Global Linear Convergence of Inexact TD Under Generalized Smoothness

Anonymous authors
Paper under double-blind review

Abstract

Recent work by Asadi et al. (2023b) gave an optimization view of TD learning with target networks and showed stability under a force-dominance condition, but their linear-rate analysis relies on global smoothness (a uniform curvature bound). This assumption can fail even when the inner problem is well posed, since curvature encountered during training can grow with the scale of TD residual–induced gradients. We retain the stabilized regime from prior theory—strong convexity in the inner variable—to isolate upper-curvature growth effects. Under generalized smoothness, where the Hessian norm may grow with gradient scale via a nondecreasing profile $\ell(\cdot)$, we analyze the inexact TD recursion with K inner gradient steps per target refresh and propose a curvature-checked constant stepsize rule that ensures global stability without a global smoothness constant. Our main result proves global linear convergence under force dominance with a single trajectory-dependent admissibility requirement governed by the maximum gradient magnitude M encountered along the run. This yields an explicit scaling law: the largest admissible constant stepsize decays as $1/\ell(cM)$ (for a universal constant c), and maintaining a fixed contraction requires K to grow proportionally to $\ell(cM)$. In the uniformly smooth case we recover Asadi et al. (2023b), while under curvature growth the worst trajectory gradient scale controls both stability and attainable convergence speed, aligning with step-control heuristics used in reinforcement learning (RL).

1 Introduction

Temporal-difference (TD) learning is widely regarded as a central algorithm in reinforcement learning (RL), and a large fraction of modern RL methods build directly on the core ideas it embodies. TD has played a key role in several landmark RL successes (Mnih et al., 2015; Silver et al., 2016), and its impact extends well beyond RL to areas such as AI, economics, and neuroscience. At the same time, TD is known to exhibit divergence in carefully constructed examples (Tsitsiklis & Van Roy, 1996; Baird, 1995; Sutton et al., 1998). Empirically, however, such divergence is uncommon even when TD is paired with highly expressive function approximators. This tension motivates a more refined theoretical understanding of TD and a strengthening of existing convergence results in directions that better reflect practice.

In this paper, we adopt this optimization perspective, framing TD as a two-level iterative optimization algorithm (Asadi et al., 2023b) that proceeds as follows:

$$\theta_{t+1} \approx \arg\min_{w} H(\theta_t, w). \tag{1}$$

The objective H is obtained by (a) selecting a function approximator (e.g., a neural network) and (b) specifying a discrepancy between successive predictions (e.g., a squared-error criterion). We provide additional examples later.

The formulation in (1) highlights two distinct parameter roles. The target parameter θ_t is held fixed at iteration t, while the optimization parameter w is updated to reduce the associated loss. In deep-RL terminology, θ_t corresponds to the target-network parameters, whereas w corresponds to the online-network parameters. A broad class of RL algorithms can be cast in this template, differing primarily in how the minimization is

carried out in practice. At one end, the original TD method (Sutton, 1988) takes a single gradient step in w and then updates θ. A commonly used intermediate regime in deep RL performs K gradient-based updates of w while freezing θ (Mnih et al., 2015). At the other end, Fitted Value Iteration (Gordon, 1995) effectively solves each inner problem exactly when closed-form solutions are available. Accordingly, understanding the iterative optimization mechanism in (1) helps clarify TD itself and the behavior of many TD-style algorithms.

The optimization process (1) has been extensively analyzed in the classical linear-function-approximation setting with a squared-error objective. In particular, when $K = 1$ (TD without a frozen target network), the seminal analysis of Tsitsiklis & Van Roy (1996) established convergence guarantees. Building on this line of work, Lee & He (2019) investigated the more general regime $K \geq 1$. Their analysis treats gradient descent with fixed K as an approximate inner solve, which induces an approximation error at each outer iteration; this per-iteration error accumulates over time and therefore appears explicitly in the final guarantee. As a consequence, (Lee & He, 2019) do not obtain convergence exactly to the TD fixed point, but rather to a neighborhood whose size is governed by the accumulated approximation error.

More recently, Asadi et al. (2023b) studied this optimization view of TD and derived sharp stability conditions that can be interpreted through a *force-dominance* lens: convergence requires the optimization force (curvature of the objective in the inner variable w) to strictly dominate the target force (the sensitivity of the induced target mapping). While such results yield fixed-point convergence for arbitrary $K \geq 1$, their guarantees hinge on a strong smoothness requirement, namely global L-smoothness. Specifically, they assume the gradient of the inner objective is uniformly Lipschitz continuous in order to certify descent for the inner optimization steps.

In modern deep RL with neural critics and value heads, the inner optimization geometry can exhibit pronounced, highly non-uniform *upper curvature*. A growing body of evidence in deep learning shows that training trajectories routinely encounter phases of *progressive sharpening* and operate near an *edge-of-stability* threshold, where the largest Hessian eigenvalues track the stepsize scale rather than remaining uniformly bounded by a global constant (Cohen et al., 2021; Gilmer et al., 2022; Damian et al., 2023; Roulet et al., 2024; Long & Bartlett, 2024). In such regimes, a single global L controlling $\|\nabla_w^2 H(\theta, w)\|_{\mathrm{op}}$ is either unavailable or so conservative that it ceases to predict the effective stepsizes that remain stable in practice, even for stabilized or regularized inner objectives. TD further amplifies this issue because the optimization problem is inherently nonstationary: in the optimization view, the function minimized at iteration t changes across outer updates since the bootstrapped targets depend on the current (or lagged) value parameters, as in target-network constructions (Mnih et al., 2015; Lee & He, 2019; Bengio et al., 2020; Asadi et al., 2023b). Empirically, this moving-target nonstationarity can destabilize gradient-based training and is associated with progressive sharpening of the loss landscape in deep RL (Lyle et al., 2023).

This motivates a convergence theory compatible with *unbounded* and *state-dependent* upper curvature in the *inner* TD objective, without requiring globally upper bounded Hessians. At the same time, analyzing fully coupled deep-TD dynamics with nonconvex function approximation is a separate challenge and is not the obstacle we target here. Instead, following the optimization-theoretic TD formulation (Asadi et al., 2023b), we work in the stabilized regime assumed by prior optimization viewpoints: for each fixed target θ, the inner problem in w is well-posed and strongly convex (Asadi et al., 2023b), yielding a unique minimizer map and enabling a clean contraction-based stability argument. This modeling choice is used only to remove nonconvex optimization pathologies; the phenomenon we isolate is *upper-curvature amplification along the trajectory*.

Within this setting, we retain the force-dominance stability lens while removing the separate global smoothness requirement used to certify inner descent. Concretely, we replace global L-smoothness with a generalized smoothness (curvature-growth) model in which the Hessian magnitude is controlled by a nondecreasing profile $\ell(\cdot)$ evaluated at the *local gradient scale* (Zhang et al., 2020; Tyurin, 2024). We quantify target drift through a cross-sensitivity (partial-gradient drift) bound on how $\nabla_w H(\theta, \cdot)$ changes with θ, which enters only through the forcing term in the outer recursion. Technically, we extend generalized-smoothness descent certificates from static objectives to the coupled TD recursion and derive a curvature-checked stepsize rule that guarantees inner descent without global curvature constants. This allows the outer force-dominance contraction argument of (Asadi et al., 2023b) to go through unchanged, while making the admissible constant

stepsize and the resulting linear rate explicitly depend on the worst gradient scale encountered along the run. In particular, the theory yields a sharp stability scaling law: as the trajectory gradient envelope grows, the largest safe constant stepsize shrinks according to $\ell(\cdot)$, and maintaining a fixed contraction requires the inner effort K to scale proportionally to the induced curvature amplification.

1.1 Contributions

We strengthen the optimization-theoretic convergence theory of temporal-difference learning with target networks by removing the global smoothness requirement on the inner objective and replacing it with a curvature-growth model that permits unbounded, trajectory-dependent upper curvature. Concretely, our contributions are:

- **Force-dominance beyond global smoothness.** Extending the force-dominance stability lens of Asadi et al. (2023b), we prove global linear convergence for inexact TD with K inner gradient steps per target refresh under generalized smoothness.
- **Constant stepsizes via curvature checks.** We analyze a curvature-checked stepsize rule for the inner updates and show that a constant baseline stepsize achieves global stability and linear convergence under generalized smoothness, provided it satisfies a single trajectory-dependent admissibility condition governed by the maximum gradient magnitude along the run.
- **Trajectory-dependent stability scaling.** We identify the trajectory gradient envelope as the key stability quantity under curvature growth, yielding an explicit scaling law: the largest admissible constant stepsize decays with curvature amplification at the worst gradient scale, and maintaining a fixed contraction requires K to scale proportionally with this amplification.
- **Classical smooth rates as a special case.** For uniformly smooth objectives (constant curvature profile) we recover the condition-number rates of (Asadi et al., 2023b); under curvature growth we quantify the stability penalty from high-curvature initialization and large early Bellman residual–induced gradient magnitudes in strongly convex, stabilized inner problems.

2 Problem Setting

Reinforcement learning (RL) studies artificial agents that learn through trial and error (Sutton et al., 1998). In this paper, we focus on the *prediction* setting, where the agent is interested in estimating the long-term goodness (value) of its states. This setting is mathematically formalized by a Markov reward process (MRP) (Puterman, 2014). We consider the discounted infinite-horizon case, specified by the tuple $\langle \mathcal{S}, R, P, \gamma \rangle$, where $\mathcal{S}$ is the set of states, $R : \mathcal{S} \to \mathbb{R}$ denotes the reward when transitioning out of a state, $P : \mathcal{S} \to \Delta(\mathcal{S})$ is the transition kernel (denoted by $P(s' \mid s)$), and $\gamma \in (0, 1)$ is the discount factor.

The goal is to approximate the state-value function, $v(s) \triangleq \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s]$. Define the Bellman operator T by $(Tv)(s) \triangleq R(s) + \sum_{s' \in \mathcal{S}} \gamma\, P(s' \mid s)\, v(s')$, equivalently $Tv = R + \gamma P v$. In large-scale RL problems, $|\mathcal{S}|$ can be enormous, making tabular approaches infeasible. We therefore focus on parameterized function approximation and seek parameters θ such that the learned value function $v(\cdot; \theta)$ approximates $v(\cdot)$ well.

A fundamental and widely used approach is temporal-difference (TD) learning (Sutton, 1988). Given a sample $\langle s, r, s' \rangle$ with $s' \sim P(\cdot \mid s)$, TD learning algorithm updates the parameters of the approximate value function as follows:

$$\theta_{t+1} \leftarrow \theta_t + \alpha \Big(r + \gamma v(s'; \theta_t) - v(s; \theta_t) \Big) \nabla_\theta v(s; \theta_t), \tag{2}$$

where $\alpha > 0$ is a stepsize and $\nabla_\theta v(s; \theta_t)$ denotes the partial gradient of $v(s; \theta)$ with respect to θ. The bootstrapped target $r + \gamma v(s'; \theta_t)$ is a one-step lookahead estimate and can be viewed as a sample of the right-hand side of the Bellman equation. TD is central not only as a standalone prediction algorithm but also because many modern RL methods inherit this bootstrapping structure, often combined with function approximation and target networks.

To place stability questions in context, let $d(\cdot)$ denote the stationary distribution of the Markov chain, defined by $\forall s' \in \mathcal{S}, \quad \sum_{s\in\mathcal{S}} d(s)\, P(s' \mid s) = d(s')$. Classical results show that, under linear function approximation, TD can be convergent under appropriate sampling conditions, notably when states are sampled from $d(\cdot)$ (Tsitsiklis & Van Roy, 1996). At the same time, TD can exhibit divergence when states are sampled from an arbitrary distribution rather than the stationary distribution of the Markov chain. First identified by Tsitsiklis & Van Roy (1996), this example is a very simple Markov chain with two non-terminal states and zero rewards. In this paper, we focus on the Markov reward process (MRP) setting, which could be thought of as a Markov decision process (MDP) with a single action. TD can display divergence even in the MRP setting (Tsitsiklis & Van Roy, 1996; Sutton et al., 1998), which indicates that the presence of multiple actions is not the core reason behind TD misbehavior (Sutton et al., 1998) – Divergence can manifest itself even in the more specific case of single action.

Moreover, little is known about the convergence guarantees of TD under alternative function approximators, or even when the TD update rule is slightly modified. These observations motivate studying TD in a way that explicitly exposes the algorithmic structure used in practice, rather than restricting attention to special linear regimes.

2.1 TD Learning as Two-level Optimization

A recent optimization-perspective analysis Asadi et al. (2023b) recasts practical TD-style procedures as a two-level scheme with two parameter roles: a *target* parameter θ_t is held fixed while an *online* parameter w is updated (to approximately minimize a target-induced objective), after which the target is refreshed by setting $\theta_{t+1} \approx w$. More precisely, following Asadi et al. (2023b), we start from the single-parameter TD update in (2) and adopt the standard target-network decoupling (Mnih et al., 2015) that separates target parameters θ from online optimization parameters w and updates θ less frequently. In particular, at outer iteration t we freeze θ_t and perform K gradient steps on w while keeping θ_t fixed:

$$w^{t,k+1} \leftarrow w^{t,k} + \alpha\Big(r + \gamma\, v(s', \theta^t) - v(s; w^{t,k})\Big)\, \nabla_w v(s; w^{t,k}) \tag{3}$$

and then updates the target parameter $\theta^{t+1} \leftarrow w^{t,K}$ before moving to the next iteration $t+1$. Here K is a hyper-parameter, where $K = 1$ takes us back to the original TD update (2). Observe that the dependence of $v(s'; \theta^t)$ to the optimization parameter w is ignored in this update, despite the fact that an implicit dependence is present due to the final step $\theta^{t+1} \leftarrow w^{t,K}$. This means that the objective function being optimized is made up of two separate input variables. We now define:

$$H(\theta, w) \;\triangleq\; \frac{1}{2}\sum_{s} d(s)\Big(\mathbb{E}_{r,s'}\big[r + \gamma v(s'; \theta)\big] - v(s; w)\Big)^2, \tag{4}$$

where $d(.)$ is allowed to be an arbitrary distribution, not just the stationary-state distribution of the Markov chain. Observe that the update direction in (3) is an unbiased stochastic estimate of the negative partial gradient $-\nabla_w H(\theta, w)$. Moreover, the scalar term $r + \gamma\, v(s', \theta^t) - v(s; w^{t,k})$ in (3) is the one-step TD error (Bellman residual), so $\nabla_w H(\theta, w)$ aggregates Bellman residuals weighted by the critic Jacobian $\nabla_w v(s; w)$; accordingly, $\|\nabla_w H(\theta, w)\|$ should be read as a Jacobian-weighted residual scale. Hence, TD, DQN, and similar algorithms could best be thought of as learning algorithms that proceed by approximately solving for the following sequence of optimization problems, $\theta_{t+1} \approx \arg\min_w H(\theta_t, w)$, using first-order optimization techniques. This optimization perspective is useful conceptually because it accentuates the unusual property of this iterative process, namely that the first argument of the objective H hinges on the

Algorithm 1: Inexact TD via K Inner Gradient Steps

Input: θ^0, T, K, $\{\alpha_{t,k}\}$
for $t = 0$ **to** $T-1$ **do**
  $w^{t,0} \leftarrow \theta^t$;
  for $k = 0$ **to** $K-1$ **do**
    $w^{t,k+1} \leftarrow w^{t,k} - \alpha_{t,k}\nabla_w H(\theta^t, w^{t,k})$;
  $\theta^{t+1} \leftarrow w^{t,K}$;
return θ^T;

output of the previous iteration. It also has important practical ramifications when designing RL optimizers (Asadi et al., 2023a). Moreover, the general form of this optimization process allows for using alternative forms of loss functions such as the Huber loss (Mnih et al., 2015), the logistic loss (Bas-Serrano et al., 2021), or the entropy (Precup & Sutton, 1997), as well as various forms of function approximation such as linear functions or deep neural networks. Each combination of loss functions and function approximators yields a different H, but one that is always comprised of a function H with two inputs.

In practice, the inner problem induced by a frozen target parameter θ^t is not solved exactly. Instead, one performs a finite number $K \geq 1$ of first-order updates on an online parameter w to approximately minimize the target-induced objective $H(\theta^t, \cdot)$, and then refreshes the target by setting $\theta^{t+1} \leftarrow w^{t,K}$. Concretely, the inexact recursion is given in Algorithm (1).

The theoretical difficulty is that the inner problem is only solved approximately (finite K), so one must control how much descent the inner updates achieve under the geometry of $H(\theta^t, \cdot)$ and then propagate this control through the outer recursion $\theta^{t+1} \leftarrow w^{t,K}$.

3 Generalized-Smoothness Preliminaries

We analyze inexact TD through the inner objectives

$$f_t(w) \triangleq H(\theta_t, w), \qquad w_{t,0} \triangleq \theta_t,$$

where t indexes the outer (target) iterate. This section collects the generalized-smoothness facts we use to certify first-order descent for inner gradient steps without assuming a global L-smoothness constant. The two takeaways are: (i) gradient differences can be controlled by a curvature-growth profile evaluated at the local gradient scale, and (ii) the classical L-smooth descent lemma is replaced by an integral bound whose magnitude depends on that same scale.

Assumption 3.1 (Generalized smoothness / curvature growth (Tyurin, 2024))**.** *Fix t. The function $f_t : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is ℓ-smooth on $\mathcal{W}$ if f_t is twice differentiable on $\mathcal{W}$, continuous on the closure of $\mathcal{W}$, and there exists a non-decreasing, positive, locally Lipschitz function $\ell : [0, \infty) \to (0, \infty)$ such that*

$$\left\|\nabla^2 f_t(w)\right\|_{\mathrm{op}} \;\leq\; \ell\big(\|\nabla f_t(w)\|\big) \qquad \textit{for all } w \in \mathcal{W}. \tag{5}$$

Assumption 3.1 strictly generalizes classical L-smoothness (take $\ell \equiv L$) and polynomial curvature-growth models of the form $\|\nabla^2 f_t(w)\|_{\mathrm{op}} \leq L_0 + L_1 \|\nabla f_t(w)\|^p$. Although (5) is stated using Hessians, our analysis uses it only to derive gradient-based inequalities; in particular, we will later invoke a purely first-order consequence (Lemma 3.4) that can be taken as an alternative assumption when f_t is not twice differentiable.

Assumption 3.2 (Lower boundedness)**.** *For each fixed t, there exists $f_t^\star \in \mathbb{R}$ such that $f_t(w) \geq f_t^\star$ for all $w \in \mathcal{W}$.*

Following Tyurin (2024), we introduce a modulus that converts the curvature-growth profile $\ell(\cdot)$ into explicit bounds on gradient variation.

Definition 3.3 (q–function)**.** *Assume Assumption 3.1. For $a \geq 0$, define*

$$q(s; a) \;=\; \int_0^s \frac{dv}{\ell(a+v)}, \qquad q_{\max}(a) \;:=\; \int_0^\infty \frac{dv}{\ell(a+v)} \in (0, \infty]. \tag{6}$$

Proposition 1 (Tyurin (2024))**.** *The function $q(\cdot; a)$ is invertible, differentiable, positive, and strongly increasing, and its inverse $q^{-1}(\cdot; a) : [0, q_{\max}(a)) \to \mathbb{R}_+$ is also differentiable, positive, and strongly increasing.*

The inverse q^{-1} appears as an intermediate modulus in the generalized-smoothness bounds below. Importantly, our final stability and stepsize conditions will be expressed directly through $\ell(\cdot)$ evaluated at trajectory-dependent gradient scales.

Lemma 3.4. *Fix t. For all $w, u \in \mathcal{W}$ with $\|u - w\| \in [0, q_{\max}(\|\nabla f_t(w)\|))$, if f_t is ℓ–smooth (Assumption 3.1), then*

$$\left\|\nabla f_t(u) - \nabla f_t(w)\right\| \;\leq\; q^{-1}\big(\|u - w\|\,;\, \|\nabla f_t(w)\|\big), \tag{7}$$

where q and $q_{\max}(\cdot)$ are defined in Definition 3.3.

Remark. When f_t is not twice differentiable, one may replace Assumption 3.1 by directly assuming (7); the subsequent analysis remains valid. We keep Assumption 3.1 because it provides an interpretable curvature-growth model.

For L-smooth objectives, one has $f_t(u) \le f_t(w) + \langle \nabla f_t(w), u-w\rangle + \frac{L}{2}\|u-w\|^2$. Under generalized smoothness, the quadratic term is replaced by an integral modulus:

Lemma 3.5 (Tyurin (2024)). *Fix t. For all $w, u \in \mathcal{W}$ with $\|u-w\| \in [0, q_{\max}(\|\nabla f_t(w)\|))$, if f_t is ℓ–smooth (Assumption 3.1), then*

$$f_t(u) \le f_t(w) + \langle \nabla f_t(w),\, u-w\rangle + \int_0^{\|u-w\|} q^{-1}\big(\tau;\, \|\nabla f_t(w)\|\big)\, d\tau. \tag{8}$$

Lemmas 3.4–3.5 are the only generalized-smoothness tools we require later. They play the role of the classical smoothness inequalities, but crucially their strength depends on the local gradient scale through $\ell(\cdot)$, which is what ultimately drives the trajectory-dependent stepsize and rate behavior in our TD analysis.

4 Convergence of TD with Generalized Smoothness

All proofs are deferred to the Appendix.

Our goal is to establish TD convergence guarantees without requiring a uniform bound on the Hessian of the inner objective. We work in the standard two-level optimization view of TD with target networks, and we isolate the failure mode that is orthogonal to nonconvexity: *heterogeneous upper curvature along the training trajectory.* Accordingly, we retain the stabilized regime in which the inner problem is well-posed (strongly convex in the online variable), and we remove the separate global L-smoothness assumption used in prior force-dominance analyses to certify inner descent.

We study Algorithm 1 for a general function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ under the following assumptions.

Assumption 4.1 (F_θ-Lipschitz partial gradient in θ). *There exists a constant $F_\theta \ge 0$ such that for all θ_1, θ_2 and all w,*

$$\big\|\nabla_w H(\theta_1, w) - \nabla_w H(\theta_2, w)\big\| \le F_\theta\, \|\theta_1 - \theta_2\|.$$

Assumption 4.2 (F_w-strong convexity in w). *For every fixed θ, the function $w \mapsto H(\theta, w)$ is F_w-strongly convex. Equivalently, there exists a constant $F_w > 0$ such that for all w_1, w_2 and all θ,*

$$\big(\nabla_w H(\theta, w_1) - \nabla_w H(\theta, w_2)\big)^\top (w_1 - w_2) \ge F_w\, \|w_1 - w_2\|^2.$$

Assumption 4.3 (Fixed point). *There exists $\theta^\star \in \mathbb{R}^d$ such that $\nabla_w H(\theta^\star, \theta^\star) = 0$.*

We analyze TD under generalized smoothness (Assumption 3.1) in three stages. First, we construct a *curvature-checked* stepsize that guarantees descent without global L-smoothness. Second, we combine this descent certificate with strong convexity in w to obtain an inner contraction with a forcing term. Finally, we couple the inner and outer dynamics and obtain *global linear convergence* under the *force-dominance* condition $F_\theta < F_w$, with a constant baseline stepsize calibrated to the trajectory-dependent curvature growth.

4.1 The Curvature-Checked Stepsize

Standard analyses rely on global L-smoothness, which yields the quadratic upper bound $f_t(u) \le f_t(w) + \langle \nabla f_t(w),\, u-w\rangle + \frac{L}{2}\|u-w\|^2, \quad \forall\, w, u \in \mathcal{W}$. This inequality justifies a constant stepsize $\gamma \le 1/L$: for the update $u = w - \gamma \nabla f_t(w)$, the linear decrease $-\gamma\|\nabla f_t(w)\|^2$ dominates the quadratic curvature penalty $\frac{L}{2}\gamma^2\|\nabla f_t(w)\|^2$. Under generalized smoothness (Assumption 3.1), the effective curvature is not constant; it is governed by the curvature-growth profile $\ell(\|\nabla f_t(w)\|)$ and can become large in steep regions. Consequently, a naively fixed stepsize need not guarantee descent.

We therefore adopt the clipped constant-step rule of the form $\gamma_{t,k} = \min\{\gamma_0,\ \gamma_{t,k}^{\text{safe}}\}$, which separates optimization speed from geometric safety. The baseline γ_0 is the step we would like to use throughout the

run, since it yields a uniform inner contraction once descent is certified. The cap $\gamma_{t,k}^{\text{safe}}$ is computed from the generalized-smoothness model to guarantee a net decrease for $w_{t,k+1} = w_{t,k} - \gamma_{t,k}\nabla f_t(w_{t,k})$, even when $\|\nabla^2 f_t\|$ is unbounded. Later, we show that along stable trajectories the gradients admit a uniform envelope M, which implies a uniform positive lower bound on $\gamma_{t,k}^{\text{safe}}$; choosing γ_0 below this bound makes clipping inactive, and the method reduces to constant-stepsize gradient descent. Then we derive a uniform linear contraction.

Strategy. Assumption 3.1 provides a generalized upper model in which the remainder term is controlled through the inverse profile $q^{-1}(\cdot;\cdot)$ (Lemma 3.5). We choose $\gamma_{t,k}$ so that this remainder consumes at most a fixed fraction of the first-order decrease. To bound the integral remainder in (8), we use the monotonicity of $q^{-1}(\tau;a)$ in τ, yielding the rectangular bound $\int_0^r q^{-1}(\tau;a)\,d\tau \;\le\; r\,q^{-1}(r;a)$, $r \in [0, q_{\max}(a))$.

Definition 4.4 (Constant target schedule)**.** *For a fixed $\gamma_0 > 0$, we set $\overline{\gamma}_n \equiv \gamma_0$ for all global steps $n = tK+k$.*

Curvature-checked stepsize. Let $g_{t,k} \triangleq \|\nabla f_t(w_{t,k})\|$. When $g_{t,k} = 0$, any stepsize yields no motion, so we use the baseline schedule. When $g_{t,k} > 0$, we define a *safe cap* from the curvature-growth profile so that the generalized remainder term is controlled relative to the linear decrease.

Definition 4.5 (Curvature-checked constant stepsize)**.** *At inner step (t,k), let $g_{t,k} \triangleq \|\nabla f_t(w_{t,k})\|$ and $n = tK + k$. If $g_{t,k} = 0$, set $\gamma_{t,k} = \overline{\gamma}_n = \gamma_0$. If $g_{t,k} > 0$, define*

$$\gamma_{t,k}^{\text{safe}} \;\triangleq\; \frac{1}{g_{t,k}}\, q\Big(\frac{g_{t,k}}{2}\,;\, g_{t,k}\Big) = \frac{1}{g_{t,k}} \int_0^{g_{t,k}/2} \frac{d\tau}{\ell(g_{t,k}+\tau)} = \int_0^{1/2} \frac{dv}{\ell\big((1+v)g_{t,k}\big)}, \tag{9}$$

and set the actual stepsize to be

$$\gamma_{t,k} \;\triangleq\; \min\{\overline{\gamma}_n,\, \gamma_{t,k}^{\text{safe}}\} = \min\{\gamma_0,\, \gamma_{t,k}^{\text{safe}}\}. \tag{10}$$

Remark 1. Define the remainder ratio $c_{t,k} \triangleq \frac{q^{-1}\left(\gamma_{t,k} g_{t,k};\, g_{t,k}\right)}{g_{t,k}}$. In the classical L-smooth case, one has $q^{-1}(r;g) = Lr$, so the remainder ratio reduces to $c_{t,k} = L\gamma_{t,k}$. Requiring $c_{t,k} \le \frac{1}{2}$ therefore yields a familiar constant-stepsize constraint of order $1/L$ (e.g., $\gamma_{t,k} \lesssim 1/L$), which is sufficient to guarantee a uniform fraction of descent. In our setting, q^{-1} is nonlinear and depends on the gradient scale; the choice (9) is constructed so that the generalized curvature remainder in (8) is controlled relative to the linear term, ensuring that it consumes at most a fixed fraction of the predicted first-order decrease.

Lemma 4.6 (One-step descent certificate)**.** *Under Assumption 3.1 and choosing $\gamma_{t,k}$ by Definition 4.5, each inner update satisfies*

$$f_t(w_{t,k+1}) - f_t(w_{t,k}) \le -\frac{\gamma_{t,k}}{2}\,\|\nabla f_t(w_{t,k})\|^2 = -\frac{1}{2\gamma_{t,k}}\,\|w_{t,k+1} - w_{t,k}\|^2. \tag{11}$$

Remark 2. Lemma 4.6 replaces the smoothness-based *choose* $\gamma \le 1/L$ step used in standard analyses: it certifies descent without global Lipschitz gradients in w. In the next section, we combine this certificate with strong convexity to obtain an inner contraction, and then lift it to an outer force-dominance contraction for the inexact K-step TD scheme.

4.2 The Inner Contraction

Section 4.1 established a local descent certificate for each inner update: the inner loop decreases $f_t(w) = H(\theta_t, w)$ even when f_t does not admit a global smoothness constant in w. In this section we show that the inner iterates contract toward the TD fixed point $\theta^\star$ up to an additive forcing term. The key additional ingredient is strong convexity in w (Assumption 4.2), which converts descent into geometric contraction.

A technical subtlety is that, at outer time t, the minimizer of $w \mapsto H(\theta_t, w)$ need not coincide with the fixed point $\theta^\star$. We measure this mismatch at the reference point $\theta^\star$ via $\Delta_t \triangleq \big\|\nabla_w H(\theta_t, \theta^\star)\big\|$. By Assumption 4.3, $\nabla_w H(\theta^\star, \theta^\star) = 0$, so $\Delta_t = \|\nabla_w H(\theta_t, \theta^\star) - \nabla_w H(\theta^\star, \theta^\star)\|$.

Proposition 2 (Key suboptimality bound)**.** *Under assumptions 4.2 and 4.3 and choosing $\gamma_{t,k}$ by Definition 4.5, for every inner step (t,k),*

$$H(\theta_t,\theta^\star) - H(\theta_t, w_{t,k}) \;\le\; \frac{\Delta_t^2}{2F_w} - \frac{1}{2\gamma_{t,k}}\big\|w_{t,k+1} - w_{t,k}\big\|^2. \tag{12}$$

Proposition 2 is where the descent certificate interacts with strong convexity. It bounds the inner-loop suboptimality at $w_{t,k}$ in terms of (i) the reference mismatch Δ_t and (ii) the step displacement $\|w_{t,k+1} - w_{t,k}\|$. The key consequence is that when we expand the squared distance to $\theta^\star$ after one gradient step, the potentially harmful cross term is controlled by this suboptimality bound while the step-length term from Lemma 4.6 cancels, yielding a clean one-step contraction up to an additive forcing term.

Proposition 3 (Inner-step contraction with forcing)**.** *Define the effective contraction rate $\kappa_{t,k} \triangleq F_w\gamma_{t,k}$. Then, for all t and all k,*

$$\|w_{t,k+1} - \theta^\star\|^2 \;\le\; (1-\kappa_{t,k})\|w_{t,k} - \theta^\star\|^2 + \kappa_{t,k}\frac{\Delta_t^2}{F_w^2}. \tag{13}$$

Remark 3. The additive term $\kappa_{t,k}\Delta_t^2/F_w^2$ appears because $\theta^\star$ is a fixed point of the two-level scheme, not necessarily the minimizer of $w \mapsto H(\theta_t, w)$ for general θ_t. When $\Delta_t = 0$, the inner loop contracts directly toward $\theta^\star$.

Proposition 3 controls a single inner gradient step. The TD update runs K such steps with the target frozen, and then refreshes the target via $\theta_{t+1} \leftarrow w_{t,K}$. Iterating the one-step contraction yields an outer recursion.

Proposition 4 (Outer recursion)**.** *Define the cumulative decay*

$$\chi_{t,K} \triangleq \prod_{k=0}^{K-1}(1-\kappa_{t,k}), \qquad \eta_t^2 \triangleq \frac{\Delta_t^2}{F_w^2}.$$

Then,

$$\|\theta_{t+1} - \theta^\star\|^2 \;\le\; \chi_{t,K}\|\theta_t - \theta^\star\|^2 + \big(1-\chi_{t,K}\big)\eta_t^2. \tag{14}$$

The recursion in (14) is the central interface between inner optimization and outer stability: $\chi_{t,K}$ captures the cumulative inner contraction over K steps, while η_t captures the mismatch-driven forcing. We next bound η_t using Assumption 4.1 and obtain a uniform contraction under force dominance.

4.3 From the Outer Recursion to Fixed-Point Convergence

Proposition 4 yields the fundamental inexact recursion (14). To turn it into a uniform linear rate, we need two ingredients: (i) force dominance $F_\theta < F_w$, which controls the forcing term η_t, and (ii) a uniform inner contraction factor, which we obtain once clipping is inactive and $\gamma_{t,k} \equiv \gamma_0$.

Using Assumption 4.1 with $w = \theta^\star$ and $\theta_2 = \theta^\star$ yields $\Delta_t = \|\nabla_w H(\theta_t,\theta^\star)\| \le F_\theta\|\theta_t - \theta^\star\|$. Since $\eta_t^2 = \Delta_t^2/F_w^2$, defining $\eta \triangleq F_\theta/F_w$ gives $\eta_t^2 \le \eta^2\|\theta_t - \theta^\star\|^2$. Substituting into (14) yields

$$\|\theta_{t+1} - \theta^\star\|^2 \le \Big(\eta^2 + (1-\eta^2)\chi_{t,K}\Big)\|\theta_t - \theta^\star\|^2. \tag{15}$$

Under force dominance $F_\theta < F_w$, we have $\eta \in (0,1)$.

It remains to ensure a uniform lower bound on the safety cap. Once $\gamma_{t,k}^{\text{safe}} \ge \gamma_{\min}^{\text{safe}} > 0$ holds for all iterates, choosing $\gamma_0 \le \gamma_{\min}^{\text{safe}}$ guarantees $\gamma_{t,k} = \gamma_0$ everywhere and $\chi_{t,K} = (1 - F_w\gamma_0)^K$. Since $\gamma_{t,k}^{\text{safe}}$ depends on the local gradient scale, we obtain this by first showing that the iterates remain in a bounded region, which implies a uniform bound on the gradient norms along the trajectory.

Lemma 4.7 (Bounded iterates and a uniform gradient envelope)**.** *Assume $H \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$, Propositions 2–4 holds, the force-dominance condition $F_\theta < F_w$ holds, and the constant baseline stepsize satisfies $\gamma_0 < 1/F_w$. Let $R \triangleq \|\theta_0 - \theta^\star\|$ and define*

$$M \;\triangleq\; \sup\Big\{\|\nabla_w H(\theta,w)\| :\; \|\theta - \theta^\star\| \le R,\; \|w - \theta^\star\| \le R\Big\}. \tag{16}$$

*Then the following hold: (i) (**Bounded outer iterates**) For all $t \geq 0$, $\|\theta_t - \theta^\star\| \leq R$, (ii) (**Bounded inner iterates**) For all $t \geq 0$ and all $k \in \{0, \ldots, K\}$, $\|w_{t,k} - \theta^\star\| \leq R$, (iii) (**Uniform gradient envelope**) The constant M in (16) is finite and for all t, k, i.e, $g_{t,k} = \|\nabla f_t(w_{t,k})\| = \|\nabla_w H(\theta_t, w_{t,k})\| \leq M < \infty$.*

5 A Unifying Theory of Convergence Under ℓ-Smoothness

All proofs are deferred to the Appendix.

This section packages our constant-stepsize theory into one theorem and then specializes it across canonical curvature-growth regimes. The punchline is simple: *generalized smoothness does not preclude linear convergence*. It changes the admissibility law for a constant stepsize from a *global* curvature constant to a *trajectory-dependent* curvature amplification captured by a gradient envelope. This yields an explicit scaling law for the required inner effort K.

Setup. We study the inexact TD recursion (Algorithm 1) with $K \geq 1$ inner steps and the curvature-checked constant stepsize

$$\gamma_{t,k} = \min\{\gamma_0, \gamma_{t,k}^{\text{safe}}\}, \qquad \gamma_{t,k}^{\text{safe}} = \frac{1}{g_{t,k}} \int_0^{g_{t,k}/2} \frac{d\tau}{\ell(g_{t,k} + \tau)}, \qquad g_{t,k} = \|\nabla f_t(w_{t,k})\|.$$

The curvature check guarantees one-step descent unconditionally. The question in this section is when the check becomes *inactive*, so that the method effectively runs with a *single constant* stepsize and admits a *uniform linear rate*.

A uniform admissibility threshold from the gradient envelope. With a uniform gradient envelope in hand, we can lower bound $\gamma_{t,k}^{\text{safe}}$ uniformly over all iterates. By Definition 4.5, for $g_{t,k} > 0$ we have $\gamma_{t,k}^{\text{safe}} = \frac{1}{g_{t,k}} \int_0^{g_{t,k}/2} \frac{d\tau}{\ell(g_{t,k}+\tau)}$. Using $g_{t,k} \leq M$ from Lemma 4.7 and monotonicity of $\ell(\cdot)$, for all $\tau \in [0, g_{t,k}/2]$ we have $g_{t,k} + \tau \leq \frac{3}{2} g_{t,k} \leq \frac{3}{2} M$, hence $\ell(g_{t,k} + \tau) \leq \ell(3M/2)$ and therefore

$$\gamma_{t,k}^{\text{safe}} \;\geq\; \frac{1}{g_{t,k}} \int_0^{g_{t,k}/2} \frac{d\tau}{\ell(3M/2)} = \frac{1}{2\,\ell(3M/2)} \;\triangleq\; \gamma_{\min}^{\text{safe}}. \tag{17}$$

Consequently, choosing $\gamma_0 \leq \gamma_{\min}^{\text{safe}}$[1] implies $\gamma_{t,k} = \gamma_0$ for all t, k, i.e., the curvature check never clips and the inner loop runs with a constant step. Hence $\kappa_{t,k} = F_w \gamma_0 \triangleq \kappa \in (0, 1)$. Therefore

$$\chi_{t,K} = \prod_{k=0}^{K-1} (1 - \kappa) = (1 - \kappa)^K \;\triangleq\; \chi_K \in (0, 1), \tag{18}$$

is constant in t. Plugging $\chi_{t,K} \equiv \chi_K$ into (15) yields a uniform contraction. We now state the unifying theorem.

Theorem 5.1 (Unifying constant-stepsize convergence under ℓ-smoothness)**.** *Assume Assumptions 3.1, 4.1, 4.2, 4.3, and suppose $H \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$. Let M be the gradient envelope from Lemma 4.7. Assume force dominance $F_\theta < F_w$ and define $\eta \triangleq F_\theta / F_w \in (0, 1)$. Choose a constant baseline $\gamma_0 > 0$ satisfying*

$$\gamma_0 \leq \frac{1}{2\,\ell(3M/2)}. \tag{19}$$

Run Algorithm 1 with stepsizes $\gamma_{t,k} = \min\{\gamma_0, \gamma_{t,k}^{\text{safe}}\}$. Then the curvature check never clips, $\gamma_{t,k} \equiv \gamma_0$, and the outer iterates contract linearly:

$$\|\theta_{t+1} - \theta^\star\|^2 \;\leq\; \rho\, \|\theta_t - \theta^\star\|^2, \qquad \rho \triangleq \eta^2 + (1 - \eta^2)(1 - F_w \gamma_0)^K \in (0, 1). \tag{20}$$

Consequently, $\|\theta_t - \theta^\star\|^2 \leq \rho^t \|\theta_0 - \theta^\star\|^2$.

[1] In the C^2 regime considered throughout our generalized-smoothness analysis, Assumption 4.2 (strong convexity in first-order form) guarantees that for any fixed target, the objective satisfies $\nabla_w^2 H(\theta, \cdot) \succeq F_w I$. Recalling that $f_t(w) \triangleq H(\theta_t, w)$, Assumption 3.1 upper bounds the Hessian norm $\|\nabla_w^2 H(\theta_t, w)\|_{\text{op}}$ by the profile $\ell(\|\nabla_w H(\theta_t, w)\|)$. Since the maximum eigenvalue must bound the minimum eigenvalue, this forces $\ell(\|\nabla_w H(\theta_t, w)\|) \geq F_w$ along the trajectory. Consequently, because ℓ is nondecreasing, the admissibility condition $\gamma_0 \leq [2\,\ell(3M/2)]^{-1}$ already strictly enforces $\gamma_0 \leq 1/(2F_w) < 1/F_w$.

Theorem 5.1 establishes three key trajectory-dependent properties governed by the envelope M:

1. **Admissible constant step.** The largest globally admissible constant stepsize scales as

$$\gamma_0^{\max} \;=\; \frac{1}{2\,\ell(3M/2)}. \tag{21}$$

2. **Trajectory-dependent condition number.** Generalized smoothness replaces the classical L/F_w (Asadi et al., 2023b) by the trajectory-dependent amplification

$$\mathcal{B}(M) \;\triangleq\; \frac{\ell(3M/2)}{F_w}. \tag{22}$$

3. **The stability scaling law (inner-effort threshold).** If we saturate γ_0 at its curvature-growth limit $\gamma_0 = [2\ell(3M/2)]^{-1}$, then

$$(1 - F_w\gamma_0)^K = \Big(1 - \frac{1}{2\mathcal{B}(M)}\Big)^K \;\le\; \exp\!\Big(-\frac{K}{2\mathcal{B}(M)}\Big). \tag{23}$$

   Thus, achieving a fixed-strength inner contraction requires

$$K \;\gtrsim\; \mathcal{B}(M) \;=\; \frac{\ell(3M/2)}{F_w}. \tag{24}$$

Because M strictly bounds the gradient envelope (Lemma 4.7), and this gradient represents the expected TD update direction (a Jacobian-weighted Bellman residual), M essentially captures the worst-case geometry-weighted Bellman residual scale encountered during training. These relations make the source of initialization sensitivity explicit: large early TD errors (the residual term in (3)) inflate the trajectory gradient envelope M, which increases $\ell(3M/2)$, shrinks $\gamma_0^{\max}$, and raises the inner-effort threshold (24).

Remark 4. The constraint $\gamma_0 \le [2\ell(3M/2)]^{-1}$ is the *uniform linear-rate admissibility law*: it guarantees that one constant baseline step is valid everywhere along the trajectory, summarized by the gradient envelope M. This is exactly where generalized smoothness departs from global L-smoothness: the admissible constant step is controlled by the worst gradient scale reached during learning, rather than curvature at the optimum. When γ_0 exceeds the local stability threshold, a global descent argument for constant-step gradient methods can fail and instability can occur, closely related to edge-of-stability behavior observed in large-scale training (Cohen et al., 2021). In contrast, Definition 4.5 is *fail-safe*: if γ_0 is too aggressive for (19), the update clips to $\gamma_{t,k}^{\text{safe}}$ and preserves the one-step descent certificate. The tradeoff is that a *uniform* contraction factor need not hold, so one must analyze the resulting variable-stepsize dynamics.

To derive explicit convergence rates for distinct smoothness classes, we now specialize (20) by saturating the admissibility condition in (19), i.e., $\gamma_0 = \frac{1}{2\,\ell(3M/2)}$. This makes the dependence on curvature amplification and inner effort K explicit.

5.1 Regime I: Global L-Smoothness ($\ell(s) \equiv L$)

When $\ell(s) \equiv L$, the admissibility condition in (19) becomes $\gamma_0 \le 1/(2L)$ in our certificate, and Theorem 5.1 yields $\rho = \eta^2 + (1-\eta^2)\Big(1 - \frac{F_w}{2L}\Big)^K \;\le\; \eta^2 + (1-\eta^2)\exp\!\Big(-K\frac{F_w}{2L}\Big)$.

Insight. In the globally smooth regime, the stability geometry collapses to the classical condition-number scale L/F_w. The effect of K is transparent: each outer step composes a K-step inner contraction, and force dominance transfers that progress to the target parameter. In the special case $\ell \equiv L$, one may replace the monotone-integral remainder bound by the exact quadratic smoothness model, recovering the classical

admissible step $\gamma_0 = 1/L$ and the contraction factor $\rho = \eta^2 + (1-\eta^2)\big(1 - \frac{F_w}{L}\big)^K$, matching Proposition 5 of Asadi et al. (2023b).[2]

5.2 Regime II: Affine Growth with $\ell(s) = L_0 + L_1 s$ and Initialization-Dependent Rates

Here $\ell(3M/2) = L_0 + \frac{3}{2}L_1 M \Rightarrow \gamma_0 \le \frac{1}{2L_0 + 3L_1 M}$. Thus, Theorem 5.1 yields $\rho = \eta^2 + (1-\eta^2)\Big(1 - \frac{F_w}{2L_0+3L_1 M}\Big)^K$.

Insight. The rate is linear but explicitly trajectory-dependent: the single scalar M sets the curvature amplification $\ell(3M/2)$ and therefore caps the globally admissible constant step. The critical threshold reads $K \gtrsim (L_0 + \frac{3}{2}L_1 M)/F_w$: if M is large, a fixed K yields a weak inner contraction and the outer progress is dominated by the forcing ratio η; increasing K restores a strong linear rate.

5.3 Regime III: Polynomial Growth with $\ell(s) = L_0 + L_1 s^p$, $p \in [0, 2]$

In this regime, $\ell(3M/2) = L_0 + L_1\big(\frac{3M}{2}\big)^p \Rightarrow \gamma_0 \le \frac{1}{2\big(L_0 + L_1(3M/2)^p\big)}$. Hence, $\rho = \eta^2 + (1-\eta^2)\Big(1 - \frac{F_w}{2(L_0+L_1(3M/2)^p)}\Big)^K$.

Insight. The admissible constant step deteriorates polynomially with the envelope: $\gamma_0^{\max} = \Theta(M^{-p})$ once M is large. Equivalently, the critical inner-effort threshold is $K \gtrsim (L_0 + L_1(3M/2)^p)/F_w$. For $p \le 2$, this yields a controlled sensitivity law: curvature amplification does not explode faster than quadratically in the gradient scale, so a constant-step linear regime remains meaningful as long as the trajectory stays within a moderate envelope.

5.4 Regime IV: Superquadratic Growth with $\ell(s) = L_0 + L_1 s^p$, $p > 2$

This specialization yields the same expression for p with $p > 2$.

Insight. Superquadratic growth produces a sharp degradation in admissible constant steps: $\gamma_0^{\max} = \Theta(M^{-p})$ collapses rapidly, and the critical inner effort scales as $K \gtrsim M^p/F_w$ (up to constants). Theorem 5.1 still guarantees linear convergence, but it predicts a practical fragility: if M is large and K is fixed, the inner contraction becomes negligible and progress is limited by the forcing ratio η. This delineates when constant-step TD is *theoretically stable but practically conservative*, motivating staged strategies such as: (i) run curvature-check in a burn-in phase to reduce the envelope, then switch to a constant-step linear regime, or (ii) use adaptive steps that track local curvature growth more tightly than a worst-case bound at M.

5.5 Regime V: Exponential Growth with $\ell(s) = L_0 + L_1 s^2 e^s$

In this regime, $\ell(3M/2) = L_0 + L_1\Big(\frac{3M}{2}\Big)^2 e^{3M/2} = L_0 + \frac{9}{4}L_1 M^2 e^{3M/2}$, so, $\gamma_0 \le \frac{1}{2L_0 + \frac{9}{2}L_1 M^2 e^{3M/2}}$, $\rho = \eta^2 + (1-\eta^2)\Big(1 - \frac{F_w}{2L_0 + \frac{9}{2}L_1 M^2 e^{3M/2}}\Big)^K$.

Insight. Here the admissible constant step decays essentially like $e^{-3M/2}/M^2$, and the critical inner effort $K \gtrsim \ell(3M/2)/F_w$ becomes astronomically large once M is moderate. This regime makes the role of curvature checks concrete: a globally safe constant step can be so small as to be useless, while the curvature-checked rule remains stable in steep regions and recovers a constant-step linear regime once the trajectory enters a smaller envelope.

6 Related Work

Target networks, inexact inner solves, and force-dominance views of TD. A recent line of work has moved beyond purely operator-based analyses by treating TD with target networks as a two-level op-

[2] Our $1/(2L)$ admissibility arises from a deliberately robust remainder bound that remains valid under curvature growth; tightening it in the constant-L case recovers the classical constant exactly.

timization recursion. In particular, Asadi et al. (2023b) formalize TD with frozen targets as an iterative optimization procedure and show that stability can be characterized through a *force-dominance* condition balancing inner curvature against target drift; their linear convergence guarantees, however, certify inner descent using a global Lipschitz-smoothness assumption on the inner objective. Complementing this view, Fellows et al. (2023) analyze *partially fitted* TD methods and show that slower target updates can mitigate poor conditioning of the TD Jacobian, providing a principled explanation for the stabilizing effect of target networks. More recently, Weissmann et al. (2026) study target update frequencies in Q-learning and derive finite-time tradeoffs induced by periodic target fixing, motivating principled update schedules. Collectively, these works position TD as a coupled two-level dynamical system in which both inner optimization dynamics and target drift determine stability.

Stabilization under the deadly triad. Several works revisit stability mechanisms for TD-style updates under bootstrapping, function approximation, and off-policy sampling. Zhang et al. (2021) analyze target-network mechanisms in linear settings and provide convergent variants that incorporate projection and regularization, establishing convergence to well-defined regularized fixed points in regimes where standard methods can diverge. At the same time, Manek & Kolter (2022) show that regularization alone does not universally prevent divergence and can introduce new instability modes, underscoring that stabilization is structural rather than incidental. Gradient-TD variants provide alternative routes to soundness while remaining close to practical TD updates; for example, Ghiassian et al. (2020) propose TDRC, a regularized correction that improves stability in challenging off-policy regimes.

Optimization under curvature growth and generalized smoothness. On the optimization side, there has been substantial progress in analyzing gradient methods when curvature is non-uniform and can grow with gradient scale. This theoretical shift is largely motivated by empirical observations in deep learning, where training trajectories routinely exhibit progressive sharpening and edge-of-stability dynamics that violate global Lipschitz-gradient assumptions (Cohen et al., 2021; Lyle et al., 2023). To address this, clipping-based analyses under (L_0, L_1)-type models show that gradient-dependent stepsizes can guarantee descent even when a global smoothness constant is unavailable (Zhang et al., 2020). More general $\ell(\cdot)$-smoothness frameworks replace uniform bounds with curvature-growth profiles and obtain convergence guarantees by controlling the gradient trajectory directly (Li et al., 2023; Tyurin, 2024). For convex (L_0, L_1)-smooth objectives, refined analyses clarify when clipping-like rules are sufficient and how to avoid overly pessimistic initialization dependence (Gorbunov et al., 2025).

7 Conclusion

We developed a convergence theory for temporal-difference learning with target networks that retains the force-dominance stability lens while removing the global smoothness requirement on the inner objective. By modeling upper curvature through a generalized smoothness profile $\ell(\cdot)$ and analyzing a curvature-checked stepsize rule, we proved global linear convergence for inexact TD with K inner steps under a single trajectory-dependent admissibility condition determined by the worst gradient scale encountered along the run. This yields an explicit stability scaling law: curvature amplification along the trajectory, rather than curvature at the optimum, governs the largest admissible constant stepsize and the inner effort required to maintain a fixed contraction. While our framework cleanly recovers classical smooth rates as a special case, its primary contribution is mathematically formalizing the stability penalties induced by high-curvature initialization and large early Bellman residual–induced gradient magnitudes. Ultimately, this provides a rigorous theoretical bridge between the empirical pathologies of modern deep RL and classical optimization, demonstrating that robust stepsize fail-safes and fast geometric contraction are not mutually exclusive.

Limitations and Future Work

While our framework characterizes constant-stepsize stability under generalized smoothness, its practical scope has limits. Our rates depend on a worst-case gradient envelope M, which can make the admissible γ_0 conservative when trajectories stay in flatter regions. The stepsize check also presumes access to the curvature profile $\ell(\cdot)$, which would need to be estimated online in practice. Finally, extending the argument to stochastic regimes and introducing non-convexity in w remains future work.

References

Kavosh Asadi, Rasool Fakoor, and Shoham Sabach. Resetting the optimizer in deep rl: An empirical study. *Advances in Neural Information Processing Systems*, 36:72284–72324, 2023a.

Kavosh Asadi, Shoham Sabach, Yao Liu, Omer Gottesman, and Rasool Fakoor. Td convergence: An optimization perspective. *Advances in Neural Information Processing Systems*, 36:49169–49186, 2023b.

Leemon Baird. Residual algorithms: Reinforcement learning with function approximation. In *Machine learning proceedings 1995*, pp. 30–37. Elsevier, 1995.

Joan Bas-Serrano, Sebastian Curi, Andreas Krause, and Gergely Neu. Logistic q-learning. In *International conference on artificial intelligence and statistics*, pp. 3610–3618. PMLR, 2021.

Emmanuel Bengio, Joelle Pineau, and Doina Precup. Interference and generalization in temporal difference learning. In *International Conference on Machine Learning*, pp. 767–777. PMLR, 2020.

Jeremy Cohen, Simran Kaur, Yuanzhi Li, J Zico Kolter, and Ameet Talwalkar. Gradient descent on neural networks typically occurs at the edge of stability. In *International Conference on Learning Representations*, 2021.

Alex Damian, Eshaan Nichani, and Jason D. Lee. Self-stabilization: The implicit bias of gradient descent at the edge of stability. In *The Eleventh International Conference on Learning Representations*, 2023. URL `https://openreview.net/forum?id=nhKHA59gXz`.

Mattie Fellows, Matthew J. A. Smith, and Shimon Whiteson. Why target networks stabilise temporal difference methods. In *Proceedings of the 40th International Conference on Machine Learning (ICML)*, pp. 9886–9909. PMLR, 2023.

Sina Ghiassian, Andrew Patterson, Shivam Garg, Dhawal Gupta, Adam White, and Martha White. Gradient temporal-difference learning with regularized corrections. In *Proceedings of the 37th International Conference on Machine Learning (ICML)*, pp. 3524–3534. PMLR, 2020.

Justin Gilmer, Behrooz Ghorbani, Ankush Garg, Sneha Kudugunta, Behnam Neyshabur, David Cardoze, George Edward Dahl, Zachary Nado, and Orhan Firat. A loss curvature perspective on training instabilities of deep learning models. In *International Conference on Learning Representations*, 2022.

Eduard Gorbunov, Nazarii Tupitsa, Sayantan Choudhury, Alen Aliev, Peter Richtarik, Samuel Horváth, and Martin Takáč. Methods for convex (l_0, l_1)-smooth optimization: Clipping, acceleration, and adaptivity. In *International Conference on Learning Representations (ICLR)*, 2025.

Geoffrey J Gordon. Stable function approximation in dynamic programming. In *Machine learning proceedings 1995*, pp. 261–268. Elsevier, 1995.

Donghwan Lee and Niao He. Target-based temporal-difference learning. In *International Conference on Machine Learning*, pp. 3713–3722. PMLR, 2019.

Haochuan Li, Jian Qian, Yi Tian, Alexander Rakhlin, and Ali Jadbabaie. Convex and non-convex optimization under generalized smoothness. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2023.

Philip M Long and Peter L Bartlett. Sharpness-aware minimization and the edge of stability. *Journal of Machine Learning Research*, 25(179):1–20, 2024.

Clare Lyle, Zeyu Zheng, Evgenii Nikishin, Bernardo Avila Pires, Razvan Pascanu, and Will Dabney. Understanding plasticity in neural networks. In *International Conference on Machine Learning*, pp. 23190–23211. PMLR, 2023.

Gaurav Manek and J. Zico Kolter. The pitfalls of regularization in off-policy td learning. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2022.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A Rusu, Joel Veness, Marc G Bellemare, Alex Graves, Martin Riedmiller, Andreas K Fidjeland, Georg Ostrovski, et al. Human-level control through deep reinforcement learning. *nature*, 518(7540):529–533, 2015.

Doina Precup and Richard S Sutton. Exponentiated gradient methods for reinforcement learning. In *ICML*, pp. 272–277, 1997.

Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

Vincent Roulet, Atish Agarwala, Jean-Bastien Grill, Grzegorz Swirszcz, Mathieu Blondel, and Fabian Pedregosa. Stepping on the edge: Curvature aware learning rate tuners. *Advances in Neural Information Processing Systems*, 37:47708–47740, 2024.

David Silver, Aja Huang, Chris J Maddison, Arthur Guez, Laurent Sifre, George Van Den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, et al. Mastering the game of go with deep neural networks and tree search. *nature*, 529(7587):484–489, 2016.

Richard S Sutton. Learning to predict by the methods of temporal differences. *Machine learning*, 3(1):9–44, 1988.

Richard S Sutton, Andrew G Barto, et al. *Reinforcement learning: An introduction*, volume 1. MIT press Cambridge, 1998.

John Tsitsiklis and Benjamin Van Roy. Analysis of temporal-diffference learning with function approximation. *Advances in neural information processing systems*, 9, 1996.

Alexander Tyurin. Toward a unified theory of gradient descent under generalized smoothness. *arXiv preprint arXiv:2412.11773*, 2024.

Simon Weissmann, Tilman Aach, Benedikt Wille, Sebastian Kassing, and Leif Döring. The role of target update frequencies in q-learning. arXiv preprint, 2026. arXiv:2602.03911.

Bohang Zhang, Jikai Jin, Cong Fang, and Liwei Wang. Improved analysis of clipping algorithms for non-convex optimization. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2020. arXiv:2010.02519.

Shangtong Zhang, Hengshuai Yao, and Shimon Whiteson. Breaking the deadly triad with a target network. In *Proceedings of the 38th International Conference on Machine Learning (ICML)*, pp. 12621–12631. PMLR, 2021.

A Appendix

A.1 Proofs of Technical Lemmas, Propositions, and Theorem(s)

We will start by proving Lemma 4.6. Before that, we introduce the following lemma.

Lemma A.1 (Curvature check implies $c_{t,k} \leq \frac{1}{2}$)**.** *Assume $g_{t,k} > 0$ and choose $\gamma_{t,k}$ by Definition 4.5. Then*

$$q^{-1}(\gamma_{t,k} g_{t,k};\, g_{t,k}) \leq \frac{g_{t,k}}{2}. \tag{25}$$

Proof. If $\gamma_{t,k} = \bar{\gamma}_n$, then $\gamma_{t,k} \leq \gamma_{t,k}^{\text{safe}}$ by Definition 4.5, so in all cases we have $\gamma_{t,k} \leq \gamma_{t,k}^{\text{safe}}$. Multiply by $g_{t,k} > 0$ and use the definition in (10):

$$\gamma_{t,k} g_{t,k} \leq q\Big(\frac{g_{t,k}}{2};\, g_{t,k}\Big)\,.$$

Since $q(\cdot\,; g_{t,k})$ is increasing, applying $q^{-1}(\cdot\,; g_{t,k})$ yields $q^{-1}(\gamma_{t,k} g_{t,k}; g_{t,k}) \leq g_{t,k}/2$. □

Lemma A.2 ((Lemma 4.5) One-step descent certificate)**.** *Under Assumption 3.1 and choosing $\gamma_{t,k}$ by Definition 4.5, each inner update satisfies*

$$f_t(w_{t,k+1}) - f_t(w_{t,k}) \le -\frac{\gamma_{t,k}}{2}\,\|\nabla f_t(w_{t,k})\|^2 = -\frac{1}{2\gamma_{t,k}}\,\|w_{t,k+1} - w_{t,k}\|^2. \tag{26}$$

Proof. If $g_{t,k} = 0$, then $w_{t,k+1} = w_{t,k}$ and the claim holds trivially. Assume $g_{t,k} > 0$ and set $x = w_{t,k}$, $y = w_{t,k+1} = x - \gamma_{t,k}\nabla f_t(x)$. By Lemma 3.5 and using the property of monotone integral $\int_0^r q^{-1}(\tau; a)\, d\tau \le r\, q^{-1}(r; a)$, $r \in [0, q_{\max}(a))$,

$$f_t(y) \le f_t(x) + \langle \nabla f_t(x), y - x\rangle + \|y - x\|\, q^{-1}(\|y - x\|; \|\nabla f_t(x)\|).$$

Here $\langle \nabla f_t(x), y - x\rangle = -\gamma_{t,k} g_{t,k}^2$ and $\|y - x\| = \gamma_{t,k} g_{t,k}$, so

$$f_t(y) - f_t(x) \le -\gamma_{t,k} g_{t,k}^2 + (\gamma_{t,k} g_{t,k})\, q^{-1}(\gamma_{t,k} g_{t,k}; g_{t,k}).$$

By Lemma A.1, $q^{-1}(\gamma_{t,k} g_{t,k}; g_{t,k}) \le g_{t,k}/2$, hence

$$f_t(y) - f_t(x) \le -\gamma_{t,k} g_{t,k}^2 + (\gamma_{t,k} g_{t,k})\frac{g_{t,k}}{2} = -\frac{\gamma_{t,k}}{2} g_{t,k}^2,$$

which is the first inequality in (26). The second equality follows from $w_{t,k+1} - w_{t,k} = -\gamma_{t,k}\nabla f_t(w_{t,k})$. □

Proposition 5 ((Proposition 2) Key suboptimality bound)**.** *Under assumptions 4.2 and 4.3 and choosing $\gamma_{t,k}$ by Definition 4.5, for every inner step (t, k),*

$$H(\theta_t, \theta^\star) - H(\theta_t, w_{t,k}) \;\le\; \frac{\Delta_t^2}{2F_w} - \frac{1}{2\gamma_{t,k}}\big\|w_{t,k+1} - w_{t,k}\big\|^2. \tag{27}$$

Proof. Fix t and write $f_t(w) = H(\theta_t, w)$. By F_w-strong convexity of f_t, for any y,

$$f_t(y) \ge f_t(\theta^\star) + \langle \nabla f_t(\theta^\star), y - \theta^\star\rangle + \frac{F_w}{2}\|y - \theta^\star\|^2.$$

Rearranging with $y = w_{t,k+1}$ gives

$$f_t(\theta^\star) - f_t(w_{t,k+1}) \le \langle \nabla f_t(\theta^\star), \theta^\star - w_{t,k+1}\rangle - \frac{F_w}{2}\|w_{t,k+1} - \theta^\star\|^2. \tag{28}$$

Apply Young's inequality $\langle a, b\rangle \le \frac{1}{2F_w}\|a\|^2 + \frac{F_w}{2}\|b\|^2$ with $a = \nabla f_t(\theta^\star) = \nabla_w H(\theta_t, \theta^\star)$ and $b = \theta^\star - w_{t,k+1}$. The quadratic term cancels the negative term in (28), yielding

$$f_t(\theta^\star) - f_t(w_{t,k+1}) \le \frac{1}{2F_w}\|\nabla_w H(\theta_t, \theta^\star)\|^2 = \frac{\Delta_t^2}{2F_w}.$$

Next, by Lemma 4.6,

$$f_t(w_{t,k+1}) - f_t(w_{t,k}) \le -\frac{1}{2\gamma_{t,k}}\|w_{t,k+1} - w_{t,k}\|^2.$$

Add the last two inequalities and cancel $f_t(w_{t,k+1})$ to obtain (12). □

Proposition 6 ((Proposition 3) Inner-step contraction with forcing)**.** *Define the effective contraction rate $\kappa_{t,k} \triangleq F_w \gamma_{t,k}$. Then, for all t and all k,*

$$\|w_{t,k+1} - \theta^\star\|^2 \;\le\; (1 - \kappa_{t,k})\|w_{t,k} - \theta^\star\|^2 + \kappa_{t,k}\frac{\Delta_t^2}{F_w^2}. \tag{29}$$

Proof. Fix t and write $f_t(w) = H(\theta_t, w)$. Expand the square using the update $w_{t,k+1} = w_{t,k} - \gamma_{t,k}\nabla f_t(w_{t,k})$:

$$\|w_{t,k+1} - \theta^\star\|^2 = \|w_{t,k} - \theta^\star\|^2 + 2\gamma_{t,k}\langle \nabla f_t(w_{t,k}), \theta^\star - w_{t,k}\rangle + \|w_{t,k+1} - w_{t,k}\|^2. \tag{30}$$

By strong convexity of f_t,

$$\langle \nabla f_t(w_{t,k}), \theta^\star - w_{t,k}\rangle \le f_t(\theta^\star) - f_t(w_{t,k}) - \frac{F_w}{2}\|w_{t,k} - \theta^\star\|^2. \tag{31}$$

Substitute (31) into (30):

$$\|w_{t,k+1} - \theta^\star\|^2 \le \big(1 - \gamma_{t,k}F_w\big)\|w_{t,k} - \theta^\star\|^2 + 2\gamma_{t,k}\big(f_t(\theta^\star) - f_t(w_{t,k})\big) + \|w_{t,k+1} - w_{t,k}\|^2.$$

Now apply Proposition 2:

$$f_t(\theta^\star) - f_t(w_{t,k}) \le \frac{\Delta_t^2}{2F_w} - \frac{1}{2\gamma_{t,k}}\|w_{t,k+1} - w_{t,k}\|^2.$$

Multiplying by $2\gamma_{t,k}$ cancels the $\|w_{t,k+1} - w_{t,k}\|^2$ term exactly, leaving

$$\|w_{t,k+1} - \theta^\star\|^2 \le \big(1 - \gamma_{t,k}F_w\big)\|w_{t,k} - \theta^\star\|^2 + \gamma_{t,k}\frac{\Delta_t^2}{F_w}.$$

Finally, set $\kappa_{t,k} = F_w\gamma_{t,k}$ and rewrite the forcing term as $\gamma_{t,k}\frac{\Delta_t^2}{F_w} = \kappa_{t,k}\frac{\Delta_t^2}{F_w^2}$, which yields (29). □

Proposition 7 ((Proposition 4) Outer recursion)**.** *Define the cumulative decay*

$$\chi_{t,K} \triangleq \prod_{k=0}^{K-1}(1 - \kappa_{t,k}), \qquad \eta_t^2 \triangleq \frac{\Delta_t^2}{F_w^2}.$$

Then,

$$\|\theta_{t+1} - \theta^\star\|^2 \;\le\; \chi_{t,K}\|\theta_t - \theta^\star\|^2 + \big(1 - \chi_{t,K}\big)\eta_t^2. \tag{32}$$

Proof. Fix t and let $x_k = \|w_{t,k} - \theta^\star\|^2$. From Proposition 2,

$$x_{k+1} \le (1 - \kappa_{t,k})x_k + \kappa_{t,k}\eta_t^2.$$

Let $d_k = x_k - \eta_t^2$. Then $d_{k+1} \le (1 - \kappa_{t,k})d_k$. Iterating gives $d_K \le \chi_{t,K}d_0$. Undoing the shift and using $w_{t,0} = \theta_t$, $w_{t,K} = \theta_{t+1}$ yields (32). □

Lemma A.3 ((Lemma 4.7) Bounded iterates and a uniform gradient envelope)**.** *Assume $H \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$, Propositions 2– 4 holds, the force-dominance condition $F_\theta < F_w$ holds, and the constant baseline stepsize satisfies $\gamma_0 < 1/F_w$. Let $R \triangleq \|\theta_0 - \theta^\star\|$ and define*

$$M \;\triangleq\; \sup\Big\{\|\nabla_w H(\theta, w)\| :\; \|\theta - \theta^\star\| \le R,\; \|w - \theta^\star\| \le R\Big\}. \tag{33}$$

Then the following hold:

1. ***(Bounded outer iterates)*** *For all $t \ge 0$, $\|\theta_t - \theta^\star\| \le R$.*
2. ***(Bounded inner iterates)*** *For all $t \ge 0$ and all $k \in \{0, \dots, K\}$, $\|w_{t,k} - \theta^\star\| \le R$.*
3. ***(Uniform gradient envelope)*** *The constant M in (33) is finite and for all t, k,*

$$g_{t,k} = \|\nabla f_t(w_{t,k})\| = \|\nabla_w H(\theta_t, w_{t,k})\| \le M < \infty.$$

Proof. Throughout the proof, recall that $f_t(w) = H(\theta_t, w)$ and that $\theta_{t+1} = w_{t,K}$.

Step 1: outer contraction and boundedness of $\{\theta_t\}$. From the outer recursion (Proposition 4) and the cross-gradient Lipschitz bound $\Delta_t = \|\nabla_w H(\theta_t, \theta^\star)\| \le F_\theta \|\theta_t - \theta^\star\|$, we have

$$\eta_t^2 = \frac{\Delta_t^2}{F_w^2} \le \eta^2 \|\theta_t - \theta^\star\|^2, \qquad \eta \triangleq \frac{F_\theta}{F_w} \in (0,1).$$

Substituting into (32) yields

$$\|\theta_{t+1} - \theta^\star\|^2 \le \Big(\eta^2 + (1-\eta^2)\chi_{t,K}\Big)\|\theta_t - \theta^\star\|^2. \tag{34}$$

We now show $\chi_{t,K} \in (0,1)$ for all t. By construction, $\gamma_{t,k} = \min\{\gamma_0, \gamma_{t,k}^{\text{safe}}\} > 0$, and since $\gamma_0 < 1/F_w$ we have $0 < F_w\gamma_{t,k} \le F_w\gamma_0 < 1$. Hence each factor $(1 - F_w\gamma_{t,k}) \in (0,1)$ and therefore $\chi_{t,K} = \prod_{k=0}^{K-1}(1 - F_w\gamma_{t,k}) \in (0,1)$. Consequently, the multiplier in (34) satisfies

$$0 < \eta^2 + (1-\eta^2)\chi_{t,K} < \eta^2 + (1-\eta^2)\cdot 1 = 1.$$

Thus $\|\theta_{t+1} - \theta^\star\|^2 \le \|\theta_t - \theta^\star\|^2$ for all t, and by induction

$$\|\theta_t - \theta^\star\| \le \|\theta_0 - \theta^\star\| = R, \qquad \forall t \ge 0,$$

proving (i).

Step 2: boundedness of the inner iterates $\{w_{t,k}\}$. Fix an outer iteration t and define $x_k \triangleq \|w_{t,k} - \theta^\star\|^2$ for $k = 0, \dots, K$. From Proposition 3,

$$x_{k+1} \le (1 - \kappa_{t,k})x_k + \kappa_{t,k}\eta_t^2, \qquad \kappa_{t,k} = F_w\gamma_{t,k} \in (0,1). \tag{35}$$

By the mismatch bound and Step 1, we have

$$\eta_t^2 = \frac{\Delta_t^2}{F_w^2} \le \eta^2\|\theta_t - \theta^\star\|^2 \le \eta^2 R^2 \le R^2,$$

since $\eta \in (0,1)$. Moreover $w_{t,0} = \theta_t$ implies

$$x_0 = \|w_{t,0} - \theta^\star\|^2 = \|\theta_t - \theta^\star\|^2 \le R^2.$$

We claim that $x_k \le R^2$ for all k. Indeed, assume $x_k \le R^2$; then from (35) and $\eta_t^2 \le R^2$,

$$x_{k+1} \le (1 - \kappa_{t,k})R^2 + \kappa_{t,k}R^2 = R^2.$$

Thus, by induction, $x_k \le R^2$ for all k, i.e.,

$$\|w_{t,k} - \theta^\star\| \le R \qquad \forall\, k \in \{0, \dots, K\}.$$

Since t was arbitrary, this proves (ii).

Step 3: finiteness of M and the uniform gradient bound. By Steps 1–2, every pair $(\theta_t, w_{t,k})$ lies in the compact product ball

$$\mathcal{B} \triangleq \big\{(\theta, w) : \|\theta - \theta^\star\| \le R,\ \|w - \theta^\star\| \le R\big\}.$$

Because $H \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$, the map $(\theta, w) \mapsto \|\nabla_w H(\theta, w)\|$ is continuous. A continuous function attains its supremum on a compact set, so the quantity M defined in (33) is finite. Finally, since $(\theta_t, w_{t,k}) \in \mathcal{B}$ for all t, k, we have

$$g_{t,k} = \|\nabla_w H(\theta_t, w_{t,k})\| \le M,$$

which proves (iii). □

Theorem A.4 ((Theorem 1) Unifying constant-stepsize convergence under ℓ-smoothness)**.** *Assume Assumptions 3.1, 4.1, 4.2, 4.3, and suppose $H \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$. Let M be the gradient envelope from Lemma 4.7. Assume force dominance $F_\theta < F_w$ and define $\eta \triangleq F_\theta / F_w \in (0,1)$. Choose a constant baseline $\gamma_0 > 0$ satisfying*

$$\gamma_0 \leq \frac{1}{2\,\ell(3M/2)}. \tag{36}$$

Run Algorithm 1 with stepsizes $\gamma_{t,k} = \min\{\gamma_0, \gamma_{t,k}^{\text{safe}}\}$. Then the curvature check never clips, $\gamma_{t,k} \equiv \gamma_0$, and the outer iterates contract linearly:

$$\|\theta_{t+1} - \theta^\star\|^2 \;\leq\; \rho\,\|\theta_t - \theta^\star\|^2, \qquad \rho \triangleq \eta^2 + (1-\eta^2)(1 - F_w\gamma_0)^K \in (0,1). \tag{37}$$

Consequently, $\|\theta_t - \theta^\star\|^2 \leq \rho^{\,t}\|\theta_0 - \theta^\star\|^2$.

Proof. By (17) and $\gamma_0 \leq \gamma_{\min}^{\text{safe}}$, we have $\gamma_0 \leq \gamma_{t,k}^{\text{safe}}$ for all (t,k), so $\gamma_{t,k} = \min\{\gamma_0, \gamma_{t,k}^{\text{safe}}\} = \gamma_0$ and no clipping occurs. With $\gamma_{t,k} \equiv \gamma_0$, we have $\kappa_{t,k} = F_w\gamma_0 \triangleq \kappa \in (0,1)$ since $\gamma_0 < 1/2\ell(3M/2)$ (this is sufficient to ensure that $\gamma_0 < 1/F_w$), and hence

$$\chi_{t,K} = \prod_{k=0}^{K-1}(1-\kappa) = (1-\kappa)^K = (1 - F_w\gamma_0)^K \qquad \text{for all } t.$$

By Assumption 4.1 and the fixed-point condition, $\Delta_t = \|\nabla_w H(\theta_t, \theta^\star)\| \leq F_\theta\|\theta_t - \theta^\star\|$, so $\eta_t^2 = \Delta_t^2 / F_w^2 \leq \eta^2\|\theta_t - \theta^\star\|^2$. Substituting into the outer recursion equation 14 yields

$$\|\theta_{t+1} - \theta^\star\|^2 \leq \chi_{t,K}\|\theta_t - \theta^\star\|^2 + (1-\chi_{t,K})\eta^2\|\theta_t - \theta^\star\|^2 = \big(\eta^2 + (1-\eta^2)\chi_{t,K}\big)\|\theta_t - \theta^\star\|^2,$$

and inserting $\chi_{t,K} = (1 - F_w\gamma_0)^K$ gives (37). Since $\eta \in (0,1)$ and $(1 - F_w\gamma_0)^K \in (0,1)$, we have $\rho \in (0,1)$. Iterating yields $\|\theta_t - \theta^\star\|^2 \leq \rho^{\,t}\|\theta_0 - \theta^\star\|^2$. □