &= -\tfrac{1}{2}\mathcal{G}_{t-1}^\top P\,\mathcal{G}_{t-1} + g_t^\top H_t^{-1}\mathcal{G}_{t-1} + \tfrac{1}{2}\|g_t\|^2_{H_t^{-1}}.\end{aligned}$$

Adding $\langle g_t, w_t\rangle = -g_t^\top H_{t-1}^{-1}\mathcal{G}_{t-1} = -g_t^\top (H_t^{-1} + P)\mathcal{G}_{t-1}$ to the above:

$$\langle g_t, w_t\rangle + \Phi_t - \Phi_{t-1} \;=\; -g_t^\top P\,\mathcal{G}_{t-1} - \tfrac{1}{2}\mathcal{G}_{t-1}^\top P\,\mathcal{G}_{t-1} + \tfrac{1}{2}\|g_t\|^2_{H_t^{-1}}.$$

Completing the square in $\mathcal{G}_{t-1}$:

$$-g_t^\top P\,\mathcal{G}_{t-1} - \tfrac{1}{2}\mathcal{G}_{t-1}^\top P\,\mathcal{G}_{t-1} \;=\; -\tfrac{1}{2}(\mathcal{G}_{t-1} + g_t)^\top P\,(\mathcal{G}_{t-1} + g_t) + \tfrac{1}{2}g_t^\top P\,g_t \;=\; -\tfrac{1}{2}\mathcal{G}_t^\top P\,\mathcal{G}_t + \tfrac{1}{2}g_t^\top P\,g_t.$$

Since $P \succeq 0$, the term $\mathcal{G}_t^\top P\,\mathcal{G}_t \geq 0$ can be dropped, yielding

$$\langle g_t, w_t\rangle + \Phi_t - \Phi_{t-1} \;\leq\; \tfrac{1}{2}g_t^\top P\,g_t + \tfrac{1}{2}\|g_t\|^2_{H_t^{-1}} \;=\; \tfrac{1}{2}g_t^\top (P + H_t^{-1})g_t \;=\; \tfrac{1}{2}\|g_t\|^2_{H_{t-1}^{-1}},$$

where the last equality uses $P + H_t^{-1} = H_{t-1}^{-1}$ by definition of $P$.

Combining the above with equation 17 gives the per-round regret bound

$$\langle g_t,\, w_t - w^*\rangle \;\leq\; \tfrac{1}{2}\|g_t\|^2_{H_{t-1}^{-1}} + \tfrac{1}{2}\|w^*\|^2_{H_t - H_{t-1}}.$$

Summing over $t = 1, \ldots, T$:

$$\sum_{t=1}^{T}\langle g_t,\, w_t - w^*\rangle \;\leq\; \tfrac{1}{2}\sum_{t=1}^{T}\|g_t\|^2_{H_{t-1}^{-1}} \;+\; \tfrac{1}{2}\sum_{t=1}^{T}\|w^*\|^2_{H_t - H_{t-1}}.$$

The second sum telescopes: $\sum_{t=1}^{T} w^{*\top}(H_t - H_{t-1})w^* \;=\; w^{*\top}(H_T - H_0)w^* \;\leq\; \|w^*\|^2_{H_T}$, completing the proof. □

### B.2 Proof of Lemma 2 (Trace-of-Square-Root Bound)

*Proof.* Define $A_t = \delta I + \sum_{\tau=1}^{t} v_\tau v_\tau^\top$ and $A_0 = \delta I$, so that $A_t = A_{t-1} + v_t v_t^\top$ and $H_t = A_t^{1/2}$.

The function $\phi(X) = \mathrm{tr}(X^{1/2})$ is concave on the positive definite cone $\mathbb{S}^d_{++}$. To see this, note that $X \mapsto X^{1/2}$ is operator concave (a classical result in matrix analysis), and the trace is a positive linear functional, so their composition is concave. The Fréchet derivative of $\phi$ at a point $X \in \mathbb{S}^d_{++}$ can be computed via spectral calculus: writing $X = \sum_i \lambda_i u_i u_i^\top$,

$$\frac{d}{d\varepsilon}\,\mathrm{tr}\big((X+\varepsilon\Delta)^{1/2}\big)\Big|_{\varepsilon=0} \;=\; \sum_i \frac{1}{2\lambda_i^{1/2}}\, u_i^\top \Delta\, u_i \;=\; \tfrac{1}{2}\,\mathrm{tr}(X^{-1/2}\Delta).$$

Applying the first-order concavity condition $\phi(Y) \le \phi(X) + D\phi(X)[Y - X]$ at the point $X = A_t$ (the larger matrix), evaluated at $Y = A_{t-1} = A_t - v_t v_t^\top$:

$$\mathrm{tr}(A_{t-1}^{1/2}) \;\le\; \mathrm{tr}(A_t^{1/2}) + D\phi(A_t)[A_{t-1} - A_t] \;=\; \mathrm{tr}(A_t^{1/2}) - \tfrac{1}{2}\, v_t^\top A_t^{-1/2} v_t.$$

Rearranging:

$$\tfrac{1}{2}\|v_t\|^2_{H_t^{-1}} \;=\; \tfrac{1}{2}\, v_t^\top A_t^{-1/2} v_t \;\le\; \mathrm{tr}(A_t^{1/2}) - \mathrm{tr}(A_{t-1}^{1/2}).$$

Summing over $t = 1, \ldots, T$ and telescoping:

$$\sum_{t=1}^{T} \tfrac{1}{2}\|v_t\|^2_{H_t^{-1}} \;\le\; \mathrm{tr}(A_T^{1/2}) - \mathrm{tr}(A_0^{1/2}) \;=\; \mathrm{tr}\left(\left(\delta I + \sum_{t=1}^{T} v_t v_t^\top\right)^{1/2}\right) - d\,\delta^{1/2}.$$

Multiplying both sides by 2 yields the first bound.

For the preconditioner-lag correction, define $P_t := H_{t-1}^{-1} - H_t^{-1} = A_{t-1}^{-1/2} - A_t^{-1/2}$, which satisfies $P_t \succeq 0$ by operator anti-monotonicity of $X \mapsto X^{-1/2}$. We decompose

$$\sum_{t=1}^{T} \|v_t\|^2_{H_{t-1}^{-1}} \;=\; \sum_{t=1}^{T} \|v_t\|^2_{H_t^{-1}} \;+\; \sum_{t=1}^{T} v_t^\top P_t v_t.$$

For each $t$, we bound $v_t^\top P_t v_t$ using the trace–operator-norm duality. Since $P_t \succeq 0$ and $v_t v_t^\top \succeq 0$:

$$v_t^\top P_t v_t \;=\; \mathrm{tr}(P_t\, v_t v_t^\top) \;\le\; \mathrm{tr}(P_t)\, \|v_t v_t^\top\|_{\mathrm{op}} \;=\; \mathrm{tr}(P_t)\, \|v_t\|^2 \;\le\; G^2\, \mathrm{tr}(P_t),$$

where the first inequality is the standard bound $\mathrm{tr}(AB) \le \mathrm{tr}(A)\, \|B\|_{\mathrm{op}}$ for PSD matrices $A, B$, and the last step uses $\|v_t\| \le G$. Summing over $t$ and telescoping the trace:

$$\sum_{t=1}^{T} v_t^\top P_t v_t \;\le\; G^2\, \mathrm{tr}\left(\sum_{t=1}^{T} P_t\right) = G^2\, \mathrm{tr}(A_0^{-1/2} - A_T^{-1/2}) \;\le\; G^2\, \mathrm{tr}(A_0^{-1/2}) \;=\; G^2 d\, \delta^{-1/2}.$$

Adding this to the first bound completes the proof. □

### B.3 Proof of Lemma 3 (Diagonal AdaGrad Bound on Sparse Vectors)

*Proof.* Since $v_{t,j} = 0$ for all $j \notin \mathcal{J}$, the Mahalanobis norm reduces to a sum over the active support:

$$\|v_t\|^2_{D_t^{-1}} \;=\; \sum_{j \in \mathcal{J}} \frac{v_{t,j}^2}{\big(\delta + \sum_{\tau=1}^{t} v_{\tau,j}^2\big)^{1/2}}.$$

For each coordinate $j \in \mathcal{J}$, define the accumulator $a_{t,j} = \delta + \sum_{\tau=1}^{t} v_{\tau,j}^2$, so that $a_{t,j} - a_{t-1,j} = v_{t,j}^2$.

By the algebraic identity $a^{1/2} - b^{1/2} = (a-b)\,/\,(a^{1/2} + b^{1/2}) \ge (a-b)\,/\,(2a^{1/2})$ valid for $a \ge b > 0$, we obtain

$$a_{t,j}^{1/2} - a_{t-1,j}^{1/2} \;\ge\; \frac{v_{t,j}^2}{2\, a_{t,j}^{1/2}}.$$

Rearranging gives $v_{t,j}^2/a_{t,j}^{1/2} \le 2(a_{t,j}^{1/2} - a_{t-1,j}^{1/2})$. Summing over all rounds $t = 1, \dots, T$ and all coordinates $j \in \mathcal{J}$, the right-hand side telescopes:

$$\sum_{t=1}^{T} \|v_t\|_{D_t^{-1}}^2 \;\le\; 2\sum_{j\in\mathcal{J}} \big(a_{T,j}^{1/2} - a_{0,j}^{1/2}\big) \;=\; 2\sum_{j\in\mathcal{J}} \Big(\delta + \sum_{t=1}^{T} v_{t,j}^2\Big)^{1/2} - 2|\mathcal{J}|\,\delta^{1/2},$$

since $a_{0,j} = \delta$ for all $j$. □

### B.4 Proof of Lemma 4 (Trace Bound under Low-Rank Structure)

*Proof.* Let $U \in \mathbb{R}^{d\times r}$ be an orthonormal basis for the subspace, so that $L_t = U\hat{L}_t$ for some $\hat{L}_t \in \mathbb{R}^r$. Then

$$A \;=\; U\Big(\sum_{t=1}^{T} \hat{L}_t \hat{L}_t^\top\Big)U^\top,$$

and the eigenvalues of $A$ are the $r$ eigenvalues $\lambda_1 \ge \cdots \ge \lambda_r \ge 0$ of the $r \times r$ matrix $\hat{A} := \sum_t \hat{L}_t\hat{L}_t^\top$, together with $(d-r)$ zero eigenvalues (since $A$ has rank at most $r$).

Expressing the trace via the eigendecomposition:

$$\mathrm{tr}\big((\delta I + A)^{1/2}\big) \;=\; \sum_{i=1}^{r} (\delta + \lambda_i)^{1/2} + (d-r)\,\delta^{1/2}.$$

Since $\lambda_i \le \lambda_1 = \sigma_1(A)$ for all $i \in [r]$, each term in the first sum satisfies $(\delta+\lambda_i)^{1/2} \le (\delta+\lambda_1)^{1/2}$, giving

$$\sum_{i=1}^{r} (\delta + \lambda_i)^{1/2} \;\le\; r\,(\delta + \sigma_1(A))^{1/2},$$

which yields equation 8.

For the Cauchy–Schwarz bound equation 9, write $\mathrm{tr}(A^{1/2}) = \sum_{i=1}^{R} \lambda_i^{1/2}$ and apply the Cauchy–Schwarz inequality with the all-ones vector:

$$\Big(\sum_{i=1}^{R} 1\cdot\lambda_i^{1/2}\Big)^2 \;\le\; R\cdot\sum_{i=1}^{R}\lambda_i \;=\; R\cdot\mathrm{tr}(A),$$

and $\mathrm{tr}(A) = \mathrm{tr}(\sum_t L_t L_t^\top) = \sum_t \|L_t\|_2^2$. □

### B.5 Proof of Lemma 5 (Coin-Betting Meta-Algorithm Regret)

*Proof.* By linearity of the inner product,

$$\langle g_t, w_t\rangle \;=\; \alpha_t\langle g_t, w_t^L\rangle + (1-\alpha_t)\langle g_t, w_t^S\rangle.$$

Define the per-round difference $\ell_t := \langle g_t, w_t^L - w_t^S\rangle$ and the normalized coin $c_t := \ell_t/L$. By Cauchy–Schwarz and Assumptions 2–6, $|\ell_t| \le \|g_t\|\,\|w_t^L - w_t^S\| \le G\cdot 2B_w = L$, so $|c_t| \le 1$.

The meta-problem reduces to one-dimensional online linear optimization on $[0,1]$ with per-round loss $\alpha \mapsto \alpha\ell_t$. By the Orabona & Pál (2016) reduction of OCO on $[0,1]$ to coin-betting, the KT strategy produces $\alpha_t \in [0,1]$ satisfying, for any $\alpha^* \in [0,1]$:

$$\sum_{t=1}^{T} (\alpha_t - \alpha^*)\,c_t \;\le\; |\alpha^*|\sqrt{2\textstyle\sum_t c_t^2\cdot\ln(T{+}1)} + O(\ln T).$$

Since $L$ is a *fixed* constant (not time-varying), the un-normalization is an exact algebraic identity:

$$\sum_{t=1}^{T} (\alpha_t - \alpha^*)\,\ell_t \;=\; L\cdot\sum_{t=1}^{T}(\alpha_t - \alpha^*)\,c_t \;\le\; |\alpha^*|\sqrt{2V_T\ln(T{+}1)} + O(L\ln T),$$

where $V_T = \sum_t \ell_t^2 = L^2 \sum_t c_t^2$.

To assemble the final bound, write

$$\sum_{t=1}^{T} \langle g_t, w_t \rangle \;=\; \sum_{t=1}^{T} \langle g_t, w_t^S \rangle + \sum_{t=1}^{T} \alpha_t \, \ell_t \;=\; \sum_{t=1}^{T} \langle g_t, w_t^S \rangle + \alpha^* \sum_{t=1}^{T} \ell_t + \sum_{t=1}^{T} (\alpha_t - \alpha^*) \, \ell_t.$$

Noting that $\sum_t \langle g_t, w_t^S \rangle + \alpha^* \sum_t \ell_t = \alpha^* \sum_t \langle g_t, w_t^L \rangle + (1-\alpha^*) \sum_t \langle g_t, w_t^S \rangle$ and using $|\alpha^*| \le 1$ gives equation 10. □

### B.6 Proof of Theorem 1 (Main Result)

*Proof.* By Assumption 1 and the linearized regret inequality equation 2,

$$\mathrm{Regret}_T(w^*) \;\le\; \sum_{t=1}^{T} \langle g_t, \, w_t - w^* \rangle.$$

Applying Lemma 5 (valid under Assumptions 2 and 6), for any $\alpha^* \in [0, 1]$:

$$\mathrm{Regret}_T(w^*) \;\le\; \alpha^* \sum_{t=1}^{T} \langle g_t, w_t^L - w^* \rangle + (1-\alpha^*) \sum_{t=1}^{T} \langle g_t, w_t^S - w^* \rangle + \sqrt{2 V_T \ln(T+1)} + O(G B_w \ln T). \tag{18}$$

It remains to bound the regret of each sub-algorithm.

Consider the low-rank sub-algorithm $\mathcal{A}^L$, which is FTRL with preconditioner $H_t = M_t = (\delta I + \sum_{\tau \le t} \hat{L}_\tau \hat{L}_\tau^\top)^{1/2}$, receiving the full gradient $g_t$ as its loss. By Lemma 1:

$$R_T^L \;:=\; \sum_{t=1}^{T} \langle g_t, w_t^L - w^* \rangle \;\le\; \tfrac{1}{2} \|w^*\|_{M_T}^2 + \tfrac{1}{2} \sum_{t=1}^{T} \|g_t\|_{M_{t-1}^{-1}}^2 .$$

To bound the stability term $\sum_t \|g_t\|_{M_{t-1}^{-1}}^2$, write $g_t = \hat{L}_t + (\hat{S}_t + E_t)$ where $E_t = g_t - \hat{S}_t - \hat{L}_t$ is the decomposition error, and use the inequality $\|a+b\|_P^2 \le 2\|a\|_P^2 + 2\|b\|_P^2$ (valid for any PSD $P$) to obtain

$$\|g_t\|_{M_{t-1}^{-1}}^2 \;\le\; 2\|\hat{L}_t\|_{M_{t-1}^{-1}}^2 + 2\|\hat{S}_t + E_t\|_{M_{t-1}^{-1}}^2 .$$

The first term captures the "matched" component, i.e., the low-rank gradient measured by the low-rank preconditioner. Lemma 2 (with preconditioner-lag correction equation 6) gives

$$\sum_{t=1}^{T} \|\hat{L}_t\|_{M_{t-1}^{-1}}^2 \;\le\; 2 \, \mathrm{tr}\big((\delta I + \hat{\mathbf{G}}_T^L)^{1/2}\big) - 2d \, \delta^{1/2} + G^2 d \, \delta^{-1/2}.$$

The second term captures the "cross-contamination," i.e., the sparse-plus-error component measured by the low-rank preconditioner. Since $A_{t-1}^L = \delta I + \sum_{\tau < t} \hat{L}_\tau \hat{L}_\tau^\top \succeq \delta I$, we have $M_{t-1} = (A_{t-1}^L)^{1/2} \succeq \delta^{1/2} I$, and hence $M_{t-1}^{-1} \preceq \delta^{-1/2} I$. Therefore,

$$\|\hat{S}_t + E_t\|_{M_{t-1}^{-1}}^2 \;\le\; \delta^{-1/2} \, \|\hat{S}_t + E_t\|^2 \;\le\; 2 \, \delta^{-1/2} \big(\|\hat{S}_t\|^2 + \|E_t\|^2\big),$$

and summing over $t$:

$$\sum_{t=1}^{T} \|\hat{S}_t + E_t\|_{M_{t-1}^{-1}}^2 \;\le\; \frac{2}{\delta^{1/2}} \Big( \sum_{t=1}^{T} \|\hat{S}_t\|^2 + \mathcal{E}_T^2 \Big).$$

Combining these estimates (applying the factor $\frac{1}{2}$ from Lemma 1 to the two terms, each carrying the factor 2 from the triangle inequality) yields equation 12.

An analogous argument bounds the sparse sub-algorithm $\mathcal{A}^S$, which is FTRL with diagonal preconditioner $H_t = D_t$, also receiving the full gradient $g_t$. By Lemma 1:

$$R_T^S \;:=\; \sum_{t=1}^{T} \langle g_t, w_t^S - w^* \rangle \;\le\; \tfrac{1}{2}\|w^*\|_{D_T}^2 + \tfrac{1}{2}\sum_{t=1}^{T} \|g_t\|_{D_{t-1}^{-1}}^2 .$$

Writing $g_t = \hat{S}_t + (\hat{L}_t + E_t)$ and applying the same decomposition:

$$\|g_t\|_{D_{t-1}^{-1}}^2 \;\le\; 2\|\hat{S}_t\|_{D_{t-1}^{-1}}^2 + 2\|\hat{L}_t + E_t\|_{D_{t-1}^{-1}}^2 .$$

The diagonal analogue of Lemma 2 (obtained by the same concavity argument applied coordinate-wise, with lag correction $G^2 d\, \delta^{-1/2}$) combined with Lemma 3 gives

$$\sum_{t=1}^{T} \|\hat{S}_t\|_{D_{t-1}^{-1}}^2 \;\le\; 2\sum_{j\in\mathcal{J}} \big(\delta + (\hat{\mathbf{G}}_T^S)_{jj}\big)^{1/2} - 2|\mathcal{J}|\,\delta^{1/2} + G^2 d\, \delta^{-1/2}.$$

Since $D_{t-1} \succeq \delta^{1/2} I$, the cross-contamination term satisfies

$$\sum_{t=1}^{T} \|\hat{L}_t + E_t\|_{D_{t-1}^{-1}}^2 \;\le\; \frac{2}{\delta^{1/2}}\Big(\sum_{t=1}^{T} \|\hat{L}_t\|^2 + \mathcal{E}_T^2\Big).$$

Combining yields equation 13.

Substituting the bounds equation 12 and equation 13 into the meta-algorithm decomposition equation 18 and taking the minimum over $\alpha^* \in [0,1]$ yields

$$\mathrm{Regret}_T(w^*) \;\le\; \min_{\alpha^*\in[0,1]} \big\{\alpha^* R_T^L + (1-\alpha^*) R_T^S\big\} + \sqrt{2V_T \ln(T{+}1)} + O(GB_w \ln T),$$

which is equation 11.

We record explicit bounds on the cross-contamination terms for later use. By the definition of Algorithm 1, $\hat{L}_t = \mathrm{Proj}_r(g_t; U_{t-1})$ is an orthogonal projection, so $\|\hat{L}_t\| \le \|g_t\| \le G$ by Assumption 2, giving $\sum_t \|\hat{L}_t\|^2 \le G^2 T$. For $\hat{S}_t = \mathrm{Hard}_s(g_t - \hat{L}_t)$, hard thresholding is a contraction in Euclidean norm, so $\|\hat{S}_t\| \le \|g_t - \hat{L}_t\| \le \|g_t\| + \|\hat{L}_t\| \le 2G$, giving $\sum_t \|\hat{S}_t\|^2 \le 4G^2 T$. Therefore, the cross-contamination contributions in $R_T^L$ and $R_T^S$ are bounded by $O(G^2 T/\delta^{1/2})$, and the preconditioner-lag corrections contribute $O(G^2 d/\delta^{1/2})$ each. □

### B.7 Proof of Proposition 2 (Unavoidable Range Dependence)

*Proof.* It suffices to prove the claim for a randomized adversary; a deterministic sequence with at least the same regret then exists by the probabilistic method. Work in dimension $d = 1$ and set the two base predictions to the fixed values $w_t^L = B$ and $w_t^S = -B$ for all $t$. Let $g_t = G\varepsilon_t$, where $\varepsilon_t$ are independent Rademacher random variables. Any meta-algorithm chooses $\alpha_t$ before observing $\varepsilon_t$, and therefore $\mathbb{E}[\alpha_t g_t] = 0$.

The loss of the meta-algorithm is

$$\sum_{t=1}^{T} g_t\big(\alpha_t B + (1-\alpha_t)(-B)\big) = -B\sum_{t=1}^{T} g_t + 2B\sum_{t=1}^{T} \alpha_t g_t.$$

The loss of a fixed mixture $\alpha \in [0,1]$ is $-B\sum_t g_t + 2B\alpha\sum_t g_t$, so the best fixed mixture in hindsight chooses $\alpha = 0$ when $\sum_t g_t \ge 0$ and $\alpha = 1$ otherwise. Hence the expected regret of any meta-algorithm is

$$\mathbb{E}\left[2B\sum_{t=1}^{T} \alpha_t g_t - 2B \min_{\alpha\in[0,1]} \alpha \sum_{t=1}^{T} g_t\right] = 2B\,\mathbb{E}\left[-\min\left\{0, \sum_{t=1}^{T} g_t\right\}\right] = B\,\mathbb{E}\left|\sum_{t=1}^{T} g_t\right|.$$

Since $\sum_t g_t = G\sum_t \varepsilon_t$ and $\mathbb{E}|\sum_t \varepsilon_t| = \Omega(\sqrt{T})$, the expected regret is $\Omega(GB\sqrt{T})$. This establishes the lower bound. Taking $B = B_w^\star$ shows that if the realized base-learner range grows as $\Theta(T)$, the unavoidable meta-regret can grow as $\Omega(T^{3/2})$. □

### B.8 Proof of Theorem 3 (Per-Term Lower Bound)

*Proof.* We construct two separate adversaries and lower-bound the regret against the worse of the two.

Consider first the purely low-rank adversary. Set $S_t = 0$ and $L_t = g_t$, with all gradients lying in the fixed subspace $V = \mathrm{span}(e_1, \ldots, e_r)$. Concretely, for each round $t$ and each coordinate $j \in \{1, \ldots, r\}$, draw $g_{t,j} \in \{+G/\sqrt{r}, -G/\sqrt{r}\}$ independently and uniformly (i.i.d. Rademacher), and set $g_{t,j} = 0$ for $j > r$. By construction, $\|g_t\|^2 = r \cdot G^2/r = G^2$, so Assumption 2 is satisfied.

The gradient outer-product matrix is $\mathbf{G}_T^L = \sum_t g_t g_t^\top$. Since the entries $g_{t,j}$ are i.i.d. with $\mathbb{E}[g_{t,j}^2] = G^2/r$, by the law of large numbers all $r$ eigenvalues of $\mathbf{G}_T^L$ concentrate around $G^2T/r$, so $\sigma_1(\mathbf{G}_T^L) = G^2T/r$ in expectation.

By the standard minimax lower bound for online linear optimization on $\mathbb{R}^r$ with bounded gradients (Cutkosky et al., 2023; Duchi et al., 2011), which follows from the minimax theorem applied to the zero-sum game between the learner and the Rademacher adversary, for any online algorithm there exists a gradient sequence such that

$$\mathbb{E}[\mathrm{Regret}_T] \;\geq\; \tfrac{1}{2}\,\|w^*\|\,G\sqrt{T}.$$

Substituting $G\sqrt{T} = \sqrt{r \cdot (G^2T/r)} = \sqrt{r \cdot \sigma_1(\mathbf{G}_T^L)}$:

$$\mathbb{E}[\mathrm{Regret}_T] \;\geq\; \tfrac{1}{2}\,\|w^*\|\sqrt{r \cdot \sigma_1(\mathbf{G}_T^L)}.$$

Consider next the purely sparse adversary. Set $L_t = 0$ and $S_t = g_t$, with $s$-sparse Rademacher gradients supported on coordinates $\{1, \ldots, s\}$: $g_{t,j} \in \{+G/\sqrt{s}, -G/\sqrt{s}\}$ i.i.d. Rademacher for $j \leq s$, and $g_{t,j} = 0$ for $j > s$. Again $\|g_t\|^2 = G^2$, and each coordinate accumulates $\sum_t g_{t,j}^2 = G^2T/s$, so $\|\mathbf{G}_T^S\|_\infty = \max_j \sum_t g_{t,j}^2 = G^2T/s$.

Choose the comparator $w^*$ with $|w_j^*| = \|w^*\|/\sqrt{s}$ for $j \leq s$ and $w_j^* = 0$ otherwise (so that $\|w^*\|^2 = s \cdot \|w^*\|^2/s = \|w^*\|^2$). Since the problem decomposes into $s$ independent one-dimensional sub-problems, the standard one-dimensional minimax lower bound $\Omega(|w_j^*|\sqrt{\sum_t g_{t,j}^2})$ applies to each coordinate. Summing over coordinates:

$$\mathbb{E}[\mathrm{Regret}_T] \;\geq\; \sum_{j=1}^{s} \tfrac{1}{2}\,|w_j^*|\sqrt{\sum_{t=1}^{T} g_{t,j}^2} \;=\; \sum_{j=1}^{s} \tfrac{1}{2}\,\frac{\|w^*\|}{\sqrt{s}} \cdot \frac{G\sqrt{T}}{\sqrt{s}} \;=\; \tfrac{1}{2}\,\|w^*\|\,G\sqrt{T}.$$

Substituting $G\sqrt{T} = \sqrt{s \cdot (G^2T/s)} = \sqrt{s \cdot \|\mathbf{G}_T^S\|_\infty}$:

$$\mathbb{E}[\mathrm{Regret}_T] \;\geq\; \tfrac{1}{2}\,\|w^*\|\sqrt{s \cdot \|\mathbf{G}_T^S\|_\infty}.$$

Since any algorithm must face both adversaries, its worst-case expected regret is at least

$$\max\Big(\tfrac{1}{2}\,\|w^*\|\sqrt{r\,\sigma_1(\mathbf{G}_T^L)},\;\; \tfrac{1}{2}\,\|w^*\|\sqrt{s\,\|\mathbf{G}_T^S\|_\infty}\Big).$$

Applying the elementary inequality $\max(a, b) \geq (a + b)/2$ with $a = \frac{1}{2}\|w^*\|\sqrt{r\sigma_1}$ and $b = \frac{1}{2}\|w^*\|\sqrt{s\|\mathbf{G}_T^S\|_\infty}$ yields

$$\max(a, b) \;\geq\; \tfrac{1}{4}\,\|w^*\|\Big(\sqrt{r\,\sigma_1(\mathbf{G}_T^L)} + \sqrt{s\,\|\mathbf{G}_T^S\|_\infty}\Big),$$

which is the claimed bound. □

### B.9 Proof of Corollary 4 (Pure-Structure Special Cases)

*Proof.* Both cases follow by direct substitution into Theorem 1 with the indicated choice of $\alpha^*$.

For part (a), when $r = 0$ we have $\hat{L}_t = 0$ and $\hat{S}_t = g_t$ for all $t$, so $E_t = 0$ and $\mathcal{E}_T = 0$. Setting $\alpha^* = 0$ selects the sparse sub-algorithm exclusively, and the low-rank sub-algorithm's regret $R_T^L$ drops out of the bound. Within $R_T^S$, the cross-contamination term $\sum_t \|\hat{L}_t\|^2 = 0$ vanishes since $\hat{L}_t = 0$, yielding equation 15. The remaining terms (the comparator penalty $\|w^*\|_{D_T}^2$, the stability sum $\sum_{j\in\mathcal{J}}(\delta + (\hat{\mathbf{G}}_T^S)_{jj})^{1/2}$, and the preconditioner-lag correction $G^2 d\,\delta^{-1/2}$) are precisely the standard diagonal AdaGrad bound.

For part (b), the argument is symmetric: when $s = 0$ we have $\hat{S}_t = 0$ and $\hat{L}_t = g_t$, and setting $\alpha^* = 1$ selects the low-rank sub-algorithm. The cross-contamination term $\sum_t \|\hat{S}_t\|^2 = 0$ vanishes, yielding equation 16. Under stable rank-$r$ structure, Lemma 4 further simplifies the trace term $\mathrm{tr}((\delta I + \hat{\mathbf{G}}_T^L)^{1/2})$ to $r(\delta+\sigma_1(\hat{\mathbf{G}}_T^L))^{1/2}+(d-r)\delta^{1/2}$, recovering the full-matrix AdaGrad bound specialized to rank-$r$ gradients. □

### B.10 Proof of Corollary 5 (Adaptive Interpolation)

*Proof.* By Lemma 5 with the fixed normalization $L = 2GB_w$, the variance proxy satisfies

$$V_T \;=\; \sum_{t=1}^{T} \langle g_t,\, w_t^L - w_t^S\rangle^2 \;\le\; L^2 T \;=\; 4G^2 B_w^2 T.$$

Therefore,

$$\sqrt{2V_T \ln(T{+}1)} \;\le\; 2\,GB_w\,\sqrt{2T\ln(T{+}1)} \;=\; \widetilde{O}(GB_w\sqrt{T}).$$

The un-normalization in the proof of Lemma 5 is an exact algebraic identity (since $L$ is a fixed constant, not time-varying), so no approximation gap is introduced. □

## C Discussion of Assumption 6 in Unconstrained Settings

Assumption 6 requires a known constant $B_w$ such that $\|w_t^S\|_2, \|w_t^L\|_2 \le B_w$ for all $t$. It is used in exactly one place in our analysis, Lemma 5, where it ensures that the scaled coin outcome $c_t = \langle g_t, w_t^L - w_t^S\rangle/(2GB_w)$ lies in $[-1, 1]$, so that the Krichevsky–Trofimov coin-betting reduction applies with a fixed normalization constant. When $\mathcal{W}$ is bounded, the assumption is automatic with $B_w$ equal to the domain diameter. In the unconstrained setting $\mathcal{W} = \mathbb{R}^d$, the FTRL updates $w_{t+1}^S = -D_t^{-1}\Sigma_t$ and $w_{t+1}^L = -M_t^{-1}\Sigma_t$, together with $D_t, M_t \succeq \delta^{1/2} I$, give the deterministic a priori bound

$$\|w_{t+1}^S\|_2, \|w_{t+1}^L\|_2 \le \delta^{-1/2}\|\Sigma_t\|_2 \le \delta^{-1/2} G t.$$

Thus $B_w := \delta^{-1/2} GT$ always satisfies the assumption, but substituting this worst-case value into Lemma 5 gives the super-linear meta-overhead $\widetilde{O}(\delta^{-1/2} G^2 T^{3/2})$. This is precisely why the main theorem displays the data-dependent term $\mathcal{M}_T$ instead of hiding it as $\widetilde{O}(\sqrt{T})$.

If $B_w$ is unknown, an implementation can use a doubling trick on the normalization $L = 2GB_w$: start from $\hat{L}_0 > 0$, restart only the one-dimensional meta-algorithm whenever $|\langle g_t, w_t^L - w_t^S\rangle| > \hat{L}_k$, and double $\hat{L}_k$. This removes the need to know $B_w$ before the run. The resulting bound replaces the supplied $B_w$ by the realized supremum $B_w^\star := \sup_t \max(\|w_t^L\|, \|w_t^S\|)$, up to logarithmic factors. Likewise, a scale-adaptive one-dimensional coin-betting or parameter-free OLO routine can be used for the scalar losses $\ell_t = \langle g_t, w_t^L - w_t^S\rangle$ without a prior bound on $|\ell_t|$, again yielding an overhead of order $\widetilde{O}(\sqrt{\sum_t \ell_t^2}) \le \widetilde{O}(GB_w^\star\sqrt{T})$. These variants remove the algorithm-level requirement to provide $B_w$, but not the regret dependence on the realized range $B_w^\star$.

This remaining dependence is unavoidable in the black-box meta-aggregation setting. Proposition 2 gives a one-dimensional lower bound of $\Omega(GB_w^\star\sqrt{T})$ for competing with the better fixed mixture of two bounded base predictions. Therefore a doubling trick cannot guarantee a universal $\widetilde{O}(\sqrt{T})$ overhead when $B_w^\star$ is allowed

to grow with $T$; if $B_w^\star = \Theta(T)$, an $\Omega(T^{3/2})$ term is possible. The positive interpretation of Theorem 1 is consequently conditional and explicit: SLR-FTRL is useful when the structural gain over the worse specialized preconditioner exceeds the meta-aggregation cost $\mathcal{M}_T$.

## D Approximation Methods for Higher-Dimensional and Higher-Rank Workloads

The per-round cost of Algorithm 1, $O(dr^2 + d)$ time and $O(dr)$ memory, is dominated by maintaining the rank-$r$ subspace basis $U_t$ and the low-rank eigendecomposition of $A_t^L = \delta I + \sum_{\tau \le t} \hat{L}_\tau \hat{L}_\tau^\top$. For large-scale workloads where $d$ is in the millions and $r$ in the hundreds, several approximation strategies preserve the structural form of our regret bound up to an additive sketch-error term. A streaming randomized SVD (Halko et al., 2011) of rank $r+p$ with oversampling $p = O(\log r)$ drops the per-round cost to $O(dr)$ and introduces an additive term proportional to $\sqrt{T \cdot \sigma_{r+1}(\mathbf{G}_T^L)/\delta^{1/2}}$ that is negligible when $\sigma_{r+1} \ll \sigma_1$. A Frequent-Directions sketch (Ghashami et al., 2016) of width $\ell \ge r$ runs in $O(d\ell)$ time and memory and preserves the trace bound in Lemma 2 up to a multiplicative $(1 + \epsilon_{\mathrm{FD}})$ factor with $\epsilon_{\mathrm{FD}} = O(\mathrm{tr}(A_t^L)/((\ell - r)\sigma_1))$, giving a smooth interpolation between the diagonal ($\ell = 0$) and full-matrix ($\ell = d$) regimes. For deep-learning workloads organized as a sequence of weight tensors $W^{(1)}, \ldots, W^{(K)}$, the low-rank preconditioner $M_t$ can be replaced by a block-diagonal Kronecker-factored preconditioner with one block per layer in the style of Shampoo (Gupta et al., 2018), reducing the per-layer cost from $O((mn)r^2)$ to $O((m^2 + n^2)r)$ at the price of a Kronecker approximation factor bounded under standard layer-wise gradient assumptions. Updating $M_t$ only every $k$ rounds (typically $k = 10$ to 100) reduces the amortized per-round cost from $O(dr^2)$ to $O(dr^2/k + dr)$ at the price of an additional cross-contamination term $O(k \cdot G^2/\delta^{1/2})$ per interval, and choosing $k = \Theta(\sqrt{T})$ gives an additive $\widetilde{O}(\sqrt{T})$ correction.

These four strategies are largely independent and can be combined, for example by using Kronecker-factored blocks together with Frequent-Directions sketching per block, refreshed every $k$ rounds. As a practical recommendation, we suggest the exact algorithm for $d \le 10^4$ and $r \le 100$; Frequent-Directions sketching with $\ell = 2r$ combined with lazy updates every $k = O(\sqrt{T})$ rounds for $d \in [10^4, 10^6]$; and Kronecker-factored blocks with per-block Frequent-Directions sketching for $d > 10^6$, which has the additional benefit of preserving the natural parameter layout of the model. A full empirical study of these approximations on production-scale neural-network training is an important direction for future work.