Foundations of Top-k Decoding for Language Models

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Abstract

Top-k decoding is a widely used method for sampling from LLMs: at each token, only the largest k next-token-probabilities are kept, and the next token is sampled after re-normalizing them to sum to unity. Top-k and other sampling methods are motivated by the intuition that true next-token distributions are sparse, and the noisy LLM probabilities need to be truncated. However, to our knowledge, a precise theoretical motivation for the use of top-k decoding is missing. In this work, we develop a theoretical framework that both explains and generalizes top-k decoding. We view decoding at a fixed token as the recovery of a sparse probability distribution. We introduce *Bregman decoders* obtained by minimizing a separable Bregman divergence (for both the *primal* and *dual* cases) with a sparsity-inducing ℓ_0-regularization; in particular, these decoders are *adaptive* in the sense that the sparsity parameter k is chosen depending on the underlying token distribution. Despite the combinatorial nature of the sparse Bregman objective, we show how to optimize it efficiently for a large class of divergences. We prove that (i) the optimal decoding strategies are greedy, and further that (ii) the objective is discretely convex in k, such that the optimal k can be identified in logarithmic time. We note that standard top-k decoding arises as a special case for the KL divergence, and construct new decoding strategies with substantially different behaviors (e.g., non-linearly up-weighting larger probabilities after re-normalization).

1. Introduction

Large language models (LLMs) are powerful generative AI tools for producing text. When pre-trained on large text corpora and aligned according to human preferences, they can be used for a wide range of tasks. On a technical level, they are probability distributions over text: given any user text prompt x, an LLM samples an answer $Y \sim \pi(\cdot|x)$ from a probability distribution $\pi(\cdot|x)$ over text. However, even after obtaining a pre-trained, fine-tuned, and human preference-aligned model π, it is rare to directly sample from the model. Instead, several sampling/decoding methods are commonly used, including top-k [21] or top-p sampling [32]. These are widely used either by

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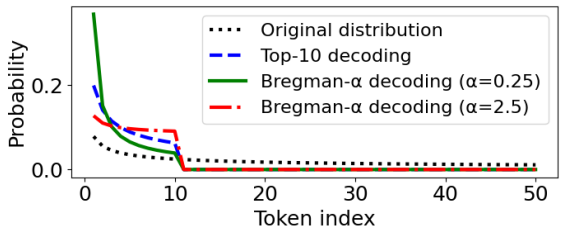
default or as an option in many popular LLMs, including the GPT series, Gemini, and Claude. In addition to other decoding methods such as beam search, temperature scaling, best-of-N, etc., top-k, top-p and related methods are known to improve performance in a broad range of settings compared to direct sampling, see e.g., [12, 21, 32].

In this paper, we focus on decoding methods that modify each next-token-probability distribution to induce *sparsity*, i.e., to keep only a small number of tokens with a nonzero probability. This includes the widely used top-k [21] and top-p [32] sampling methods, among others. These methods are motivated by the intuition that the noisy LLMs probabilities need to be truncated to denoise the "unreliable tail" [32]. In particular, we focus on the popular top-k decoding method, which keeps only the largest k next-token-probabilities at each decoding step. These are re-normalized—via dividing by their sum—to a probability distribution from which the next token is sampled.

Despite the wide use and rich intuition behind top-k decoding, to our knowledge, a precise theoretical understanding of top-k decoding is not available—see Section C for a discussion of related work. In this work, we develop a theoretical framework that enables a flexible range of generalizations of top-k decoding. For a fixed token, we view decoding as recovering a sparse probability distribution. We consider denoisers obtained by minimizing a Bregman divergence (such as a KL divergence or Brier score) with a sparsity-inducing ℓ_0 regularization. This approach is motivated by a rich literature of both Bregman divergences and sparsity, see Section C for details.



Our approach leads to new decoding methods. As an example, we consider Bregman divergences generated by the α-entropies $x \mapsto \frac{x^\alpha}{\alpha(\alpha-1)}$ [29, 51]. Top-k decoding arises as an instance of this class for $\alpha \to 1$, corresponding to the KL divergence. We also identify new decoding strategies with distinct behavior. The figure on the left shows an example of a distribution over 100 tokens, the result of top-10 decoding, and results for our Bregman-α decoding with $k = 10$: for $\alpha = 1/4$, Bregman decoding places relatively more mass on larger probabilities, while for $\alpha = 2.5$, the situation is reversed. In applications, either behavior could be desired.

1.1. A roadmap of the paper and our contributions

In Section 2 we introduce some fundamentals. We decompose a decoding strategy into two steps: selecting a subset of tokens, and re-normalizing their weights to a probability distribution. We introduce decoding strategies obtained by ℓ_0-regularized Bregman divergence-minimization. We consider both *primal* and *dual* decoding, minimizing the Bregman divergence in its first and second arguments, respectively; both are well-motivated choices [see e.g., 1, 10, 24, 56, etc].

In general, ℓ_0-regularization leads to combinatorial optimization problems, for which there are no known polynomial-time algorithms [11, 42]. Our main contribution is to show that, despite this, sparse Bregman decoding can be optimized efficiently for a large class of divergences. Specifically, in Sections 3.2, 3.3, we show these two properties under certain natural and broad assumptions:

1. *Greedy selection*—choosing k largest probabilities for some k—is optimal (Theorems 3, 4);
2. *Discrete convexity of the loss in k* — enabling very fast finding of the optimal k^* (Theorem 5).

Both properties turn out to be non-trivial to show, and require several novel structural insights into the sparse Bregman objective that we believe may be of independent interest.

In Section A, we discuss an example: α-Bregman decoding strategies, generated by α-entropies $x \mapsto \frac{x^\alpha}{\alpha(\alpha-1)}$; we show that primal renormalization can be solved exactly in several cases of interest, and converges to *water-filling* as $\alpha \to \infty$. Finally, in Appendix B we empirically evaluate several α-decoding schemes on open-ended text generation and mathematical problem-solving tasks with LLMs, where we find them to perform competitively with top-k decoding.

2. A sparse optimization formulation for top-k decoding

Top-k decoding. Given a probability distribution $p = (p_1, \ldots, p_V)$ (where V stands for "vocabulary size"), and some $1 \leqslant k \leqslant V$, **top-$k$ decoding** first selects the indices $S_k = (i_1, \ldots, i_k)$ of the largest k probabilities, breaking ties arbitrarily. Setting all other coordinates to zero in p, one obtains the vector $p[1:k]$ of the k largest entries. Then, it re-normalizes this vector by dividing it by its sum. Letting $(p_{(1)}, p_{(2)}, \ldots, p_{(k)}) = (p_{i_1}, \ldots, p_{i_k})$ be the largest k entries of p, we have: top-$k(p) = p[1:k]/\sum_{j=1}^k p_{(j)}$. One then draws a sample from the distribution top-$k(p)$.

Decoding strategies. Next, we aim to generalize top-k decoding. We will refer to any operator Dec on probability distributions as a *decoding strategy*; formally $\mathrm{Dec} : \Delta_V \to \Delta_V$, where $\Delta_V = \{x \in [0,1]^V : \sum_{i=1}^V x_i = 1\}$ is the simplex of V-dimensional probability distributions. Observe that top-k decoding consists of two steps: selecting the largest coordinates and re-normalizing them.

The second step can be viewed as "re-distributing" among the remaining indices the probability mass that has been thresholded away by selection. This step can be performed in a lot of other meaningful ways besides division by the sum. Motivated by this, we call, given $k \geq 1$, any $T : \Delta_{\mathrm{sub},k} \to \Delta_k$ a *renormalization map*, where we define the *sub-probability simplex* $\Delta_{\mathrm{sub},k} = \{x \in [0,1]^k : \sum_{i=1}^k x_i \leqslant 1\}$. We can now define generalized top-k decoding:

Definition 1 (Generalized top-k decoding) *For a fixed k and renormalization map T, a generalized top-k decoding strategy $\mathrm{Dec}_{k,T} : \Delta_V \to \Delta_V$ takes any V-class probability vector p, thresholds it to the sub-vector $p[1:k]$ consisting of its top-k elements, and renormalizes it to $T(p[1:k]) \in \Delta_V$.*

Adaptivity. A natural extension is to choose k adaptively based on p. For this, we consider a k-selector map $\hat{k} : \Delta_V \to [V] := \{1, \ldots, V\}$, and a collection of renormalization maps $T_k : \Delta_{\mathrm{sub},k} \to \Delta_k$, $k = 1, \ldots, V$. We define an *adaptive generalized top-k decoding strategy* $\mathrm{Dec}_T : \Delta_V \to \Delta_V$ via $p \mapsto T_{\hat{k}(p)}(p[1:\hat{k}(p)])$. Below, we will design specific renormalizers T and ways to choose k.

2.1. Regularized sparse Bregman decoding

Decoding via sparse divergence minimization. Consider a divergence $\mathrm{Div}(\cdot,\cdot) : \Delta_V \times \Delta_V \to \mathbb{R}$ between two distributions. Classical examples include the squared error $\mathrm{Div}(p,q) = \|p-q\|_2^2$ and the KL divergence $\mathrm{Div}(p,q) = \sum_{j=1}^V p_j \ln(p_j/q_j)$. We define the decoding strategy $\mathrm{Dec}_{\mathrm{Div}}$, via sparsity-regularized divergence minimization[1] under divergence Div, for any probability vector p as:

$$\mathrm{Dec}_{\mathrm{Div}}(p) \in \operatorname*{arg\,min}_{\hat{p} \in \Delta_V} \left\{ \mathrm{Div}(\hat{p}, p) + \lambda \|\hat{p}\|_0 \right\} \quad \textbf{(sparsity-regularized decoding)}. \tag{1}$$

Here, the ℓ_0-pseudonorm $\|\hat{p}\|_0$ is the number of nonzero entries of $\hat{p}$, and $\lambda \geqslant 0$ is a *sparsity cost* hyperparameter. As λ increases, the optimal solution $\hat{p} = p^*$ gets increasingly more sparse.

1. In our examples of interest, we will show that this optimization problem is well-defined. When there are multiple minimizers, we assume that one is selected in an arbitrary measurable way.

Bregman divergences. In this work, we shall instantiate Div in Problem 1 with separable Bregman divergences [1, 10]. This class is very expressive, inducing top-k decoding and many fruitful generalizations thereof. For a convex $\mathrm{Dom} \subseteq \mathbb{R}$ and a convex differentiable $\phi : \mathrm{Dom} \to \mathbb{R}$, the one-dimensional Bregman ϕ-divergence d_ϕ is defined as: $\mathrm{d}_\phi(x, y) = \phi(x) - \phi(y) - \phi'(y)(x - y)$, for $x, y \in \mathrm{Dom}$. The separable V-dimensional Bregman ϕ-divergence $\mathrm{D}_\phi : \mathrm{Dom}^V \to \mathbb{R}$ is defined as: $\mathrm{D}_\phi(x, y) = \sum_{i \in [V]} \mathrm{d}_\phi(x_i, y_i), \quad \text{for } x, y \in \mathrm{Dom}^V$.

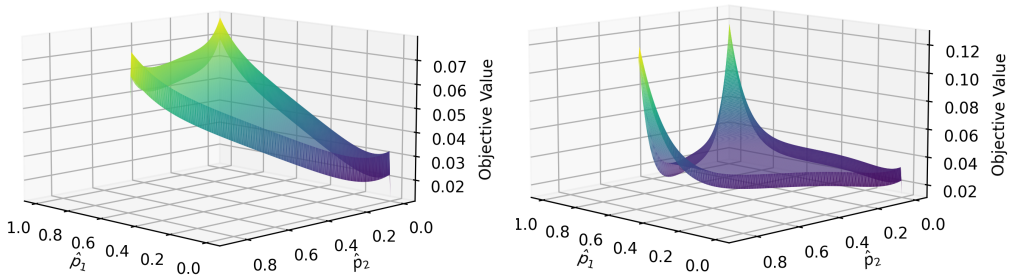


Figure 1: Sparse α-Bregman primal (left) and dual (right) landscape ($p = \frac{(0.1, 0.01, 0.001)}{0.111}$, $\alpha = 10$, $\lambda = 0.01$).

Primal and dual Bregman decoding. Since Bregman divergences are generally non-symmetric in their arguments, we may instantiate the sparse Bregman decoding Problem 1 in two substantially distinct ways: by placing the estimand $\hat{p}$ in the first (*primal*) or second (*dual*) argument:

$$\mathrm{Div}(\hat{p}, p) := \mathrm{D}_\phi(\hat{p}, p) \quad \textbf{(primal decoding)}, \qquad \mathrm{Div}(\hat{p}, p) := \mathrm{D}_\phi(p, \hat{p}) \quad \textbf{(dual decoding)}. \tag{2}$$

Both formulations possess a sound theoretical motivation. *Bregman projections* are commonly defined as minimization in the first argument, while Bregman-based *proper scoring rules* for mean elicitation correspond to minimization in the second argument [see e.g., 24, 39, etc].

The landscapes of primal and dual decoding are illustrated in Figure 1. The dual objective can be non-convex even in the interior of the simplex, but we will soon show that this can be circumvented.

3. The algorithmic structure of primal and dual Bregman decoding

We now show that under mild assumptions on the divergence D_ϕ, both primal and dual decoding give **algorithmically efficient** *adaptive generalized top-k decoding* in the sense of Definition 1.

In Section 3.2 we demonstrate for any $p \in \Delta_V$ that out of the (a-priori) 2^V possible sparsity patterns $S \subseteq [V]$, the optimal one must consist of the top-k entries of p for some $k \in [V]$.

Next, in Section 3.3 we will establish that finding the optimal $k^* = k^*(p)$ is in fact a (discretely) *convex* optimization problem in $k \in [V]$. The ability to perform binary or exponential search for k^* is critical in modern settings where vocabulary sizes upwards of $V \sim 10^5$ are common.

3.1. Primal and dual renormalization for a fixed sparsity pattern

Once an optimal sparsity pattern $S \subseteq [V]$ (of some size $|S| = k$) has been identified, the vector x — which denotes the sub-vector of p restricted to indices in S — needs to be projected onto Δ_k. The ℓ_0-term in Problem (1) gets fixed to λk, reducing it to Bregman projection: $\arg\min_{\hat{p} \in \Delta_k} \mathrm{Div}(\hat{p}, x)$.

Primal renormalization We impose the following mild condition on the Bregman generator ϕ.

Assumption 3.1 (Primal validity) *Map ϕ is convex and C^1 on $[0, 1]$ and strictly convex on $(0, 1)$.*

Existing results [33, 34] imply that for a primal valid potential ϕ, denoting[2] $f = \phi'$, the **primal renormalization** map T_ϕ is given for $x \in \Delta_{\text{sub},k}$ by:

$$[T_\phi(x)]_i = f^{-1}(f(x_i) + \nu) \quad \text{for all } i \in [k], \text{ where } \nu \in \mathbb{R} \text{ is chosen s.t. } \sum_{i=1}^{k} [T_\phi(x)]_i = 1. \quad (3)$$

Since $\nu \mapsto f^{-1}(f(x_i) + \nu)$ is non-decreasing[3] in ν, the solution can be found efficiently using off-the-shelf root-finding algorithms such as Brent's method.

Dual renormalization While primal projections are well-studied in prior work [33, 34], we are not aware of a direct derivation of dual Bregman projections. Indeed, Bregman divergences are convex in the first [3] but generally not the second argument, which can interfere with the uniqueness of dual projections. To pave the road towards dual Bregman projections, we will therefore rely on additional structure in ϕ and d_ϕ, which we express as the following dual validity condition.

Assumption 3.2 (Dual validity) *The map ϕ is thrice differentiable on $(0, 1]$ with $\lim_{x \to 0^+} x\phi''(x) = 0$. For $x \in (0,1]$, $y \mapsto \mathrm{d}_\phi(x,y)$ is strictly convex for $y \in [x,1]$, and $y \mapsto \mathrm{d}_\phi(0,y)$ is strictly convex for $y \in (0,1]$.*

We establish in Theorem 9 (see Appendix D) that subject to dual validity, the **dual renormalization** map T^*_ϕ is uniquely defined for any $x \in \Delta_{\text{sub},k}$ with $x \neq 0_k$ by the following implicit equations:

$$[T^*_\phi(x)]_i = x_i + \nu^* / f'([T^*_\phi(x)]_i) \text{ for } i \in [k], \text{ with } \nu^* \in \mathbb{R} \text{ chosen so that } \sum_{i=1}^{k} [T^*_\phi(x)]_i = 1. \quad (4)$$

Assumption 3.2, short of requiring global convexity of $\mathrm{d}_\phi(x, \cdot)$ on $[0, 1]$, only enforces it for $y \in [x, 1]$. To enable this relaxation, the proof of Theorem 9 carefully excludes optimal solutions belonging to the region $y \leq x$ or to the simplex boundary. Rather than a mere curiosity, this refinement substantially expands the scope of dual decoding. In particular, in our later specialization, it is essential for ensuring that dual α-decoding is uniquely defined for all $\alpha > 1$, not just $\alpha \in (1, 2]$: as plots in Appendix J.4 demonstrate, α-Bregman divergences are nonconvex for $y \leq x$ for $\alpha > 2$.

See Section I for algorithmic details on computing the dual map and the pseudocode.

3.2. Greedy property: Justifying top-k selection

The viewpoint that lower-probability tokens can be considered as noisy [32] suggests that it would be natural and indeed desirable for a decoding strategy to be "greedy"—dictating that it is optimal to renormalize over the top-k-probability tokens, for some $k \in [V]$. We formalize this as follows.

Definition 2 (Greedy decoding) *A decoding strategy $\mathrm{Dec} : \Delta_V \to \Delta_V$ is called* greedy *if for every $p \in \Delta_V$, the set of nonzero entries of $\mathrm{Dec}(p)$ is a set of top-$\hat{k}$ entries of p, for some $\hat{k} = \hat{k}(p)$.*

Many popular decoding methods are greedy [12, 21, 32, 38], but not all [22, 36]: non-greediness, i.e., excluding some of the top-k tokens, has been justified by e.g. suggesting that it could help generate more "typical" text. As such, our assertion that the primal and dual Bregman decoding strategies are greedy is not trivial and requires proof. First, we state our result for primal decoding.

Theorem 3 (Primal Bregman decoding is greedy) *The primal Bregman decoding strategy from (1) is greedy for any primal valid potential ϕ.*

The proof is provided in Appendix E. It proceeds by decomposing the Bregman objective into several terms, see Lemma 11, and bounding them with the help of the primal renormalization equations (3).

2. To make f^{-1} continuous and increasing on $\mathbb{R}$, let $f^{-1}(x) = 0$ for $x < f(0)$ and $f^{-1}(x) = 1$ for $x > f(1)$.
3. It is strictly increasing for $\nu \in [-f(x_i), 1 - f(x_i)]$, but the required ν may lie outside this range.

The dual case, owing i.a. to the implicit form of the dual renormalization formulas (4), is correspondingly more complex to handle. Unlike in Theorem 3, our next result requires further conditions, which we state as a menu of two options. The relationship between the extra assumptions is intricate; Assumption (A2) is implied by, but is strictly weaker than, log-convexity of ϕ'.

Theorem 4 (Dual Bregman decoding is greedy) *The dual Bregman decoding strategy from* (1) *is greedy for any dual-valid ϕ with $\phi'(0) = 0$ that further satisfies either of the following conditions:*
(A1) ϕ' is convex;
(A2) The maps[4] *u defined as $u(x) := x\phi''(x)/\phi'(x)$ for $x \in (0, 1]$ and ϕ are nondecreasing.*

The proof is provided in Appendix F. In it, we use two different proof techniques for both conditions: For Condition (A1), our proof in Appendix F.1 leverages the decomposition from the primal case along with the change of variables $\mathrm{d}_\phi(x, y) = \mathrm{d}_{\phi^*}(\phi'(y), \phi'(x))$, where ϕ^* is the convex conjugate of ϕ. For Condition (A2), we develop a saddle-point proof approach in Appendix F.2. For that, we perform a sensitivity analysis of both the renormalized values $[T^*_\phi(p)]_i$ and of the per-coordinate Bregman loss terms, relative to hypothetical changes in the dual Lagrange multiplier ν^* and in the entries p_i of p; we carry this out via implicit differentiation of the defining equations (4).

3.3. Discrete k-convexity: Fast search for optimal adaptive k

Next, we show that when restricted to the greedy (top-k) selection, the primal and dual decoding objectives both enjoy discrete convexity with respect to the sparsity parameter k. First, for a general divergence Div, denote the ℓ_0-regularized cost of each greedy (top-k) choice by $\mathrm{cost}(k)$:

$$\mathrm{cost}(k) := \min_{\hat{p} \in \Delta_k} \left\{ \mathrm{Div}\left((\hat{p}, 0_{V-k}), p\right) + \lambda k \right\}. \tag{5}$$

Recall that a function $h : [V] \to \mathbb{R}$ is *discretely convex* if for all $k \in [V-1] - \{1\}$, its discrete second derivative $\Delta^2 h(k) := \Delta h(k+1) - \Delta h(k) := \{h(k+1) - h(k)\} - \{h(k) - h(k-1)\} \geqslant 0$.

Theorem 5 (Discrete primal and dual cost convexity) $\mathrm{cost}(\cdot)$ *is discretely convex in $k \in [V]$ for:*

1. $\mathrm{Div}(\hat{p}, p) = \mathrm{D}_\phi(\hat{p}, p)$*, if ϕ is primal valid;*
2. $\mathrm{Div}(\hat{p}, p) = \mathrm{D}_\phi(p, \hat{p})$*, if ϕ is dual valid.*

In Figure 5 (see Appendix J.5), we illustrate the result of Theorem 5 by plotting the $\mathrm{cost}(\cdot)$ functions for primal and dual Bregman α-decoding (defined in Section A below) for assorted α.

Provable binary search over k: As a direct consequence of Theorem 5, the cost increments $\Delta\mathrm{cost}(k) = \mathrm{cost}(k+1) - \mathrm{cost}(k)$ increase with k, so binary search over k will efficiently identify an optimal sparsity parameter k^* — as one for which $\Delta\mathrm{cost}(k^*) \leqslant 0$ and $\Delta\mathrm{cost}(k^*+1) \geqslant 0$.

The proof of Theorem 5 requires very distinct techniques in the primal and dual cases.

Primal k-convexity. The proof is developed in Appendix G. As its cornerstone, we use the Legendre dual mapping ϕ^* of the generator ϕ to establish and leverage the following cost structure: for any k, cost(k), up to additional terms, can be represented as $\max_{\nu \geq 0} \left[\nu - \sum_{i=1}^k \phi^*(\phi'(p_i) + \nu)\right]$, where the objective is concave in ν and has ν_k, the optimal Lagrange multiplier for renormalizing the top k probabilities of p from (3), as it unique optimizer. From here, we establish $\Delta^2\mathrm{cost}(k) \geq 0$.

Dual k-convexity. The proof is in Appendix H. The above dualization strategy does not directly apply. Instead, we lower bound $\Delta^2\mathrm{cost}^*(k)$ by regrouping the loss contributions of the indices $i \in [k+1]$, and —via intricate rearrangement and bounding—reduce to proving the local concavity of a special transformation (Equation 19) that turns out to hold by our dual-validity assumption.

4. In the economics literature, $u(x) = x\phi''(x)/\phi'(x)$ is referred to as the *elasticity* of the function ϕ'.

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Appendix A. Example: Bregman α-decoding

We now consider, as an illustration, a single-parameter family of Bregman decoding strategies, which arises via the generators of the Havrda-Charvát-Tsallis α-entropies [8, 29, 45, 51, 52]:

$$\phi_\alpha(x) = x^\alpha/[\alpha(\alpha-1)], x \in [0,1], \quad \text{for } \alpha \in J := (-\infty, 0) \cup (0,1) \cup (1, \infty).$$

When $\alpha < 0$ and $x = 0$, we set $x^\alpha := +\infty$ so that $\phi_\alpha(0) = \infty$. For $\alpha = 1$, one defines $\phi_1(x) = x\log(x)$, which corresponds to the Shannon entropy, arising in the limit[5] as $\alpha \to 1$. Observe that ϕ_α is *primal valid* for all $\alpha \neq 0$, as $\phi''_\alpha(x) = x^{\alpha-2}$. This yields the following primal family of renormalizations, which we will index by α rather than ϕ:

Definition 6 (Primal Bregman α-decoding) *Fix $\alpha \in J, k \in [V]$. The renormalization map T_α is given for $p \in \Delta_{\text{sub},k}$ as: $[T_\alpha(p)]_i = (p_i^{\alpha-1} + \nu)^{\frac{1}{\alpha-1}}$ for $i \in [k]$, with $\nu \in \mathbb{R}$ chosen so that $\sum_{i\in[k]} [T_\alpha(p)]_i = 1$.*

Note that for $\alpha = 1$, we have $\phi'_1(x) = \log x + 1$. Hence, (3) implies $e^\nu \sum_{i=1}^k p_i = 1$, and we obtain the "standard" renormalization: $[T_1(p)]_i = p_i/(\sum_{j=1}^k p_j)$, for $i \in [k]$. Therefore, *primal Bregman 1-decoding is top-k decoding*, showing how one recovers top-k in our framework. It turns out that some further values of α also lead to renormalization maps of special interest. For any fixed p, we let $T_{-\infty}(p) = \liminf_{\alpha\to-\infty} T_\alpha(p)$ and $T_\infty(p) = \liminf_{\alpha\to\infty} T_\alpha(p)$, where the limits are entrywise.

Proposition 7 (Special primal α-renormalization maps)
We have the following special instances[6] of the primal Bregman α-renormalization map, defined for all $i \in [k]$ as follows:
$[T_{-\infty}(p)]_i = p_i + \mathbb{1}[i = i^*] \cdot \left(1 - \sum_{j=1}^k p_j\right)$, *assuming that* $\arg\max_i p_i = \{i^*\}$.
$[T_{1.5}(p)]_i = \left(\sqrt{p_i} + \left[\sqrt{r^2 + k\left(1-s\right)} - r\right]/k\right)^2$, *where* $r = \sum_{j=1}^k \sqrt{p_j}$ *and* $s = \sum_{j=1}^k p_j$.
$[T_2(p)]_i = p_i + (1 - \sum_{j=1}^k p_j)/k$.
$[T_\infty(p)]_i = \max\{p_i, \nu\}$, *where $\nu \in \mathbb{R}$ is the "water level" for which* $\sum_{i=1}^k [T_\infty(p)]_i = 1$.

Along with the primal family, the dual α-decoding family can also be defined based on ϕ_α. Unlike α-decoding, the dual Bregman sparse decoding Problem 1 can be non-convex, see Figure 1 above. Figure 4 in Appendix J.4 further demonstrates the nonconvexity of D_{ϕ_α} on the unit square for some α. Yet, we still show that any dual α-decoding with $\alpha > 1$ is valid, greedy and k-convex:

Lemma 8 *All generator functions ϕ_α, $\alpha > 1$, are dual-valid and satisfy Assumption (A2).*

We give an illustration contrasting primal and dual α-decoding for various $\alpha > 1$ in Appendix J.3.

Appendix B. Experiments

We now illustrate some of the decoding schemes described in our paper in the context of LLMs. Since our goal is to develop the theoretical foundations of top-k decoding, our aim in this section is simply to illustrate that the performance of our novel decoding schemes can be competitive with standard top-k decoding. In particular, we do not aim to compare or compete with other popular and established decoding methods, which is beyond the scope of our theory-focused paper.

5. One conventionally defines the entropies via $(x^\alpha - x)/[\alpha(\alpha-1)]$, in which case the Shannon entropy is obtained in the limit as $\alpha \to 1$. In our case, we use the definition $\phi_\alpha(x) = x^\alpha/[\alpha(\alpha-1)]$ so that some technical conditions (such as $\phi'_\alpha(0) = 0$) hold in the proofs. Both definitions lead to the same decoding strategies in (3).
6. In particular, $T_{-\infty}(p), T_{1.5}(p), T_2(p)$ do not require solving for ν in Definition 6, enabling a fast implementation just like in the case of the canonical top-k renormalization.

B.1. Experimental Setup

Method. In addition to standard top-k decoding, which coincides with the $\alpha = 1$ case of our primal α-decoding family described in Section A, we illustrate primal α-decoding strategies for $\alpha = 1.5$ and $\alpha = 2$. These have closed-form renormalization maps that are as fast as standard renormalization.

Full and partial evaluation. Further, we perform two types of experiments: (1) For the evaluation of our *full* decoding strategy, we decode by adaptively selecting the optimal sparsity parameter k^* by optimizing our sparse Bregman objective. In this approach, we aim to observe the behavior when adaptively choosing k^*. Since practical choices of k^* are always upper bounded, we set a maximum $k^* \leqslant k_{\max} := 50$. (2) In the *partial* evaluation approach, we instead directly evaluate—for each fixed choice of k in the grid $k \in \{5, 10, \ldots, 50\}$—our proposed renormalization strategies along with standard top-k renormalization.

Models and benchmarks. We conduct experiments using the GPT-2 Large [43] and Llama 3.1 8B [25] models. We evaluate on two benchmarks: (1) open-ended text generation using the WebText test set from the GPT-2 output dataset [40], and (2) grade school math reasoning using the GSM8K Chain-of-Thought benchmark [13].

Evaluation metrics. For open-ended text generation, following Chen et al. [12], we use the first 35 tokens of each WebText test sample as a prompt and generate up to 256 tokens. We evaluate the following standard metrics [see e.g., 12, 32, 38, etc]:

(1) *Perplexity difference*, which measures the perplexity (according to base model p_{base}) of human text compared to that obtained from a decoding strategy p_{decoding} derived from the base model, where lower is better. This equals $\mathbb{E}_{X\sim\mathcal{D}}[\mathbb{E}_{Y\sim\mathcal{D}(\cdot|\mathcal{X})}(p_{\text{base}}(Y \mid X)^{-1/|Y|}) - \mathbb{E}_{Y\sim p_{\text{decoding}}(\cdot|X)}(p_{\text{base}}(Y \mid X)^{-1/|Y|})]$, where $X \sim \mathcal{D}$ is a prompt drawn from the dataset, $Y \sim \mathcal{D}(\cdot|\mathcal{X})$ denotes a human-written continuation drawn from the dataset, and $Y \sim p_{\text{decoding}}(\cdot \mid X)$ denotes a model-generated continuation using a specific decoding strategy. Here, $|Y|$ is the length of the continuation.

(2) *Repetition difference*: $\mathbb{E}_{X\sim\mathcal{D}}\left[\mathbb{P}_{Y\sim p_{\text{decoding}}(\cdot|X)}(\text{rep}(Y)) - \mathbb{P}_{Y\sim\mathcal{D}(\cdot|X)}(\text{rep}(Y))\right]$, where $\text{rep}(Y)$ is the event that Y contains two contiguous and identical token spans of length $\geqslant 2$; lower is better.

B.2. Results

Open-ended text generation. Using the *partial* evaluation setup with temperature fixed at 1.0, Figure 2 reports the differences in perplexity and repetition frequency between model-generated and human-written text across a range of k values. Primal decoding strategies are competitive with top-k in terms of both metrics. In particular $\alpha = 2.0$ has the smallest gaps in perplexity.

GSM8K dataset. Using the *full* decoding strategy, we evaluate the LLaMA 3.1 8B model using 8-shot CoT prompting. We test various temperatures, regularization strengths $\lambda \in \{0.01, 0.0001\}$ and primal decoding parameters $\alpha \in \{1.5, 2.0\}$. Results for other settings are in Appendix K. To ensure a matched comparison, we run top-k with $k = k^*$ for the Bregman decoding run with the same temperature, λ, and α, rounded to the nearest integer, see Table 11 in Appendix K. As seen in Table 1, across all temperature settings, primal decoding with adaptive k^* achieves accuracy comparable to top-k. At higher temperatures (such as 1.5), the performance of top-k decoding degrades more rapidly than that of primal decoding.

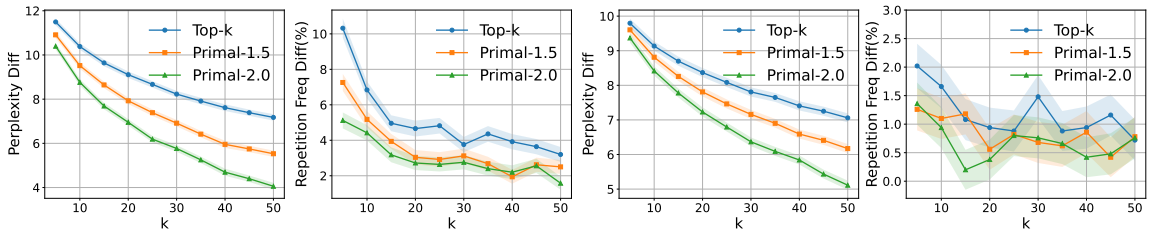

Figure 2: Perplexity and repetition frequency differences between generated and human-written text for GPT2-large (left two panels) and LLaMA 3.1 8B (right two panels), for various k values. We show top-k decoding and primal decoding with $\alpha \in \{1.5, 2.0\}$. Standard deviations are estimated using 1000 bootstrap resamples.

Table 1: Accuracy on GSM8K for LLaMA 3.1 8B using Bregman primal decoding ($\lambda \in \{0.01, 0.0001\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.01$		Top-k ($\lambda = 0.01$)		$\lambda = 0.0001$		Top-k ($\lambda = 0.0001$)	
	$\alpha = 1.5$	$\alpha = 2.0$			$\alpha = 1.5$	$\alpha = 2.0$		
0.3	$85.14_{\pm 0.80}$	$84.38_{\pm 1.00}$	$83.62_{\pm 1.02}$	$84.69_{\pm 0.99}$	$84.69_{\pm 0.99}$	$84.46_{\pm 1.00}$	$85.14_{\pm 0.98}$	$83.62_{\pm 1.02}$
0.7	$83.24_{\pm 1.02}$	$81.73_{\pm 1.06}$	$83.78_{\pm 1.02}$	$84.69_{\pm 0.99}$	$82.03_{\pm 1.06}$	$82.03_{\pm 1.06}$	$82.11_{\pm 1.06}$	$83.78_{\pm 1.02}$
1.0	$81.20_{\pm 1.08}$	$80.97_{\pm 1.08}$	$81.20_{\pm 1.08}$	$81.20_{\pm 1.08}$	$77.41_{\pm 1.15}$	$77.26_{\pm 1.15}$	$79.23_{\pm 1.12}$	$78.54_{\pm 1.13}$
1.5	$79.00_{\pm 1.12}$	$80.06_{\pm 1.10}$	$75.97_{\pm 1.18}$	$75.97_{\pm 1.18}$	$57.24_{\pm 1.36}$	$64.97_{\pm 1.31}$	$43.21_{\pm 1.36}$	$58.53_{\pm 1.36}$

Appendix C. Related work

Bregman projection. Michelot [37] considered the Brier score projection problem and derived an efficient algorithm. Later, Shalev-Shwartz et al. [48] revisited the properties of optimal Brier projection, and Duchi et al. [17] gave and analyzed the explicit algorithm that we discuss in what follows. Wang and Carreira-Perpinán [53] simplified and distilled the proof. [35] further studied the projection as a method for generating sparse probability predictions in multiclass prediction problems. [33, 34] developed methods for efficient Bregman projections to the simplex; for a fixed support, these results characterize our primal decoding. [44, 46] developed differentiable variants of top-k decoding. In contrast to these works, we: (1) consider Bregman projections under ℓ_0 regularization, and (2) offer, to the best of our knowledge, novel analyses of *dual* Bregman projections.

ℓ_0 **regularization.** Regularization via the ℓ_0-pseudonorm has been studied widely, with various approximate algorithms (based on surrogates, integer programming, branch-and-bound methods, etc.) developed for problems ranging from linear regression to more general learning tasks [see e.g., 2, 6, 9, 15, 18–20, 30, 41, 49, 50, 58, 61, etc]. In contrast, the algorithms we propose are exact within numerical precision for the specific class of problems we consider.

Bregman divergences. The properties of Bregman divergences [10] have been widely studied; see, e.g., [1, 3, 5, 8, 27, 39, 47, 55, 57], etc. In particular, there are a number of relations between Bregman divergences and their versions with reversed arguments, motivated by the fact that convexity in the first parameter allows for minimization, making it useful to switch the order of the variables, see e.g., [1, 26] etc. We both leverage some of these results in our work, and contribute some, to the best of our knowledge, novel proof techniques and insights into the (primal and dual) Bregman geometry.

LLM decoding. There is a vast range of work on LLM sampling (or decoding), see e.g., [54] and references therein. Classical methods include greedy sampling and beam search. Sparse sampling methods such as top-k sampling [21] are motivated by intuition that the "unreliable tail" of low-probability tokens is mis-estimated [32]. In particular, [32] propose top-p sampling, and [38] propose min-p sampling. Other sampling methods were proposed in [4, 22, 31, 36]. [12] propose the decoding game, a two-player game between a generator/LLM and an adversary that distorts the true distribution. They show that certain sparse truncated sampling methods are approximately minimax optimal. There have also been various approaches to explicitly make language model output probabilities sparse, see e.g., [14, 59, 60]. In contrast, our goal is to develop a deeper theoretical understanding of the popular top-k decoding method, placing it into a broader framework.

General motivation. The motivation for our general approach is two-fold: (1) Without sparsity considerations, Bregman divergences are known to have a close correspondence to proper scoring rules, and are minimized at the true probability distribution, see e.g., [10, 24]. This property is highly desirable in probabilistic forecasting and prediction, ensuring that the forecaster is incentivized to predict the true distribution in order to minimize their loss. (2) The ℓ_0-"norm", i.e., the number of nonzero entries of a sparse vector, has been widely argued to both be a reasonable measure of sparsity, and to have good properties as a regularizer in certain sparse estimation problems such as sparse regression [see e.g., 7, 16, 23, 28, etc]. Combining these two lines of thought provides the motivation for studying ℓ_0-regularized Bregman divergence minimization.

Appendix D. Existence and uniqueness of dual Bregman decoding

Theorem 9 (Uniqueness and formula for dual Bregman renormalization) *Fix a dual valid potential ϕ. Then, for any $x \in \Delta_{\mathrm{sub},k}$ with $\sum_i x_i > 0$, the renormalization map T^*_ϕ is uniquely defined by:*

$$[T^*_\phi(x)]_i = x_i + \nu^* / f'([T^*_\phi(x)]_i) \quad \textit{for all } i \in [k], \textit{ where } \nu^* \in \mathbb{R} \textit{ is chosen so that } \sum_{i=1}^{k} [T^*_\phi(x)]_i = 1.$$

Proof

First, assume without loss of generality that $0 < \sum_{i\in[k]} x_i < 1$. Otherwise, if $\sum_{i\in[k]} x_i = 1$ then $x \in \Delta_k$, so the unique unconstrained optimum, which is at x by the standard property of Bregman divergences, is also the unique optimum of our constrained projection problem.

Note that Slater's condition is satisfied for this projection problem as we are optimizing over the simplex (whose relative interior is nonempty). Therefore, in this differentiable problem, its optimal solutions can be characterized via its KKT conditions.

Introduce a Lagrange multiplier $\nu \in \mathbb{R}$ for the simplex constraint, and Lagrange multipliers $(\lambda_i)_{i\in[k]}$ for the nonnegativity constraints. Then, the Lagrangian is as follows:

$$\mathcal{L}(\hat{p}, \nu) = \sum_{i=1}^{k} \big[\phi(x_i) - \phi(\hat{p}_i) - \phi'(\hat{p}_i)\,(x_i - \hat{p}_i)\big] - \nu\Big(\sum_{i=1}^{k} \hat{p}_i - 1\Big) - \sum_{i=1}^{k} \lambda_i \hat{p}_i.$$

Here, $\lambda_i \geqslant 0$ for all i, and by complementary slackness, at optimality $\lambda_i = 0$ whenever $\hat{p}_i > 0$.

For each $i \in [k]$, the stationarity condition reads (except possibly when $\hat{p}_i = 0$, where the second derivative could be infinite):

$$0 = \frac{\partial \mathcal{L}}{\partial \hat{p}_i} = -\phi''(\hat{p}_i)\,(x_i - \hat{p}_i) - \nu - \lambda_i \iff \phi''(\hat{p}_i)(\hat{p}_i - x_i) = \nu + \lambda_i.$$

In particular, for each coordinate i for which the optimal $\hat{p}_i \in (0,1)$, the stationarity condition is:

$$\phi''(\hat{p}_i)(\hat{p}_i - x_i) = \nu \implies \hat{p}_i = x_i + \frac{\nu}{\phi''(\hat{p}_i)} = x_i + \frac{\nu}{f'(\hat{p}_i)}. \tag{6}$$

Now, we show that $\nu > 0$. Indeed, observe that there must be at least one index i for which $\hat{p}_i > x_i$. If that was not the case, we would get $\sum_{i\in[k]} \hat{p}_i \leqslant \sum_{i\in[k]} x_i < 1$ by our assumption, contradicting that $\hat{p} \in \Delta_k$. In particular, then, $\hat{p}_i > x_i \geqslant 0$, and therefore we have $\phi''(\hat{p}_i)(\hat{p}_i - x_i) = \nu$. Since $\phi''(\hat{p}_i) > 0$ and $\hat{p}_i - x_i > 0$, we thus conclude that $\nu > 0$.

Having shown that $\nu > 0$, we now proceed to show that all $\hat{p}_i > 0$ at optimality. Note that $\frac{\partial}{\partial y}\mathrm{d}_\phi(x,y) = \phi''(y)\,(y-x)$ for $y > 0$. We will now consider two cases:

1. $\phi''(0)$ is finite;
2. $\lim_{y\to 0} \phi''(y) = +\infty$.

If $\phi''(0)$ is finite, $\hat{p}_i > 0$ for all i. Indeed, suppose that was not the case, and $\hat{p}_i = 0$ for some i. Then we would have: $\phi''(0)(0 - x_i) = \nu + \lambda_i$, or equivalently, $\phi''(0) \cdot x_i + \nu + \lambda_i = 0$. Each of the three terms is nonnegative, and $\nu > 0$, so we arrive at a contradiction.

Next, consider the case in which $\lim_{y\to 0} \phi''(y) = +\infty$. Then, $\lim_{y\to 0} \frac{\partial}{\partial y}\mathrm{d}_\phi(x,y) = -\infty$ for all $x \in (0,1]$. Then, since $\lim_{y\to 0} \frac{\partial}{\partial y}\mathrm{d}_\phi(x,y) = -\infty$ for all $x \in (0,1]$, for any i such that $x_i > 0$, setting $\hat{p}_i = 0$ would lead to $\nu = -\infty$, hence necessarily $\hat{p}_i > 0$. On the other hand, for any i for which $x_i = 0$, since $\lim_{y\to 0} y\phi''(y) = 0$, setting $\hat{p}_i = 0$ would lead to $\nu = 0$, which is a contradiction.

In all cases, the optimal $\hat{p}$ is in the strict interior of the simplex, so it suffices to solve (6) over this range. To show that the solution exists and is unique, we collect together the following information about Ψ from (12) with $\Psi(x,y,\nu) := \phi''(y)(y-x) - \nu$ for all x, y, ν. Then, for a fixed ν, (6) is equivalent to solving $\Psi(x_i, \hat{p}_i, \nu) = 0$. First, consider $x > 0$. Then, we have the following:

1. Since the map $y \mapsto \mathrm{d}_\phi(x,y)$ is strictly convex for $y \in [x,1]$, it follows that $\frac{\partial}{\partial y}\mathrm{d}_\phi(x,y) = \Psi(x,y,0)$ is strictly increasing for $y \in [x,1]$, and so is $\Psi(x,y,\nu)$.

2. We have $\Psi(x,x,\nu) = -\nu \leqslant 0$. Further, $\Psi(x,1,\nu) = \phi''(1)(1-x) - \nu \geqslant 0$, whenever $\nu \leqslant \phi''(1)(1-x)$.

Hence, the map $y \mapsto \Psi(x,y,\nu)$ has a unique zero on the interval $[x,1]$, as long as $0 < \nu \leqslant \phi''(1)(1-x)$.

Next, consider $x = 0$, in which case we need to solve the equation $\phi''(y)y = \nu$. Then, we have the following:

1. Since the map $y \mapsto \mathrm{d}_\phi(0,y)$ is strictly convex for $y \in (0,1]$, it follows that $\frac{\partial}{\partial y}\mathrm{d}_\phi(0,y) = \Psi(0,y,0) = \phi''(y)y$ is strictly increasing for $y \in (0,1]$, and so is $\Psi(0,y,\nu)$.

2. By assumption, $\lim_{y\to 0^+} y\phi''(y) = 0$, hence we have $\lim_{y\to 0^+} \Psi(x,x,\nu) = -\nu \leqslant 0$. Further, $\Psi(0,1,\nu) = \phi''(1)(1-x) - \nu \geqslant 0$, whenever $\nu \leqslant \phi''(1)$.

Hence, the map $y \mapsto \Psi(0,y,\nu)$ has a unique zero on the interval $(0,1]$, as long as $0 < \nu \leqslant \phi''(1)$.

Now define $M := \min_i \phi''(1)(1-x_i) = \phi''(1)(1 - \max_i x_i)$. Since by assumption $\sum_i x_i < 1$, it follows that $M > 0$. From the above analysis, it follows that, as long as $\nu \in (0,M]$, for each i, the equation $\phi''(y_i)(y_i - x_i) = \nu$. has a unique solution $y_i(\nu) \in (x_i, 1]$.

Furthermore, as we establish in Lemma 13, the map $\nu \mapsto y_i(\nu)$ is strictly increasing for $\nu > 0$, also owing to the assumed second-argument convexity of d_ϕ. In particular, define $G(\nu) = \sum_{i=1}^k y_i(\nu)$ for $\nu > 0$; then G is continuous and strictly increasing, and satisfies $\lim_{\nu \to 0} G(\nu) = \sum_i x_i < 1$ and $G(M) \geqslant y_{i^*}(M) = 1$, where i^* is any index achieving the maximum among the coordinates of x. Hence there is a unique $\nu^* \in (0, M]$ with $G(\nu^*) = 1$. Setting $\hat{p}_i = y_i(\nu^*)$ yields a vector in Δ_k that satisfies the KKT stationarity.

Finally, note that the solution $\hat{p}$ that we just identified is unique. Indeed, we have earlier excluded boundary solutions from consideration, and then further excluded any solutions in which $\hat{p}_i < x_i$ for any $i \in [k]$; thus, it suffices to recall that the Bregman objective is assumed to be strictly convex in the interior of the region of the simplex given by $\{\hat{p} \in \Delta_k : \hat{p}_i \geqslant x_i \text{ for all } i \in [k]\}$, thus concluding the proof. ■

Appendix E. Proof of the primal greedy property in Theorem 3

We will first fix some notations. Henceforth, we will assume that the vector p has been sorted, i.e., $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_V$. For any subset $Q = \{i_1, \ldots, i_k\} \subseteq [V]$ of size k, let $Q^{\mathsf{c}} = [V] \backslash Q$. Let p_Q denote the sub-probability vector with the entries of p whose indices are in Q. We define the loss $L(Q)$ as

$$L(Q) = \min_{\hat{p} \in \Delta_k} \mathrm{D}_\phi((\hat{p}, 0_{V-k}), (p_Q, p_{Q^{\mathsf{c}}})) = \min_{\hat{p} \in \Delta_k} \sum_{j=1}^k \mathrm{d}_\phi(\hat{p}_j, p_{i_j}) + S_{Q^{\mathsf{c}}}. \tag{7}$$

Here, $S_{Q^{\mathsf{c}}} = \sum_{j \notin Q} \mathrm{d}_\phi(0, p_j)$. To prove Theorem 3, we will show that $L(S') \geqslant L(S)$ for any $S' \subseteq [V]$ of size k, where $S = [k]$ consists of the top-k indices. We will further show that strict inequality always holds if $p_{S'} \neq p_S$. To do this, we proceed in three steps: (1) We first simplify the form of the loss function $L(Q)$ in Lemma 10, (2) For any two subsets S, S', we decompose the loss difference $L(S') - L(S)$ into three terms in Lemma 11, (3) We individually analyze each of the terms in this decomposition and prove they are non-negative.

E.1. Decomposing the Bregman cost function on subsets

Lemma 10 *For any $Q = \{i_1, i_2, \ldots i_k\} \subseteq [V]$ of size k, the loss function as defined in (7) simplifies to:*

$$L(Q) = \sum_{j=1}^k [\phi([T_Q(p)]_j) - \phi'(p_{i_j})[T_Q(p)]_j] + S_{[V]} - |Q|\phi(0). \tag{8}$$

Proof Observe that:

$$\begin{aligned} L(Q) &= \mathrm{D}_\phi((\hat{p}_Q, 0_{V-k}), (p_Q, p_{Q^{\mathsf{c}}})) = \sum_{j=1}^k \mathrm{d}([T_Q(p)]_j, p_{i_j}) + S_{Q^{\mathsf{c}}} \\ &= \sum_{j=1}^k [\phi([T_Q(p)]_j) - \phi(p_{i_j}) - \phi'(p_{i_j})([T_Q(p)]_j - p_{i_j})] + S_{Q^{\mathsf{c}}} \\ &= \sum_{j=1}^k [\phi([T_Q(p)]_j) - \phi'(p_{i_j})[T_Q(p)]_j] + \sum_{j=1}^k [-\phi(p_{i_j}) + f(p_{i_j})p_{i_j}] + S_{Q^{\mathsf{c}}}. \end{aligned}$$

This further equals

$$\begin{aligned}
&\sum_{j=1}^{k}[\phi([T_Q(p)]_j) - \phi'(p_{i_j})[T_Q(p)]_j] + \sum_{j\in Q} \mathrm{d}_\phi(0, p_j) + S_{Q^c} - |Q|\phi(0) \\
&= \sum_{j=1}^{k}[\phi([T_Q(p)]_j) - \phi'(p_{i_j})[T_Q(p)]_j] + S_Q + S_{Q^c} - |Q|\phi(0) \\
&= \sum_{j=1}^{k}[\phi([T_Q(p)]_j) - \phi'(p_{i_j})[T_Q(p)]_j] + S_{[V]} - |Q|\phi(0).
\end{aligned}$$

This finishes the proof. ■

Let $T_Q(p)$ denote a minimizer of the above loss $L(Q)$, i.e.,

$$T_Q(p) \in \arg\min_{\hat{p}\in\Delta_k} \mathrm{D}_\phi((\hat{p}, 0_{V-k}), (p_Q, p_{Q^c})) \overset{(a)}{=} \arg\min_{\hat{p}\in\Delta_k} \sum_{j=1}^{k} \mathrm{d}_\phi(\hat{p}_j, p_{i_j}).$$

Note that (a) holds above as the term S_{Q^c} does not play any role in the location of the minimizer. However, it does contribute to the final loss $L(Q)$. Also, as the divergence is separable, once we have selected a subset Q, the ordering of its elements does not matter for the calculation of the above loss and minimizer. Thus, without loss of generality, we may assume $i_1 < i_2 < \ldots < i_k$ for $k \in [V]$. By forming the Lagrangian and differentiating it, we obtain the primal thresholding from (3):

$$\phi'([T_Q(p)]_j) = \phi'(p_{i_j}) + \nu_Q \;\forall\, j \in [k]. \tag{9}$$

Here, ν_Q is chosen such that $\sum_{j=1}^{k}[T_Q(p)]_j = 1$.

Lemma 11 *Let $S = \{i_1, \ldots, i_k\}, S' = \{i'_1, \ldots, i'_k\} \subseteq [V]$ and $T_S(p)$ and $T_{S'}(p)$ be the corresponding minimizers. Then, the following decomposition holds:*

$$\begin{aligned}
L(S') - L(S) = \mathrm{D}_\phi(T_{S'}(p), T_S(p)) &+ \sum_{j=1}^{k} ([T_{S'}(p)]_j - [T_S(p)]_j)\left(\phi'([T_S(p)]_j) - \phi'(p_{i_j})\right) \\
&+ \sum_{j=1}^{k}[T_{S'}(p)]_j \left(\phi'(p_{i_j}) - \phi'(p_{i'_j})\right).
\end{aligned} \tag{10}$$

Proof We have from Lemma 10 that

$$\begin{aligned}
L(S') - L(S) &= \sum_{j=1}^{k}[\phi([T_{S'}(p)]_j) - \phi'(p_{i'_j})[T_{S'}(p)]_j] - \sum_{j=1}^{k}[\phi([T_S(p)]_j) - \phi'(p_{i_j})[T_S(p)]_j] \\
&= \sum_{j=1}^{k}[\phi([T_{S'}(p)]_j) - \phi([T_S(p)]_j)] + \phi'(p_{i_j})[T_S(p)]_j - \phi'(p_{i'_j})[T_{S'}(p)]_j.
\end{aligned}$$

This further equals

$$\begin{aligned}
&\sum_{j=1}^{k}[\phi([T_{S'}(p)]_j) - \phi([T_S(p)]_j) - \phi'([T_S(p)]_j)([T_{S'}(p)]_j - [T_S(p)]_j)] \\
&\qquad + \sum_{j=1}^{k}\left([T_{S'}(p)]_j\left[\phi'([T_S(p)]_j) - \phi'(p_{i'_j})\right] - [T_S(p)]_j\left[\phi'([T_S(p)]_j) - \phi'(p_{i_j})\right]\right) \\
&= \mathrm{D}_\phi(T_{S'}(p), T_S(p)) + \sum_{j=1}^{k}\left([T_{S'}(p)]_j - [T_S(p)]_j\right)\left(\phi'([T_S(p)]_j) - \phi'(p_{i_j})\right) \\
&\qquad + \sum_{j=1}^{k}[T_{S'}(p)]_j\left(\phi'(p_{i_j}) - \phi'(p_{i'_j})\right).
\end{aligned}$$

■

Now, returning to our proof, suppose $S = [k]$ and $S' = \{i'_1, \dots i'_k\}$. We know from Lemma 11 that

$$\begin{aligned}
L(S') - L(S) = \underbrace{\mathrm{D}_\phi(T_{S'}(p), T_S(p))}_{\mathbf{I}} &+ \underbrace{\sum_{j=1}^{k}\left([T_{S'}(p)]_j - [T_S(p)]_j\right)\left(\phi'([T_S(p)]_j) - \phi'(p_{i_j})\right)}_{\mathbf{II}} \\
&+ \underbrace{\sum_{j=1}^{k}[T_{S'}(p)]_j\left(\phi'(p_{i_j}) - \phi'(p_{i'_j})\right)}_{\mathbf{III}}.
\end{aligned}$$

Now, consider the term **II**. Using (9), we can simplify this further as follows:

$$\mathbf{II} = \sum_{j=1}^{k}\left([T_{S'}(p)]_j - [T_S(p)]_j\right)\nu_S = \nu_S\left(\sum_{j=1}^{k}[T_{S'}(p)]_j - \sum_{j=1}^{k}[T_S(p)]_j\right) \overset{(a)}{=} 0,$$

where (a) follows as $\sum_{j=1}^{k}[T_{S'}(p)]_j = \sum_{j=1}^{k}[T_S(p)]_j = 1$. Also, $\mathbf{I} \geqslant 0$ as D_ϕ is a divergence measure.

Finally, to conclude our proof, we show that $\mathbf{III} \geqslant 0$. Since the entries of p are sorted in a non-decreasing order and as the indices in $S = [k]$ and S' are sorted in ascending order, we have

$$\begin{aligned}
\forall\, j \in [k],\ j = i_j \leqslant i'_j \Rightarrow &\forall\, j \in [k],\ p(i_j) \geqslant p(i'_j) \\
&\Rightarrow \sum_{j=1}^{k}[T_{S'}(p)]_j\left(\phi'(p_{i_j}) - \phi'(p_{i'_j})\right) = \mathbf{III} \geqslant 0.
\end{aligned}$$

Strict inequality holds as long as some $p_{i'_j}$ is not among the top-k indices of p.

Appendix F. Proof of the dual greedy property in Theorem 4

To prove the greedy property for the two alternate conditions in Theorem 4, we will provide two distinct proof techniques for the two cases (A1) and (A2). The first one uses duality and the second one uses a saddle point argument. We will now recall the definition of the Legendre dual of a convex function—in this case, of the generator function ϕ—and its defining property that will help us. Below, $f([0,1])$ denotes the image of $[0,1]$ under f.

Lemma 12 (Classical) *For a valid ϕ, let $\phi^*(x) = \sup_{p\geqslant 0}\{px - \phi(p)\}$ be the Legendre dual of ϕ, defined for all $x \in f([0,1])$. Then, we have for every $x \in f([0,1])$ the identity: $\phi(f^{-1}(x)) = xf^{-1}(x) - \phi^*(x)$. Moreover $(\phi^*)' = f^{-1}$, and ϕ^* is strictly increasing.*

Proof Since the map $p \mapsto R(p) := px - \phi(p)$ is continuous, it achieves a maximum on $[0,1]$. From the first order condition of the defining equation for ϕ^*, if the maximum is achieved in $(0,1)$, we have:

$$\frac{\partial R}{\partial p} = x - \phi'(p) = x - f(p) = 0,$$

so for the maximizer $p_{\max}$ we have $f(p_{\max}) = x \Rightarrow p_{\max} = f^{-1}(x)$. Now, since f is increasing and $x \in f([0,1])$, we have $R'(0) = x - f(0) \geqslant 0$, with equality if $x = f(0)$. Similarly, $R'(1) = x - f(1) \leqslant 0$, with equality if $x = f(1)$. Hence, it follows that the above characterization for the maximizer $p_{\max}$ also applies on the boundaries of $[0,1]$. To conclude the proof of the identity, it suffices to observe that $\phi^*(x) = p_{\max}x - \phi(p_{\max}) = xf^{-1}(x) - \phi(f^{-1}(x))$. The expression for $(\phi^*)'$ follows by direct calculation. ■

F.1. Proof under Assumption (A1)

With the dual convex conjugate ϕ^* as per Lemma 12, the divergence measure satisfies:

$$\mathrm{d}_\phi(p,q) = \mathrm{d}_{\phi^*}(\phi'(q), \phi'(p)). \tag{11}$$

Let the loss for the dual problem be denoted as L^*, (the divergence measure with the arguments swapped), and let T^*_Q be the dual renormalization map from Lemma 9 applied to p_Q, i.e.,

$$L^*(Q) = \min_{\hat{p}\in\Delta_k} \mathrm{D}_\phi((p_Q, p_{Q^c}), (\hat{p}, 0_{V-k})) = \min_{\hat{p}\in\Delta_k} \sum_{j=1}^{k} \mathrm{d}_\phi(p_{i_j}, \hat{p}_j) + S^*_{Q^c}, \text{ where } S^*_{Q^c} = \sum_{j\notin Q} \mathrm{d}_\phi(p_j, 0)$$
$$= \sum_{j=1}^{k} \mathrm{d}_\phi(p_{i_j}, [T^*_Q(p)]_j) + S^*_{Q^c}.$$

F.1.1. DECOMPOSITION OF THE LOSS DIFFERENCE

Using the form of the loss difference in Lemma (11) and (11), we can compute the loss difference for the dual problem as follows:

$$L^*(S') - L^*(S) = \sum_{j=1}^{V} \mathrm{d}_\phi(p_{i'_j}, [T^*_{S'}(p)]_j) - \sum_{j=1}^{V} \mathrm{d}_\phi(p_{i_j}, [T^*_S(p)]_j)$$
$$\overset{\text{(due to (11))}}{=} \sum_{i=1}^{V} \mathrm{d}_{\phi^*}(\phi'([T^*_{S'}(p)]_j), \phi'(p_{i'_j})) - \sum_{i=1}^{V} \mathrm{d}_{\phi^*}(\phi'([T^*_S(p)]_j), \phi'(p_{i_j}))$$

Indeed, changing the potential ϕ to ϕ^*, and changing all the arguments $p_{i_j}, p_{i'_j}, T^*_S, T^*_{S'}$ to $\phi'(p_{i_j}), \phi'(p_{i'_j}), \phi'(T^*_S), \phi'(T^*_{S'})$ respectively in Lemma (11) suffices. Thus, under the same setup of the two subsets $S = [k]$ and S' and denoting $\phi' = f$, we obtain:

$$\begin{aligned} L^*(S') - L^*(S) = {} & \mathrm{D}_{\phi^*}\left(f(T^*_{S'}(p)), f(T^*_S(p))\right) \\ & + \sum_{j=1}^{k} \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \left((\phi^*)'\left(f([T^*_S(p)]_j)\right) - (\phi^*)'\left(f(p_{i_j})\right)\right) \\ & + \sum_{j=1}^{k} f\left([T^*_{S'}(p)]_j\right) \left((\phi^*)'(f(p_{i_j})) - (\phi^*)'(f(p_{i'_j}))\right). \end{aligned}$$

Since $(\phi^*)' = f^{-1}$, this further equals

$$\begin{aligned} & \underbrace{\mathrm{Div}_{\phi^*}\left(f(T^*_{S'}(p)), f(T^*_S(p))\right)}_{\mathbf{I}'} \\ & + \underbrace{\sum_{j=1}^{k} \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \left([T^*_S(p)]_j - p_{i_j}\right)}_{\mathbf{II}'} + \underbrace{\sum_{j=1}^{k} f\left([T^*_{S'}(p)]_j\right) \left(p_{i_j} - p_{i'_j}\right)}_{\mathbf{III}'}. \end{aligned}$$

F.1.2. ANALYSIS OF TERMS BASED ON THE DUAL SOLUTION

Similar to the proof for the primal case, the term $\mathbf{I}' \geqslant 0$, as D_{ϕ^*} is a divergence, and $\mathbf{III}' \geqslant 0$ as $\phi' = f \geqslant 0$, as $f(0) = 0$ and f is increasing. Moreover, as f is strictly increasing, if any of the $p_{i'_j}$ are not among the top-k entries, then strict inequality holds.

To analyze **II**, we have

$$\begin{aligned} \mathbf{II} &= \sum_{j=1}^{k} \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \left([T^*_S(p)]_j - p_{i_j}\right) \\ &\overset{\text{from Lemma 9}}{=} \sum_{j=1}^{k} \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \frac{\nu^*_S}{f'\left([T^*_S(p)]_j\right)}. \end{aligned}$$

Since f is convex,

$$\begin{aligned} & \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \geqslant f'\left([T^*_S(p)]_j\right) \left([T^*_{S'}(p)]_j - [T^*_S(p)]_j\right) \\ & \overset{(a)}{\Rightarrow} \frac{1}{f'\left([T^*_S(p)]_j\right)} \cdot \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \geqslant [T^*_{S'}(p)]_j - [T^*_S(p)]_j \\ & \overset{(b)}{\Rightarrow} \sum_{j=1}^{k} \frac{1}{f'\left([T^*_S(p)]_j\right)} \cdot \left(f([T^*_{S'}(p)]_j) - f([T^*_S(p)]_j)\right) \geqslant \sum_{j=1}^{k} \left([T^*_{S'}(p)]_j - [T^*_S(p)]_j\right) = 0. \end{aligned}$$

In the above steps, (a) follows as $f' > 0$ as f is strictly increasing and (b) follows as $\sum_{j=1}^{k} [T^*_{S'}(p)]_j = \sum_{j=1}^{k} [T^*_S(p)]_j = 1$. This implies $\mathbf{II}' \geqslant 0$, finishing the proof.

F.2. Proof under Assumption (A2)

F.2.1. EXTRA NOTATION

Since $\frac{\partial}{\partial y}\mathrm{d}_\phi(x,y) = \phi''(y)\,(y-x)$ for $y>0$, we define for $(x,y,\nu)\in D := [0,1]\times(0,1]\times(0,\infty)$,

$$\Psi(x,y,\nu) := \phi''(y)(y-x) - \nu. \tag{12}$$

Define the mapping derived from solving $\Psi(x,y,\nu)=0$ over y by:

$$\xi(x,\nu) : [0,1]\times(0,\infty)\to(0,1], \text{ such that } [T(p)]_i = \xi(p_i,\nu) \text{ for all } i, \text{ and for optimal } \nu.$$

It follows from the proof of Lemma 9 that the solution ξ is well-defined. Define two auxiliary functions ψ, h that will be used in the computation of the Bregman costs below, such that for all $(x,y,\nu)\in D$:

$$\psi(x,y) := \phi(y) - \phi'(y)(y-x), \qquad \text{and } h(x,\nu) := \psi(x,\xi(x,\nu)).$$

F.2.2. PROPERTIES OF THE AUXILIARY FUNCTIONS

Lemma 13 (Derivatives $\frac{\partial\xi}{\partial x}$, $\frac{\partial\xi}{\partial\nu}$) *Define $v : [0,1]\times(0,1]\to[0,\infty)$ as $v(x,y) = \phi''(y) + \phi'''(y)(y-x)$. We have for all $(x,\nu)\in[0,1]\times(0,\infty)$:*

$$\frac{\partial\xi}{\partial\nu}(x,\nu) = \frac{1}{v(x,\xi(x,\nu))}, \quad \textit{and} \quad \frac{\partial\xi}{\partial x}(x,\nu) = \frac{\phi''(\xi(x,\nu))}{v(x,\xi(x,\nu))}. \tag{13}$$

Proof The proof of either identity follows by applying implicit differentiation to the function Ψ. Fix $x\in[0,1]$ and consider

$$F(y,\nu) \;=\; \Psi(x,y,\nu) \;=\; \phi''(y)(y-x)-\nu \quad \text{for } (y,\nu)\in(0,1]\times(0,\infty).$$

Because ϕ is $\mathcal{C}^3$ on $(0,1]$, F is continuously differentiable, and

$$\frac{\partial F}{\partial y}(y,\nu) \;=\; \phi'''(y)(y-x)+\phi''(y) \;=\; v(x,y) \;>\; 0$$

by Assumption 3.2. Hence, by the implicit function theorem, the map $\nu\mapsto\xi(x,\nu)$ is $\mathcal{C}^1$ with

$$\frac{\partial\xi}{\partial\nu}(x,\nu) = -\frac{\partial F/\partial\nu}{\partial F/\partial y} = \frac{1}{v(x,\xi(x,\nu))}\ .$$

For the latter identity, fix $\nu>0$ and define

$$G(x,y) := \Psi(x,y,\nu) = \phi''(y)(y-x)-\nu, \qquad (x,y)\in[0,1]\times(0,1].$$

For each $x_0\in(0,1]$ let $y_0 := \xi(x_0,\nu)\in(0,1]$ satisfy $G(x_0,y_0)=0$. We have $\frac{\partial G}{\partial y}(x,y) = v(x,y)$. Assumption 3.2 gives $v(x,y)>0$ for all $0<y\leqslant 1$ and $0\leqslant x\leqslant y$. Hence $\partial G/\partial y(x_0,y_0)\neq 0$.

Since G is continuously differentiable and $\partial G/\partial y\neq 0$ at (x_0,y_0), the implicit-function theorem guarantees a C^1 map $x\mapsto\xi(x,\nu)$ in a neighborhood of x_0 with $G\big(x,\xi(x,\nu)\big)=0$.

Differentiating $G\big(x,\xi(x,\nu)\big)\equiv 0$ with respect to x and using $\partial G/\partial x = -\phi''(y)$ gives

$$0 = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\,\frac{\partial\xi}{\partial x} = -\phi''\big(\xi(x,\nu)\big) + v\big(x,\xi(x,\nu)\big)\frac{\partial\xi}{\partial x},$$

so

$$\frac{\partial \xi}{\partial x}(x,\nu) = \frac{\phi''\big(\xi(x,\nu)\big)}{v\big(x,\xi(x,\nu)\big)}.$$

When $x = 0$, the same argument applies, because $\frac{\partial G}{\partial y}(0,y) = v(0,y) > 0$ and $\partial G/\partial x|_{(0,y)} = -\phi''(y)$ is finite (the solution $y = \xi(0,\nu)$ is strictly positive, so $\phi''(y)$ is finite even if $\phi''(y) \to \infty$ as $y \downarrow 0$). Thus $\partial\xi/\partial x|_{(0,\nu)}$ exists and the same formula holds. This completes the proof. ■

Lemma 14 (Derivative $\frac{\partial h}{\partial \nu}$) *Under the condition that $x \mapsto u(x) := x\phi''(x)/\phi'(x)$ is non-decreasing from Assumption (A2), we have $\frac{\partial h}{\partial \nu}(x,\nu) \leqslant 0$ for all $x \in [0,1]$ and $\nu > 0$.*

Proof For the derivative with respect to ν, observe first that

$$\frac{\partial \psi}{\partial y}(x,y) = \phi'(y) - \big[\phi''(y)\,y + \phi'(y)\big] + x\,\phi''(y) = \phi''(y)\,(x-y).$$

Hence, by the chain rule,

$$\frac{\partial}{\partial \nu}\psi\big(x,\xi(x,\nu)\big) = \frac{\partial \psi}{\partial y}\big(x,\xi(x,\nu)\big)\,\frac{\partial \xi}{\partial \nu}(x,\nu) = \phi''\big(\xi(x,\nu)\big)\,[x-\xi(x,\nu)]\,\frac{\partial \xi}{\partial \nu}(x,\nu).$$

Due to the defining equation $\phi''(\xi)\,(\xi - x) = \nu$, this simplifies to

$$\frac{\partial h}{\partial \nu}(x,\nu) = \frac{\partial}{\partial \nu}\psi\big(x,\xi(x,\nu)\big) = -\nu\,\frac{\partial \xi}{\partial \nu}(x,\nu) = -\frac{\nu}{v\big(x,\xi(x,\nu)\big)} \quad \leqslant\ 0,$$

where the last equality uses $\dfrac{\partial \xi}{\partial \nu}(x,\nu) = \dfrac{1}{v\big(x,\xi(x,\nu)\big)}$ and $\nu > 0$. ■

Lemma 15 (Derivative $\frac{\partial h}{\partial x}$) *Assumption (A2) implies $\frac{\partial h}{\partial x}(x,\nu) \geqslant 0$ for all $x \in [0,1]$ and $\nu > 0$.*

Proof First recall that

$$\psi(x,y) = \phi(y) - \phi'(y)\,(y-x) \quad \Longrightarrow \quad \frac{\partial \psi}{\partial x}(x,y) = \phi'(y), \qquad \frac{\partial \psi}{\partial y}(x,y) = \phi''(y)\,(x-y).$$

Hence, with $y = \xi(x,\nu)$,

$$\frac{\partial h}{\partial x}(x,\nu) = \frac{\partial \psi}{\partial x}\big(x,\xi\big) + \frac{\partial \psi}{\partial y}\big(x,\xi\big)\,\frac{\partial \xi}{\partial x}(x,\nu) = \phi'(\xi) + \phi''(\xi)\,[x-\xi]\,\frac{\partial \xi}{\partial x}(x,\nu).$$

Because $\xi = \xi(x,\nu)$ satisfies $\phi''(\xi)\,(\xi - x) = \nu$, we have

$$\frac{\partial h}{\partial x}(x,\nu) = \phi'(\xi) - \nu\,\frac{\partial \xi}{\partial x}(x,\nu) = \phi'(\xi) - \nu\,\frac{\phi''(\xi)}{v\big(x,\xi\big)}.$$

Write

$$N(x,\nu) \ = \ \phi'(\xi)\,\phi''(\xi) \ + \ (\xi - x)\,\big[\,\phi'(\xi)\,\phi'''(\xi) - \phi''(\xi)^2\big] \ = \ \phi'(\xi)\,\phi''(\xi) + (\xi - x)\,A(\xi),$$

where $A(t) := \phi'(t)\phi'''(t) - \phi''(t)^2$.

Case 1: $A(\xi) \geqslant 0$. Because $\xi \geqslant x$ from Lemma 9, the second term is non-negative; with $\phi', \phi'' \geqslant 0$ the first term is also non-negative, so $N \geqslant 0$.

Case 2: $A(\xi) < 0$. Since $\xi \geqslant x$, we have

$$N(x,\nu) \;\geqslant\; \phi'(\xi)\phi''(\xi) + \xi\, A(\xi) \;=\; \phi'(\xi)^2\, u'(\xi),$$

where $u(t) := t\,\phi''(t)/\phi'(t)$. Indeed,

$$u'(t)\,\phi'(t)^2 = \phi'(t)\big[\phi''(t) + t\,\phi'''(t)\big] - t\,\phi''(t)^2 = \phi'(t)\phi''(t) + t\big[\phi'(t)\phi'''(t) - \phi''(t)^2\big].$$

By Assumption (A2), u is non-decreasing, so $u'(\xi) \geqslant 0$; hence $N(x,\nu) \geqslant 0$ in this case as well.

Because $v(x,\xi) > 0$ and $N(x,\nu) \geqslant 0$ in both cases, we conclude $\partial h(x,\nu)/\partial x \geqslant 0$ for all $x \in [0,1]$ and $\nu > 0$, thereby proving the lemma. ■

F.2.3. PROVING THE DUAL GREEDY PROPERTY

Denote an arbitrary subset of the indices by: $S \subseteq [J]$. Let ν_S be the corresponding Lagrange multiplier. Below, for a vector $x \in \mathbb{R}^V$ and a set $S \subset [V]$, we denote by $x[S]$ the sub-vector of x restricted to the coordinates in S. Since $\phi'(0) = 0$ by the assumptions of Theorem 4, denoting $\Gamma = \sum_{m=1}^{J} \mathrm{d}_\phi(p_m, 0) + \phi(0)|S|$ we can write for every S:

$$\begin{aligned}
\mathrm{D}_\phi(p, \hat{p}[S]) &= \sum_{m\in S} \phi(p_m) - \phi([T(p)]_m) - \phi'([T(p)]_m)\cdot(p_m - [T(p)]_m) + \sum_{m\in[J]\setminus S} \mathrm{d}_\phi(p_m, 0) \\
&= \sum_{m\in S} -\big(\phi([T(p)]_m) - \phi'([T(p)]_m)\cdot([T(p)]_m - p_m)\big) + \Gamma \\
&= \sum_{m\in S} -\psi(p_m, [T(p)]_m) + \Gamma = \sum_{m\in S} -h(p_m, \nu_S) + \Gamma.
\end{aligned}$$

Now, let us prove that the greedy property holds. Suppose S is optimal among all subsets of indices of size k but does not consist of some of the top k probability tokens. Then there exist some $i \neq j$ such that $i \in S$, $j \notin S$, and $p_j > p_i$. Denote $S' = S \setminus \{i\} \cup \{j\}$.

Let $\nu_S, \nu_{S'}$ denote the choice of ν that makes the projected probabilities sum to unity. Now since S' only differs from S in that it includes the larger $p_j > p_i$, we can conclude that $\nu_S > \nu_{S'}$.

Then, using the above formula for the value of the objective function on an arbitrary subset, we have:

$$\mathrm{D}_\phi(p, \hat{p}[S]) - \mathrm{D}_\phi(p, \hat{p}[S']) = h(p_j, \nu_{S'}) - h(p_i, \nu_S) + \sum_{m\in S\setminus\{i\}} \left(h(p_m, \nu_{S'}) - h(p_m, \nu_S)\right).$$

Now, since h decreases in ν by Lemma 14, we have that the sum is nonnegative since $\nu_{S'} < \nu_S$. As for the remaining term, we have:

$$h(p_j, \nu_{S'}) \geqslant h(p_j, \nu_S) \geqslant h(p_i, \nu_S),$$

where the first inequality is by the fact that $\nu_{S'} < \nu_S$ and Lemma 14, and the second inequality is by the fact that $p_j > p_i$ and Lemma 15. This concludes the proof of the dual greedy property under Assumption (A2).

Appendix G. Proof of discrete convexity for primal Bregman projection

We follow the notations that were introduced in the beginning of the proof in Section E. To show that the cost function is discretely convex in k for the primal, it suffices to show that

$$L([k]) := \min_{\hat{p}\in\Delta_k} \mathrm{D}_\phi((\hat{p}, 0_{V-k}), p) = \mathrm{D}_\phi((T_{[k]}(p), 0_{V-k}), p)$$

is discretely convex in k. Indeed, the difference $\mathrm{cost}(k) - L([k]) = \lambda k$ is linear in k.

To simplify notation, let us denote $L([k])$ by $L(k)$ and $T_{[k]}$ by T_k. From Lemma (10) we know that with $\tilde{S}_V := S_{[V]} - k\phi(0)$

$$L(k) = \sum_{j=1}^{k} \left\{\phi([T_k(p)]_j) - \phi'(p_j)[T_k(p)]_j\right\} + \tilde{S}_V.$$

Using (9), we know that $f([T_k(p)]_j) = f(p_j) + \nu_{[k]}\ \forall\, j \in [k]$. Again, we simply denote $\nu_{[k]}$ as ν_k. For $j \in [k]$, letting $x = f(p_j) + \nu_k$ in Lemma 12, we have:

$$\begin{aligned}
&\phi([T_k(p)]_j) - \phi'(p_j)[T_k(p)]_j = \phi(f^{-1}(f(p_j) + \nu_k)) - f(p_j)f^{-1}(f(p_j) + \nu_k)\\
&= \phi(f^{-1}(x)) - f(p_j)f^{-1}(x) = xf^{-1}(x) - \phi^*(x) - f(p_j)f^{-1}(x)\\
&= (x - f(p_j))f^{-1}(x) - \phi^*(x) = \nu_k[T_k(p)]_j - \phi^*(f(p_j) + \nu_k).
\end{aligned}$$

But now, using that the nonzero entries of $T_k(p)$ must sum to unity, we find the following simplification:

$$\begin{aligned}
L(k) &= \sum_{j=1}^{k} \{\nu_k[T_k(p)]_j - \phi^*(f(p_j) + \nu_k)\} + \tilde{S}_V\\
&= \nu_k \sum_{j=1}^{k} [T_k(p)]_j - \sum_{j=1}^{k} \phi^*(f(p_j) + \nu_k) + \tilde{S}_V = \nu_k - \sum_{j=1}^{k} \phi^*(f(p_j) + \nu_k) + \tilde{S}_V. \qquad (14)
\end{aligned}$$

Now, define the auxiliary function W for all j, ν for which the expression below is well defined:

$$W(k, \nu) := \nu - \sum_{j=1}^{k} \phi^*(f(p_j) + \nu), \qquad (15)$$

where p is implicitly kept fixed. From the above calculation, we thus obtain after canceling out terms:

$$L(k+1) - 2L(k) + L(k-1) = W(k+1, \nu_{k+1}) - 2W(k, \nu_k) + W(k-1, \nu_{k-1}).$$

To prove that this is nonnegative, we leverage that $W(k, \cdot)$ is strictly concave in ν for each k, which follows as the Legendre dual mapping ϕ^* is strictly convex since so is ϕ. Then, observe that for every j,

$$\frac{\partial}{\partial \nu} W(k, \nu) = 1 - \sum_{j=1}^{k} (\phi^*)'(f(p_i) + \nu) = 1 - \sum_{j=1}^{k} f^{-1}(f(p_j) + \nu). \qquad (16)$$

Thus,

$$\frac{\partial}{\partial \nu} W(k,\nu) \mid_{\nu=\nu_k} = 1 - \sum_{j=1}^{k} f^{-1}(f(p_j)+\nu_k) = 1 - \sum_{j=1}^{k} [T_k(p)]_j = 0.$$

As $W(k,\cdot)$ is strictly concave in ν, $W(k,\cdot)$ is maximized at ν_k. Thus, we have: (1) $W(k+1,\nu_{k+1}) \geqslant W(k+1,\nu_k)$, and (2) $W(k-1,\nu_{k-1}) \geqslant W(k-1,\nu_k)$. With these in hand, we have:

$$\begin{aligned} L(k+1) - 2L(k) + L(k-1) &= W(k+1,\nu_{k+1}) - 2W(k,\nu_k) + W(k-1,\nu_{k-1}) \\ &\geqslant [W(k+1,\nu_k) - W(k,\nu_k)] - [W(k,\nu_k) - W(k-1,\nu_k)]. \end{aligned} \tag{17}$$

Now, due to the definition of W, the last display equals

$$-\phi^*(f(p_{k+1})+\nu_k) + \phi^*(f(p_k)+\nu_k) \geqslant 0, \tag{18}$$

the inequality holding as $p_k \geqslant p_{k+1}$, and as the mapping $p \mapsto \phi^*(f(p)+\nu_k)$ is increasing in p since so are ϕ^* and f. This concludes the proof.

Appendix H. Proof of discrete convexity for dual Bregman projection

We denote $\theta_x(y) = \phi''(y)(y-x)$. As observed before, we have for all admissible x, y that

$$\frac{\partial}{\partial y} \mathrm{d}_\phi(x,y) = \theta_x(y),$$

and the convexity condition for the second argument of d_ϕ of Assumption 3.2 is given by:

$$\frac{\partial}{\partial y}\theta_x(y) \geqslant 0 \Leftrightarrow \phi''(y) + \phi'''(y)(y-x) \geqslant 0 \quad \text{for all } y \geqslant x \geqslant 0.$$

The dual projection for any $1 \leqslant i \leqslant j \leqslant V$ is given (for optimal Lagrange multiplier ν_j) by:

$$\theta_{p_i}([T_j^*(p)]_i) = \nu_j \Leftrightarrow \phi''([T_j^*(p)]_i)([T_j^*(p)]_i - p_i) = \nu_j.$$

Denote the dual Bregman objective, as a function of the selected sparsity k, as:

$$\mathrm{cost}^*(k) = \mathrm{D}_\phi\left(p, (T_k^*(p), 0_{V-k})\right) + \lambda k.$$

We now demonstrate that $\mathrm{cost}^*(k)$ is discretely convex in k. For this, we will directly show that the second-order differences of this function are nonnegative at every $k \in \{2,\ldots,V-1\}$. Specifically, we can write:

$$\begin{aligned} &\Delta^{*,2}(k) := \mathrm{cost}^*(k+1) - 2\mathrm{cost}^*(k) + \mathrm{cost}^*(k-1) \\ &= \mathrm{D}_\phi\left(p, \left(T_{k+1}^*(p), 0_{V-k-1}\right)\right) - 2\mathrm{D}_\phi\left(p, (T_k^*(p), 0_{V-k})\right) + \mathrm{D}_\phi\left(p, \left(T_{k-1}^*(p), 0_{V-k+1}\right)\right) \end{aligned}$$

We now decompose this quantity into three terms corresponding to three ranges of index $i \in [V]$, namely $i \in [k-1]$, $i \in \{k, k+1\}$, and $i \in \{k+2, \ldots, V\}$. We obtain:

$$\begin{aligned}
\Delta^{*,2}(k) = &\sum_{i=1}^{k-1} \Big\{ \big\{ \mathrm{d}_\phi(p_i, [T^*_{k+1}(p)]_i) - \mathrm{d}_\phi(p_i, [T^*_k(p)]_i) \big\} + \big\{ \mathrm{d}_\phi(p_i, [T^*_{k-1}(p)]_i) - \mathrm{d}_\phi(p_i, [T^*_k(p)]_i) \big\} \Big\} \\
&+ \Big\{ \big(\phi(p_k) - \phi(0) - \phi'(0) \cdot p_k\big) - 2\left(\phi(p_k) - \phi([T^*_k(p)]_k) - \phi'([T^*_k(p)]_k) \cdot (p_k - [T^*_k(p)]_k)\right) \\
&+ \big(\phi(p_k) - \phi([T^*_{k+1}(p)]_k) - \phi'([T^*_{k+1}(p)]_k) \cdot (p_k - [T^*_{k+1}(p)]_k)\big) \\
&+ \big(\phi(p_{k+1}) - \phi(0) - \phi'(0) \cdot p_{k+1}\big) - 2\left(\phi(p_{k+1}) - \phi(0) - \phi'(0) \cdot p_{k+1}\right) \\
&+ \big(\phi(p_{k+1}) - \phi([T^*_{k+1}(p)]_{k+1}) - \phi'([T^*_{k+1}(p)]_{k+1}) \cdot (p_{k+1} - [T^*_{k+1}(p)]_{k+1})\big) \Big\} \\
&- \sum_{i=k+2}^{V} \{\mathrm{d}_\phi(p_i, 0) - 2\mathrm{d}_\phi(p_i, 0) + \mathrm{d}_\phi(p_i, 0)\}.
\end{aligned}$$

The last sum is identically zero, so we engage with the other two ranges of indices.

Range 1: $i \in [k-1]$**.** For Range 1, recall that for any convex function ψ, it holds for any two points x, y in its domain that $\psi(x) - \psi(y) \geqslant \psi'(y)(x-y)$. Now, notice that for each i in Range 1, each of the two terms in figure brackets can be bounded via the convexity of $\mathrm{d}_\phi(x, \cdot)$ in its second argument as:

$$\begin{aligned}
&\mathrm{d}_\phi(p_i, [T^*_{k+1}(p)]_i) - \mathrm{d}_\phi(p_i, [T^*_k(p)]_i) \geqslant \left(\frac{\partial}{\partial y} \mathrm{d}_\phi(p_i, y)\right)\Big|_{y=[T^*_k(p)]_i} \cdot \big([T^*_{k+1}(p)]_i - [T^*_k(p)]_i\big) \\
&= \theta_{p_i}\left([T^*_k(p)]_i\right) \cdot \big([T^*_{k+1}(p)]_i - [T^*_k(p)]_i\big) = \nu_k \cdot \big([T^*_{k+1}(p)]_i - [T^*_k(p)]_i\big)
\end{aligned}$$

and:

$$\begin{aligned}
&\mathrm{d}_\phi(p_i, [T^*_{k-1}(p)]_i) - \mathrm{d}_\phi(p_i, [T^*_k(p)]_i) \geqslant \left(\frac{\partial}{\partial y} \mathrm{d}_\phi(p_i, y)\right)\Big|_{y=[T^*_k(p)]_i} \cdot \big([T^*_{k-1}(p)]_i - [T^*_k(p)]_i\big) \\
&= \theta_{p_i}\left([T^*_k(p)]_i\right) \cdot \big([T^*_{k-1}(p)]_i - [T^*_k(p)]_i\big) = \nu_k \cdot \big([T^*_{k-1}(p)]_i - [T^*_k(p)]_i\big).
\end{aligned}$$

As a result, we may simplify the Range 1 sum as follows, using that by definition, the first j terms in the projection T^*_j for each $j \in \{k-1, k, k+1\}$ sum to unity:

$$\begin{aligned}
\text{Range 1 Sum} &\geqslant \sum_{i=1}^{k-1} \nu_k \cdot \big(\big\{[T^*_{k+1}(p)]_i - [T^*_k(p)]_i\big\} + \big\{[T^*_{k-1}(p)]_i - [T^*_k(p)]_i\big\}\big) \\
&= \nu_k \left(\sum_{i=1}^{k-1} [T^*_{k+1}(p)]_i - 2\sum_{i=1}^{k-1} [T^*_k(p)]_i + \sum_{i=1}^{k-1} [T^*_{k-1}(p)]_i\right) \\
&= \nu_k \left(\big(1 - [T^*_{k+1}(p)]_k - [T^*_{k+1}(p)]_{k+1}\big) - 2(1 - [T^*_k(p)]_k) + 1\right) \\
&= \nu_k \left(2[T^*_k(p)]_k - [T^*_{k+1}(p)]_k - [T^*_{k+1}(p)]_{k+1}\right).
\end{aligned}$$

Range 2: $i \in \{k, k+1\}$**.** For Range 2, we first note that the following three types of terms cancel out: $\phi(0)$, $\phi(p_k)$, $\phi(p_{k+1})$. Furthermore, terms involving $\phi'(0)$ vanish by assumption.

The remaining terms in the Range 2 sum can then be written as:

$$\begin{aligned}\text{Range 2 Sum} \geqslant \Big\{ &-2\left(-\phi([T_k^*(p)]_k) - \phi'([T_k^*(p)]_k)\cdot(p_k - [T_k^*(p)]_k)\right)\\ &+\left(-\phi([T_{k+1}^*(p)]_k) - \phi'([T_{k+1}^*(p)]_k)\cdot(p_k - [T_{k+1}^*(p)]_k)\right)\Big\}\\ &+\Big\{-\phi([T_{k+1}^*(p)]_{k+1}) - \phi'([T_{k+1}^*(p)]_{k+1})\cdot(p_{k+1} - [T_{k+1}^*(p)]_{k+1})\Big\}.\end{aligned}$$

Now, we can bound

$$-\phi'([T_{k+1}^*(p)]_{k+1})\cdot p_{k+1} \geqslant -\phi'([T_{k+1}^*(p)]_{k+1})\cdot p_k,$$

using that $p_k \geqslant p_{k+1}$ and the strict convexity of ϕ. We find the lower bound

$$\begin{aligned}\text{Range 2 Sum} \geqslant -2\Big\{ &-\phi([T_k^*(p)]_k) - \phi'([T_k^*(p)]_k)\cdot(p_k - [T_k^*(p)]_k)\Big\}\\ &+\Big\{-\phi([T_{k+1}^*(p)]_k) - \phi'([T_{k+1}^*(p)]_k)\cdot(p_k - [T_{k+1}^*(p)]_k)\Big\}\\ &+\Big\{-\phi([T_{k+1}^*(p)]_{k+1}) - \phi'([T_{k+1}^*(p)]_{k+1})\cdot(p_k - [T_{k+1}^*(p)]_{k+1})\Big\}.\end{aligned}$$

By adding and subtracting the term $\phi(p_k)$ twice, we have the following equivalent bound:

$$\begin{aligned}\text{Range 2 Sum} \geqslant -2\Big\{&\phi(p_k) - \phi([T_k^*(p)]_k) - \phi'([T_k^*(p)]_k)\cdot(p_k - [T_k^*(p)]_k)\Big\}\\ &+\Big\{\phi(p_k) - \phi([T_{k+1}^*(p)]_k) - \phi'([T_{k+1}^*(p)]_k)\cdot(p_k - [T_{k+1}^*(p)]_k)\Big\}\\ &+\Big\{\phi(p_k) - \phi([T_{k+1}^*(p)]_{k+1}) - \phi'([T_{k+1}^*(p)]_{k+1})\cdot(p_k - [T_{k+1}^*(p)]_{k+1})\Big\}\\ &= -2\mathrm{d}_\phi\left(p_k, [T_k^*(p)]_k\right) + \mathrm{d}_\phi\left(p_k, [T_{k+1}^*(p)]_k\right) + \mathrm{d}_\phi\left(p_k, [T_{k+1}^*(p)]_{k+1}\right).\end{aligned}$$

Returning to the main bound We can now merge the cases, resulting in the following tight lower bound of the second differential of the cost function:

$$\begin{aligned}\Delta^{*,2}(k) \geqslant\quad &\nu_k\left(2[T_k^*(p)]_k - [T_{k+1}^*(p)]_k - [T_{k+1}^*(p)]_{k+1}\right)\\ -\quad &2\mathrm{d}_\phi\left(p_k, [T_k^*(p)]_k\right) + \mathrm{d}_\phi\left(p_k, [T_{k+1}^*(p)]_k\right) + \mathrm{d}_\phi\left(p_k, [T_{k+1}^*(p)]_{k+1}\right).\end{aligned}$$

Now, define the following key auxiliary function $\psi_k : [0,1] \to \mathbb{R}$, such that for all $x \in [0,1]$:

$$\psi_k(x) = \nu_k \cdot x - \mathrm{d}_\phi(p_k, x).$$

This lets us rewrite our lower bound equivalently as:

$$\Delta^{*,2}(k) \geqslant 2\,\psi\left([T_k^*(p)]_k\right) - \psi\left([T_{k+1}^*(p)]_k\right) - \psi\left([T_{k+1}^*(p)]_{k+1}\right). \tag{19}$$

We now establish a monotonicity property for ψ_k.

Lemma 16 *For every $k \in [V]$ the function $\psi_k(x)$ is increasing on $x \in [0, [T_k^*(p)]_k]$.*

Proof We consider the derivative of the function ψ_k:

$$\frac{\partial}{\partial x}\psi_k(x) = \nu_k - \frac{\partial}{\partial x}\mathrm{d}_\phi(p_k, x) = \nu_k - \theta_{p_k}(x) = \theta_{p_k}\left([T_k^*(p)]_k\right) - \theta_{p_k}(x),$$

where we have used the connection between $\theta_x(y)$ and ν_k (see Lemma 9).

Now, recalling that by assumption, $\frac{\partial}{\partial y}\theta_x(y) \geqslant 0$ for all $y \geqslant x \geqslant 0$, and using that $[T_k^*(p)]_k \geqslant p_k$ by the properties of the dual projection method (see Lemma 9), we have that:

$$\frac{\partial}{\partial x}\psi_k(x) = \theta_{p_k}\left([T_k^*(p)]_k\right) - \theta_{p_k}(x) \geqslant 0,$$

so long as $0 \leqslant x \leqslant [T_k^*(p)]_k$. ■

Continuing, by the properties of the dual projection, we have:

$$[T_k^*(p)]_k \geqslant [T_{k+1}^*(p)]_k \geqslant [T_{k+1}^*(p)]_{k+1}.$$

In view of Lemma 16, (19) implies that

$$\Delta^{*,2}(k) \geqslant \left[\psi\left([T_k^*(p)]_k\right) - \psi\left([T_{k+1}^*(p)]_k\right)\right] + \left[\psi\left([T_k^*(p)]_k\right) - \psi\left([T_{k+1}^*(p)]_{k+1}\right)\right] \geqslant 0 + 0 = 0.$$

This concludes the proof of dual discrete convexity of the Bregman cost function.

Appendix I. Algorithmic details

I.1. Computing the dual renormalization map

Recall that when ϕ is dual valid, the renormalization map T_ϕ^* is uniquely defined for $x \in \Delta_{\mathrm{sub},k}$ with $\sum_i x_i > 0$ by the fixed point equation (see Lemma 9)

$$[T_\phi^*(x)]_i = x_i + \nu^*/f'([T_\phi^*(x)]_i) \quad \text{for all } i \in [k], \text{ where } \nu^* \in \mathbb{R} \text{ is chosen so that } \sum_{i=1}^{k}[T_\phi^*(x)]_i = 1.$$

To compute T_ϕ^*, recall from Section F.2.1 the function Ψ from (12) with $\Psi(x, y, \nu) := \phi''(y)(y - x) - \nu$ for all x, y, ν. Then, for a fixed ν, $[T(x)]_i$ satisfying the equation $[T(x)]_i = x_i + \nu/f'([T(x)]_i)$ is equivalent to solving $\Psi(x_i, y_i, \nu) = 0$ for $y_i = [T(x)]_i$. The monotonicity properties from Lemma 9 then suggest the following algorithm, consisting of a binary search over $\nu \in (0, M]$, and then over each coordinate of T solving $\phi''([T(x)]_i)([T(x)]_i - x_i) = \nu$.

Algorithm 1 Dual Renormalization Map $T^*_\phi(x)$ via Nested Binary Search

Require: Convex generator ϕ with derivatives $f = \phi'$, $f'' = \phi''$; input vector $x \in \Delta_{\text{sub},k}$ with $\sum x_i < 1$; tolerance $\varepsilon > 0$
Ensure: Renormalized vector $\hat{p} = T^*_\phi(x) \in \Delta_k$

```
function DualRenormalize(x, φ, ε)
    k ← length of x
    f'' ← φ''
    M ← φ''(1) · (1 − max_i x_i)                ▷ Upper bound on feasible ν
    Initialize ν_low ← 0, ν_high ← M
    while ν_high − ν_low > ε do
        ν ← (ν_low + ν_high)/2
        for i = 1 to k do
            x_i ← x[i]
            y[i] ← SolveRoot(x_i, ν, f'', ε)
        end for
        G ← Σ_{i=1}^{k} y[i]
        if G < 1 then
            ν_low ← ν
        else
            ν_high ← ν
        end if
    end while
    return y
end function
function SolveRoot(x_i, ν, f'', ε)
    a ← x_i, b ← 1
    while b − a > ε do
        m ← (a + b)/2
        Ψ ← f''(m) · (m − x_i) − ν
        if Ψ < 0 then
            a ← m
        else
            b ← m
        end if
    end while
    return (a + b)/2
end function
```

I.2. Pseudocode for algorithms

See Algorithm 3 and Algorithm 4 for pseudocode for sparse primal (resp. dual) Bregman decoding.

Algorithm 2 Discrete Binary Search for Unimodal Cost Minimization

Require: Callable function COMPUTECOST, maximum support size V
Ensure: Optimal support size k^* minimizing COMPUTECOST(k)

```
function BINARYSEARCH(COMPUTECOST, V)
    c_1 ← COMPUTECOST(1)
    c_2 ← COMPUTECOST(2)
    if c_2 − c_1 ⩾ 0 then
        return 1
    end if
    c_{V−1} ← COMPUTECOST(V − 1)
    c_V ← COMPUTECOST(V)
    if c_V − c_{V−1} ⩽ 0 then
        return V
    end if
    Initialize L ← 1, R ← V
    while R − L > 1 do
        m ← ⌊(L + R)/2⌋
        c_m ← COMPUTECOST(m)
        c_{m+1} ← COMPUTECOST(m + 1)
        if c_{m+1} − c_m ⩾ 0 then
            R ← m
        else
            L ← m
        end if
    end while
    return R
end function
```

Algorithm 3 Regularized Sparse Primal Bregman Decoding

Require: Probability vector $p \in \Delta_V$, valid convex generator ϕ, sparsity penalty $\lambda \geqslant 0$
Ensure: Sparse decoded distribution $\hat{p} \in \Delta_V$

function SPARSEPRIMALBREGMANDECODE(p, ϕ, λ)
  Sort p in descending order: $p_{(1)} \geqslant p_{(2)} \geqslant \cdots \geqslant p_{(V)}$
  Define $f = \phi'$
  function COMPUTERENORMALIZATION($x \in \mathbb{R}^k$)
    Solve for $\nu \in \mathbb{R}$ such that $\sum_{i=1}^{k} f^{-1}(f(x_i) + \nu) = 1$
    return $\hat{p}^{(k)}$ with $[\hat{p}^{(k)}]_i = f^{-1}(f(x_i) + \nu)$ for $i \in [k]$
  end function
  function COMPUTECOST(k)
    Let $x = p[1{:}k]$
    $\hat{p}^{(k)} \leftarrow$ COMPUTERENORMALIZATION(x)
    Pad with zeros: $\hat{p}^{(k)} \leftarrow (\hat{p}_1^{(k)}, \ldots, \hat{p}_k^{(k)}, 0, \ldots, 0)$
    Compute $\mathrm{D}_\phi(\hat{p}^{(k)}, p) = \sum_{i=1}^{V} \left[\phi(\hat{p}_i^{(k)}) - \phi(p_i) - f(p_i)(\hat{p}_i^{(k)} - p_i)\right]$
    return $\mathrm{cost}(k) = \mathrm{D}_\phi(\hat{p}^{(k)}, p) + \lambda k$
  end function
  $k^* \leftarrow$ BINARYSEARCH(ComputeCost, V)
  Recompute $\hat{p}^{(k^*)}$ using COMPUTERENORMALIZATION($p[1{:}k^*]$)
  Pad with zeros to full length V
  return $\hat{p}^{(k^*)}$
end function

Algorithm 4 Regularized Sparse Dual Bregman Decoding

Require: Probability vector $p \in \Delta_V$, valid convex generator ϕ, sparsity penalty $\lambda \geqslant 0$
Ensure: Sparse decoded distribution $\hat{p} \in \Delta_V$

function SPARSEDUALBREGMANDECODE(p, ϕ, λ)
  Sort p in descending order: $p_{(1)} \geqslant p_{(2)} \geqslant \cdots \geqslant p_{(V)}$
  Define $f = \phi', \quad f' = \phi''$
  function COMPUTEDUALRENORMALIZATION($x \in \mathbb{R}^k$)
    Solve for $\nu \in \mathbb{R}$ such that: $\sum_{i=1}^{k} [T_\phi^*(x)]_i = 1$, where $[T_\phi^*(x)]_i$ satisfies the fixed-point equation: $[T_\phi^*(x)]_i = x_i + \nu / f'([T_\phi^*(x)]_i)$.
    return $\hat{p}^{(k)} = T_\phi^*(x)$
  end function
  function COMPUTEDUALCOST(k)
    Let $x = p[1{:}k]$
    $\hat{p}^{(k)} \leftarrow$ COMPUTEDUALRENORMALIZATION(x)
    Pad with zeros: $\hat{p}^{(k)} \leftarrow (\hat{p}_1^{(k)}, \ldots, \hat{p}_k^{(k)}, 0, \ldots, 0)$
    Compute $\mathrm{D}_\phi(p, \hat{p}^{(k)}) = \sum_{i=1}^{V} \left[\phi(p_i) - \phi(\hat{p}_i^{(k)}) - f(\hat{p}_i^{(k)})(p_i - \hat{p}_i^{(k)}) \right]$
    return $\mathrm{cost}(k) = \mathrm{D}_\phi(p, \hat{p}^{(k)}) + \lambda k$
  end function
  $k^* \leftarrow$ BINARYSEARCH(ComputeDualCost, V)
  Recompute $\hat{p}^{(k^*)}$ using COMPUTEDUALRENORMALIZATION($p[1{:}k^*]$)
  Pad with zeros to full length V
  return $\hat{p}^{(k^*)}$
end function

Appendix J. Example: α-Bregman decoding

J.1. Proof of Lemma 8

We first restate the lemma.

Lemma 17 *All generator functions ϕ_α, $\alpha > 1$, are dual-valid and satisfy Assumption (A2).*

Proof For Assumption 3.2, we can explicitly write:

$$\mathrm{d}_\phi(x, y) = \frac{x^\alpha}{\alpha(\alpha-1)} - \frac{y^\alpha}{\alpha(\alpha-1)} - \frac{y^{\alpha-1}}{\alpha-1}(x-y) = \frac{y^\alpha}{\alpha} - \frac{x}{\alpha-1} y^{\alpha-1} + \frac{x^\alpha}{\alpha(\alpha-1)}.$$

Therefore, the second derivative in y of this expression is

$$(\alpha-1)y^{\alpha-2} - (\alpha-2)xy^{\alpha-3} = y^{\alpha-3}(y(\alpha-1) - x(\alpha-2)) = y^{\alpha-3}\left(y(\alpha-1) + x(2-\alpha)\right).$$

Now, if $y \geqslant x$, then using $\alpha - 1 \geqslant 0$ we have that the above expression is

$$\geqslant y^{\alpha-3}(x(\alpha-1) + x(2-\alpha)) = y^{\alpha-3}x \geqslant 0,$$

confirming the convexity in y. Now for the condition that $x \mapsto u(x) := x\phi''(x)/\phi'(x)$ is non-decreasing from Assumption (A2), we can observe that

$$\phi'(x)\phi'''(x) - \phi''(x)^2 = \frac{x^{\alpha-1}}{\alpha-1} \cdot (\alpha-2)x^{\alpha-3} - (x^{\alpha-2})^2 = -\frac{x^{2\alpha-4}}{\alpha-1}.$$

Therefore, we identically have:

$$\phi'(x)\phi''(x) + x(\phi'(x)\phi'''(x) - \phi''(x)^2) = \frac{x^{2\alpha-3}}{\alpha-1} - x\frac{x^{2\alpha-4}}{\alpha-1} = 0,$$

thus concluding the proof. ■

J.2. Proof of Proposition 7

Recall the α–renormalization map $[T_\alpha(p)]_i = \big(p_i^{\alpha-1} + \nu\big)^{\frac{1}{\alpha-1}}, i \in [k]$, where the shift parameter $\nu = \nu(\alpha, p)$ is chosen so that $\sum_{i=1}^k [T_\alpha(p)]_i = 1$. We treat each value (or limit) of α in turn.

The limit $\alpha \to -\infty$. Define

$$F_\beta(\nu) := \sum_{i=1}^k \big(p_i^\beta + \nu\big)^{1/\beta}, \qquad \beta := \alpha - 1 < 0.$$

Because $x \mapsto x^{1/\beta}$ is strictly *decreasing* and convex on $(0, \infty)$ for $\beta < 0$, F_β is strictly decreasing and continuous on the interval $\big(-\min_i p_i^\beta, \infty\big)$. Moreover, $\lim_{\nu \downarrow -\min_i p_i^\beta} F_\beta(\nu) = \infty$ and $\lim_{\nu\uparrow\infty} F_\beta(\nu) = 0$, so a unique root ν_β with $F_\beta(\nu_\beta) = 1$ exists. Because $F_\beta(0) = S := \sum_{i=1}^k p_i \leqslant 1$ and F_β is decreasing, we have $\nu_\beta \leqslant 0$.

Let $q_i^{(\alpha)} = [T_\alpha(p)]_i = \big(p_i^\beta + \nu_\beta\big)^{1/\beta}$, and i^* be the index where p_i is largest. Using the constraint $\sum_i q_i^{(\alpha)} = 1$,

$$q_{i^\star}^{(\alpha)} = 1 - \sum_{i \neq i^\star} q_i^{(\alpha)} = \delta + p_{i^\star} + \sum_{i \neq i^\star} \big(p_i - q_i^{(\alpha)}\big) \geqslant p_{i^\star} + \delta.$$

Raising $q_{i^\star}^{(\alpha)} = \big(p_{i^\star}^\beta + \nu_\beta\big)^{1/\beta}$ to the power $\beta < 0$ yields

$$\nu_\beta = \big(p_{i^\star} + \delta + R_\beta\big)^\beta - p_{i^\star}^\beta, \qquad R_\beta := \sum_{i \neq i^\star} \big(p_i - q_i^{(\alpha)}\big) \in [0, \delta]. \tag{20}$$

For $i \neq i^\star$, we have $\nu_\beta / p_i^\beta \to 0$. Indeed, (20) implies $|\nu_\beta| \leqslant p_{i^\star}^\beta (c^\beta - 1)$ with $c := (p_{i^\star} + \delta)/p_{i^\star} > 1$. Because $\beta \to -\infty$, $c^\beta \to 0$, we have $|\nu_\beta| = O\big(p_{i^\star}^\beta\big) = o\big(p_i^\beta\big)$. Then,

$$q_i^{(\alpha)} = p_i \Big(1 + \frac{\nu_\beta}{p_i^\beta}\Big)^{1/\beta} \to p_i, \qquad i \neq i^\star. \tag{21}$$

Summing (21) over $i \neq i^\star$ and using $\sum_i q_i^{(\alpha)} = 1$ gives

$$q_{i^\star}^{(\alpha)} = 1 - \sum_{i \neq i^\star} q_i^{(\alpha)} \to 1 - \sum_{i \neq i^\star} p_i = p_{i^\star} + \delta. \tag{22}$$

Equations (21) and (22) establish $q^{(\alpha)} \to T_{-\infty}(p)$ component-wise, completing the proof.

The case $\alpha = \frac{3}{2}$. Now $\alpha - 1 = \frac{1}{2}$, hence $[T_{1.5}(p)]_i = \left(\sqrt{p_i} + \nu\right)^2, \qquad i \in [k]$. Set $s := \sum_{j=1}^{k} \sqrt{p_j}$ and $A := \sum_{j=1}^{k} p_j$. The normalization condition becomes

$$1 = \sum_{i=1}^{k} (\sqrt{p_i} + \nu)^2 = A + 2s\nu + k\nu^2.$$

Solving $k\nu^2 + 2s\nu + (A - 1) = 0$ for the root that yields non–negative probabilities gives $\nu = \frac{-s+\sqrt{s^2+k\,(1-A)}}{k}$. Hence

$$[T_{1.5}(p)]_i = \left(\sqrt{p_i} + \frac{\sqrt{s^2 + k\,(1 - A)} - s}{k}\right)^2, \qquad i \in [k].$$

The case $\alpha = 2$. Here $\alpha - 1 = 1$, so Definition 6 yields $[T_2(p)]_i = p_i + \nu, \qquad i \in [k]$. The normalization condition gives $1 = \sum_{i=1}^{k}(p_i + \nu) = \sum_{i=1}^{k} p_i + k\nu$, hence $\nu = \frac{1-\sum_{j=1}^{k} p_j}{k}$. Substituting yields

$$[T_2(p)]_i = p_i + \frac{1 - \sum_{j=1}^{k} p_j}{k}, \qquad i \in [k].$$

The limit $\alpha \to +\infty$. Write $\beta := \alpha - 1 \to +\infty$. Let $\nu = c^\beta$ with $c \in [0, 1]$. Then

$$[T_\alpha(p)]_i = \left(p_i^\beta + c^\beta\right)^{1/\beta} = \exp\Big\{\tfrac{1}{\beta}\log\big(p_i^\beta + c^\beta\big)\Big\}.$$

Using $\frac{1}{\beta}\log(a^\beta + b^\beta) \to \log(\max\{a, b\})$ as $\beta \to \infty$ gives $\lim_{\alpha\to\infty}[T_\alpha(p)]_i = \max\{p_i, c\}$. Choose the *water level* c so that $\sum_{i=1}^{k} \max\{p_i, c\} = 1$. This furnishes the claimed water–filling rule.

The four cases above prove Proposition 7.

J.3. Illustrating primal and dual renormalization

We consider the peaked vector $v = [0.1, 0.001, 0.001, 0.001, 0.001]$, and plot how both of its distinct constituent values get transformed by the primal and dual Bregman α-renormalization (by symmetry, all copies of 0.001 are guaranteed to get mapped to the same value by any of our renormalizations). The resulting plots are in Figure 3. As predicted by our theory, both renormalization families coincide at three values of the parameter, namely at $\alpha \in \{1, 2, \infty\}$. Furthermore, the primal family evolves more gradually than the dual family between the endpoints of the parameter interval $\alpha \in (1, 2]$, while the reverse behavior occurs for $\alpha \in (2, \infty)$ (where both renormalizations gradually converge to the water-filling limit which, in this case, is the uniform distribution).

J.4. Illustrating general nonconvexity of dual renormalization

Figure 4 illustrates that the dual Bregman objective can in general be non-convex for large α.

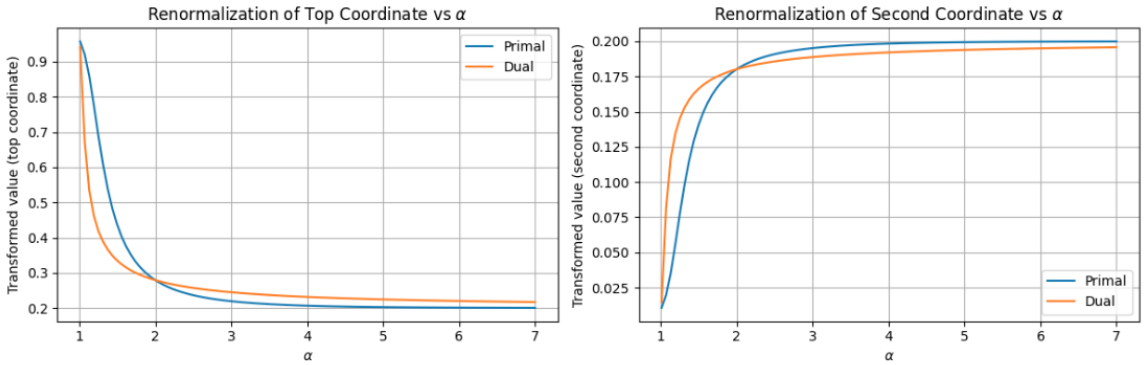

Figure 3: Comparison of primal and dual renormalization maps: The transformation of the larger value (0.1, left) and of the smaller value (0.001, right).

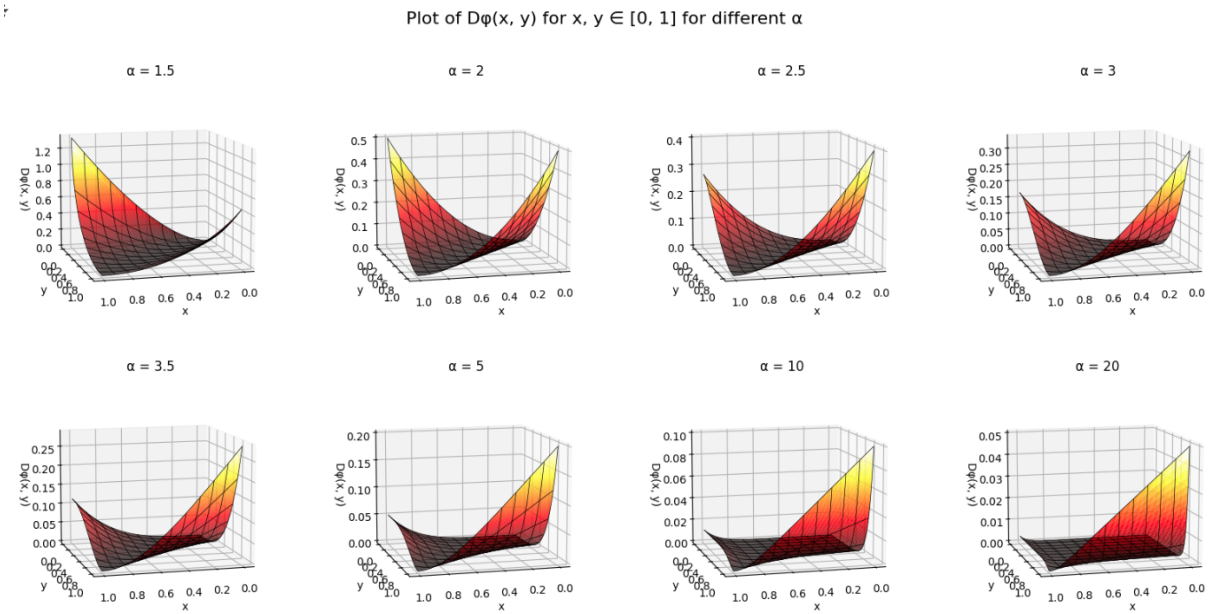

Figure 4: Nonconvexity of the Bregman dual landscape on the square $(x, y) \in [0, 1]^2$.

J.5. Illustrating discrete convexity

Figure 5 illustrates that the loss function $\mathrm{cost}(\cdot)$ defined in (5) is discretely convex for both the primal and dual decoding strategies. Here, we have chosen $V = 80$ and the regularization parameter λ as $1/80$. When k is close to V, the renormalization maps are all close to the true vector p, regardless of the value of α, and hence the loss primarily depends on the regularization term λk, which here equals $\lambda k = 1$ for $k = 80$. Thus, all curves (corresponding to different values of α) for both the primal and dual plots, asymptote to linearity and converge to this value at $k = 80$.

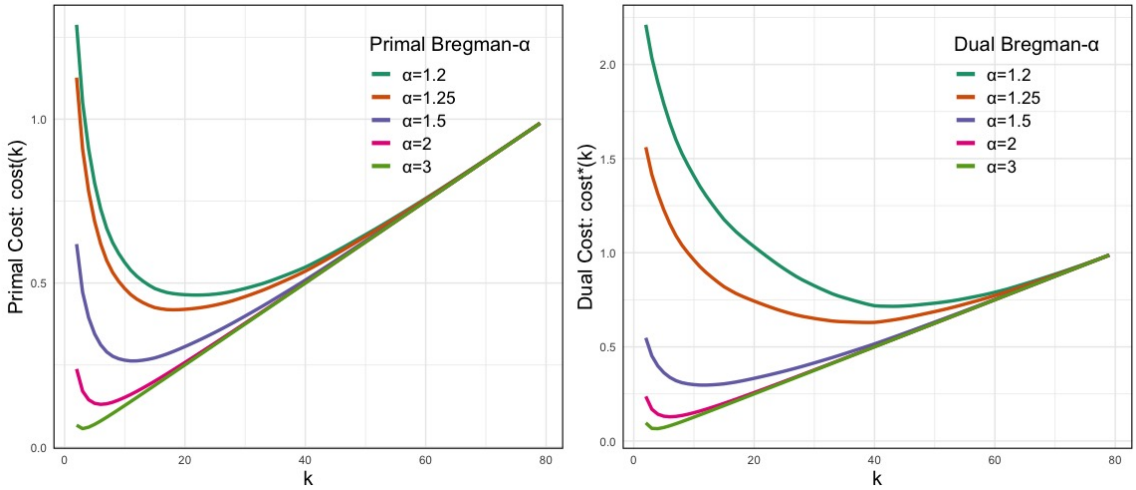

Figure 5: Discrete convexity of the function $k \mapsto \mathrm{cost}(k)$ for primal and dual Bregman α-decoding.

J.6. The simultaneous effects of Bregman decoding and temperature scaling

Here, we provide a plot to help compare the simultaneous effects of Bregman decoding and temperature scaling. We use the same simulation setting and plotting style as in our figure from the introduction (Section 1); except we only plot the nonzero probabilities (i.e., the top $k = 10$ probabilities), and we plot the *relative* sizes of the probabilities compared to the standard top-k decoding. Further, we use the same α and temperature hyperparameters used in our experiments in Table 1. The results are shown in Figure 6. Standard top-k decoding corresponds to $\alpha = 1$ and $T = 1$. From the figure, it appears that the effect of $\alpha > 1$ is to moderate/regularize the amount by which the small probabilities are pushed to zero; which could potentially be one reason why α-Bregman decoding with $\alpha > 1$ can perform better at high temperatures.

Appendix K. Supplementary experimental details

K.1. Compute resources

The experiments were conducted on a system running Rocky Linux 8.10, with 64 CPU cores of Intel(R) Xeon(R) Gold 6448Y processors at 2.10 GHz, 1 TB of RAM, and 8 NVIDIA L40S GPUs with 46 GB of memory each. All experiments can be done with only one GPU and multiple GPUs

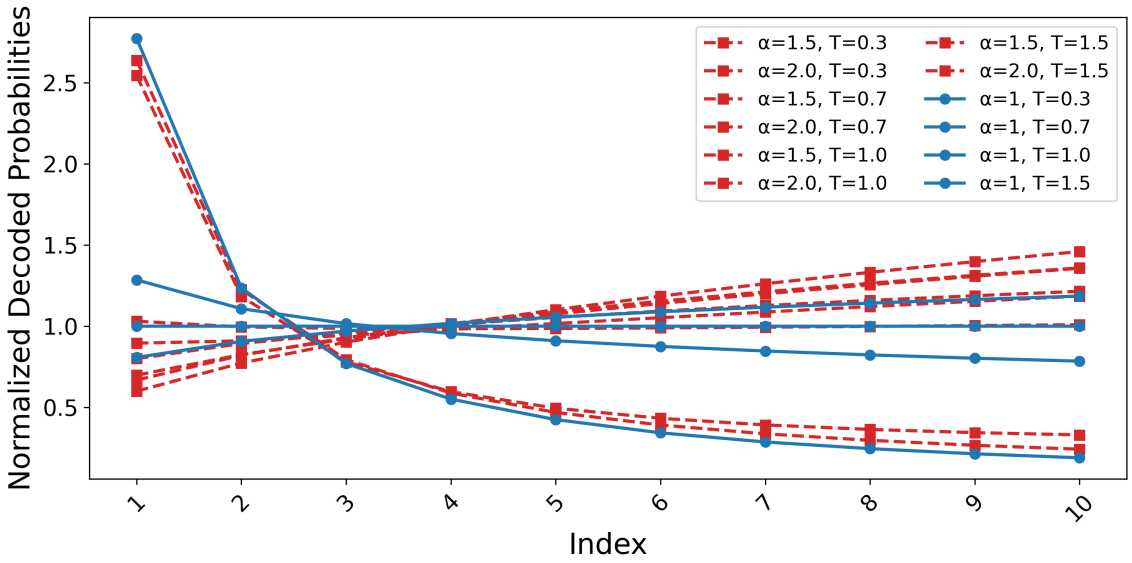

Figure 6: Comparison with changing the temperature.

were used only to parallelize experiments. The software environment used Python 3.11.11, PyTorch 2.5.1, and CUDA 12.4.

K.2. Supplementary experimental results

In this section, we provide additional experimental results to supplement those from Section B.

Table 2 shows results analogous to those in Table 1 for $\lambda \in \{0.1, 0.001\}$.

Table 2: Accuracy on GSM8K for LLaMA 3.1 8B using Bregman primal decoding ($\lambda \in \{0.1, 0.001\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, across different temperature settings. For top-k, k equals the averaged optimal k^* from the corresponding primal decoding run (matching temperature, λ, and α). Standard deviations are estimated using 1000 bootstrap resamples.

Temp	$\lambda = 0.1$		Top-k ($\lambda = 0.1$)		$\lambda = 0.001$		Top-k ($\lambda = 0.001$)	
	$\alpha = 1.5$	$\alpha = 2.0$			$\alpha = 1.5$	$\alpha = 2.0$		
0.3	$83.93_{\pm 1.01}$	$84.46_{\pm 1.00}$	$84.69_{\pm 0.99}$	$84.69_{\pm 0.99}$	$83.93_{\pm 1.01}$	$85.29_{\pm 0.98}$	$83.62_{\pm 1.02}$	$83.62_{\pm 1.02}$
0.7	$83.47_{\pm 1.02}$	$85.29_{\pm 0.98}$	$84.69_{\pm 0.99}$	$84.69_{\pm 0.99}$	$82.18_{\pm 1.05}$	$82.41_{\pm 1.05}$	$83.78_{\pm 1.02}$	$83.78_{\pm 1.02}$
1.0	$84.46_{\pm 1.00}$	$84.38_{\pm 1.00}$	$84.69_{\pm 0.99}$	$84.69_{\pm 0.99}$	$78.92_{\pm 1.12}$	$80.89_{\pm 1.08}$	$78.54_{\pm 1.13}$	$81.20_{\pm 1.08}$
1.5	$83.78_{\pm 1.02}$	$84.38_{\pm 1.00}$	$84.69_{\pm 0.99}$	$84.69_{\pm 0.99}$	$69.22_{\pm 1.23}$	$73.92_{\pm 1.21}$	$64.67_{\pm 1.32}$	$75.97_{\pm 1.18}$

Figure 7 presents the MAUVE scores comparing generated and human-written text under different decoding strategies. While primal decoding shows a slight advantage over top-k decoding, the differences are not statistically significant. We report standard deviations estimated from 50 bootstrap resamples; a higher number of resamples was not used due to the high computational cost of MAUVE score evaluation.

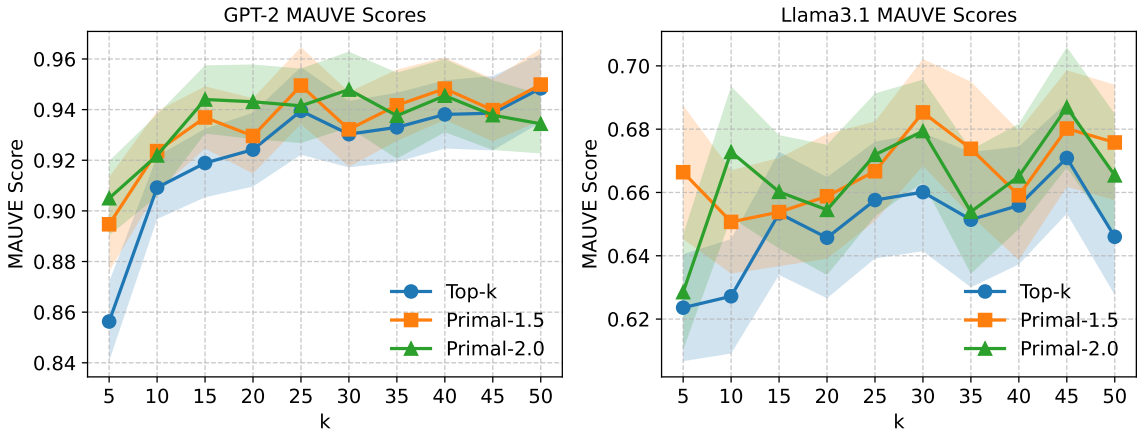

Figure 7: MAUVE scores results between generated and human-written text for GPT2-large (left panel) and LLaMA 3.1 8B (right panel), for various k values. We show top-k decoding and primal decoding with $\alpha \in \{1.5, 2.0\}$. Standard deviations are estimated using 50 bootstrap resamples

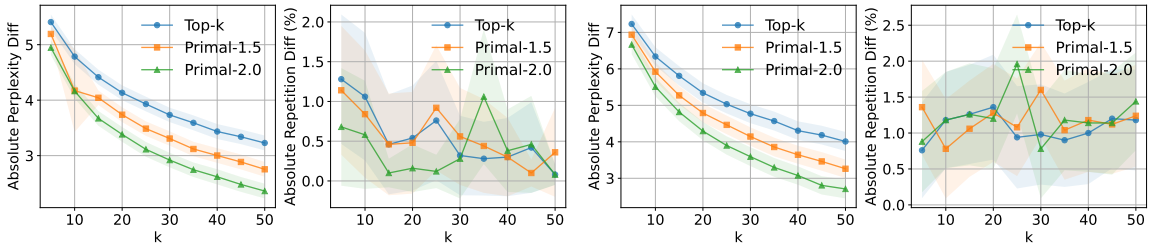

Figure 8: Perplexity and repetition frequency differences between generated and human-written text for Phi-3-medium-4k-instruct (left two panels) and Qwen2.5-14B-Instruct (right two panels), for various k values. We show top-k decoding and primal decoding with $\alpha \in \{1.5, 2.0\}$. Standard deviations are estimated using 1000 bootstrap resamples.

K.3. Experiments for Larger models: Qwen and Phi

We implement our experiments for Qwen2.5-14B-Instruct and Phi-3-medium-4k-instruct.

Figure 8 shows analogous results to Figure 2. Table 3 and 4 show the accuracy on GSM8K analogous to Table 1 and 2. Table 5 and 6 show analogous results for Phi-3-medium-4k-instruct model.

Table 3: Accuracy on GSM8K for Qwen2.5-14B-Instruct using Bregman primal decoding ($\lambda \in \{0.1, 0.01\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.1$		Top-k ($\lambda = 0.1$)		$\lambda = 0.01$		Top-k ($\lambda = 0.01$)	
	$\alpha = 1.5$	$\alpha = 2.0$			$\alpha = 1.5$	$\alpha = 2.0$		
0.3	$82.71_{\pm 1.04}$	$82.26_{\pm 1.05}$	$81.42_{\pm 1.07}$	$81.43_{\pm 1.07}$	$82.64_{\pm 1.04}$	$82.18_{\pm 1.05}$	$81.43_{\pm 1.07}$	$81.43_{\pm 1.07}$
0.7	$81.73_{\pm 1.06}$	$81.05_{\pm 1.08}$	$81.43_{\pm 1.07}$	$81.43_{\pm 1.07}$	$79.53_{\pm 1.11}$	$80.21_{\pm 1.10}$	$80.21_{\pm 1.10}$	$81.43_{\pm 1.07}$
1.0	$80.59_{\pm 1.09}$	$81.50_{\pm 1.07}$	$81.43_{\pm 1.07}$	$81.43_{\pm 1.07}$	$78.85_{\pm 1.12}$	$80.29_{\pm 1.10}$	$79.30_{\pm 1.12}$	$81.43_{\pm 1.07}$
1.5	$80.89_{\pm 1.08}$	$81.73_{\pm 1.06}$	$81.43_{\pm 1.07}$	$81.43_{\pm 1.07}$	$77.18_{\pm 1.16}$	$78.99_{\pm 1.12}$	$77.48_{\pm 1.15}$	$81.43_{\pm 1.07}$

Table 4: Accuracy on GSM8K for Qwen2.5-14B-Instruct using Bregman primal decoding ($\lambda \in \{0.001, 0.0001\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.001$		Top-k ($\lambda = 0.001$)		$\lambda = 0.0001$		Top-k ($\lambda = 0.0001$)	
	$\alpha = 1.5$	$\alpha = 2.0$			$\alpha = 1.5$	$\alpha = 2.0$		
0.3	$82.11_{\pm 1.06}$	$82.49_{\pm 1.05}$	$82.41_{\pm 1.05}$	$82.56_{\pm 1.05}$	$81.88_{\pm 1.06}$	$82.26_{\pm 1.05}$	$82.03_{\pm 1.06}$	$82.41_{\pm 1.05}$
0.7	$80.21_{\pm 1.10}$	$79.76_{\pm 1.11}$	$80.06_{\pm 1.10}$	$80.21_{\pm 1.10}$	$79.61_{\pm 1.11}$	$79.76_{\pm 1.11}$	$79.98_{\pm 1.10}$	$80.06_{\pm 1.10}$
1.0	$78.92_{\pm 1.12}$	$78.32_{\pm 1.14}$	$79.38_{\pm 1.11}$	$79.30_{\pm 1.12}$	$78.47_{\pm 1.13}$	$79.30_{\pm 1.12}$	$78.77_{\pm 1.13}$	$79.38_{\pm 1.11}$
1.5	$76.72_{\pm 1.16}$	$78.01_{\pm 1.14}$	$75.89_{\pm 1.18}$	$77.48_{\pm 1.15}$	$74.91_{\pm 1.19}$	$74.91_{\pm 1.19}$	$71.19_{\pm 1.25}$	$75.89_{\pm 1.18}$

Table 5: Accuracy on GSM8K for Phi-3-medium-4k-instruct using Bregman primal decoding ($\lambda \in \{0.1, 0.01\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, μ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.1$		Top-k ($\lambda = 0.1$)		$\lambda = 0.01$		Top-k ($\lambda = 0.01$)	
	$\alpha = 1.5$	$\alpha = 2.0$			$\alpha = 1.5$	$\alpha = 2.0$		
0.3	$86.81_{\pm 0.93}$	$87.87_{\pm 0.90}$	$85.97_{\pm 0.96}$	$85.97_{\pm 0.96}$	$87.41_{\pm 0.91}$	$87.04_{\pm 0.93}$	$87.26_{\pm 0.92}$	$87.26_{\pm 0.92}$
0.7	$86.96_{\pm 0.93}$	$88.17_{\pm 0.89}$	$85.97_{\pm 0.96}$	$85.97_{\pm 0.96}$	$85.67_{\pm 0.97}$	$86.88_{\pm 0.93}$	$88.10_{\pm 0.89}$	$88.10_{\pm 0.89}$
1.0	$86.35_{\pm 0.95}$	$87.11_{\pm 0.92}$	$85.97_{\pm 0.96}$	$85.97_{\pm 0.96}$	$84.99_{\pm 0.98}$	$83.93_{\pm 1.01}$	$85.44_{\pm 0.97}$	$85.44_{\pm 0.97}$
1.5	$87.19_{\pm 0.92}$	$86.58_{\pm 0.94}$	$85.97_{\pm 0.96}$	$85.97_{\pm 0.96}$	$82.94_{\pm 1.04}$	$83.70_{\pm 1.02}$	$80.14_{\pm 1.10}$	$80.14_{\pm 1.10}$

Table 6: Accuracy on GSM8K for Phi-3-medium-4k-instruct using Bregman primal decoding ($\lambda \in \{0.001, 0.0001\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, μ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.001$		Top-k ($\lambda = 0.001$)		$\lambda = 0.0001$		Top-k ($\lambda = 0.0001$)	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	$87.11_{\pm 0.92}$	$86.88_{\pm 0.93}$	$86.50_{\pm 0.94}$	$86.81_{\pm 0.93}$	$87.49_{\pm 0.91}$	$87.49_{\pm 0.91}$	$86.20_{\pm 0.95}$	$86.50_{\pm 0.94}$
0.7	$86.81_{\pm 0.93}$	$86.50_{\pm 0.94}$	$85.29_{\pm 0.98}$	$85.67_{\pm 0.97}$	$84.99_{\pm 0.98}$	$84.91_{\pm 0.99}$	$85.60_{\pm 0.97}$	$85.29_{\pm 0.98}$
1.0	$83.62_{\pm 1.02}$	$82.34_{\pm 1.05}$	$82.71_{\pm 1.04}$	$82.79_{\pm 1.04}$	$82.71_{\pm 1.04}$	$82.11_{\pm 1.06}$	$81.35_{\pm 1.07}$	$82.71_{\pm 1.04}$
1.5	$76.95_{\pm 1.16}$	$78.92_{\pm 1.12}$	$69.75_{\pm 1.27}$	$73.84_{\pm 1.21}$	$72.25_{\pm 1.23}$	$76.04_{\pm 1.18}$	$62.62_{\pm 1.33}$	$65.81_{\pm 1.31}$

K.4. Experiments for TriviaQA

Table 7 and 8 show accuracy on TriviaQA for LLaMA3.1-8B model. Here we choose 10% ($\approx$ 1800 questions) proportion of TriviQA validation dataset for evaluation.

Table 7: Accuracy on TriviaQA for LLaMA 3.1 8B using Bregman primal decoding ($\lambda \in \{0.1, 0.01\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.1$		Top-k ($\lambda = 0.1$)		$\lambda = 0.01$		Top-k ($\lambda = 0.01$)	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	$67.80_{\pm 1.10}$	$67.47_{\pm 1.11}$	$67.58_{\pm 1.11}$	$67.58_{\pm 1.11}$	$66.57_{\pm 1.11}$	$66.69_{\pm 1.11}$	$66.74_{\pm 1.11}$	$66.74_{\pm 1.11}$
0.7	$65.68_{\pm 1.12}$	$66.35_{\pm 1.12}$	$67.58_{\pm 1.11}$	$67.58_{\pm 1.11}$	$64.23_{\pm 1.13}$	$63.84_{\pm 1.13}$	$65.01_{\pm 1.13}$	$65.01_{\pm 1.13}$
1.0	$65.63_{\pm 1.12}$	$66.69_{\pm 1.11}$	$67.58_{\pm 1.11}$	$67.58_{\pm 1.11}$	$61.06_{\pm 1.15}$	$61.17_{\pm 1.15}$	$62.67_{\pm 1.14}$	$62.67_{\pm 1.14}$
1.5	$64.85_{\pm 1.13}$	$66.96_{\pm 1.11}$	$67.58_{\pm 1.11}$	$67.58_{\pm 1.11}$	$59.78_{\pm 1.16}$	$60.84_{\pm 1.15}$	$60.84_{\pm 1.15}$	$60.84_{\pm 1.15}$

Table 8: Accuracy on TriviaQA for LLaMA 3.1 8B using Bregman primal decoding ($\lambda \in \{0.001, 0.0001\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.001$		Top-k ($\lambda = 0.001$)		$\lambda = 0.0001$		Top-k ($\lambda = 0.0001$)	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	$66.85_{\pm 1.11}$	$67.58_{\pm 1.11}$	$67.13_{\pm 1.11}$	$67.13_{\pm 1.11}$	$66.69_{\pm 1.11}$	$67.08_{\pm 1.11}$	$67.19_{\pm 1.11}$	$67.58_{\pm 1.11}$
0.7	$63.40_{\pm 1.14}$	$63.18_{\pm 1.14}$	$64.68_{\pm 1.13}$	$64.79_{\pm 1.13}$	$62.73_{\pm 1.14}$	$62.73_{\pm 1.14}$	$63.79_{\pm 1.13}$	$63.68_{\pm 1.14}$
1.0	$59.00_{\pm 1.16}$	$59.00_{\pm 1.16}$	$60.17_{\pm 1.16}$	$62.23_{\pm 1.14}$	$57.99_{\pm 1.17}$	$59.11_{\pm 1.16}$	$58.55_{\pm 1.16}$	$60.11_{\pm 1.16}$
1.5	$55.04_{\pm 1.17}$	$55.71_{\pm 1.17}$	$52.81_{\pm 1.18}$	$56.38_{\pm 1.17}$	$49.19_{\pm 1.18}$	$52.59_{\pm 1.18}$	$50.19_{\pm 1.18}$	$51.31_{\pm 1.18}$

Table 9 and 10 show analogous accuracy results for Phi3-medium-4k-instruct on TriviaQA.

Table 9: Accuracy on TriviaQA for Phi-3-medium-4k-instruct using Bregman primal decoding ($\lambda \in \{0.1, 0.01\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.1$		Top-k ($\lambda = 0.1$)		$\lambda = 0.01$		Top-k ($\lambda = 0.01$)	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	$58.44_{\pm 1.16}$	$59.67_{\pm 1.16}$	$59.05_{\pm 1.16}$	$60.50_{\pm 1.15}$	$59.33_{\pm 1.16}$	$59.22_{\pm 1.16}$	$59.11_{\pm 1.16}$	$59.39_{\pm 1.16}$
0.7	$57.44_{\pm 1.17}$	$58.22_{\pm 1.16}$	$56.77_{\pm 1.17}$	$60.50_{\pm 1.15}$	$55.21_{\pm 1.17}$	$55.88_{\pm 1.17}$	$55.54_{\pm 1.17}$	$56.77_{\pm 1.17}$
1.0	$56.60_{\pm 1.17}$	$56.94_{\pm 1.17}$	$54.54_{\pm 1.18}$	$60.50_{\pm 1.15}$	$52.09_{\pm 1.18}$	$51.75_{\pm 1.18}$	$50.31_{\pm 1.18}$	$52.37_{\pm 1.18}$
1.5	$57.16_{\pm 1.17}$	$58.22_{\pm 1.16}$	$50.14_{\pm 1.18}$	$60.50_{\pm 1.15}$	$49.47_{\pm 1.18}$	$50.19_{\pm 1.18}$	$43.57_{\pm 1.17}$	$45.29_{\pm 1.18}$

Table 10: Accuracy on TriviaQA for Phi-3-medium-4k-instruct using Bregman primal decoding ($\lambda \in \{0.001, 0.0001\}$, $\alpha \in \{1.5, 2.0\}$) and top-k decoding, for various temperatures. For top-k, k equals the averaged k^* from primal decoding with matching temperature, λ, and α. Standard deviations are over 1000 bootstrap resamples.

Temp	$\lambda = 0.001$		Top-k ($\lambda = 0.001$)		$\lambda = 0.0001$		Top-k ($\lambda = 0.0001$)	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	$59.72_{\pm 1.16}$	$58.61_{\pm 1.16}$	$59.44_{\pm 1.16}$	$59.22_{\pm 1.16}$	$59.83_{\pm 1.16}$	$59.39_{\pm 1.16}$	$59.44_{\pm 1.16}$	$59.44_{\pm 1.16}$
0.7	$54.82_{\pm 1.17}$	$54.04_{\pm 1.18}$	$53.70_{\pm 1.18}$	$54.60_{\pm 1.18}$	$54.54_{\pm 1.18}$	$54.43_{\pm 1.18}$	$56.21_{\pm 1.17}$	$54.71_{\pm 1.18}$
1.0	$48.13_{\pm 1.18}$	$49.19_{\pm 1.18}$	$49.58_{\pm 1.18}$	$50.64_{\pm 1.18}$	$48.69_{\pm 1.18}$	$48.58_{\pm 1.18}$	$48.64_{\pm 1.18}$	$48.64_{\pm 1.18}$
1.5	$42.51_{\pm 1.17}$	$44.18_{\pm 1.17}$	$39.55_{\pm 1.15}$	$42.67_{\pm 1.17}$	$38.22_{\pm 1.15}$	$39.94_{\pm 1.16}$	$36.04_{\pm 1.13}$	$37.72_{\pm 1.14}$

K.5. Adaptivity

In this section, we consider the adaptivity of primal decoding by presenting the mean, standard deviation and entropy of the k^* chosen by our method during evaluation on GSM8K and TriviaQA datasets.

In Table 11, we show the average k^* values (and their values rounded to the nearest integer) selected by primal Bregman decoding on GSM8K with LLaMA 3.1 8B for various temperatures, α, and λ. Table 12 shows corresponding standard deviation and entropy.

Table 11: Mean (and rounded) average k^* values on GSM8K with LLaMA 3.1 8B for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	1.2231(1)	1.1537 (1)	1.6201 (2)	1.4453 (1)	2.1274 (2)	1.7964 (2)	2.8578 (3)	2.2112 (2)
0.7	1.2295 (1)	1.1554 (1)	1.6689 (2)	1.4794 (1)	2.3193 (2)	1.9048 (2)	3.2554 (3)	2.4974 (2)
1.0	1.2287 (1)	1.1594 (1)	1.7519 (2)	1.5048 (2)	2.7231 (3)	2.0234 (2)	4.6926 (5)	3.0924 (3)
1.5	1.2331 (1)	1.1566 (1)	1.8106 (2)	1.5189 (2)	4.1842 (4)	2.4067 (2)	14.2539 (14)	5.6002 (6)

Table 12: Standard deviation (and entropy) of average k^* values on GSM8K with LLaMA 3.1 8B for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	0.46 (0.82)	0.36 (0.62)	1.07 (1.55)	0.77 (1.28)	1.89 (2.08)	1.31 (1.77)	3.11 (2.58)	2.00 (2.16)
0.7	0.47 (0.84)	0.36 (0.62)	1.12 (1.62)	0.80 (1.34)	2.21 (2.24)	1.47 (1.89)	3.98 (2.78)	2.53 (2.37)
1.0	0.47 (0.84)	0.37 (0.63)	1.23 (1.72)	0.83 (1.38)	3.03 (2.49)	1.65 (2.00)	7.31 (3.21)	3.69 (2.69)
1.5	0.47 (0.85)	0.36 (0.63)	1.30 (1.79)	0.84 (1.40)	5.37 (3.13)	2.19 (2.32)	18.01 (4.04)	7.77 (3.51)

Table 13-14 show analougous adaptivity results for Qwen2.5-14B-Instruct. (Here, we only show results for $\lambda = 0.0001$, which is more important for adaptivity evidence, due to time limit, will complete afer rebuttal)

Table 13: Mean (and rounded) average k^* values on GSM8K with Qwen2.5-14B-Instruct for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	1.0973(1)	1.0660(1)	1.4899(1)	1.3425(1)	2.7614(3)	1.9317(2)	5.4537(5)	3.1726(3)
0.7	1.1010(1)	1.0672(1)	1.5043(2)	1.3534(1)	2.7778(3)	1.9522(2)	5.5047(6)	3.1911(3)
1.0	1.1000(1)	1.0666(1)	1.5171(2)	1.3591(1)	2.7985(3)	1.9723(2)	5.5603(6)	3.2493(3)
1.5	1.1008(1)	1.0662(1)	1.5211(2)	1.3628(1)	2.8761(3)	2.0028(2)	5.7831(6)	3.3285(3)

Table 14: Standard deviation (and entropy) of average k^* values on GSM8K with Qwen2.5-14B-Instruct under $\lambda = 0.0001$ and varying temperatures.

Temp	$\alpha = 1.5$	$\alpha = 2.0$
0.3	10.75 (2.81)	4.88 (2.26)
0.7	10.71 (2.86)	4.85 (2.29)
1.0	10.70 (2.90)	4.88 (2.34)
1.5	10.75 (3.03)	4.90 (2.42)

Table 15-16 show analougous adaptivity results for Phi-3-medium-4k-instruct.

In Table 17, we show the average k^* values (and their values rounded to the nearest integer) selected by primal Bregman decoding on TriviaQA with LLaMA 3.1 8B for various temperatures, α, and λ. Table 18 shows corresponding standard deviation and entropy.

Table 15: Mean (and rounded) average k^* values on GSM8K with Phi-3-medium-4k-instruct for various temperatures, α, and μ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	1.4048(1)	1.2609(1)	2.4123(2)	1.9287(2)	4.7186(5)	3.1299(3)	8.6473(9)	5.2889(5)
0.7	1.4074(1)	1.2601(1)	2.4337(2)	1.9409(2)	4.6706(5)	3.1307(3)	8.6958(9)	5.3697(5)
1.0	1.4073(1)	1.2603(1)	2.4541(2)	1.9364(2)	4.7772(5)	3.1792(3)	8.8501(9)	5.4394(5)
1.5	1.4098(1)	1.2575(1)	2.4667(2)	1.9498(2)	4.9289(5)	3.2335(3)	9.4782(9)	5.6113(6)

Table 16: Standard deviation (and entropy) of average k^* values on GSM8K with Phi-3-medium-4k-instruct under $\lambda = 0.0001$ for varying temperatures and α.

Temp	$\alpha = 1.5$	$\alpha = 2.0$
0.3	12.09 (3.83)	6.77 (3.32)
0.7	12.01 (3.89)	7.23 (3.61)
1.0	11.98 (3.98)	6.74 (3.45)
1.5	11.79 (4.24)	7.29 (3.79)

Table 17: Mean (and rounded) average k^* values on TriviaQA with LLaMA 3.1 8B for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	1.1536(1)	1.1452(1)	1.9135(2)	1.5291(2)	3.4193(3)	2.5753(3)	6.9406(7)	4.5149(5)
0.7	1.2265(1)	1.1275(1)	2.0109(2)	1.6265(2)	3.8877(4)	2.7593(3)	8.8845(9)	5.1892(5)
1.0	1.2138(1)	1.1324(1)	2.0273(2)	1.6818(2)	3.9715(4)	2.9759(3)	8.4552(8)	5.7381(6)
1.5	1.2013(1)	1.1384(1)	2.0289(2)	1.7032(2)	4.1749(4)	2.9398(3)	8.4399(8)	5.5166(6)

Table 18: Standard deviation (and entropy) of average k^* values on TriviaQA with LLaMA 3.1 8B for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	0.41 (0.65)	0.35 (0.60)	1.37 (1.90)	0.86 (1.42)	3.65 (2.85)	2.09 (2.42)	10.36 (3.63)	5.35 (3.28)
0.7	0.48 (0.83)	0.33 (0.55)	1.44 (2.00)	0.93 (1.56)	4.24 (3.09)	2.20 (2.53)	12.18 (4.10)	5.98 (3.56)
1.0	0.47 (0.81)	0.34 (0.56)	1.42 (2.01)	0.98 (1.63)	4.42 (3.03)	2.43 (2.68)	12.07 (3.77)	6.54 (3.68)
1.5	0.46 (0.78)	0.35 (0.58)	1.42 (2.01)	1.00 (1.66)	5.07 (3.02)	2.46 (2.62)	12.90 (3.34)	7.18 (3.35)

Table 19-20 show analougous adaptivity results for Phi-3-medium-4k-instruct on TriviaQA.

Table 19: Mean (and rounded) average k^* values on TriviaQA with Phi-3-medium-4k-instruct for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	1.7393(2)	1.4142(1)	3.6184(4)	2.8184(3)	9.2976(9)	5.2226(5)	18.7026(19)	10.4901(10)
0.7	1.7148(2)	1.4288(1)	3.6134(4)	2.6381(3)	8.4512(8)	4.8061(5)	16.8627(17)	9.3718(9)
1.0	1.7348(2)	1.4216(1)	3.6840(4)	2.6050(3)	8.3500(8)	4.8924(5)	16.7567(17)	9.6411(10)
1.5	1.6687(2)	1.4378(1)	3.6081(4)	2.6601(3)	8.6007(9)	5.1906(5)	18.2735(18)	9.7162(10)

Table 20: Standard deviation (and entropy) of average k^* values on TriviaQA with Phi-3-medium-4k-instruct for various temperatures, α, and λ.

Temp	$\lambda = 0.1$		$\lambda = 0.01$		$\lambda = 0.001$		$\lambda = 0.0001$	
	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 1.5$	$\alpha = 2.0$
0.3	0.87 (1.43)	0.49 (0.98)	2.84 (2.65)	1.76 (2.26)	8.19 (4.18)	3.99 (3.32)	16.54 (5.06)	8.88 (4.33)
0.7	0.87 (1.41)	0.49 (0.99)	2.69 (2.76)	1.70 (2.22)	7.50 (4.16)	3.78 (3.28)	15.26 (5.16)	8.38 (4.31)
1.0	0.87 (1.43)	0.49 (0.98)	2.68 (2.82)	1.65 (2.24)	7.04 (4.22)	3.62 (3.37)	13.81 (5.27)	7.75 (4.46)
1.5	0.84 (1.40)	0.50 (0.99)	2.58 (2.84)	1.63 (2.27)	6.60 (4.30)	3.58 (3.45)	13.94 (5.37)	7.51 (4.51)