FASTER BINARY EMBEDDINGS FOR PRESERVING EUCLIDEAN DISTANCES

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ABSTRACT

We propose a fast, distance-preserving, binary embedding algorithm to transform a high-dimensional dataset $\mathcal{T} \subseteq \mathbb{R}^n$ into binary sequences in the cube $\{\pm 1\}^m$. When $\mathcal{T}$ consists of well-spread (i.e., non-sparse) vectors, our embedding method applies a stable noise-shaping quantization scheme to $\mathbf{A}\boldsymbol{x}$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a sparse Gaussian random matrix. This contrasts with most binary embedding methods, which usually use $\boldsymbol{x} \mapsto \operatorname{sign}(\mathbf{A}\boldsymbol{x})$ for the embedding. Moreover, we show that Euclidean distances among the elements of $\mathcal{T}$ are approximated by the ℓ_1 norm on the images of $\{\pm 1\}^m$ under a fast linear transformation. This again contrasts with standard methods, where the Hamming distance is used instead. Our method is both fast and memory efficient, with time complexity $O(m)$ and space complexity $O(m)$. Further, we prove that the method is accurate and its associated error is comparable to that of a continuous valued Johnson-Lindenstrauss embedding plus a quantization error that admits a polynomial decay as the embedding dimension m increases. Thus the length of the binary codes required to achieve a desired accuracy is quite small, and we show it can even be compressed further without compromising the accuracy. To illustrate our results, we test the proposed method on natural images and show that it achieves strong performance.

1 INTRODUCTION

Analyzing large data sets of high-dimensional raw data is usually computationally demanding and memory intensive. As a result, it is often necessary as a preprocessing step to transform data into a lower-dimensional space while approximately preserving important geometric properties, such as pairwise ℓ_2 distances. As a critical result in dimensionality reduction, the Johnson-Lindenstrauss (JL) lemma (Johnson & Lindenstrauss, 1984) guarantees that every finite set $\mathcal{T} \subseteq \mathbb{R}^n$ can be (linearly) mapped to a $m = O(\epsilon^{-2}\log(|\mathcal{T}|))$ dimensional space in such a way that all pairwise distances are preserved up to an ϵ-Lipschitz distortion. Additionally, there are many significant results to speed up the JL transform by introducing fast embeddings, e.g. (Ailon & Chazelle, 2009; Ailon & Liberty, 2013; Krahmer & Ward, 2011; Nelson et al., 2014), or by using sparse matrices (Kane & Nelson, 2014; 2010). Such fast embeddings can usually be computed in $O(n\log n)$ versus the $O(mn)$ time complexity of JL transforms that rely on unstructured dense matrices.

1.1 RELATED WORK

To further reduce memory requirements, progress has been made in *nonlinearly* embedding high-dimensional sets $\mathcal{T} \subseteq \mathbb{R}^n$ to the binary cube $\{-1,1\}^m$ with $m \ll n$, a process known as binary embedding. Provided that $d_1(\cdot,\cdot)$ is a metric on $\mathbb{R}^n$, a distace preserving binary embedding is a map $f : \mathcal{T} \to \{-1,1\}^m$ and a function $d_2(\cdot,\cdot)$ on $\{-1,1\}^m \times \{-1,1\}^m$ to approximate distances, i.e.,

$$|d_2(f(\boldsymbol{x}), f(\boldsymbol{y})) - d_1(\boldsymbol{x},\boldsymbol{y})| \le \alpha, \quad \text{for } \forall \boldsymbol{x},\boldsymbol{y} \in \mathcal{T}. \tag{1}$$

The potential dimensionality reduction $(m \ll n)$ and 1-bit representation per dimension imply that storage space can be considerably reduced and downstream applications like learning and retrieval can happen directly using bitwise operations. Most existing nonlinear mappings f in (1) are generated using simple memory-less scalar quantization (MSQ). For example, given a set of unit vectors $\mathcal{T} \subseteq \mathbb{S}^{n-1}$ with finite size $|\mathcal{T}|$, consider the map

$$\boldsymbol{q}_{\boldsymbol{x}} := f(\boldsymbol{x}) = \operatorname{sign}(\mathbf{G}\boldsymbol{x}) \tag{2}$$

where $\mathbf{G} \in \mathbb{R}^{m\times n}$ is a standard Gaussian random matrix and $\operatorname{sign}(\cdot)$ returns the element-wise sign of its argument. Let $d_1(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\pi}\arccos(\|\boldsymbol{x}\|_2^{-1}\|\boldsymbol{y}\|_2^{-1}\langle \boldsymbol{x},\boldsymbol{y}\rangle)$ be the normalized angular distance and $d_2(\boldsymbol{q_x},\boldsymbol{q_y}) = \frac{1}{2m}\|\boldsymbol{q_x}-\boldsymbol{q_y}\|_1$ be the normalized Hamming distance. Then, Yi et al. (2015) show that (1) holds with probability at least $1-\eta$ if $m \gtrsim \alpha^{-2}\log(|\mathcal{T}|/\eta)$, so one can approximate geodesic distances with normalized Hamming distances. While this approach achieves optimal bit complexity (up to constants) (Yi et al., 2015), it has been observed in practice that m is usually around $O(n)$ to guarantee reasonable accuracy (Gong et al., 2013; Sánchez & Perronnin, 2011; Yu et al., 2014). Much like linear JL embedding techniques admit fast counterparts, fast binary embedding algorithms have been developed to significantly reduce the runtime of binary embeddings (Gong et al., 2012b; Liu et al., 2011; Gong et al., 2012a; 2013; Li et al., 2011; Raginsky & Lazebnik, 2009). Indeed, fast JL transforms (FJLT) and Gaussian Toeplitz matrices (Yi et al., 2015), structured hashed projections (Choromanska et al., 2016), iterative quantization (Gong et al., 2012b), bilinear projection (Gong et al., 2013), circulant binary embedding (Yu et al., 2014; Dirksen & Stollenwerk, 2018; 2017; Oymak et al., 2017; Kim et al., 2018), sparse projection (Xia et al., 2015), and fast orthogonal projection (Zhang et al., 2015) have all been considered.

These methods can decrease time complexity to $O(n\log n)$ operations per embedding, but still suffer from some important drawbacks. Notably, due to the sign function, these algorithms completely discard all magnitude information, as $\operatorname{sign}(\mathbf{A}\boldsymbol{x}) = \operatorname{sign}(\mathbf{A}(\alpha\boldsymbol{x}))$ for all $\alpha > 0$. So, all points in the same direction embed to the same binary vector and cannot be distinguished. Even if one settles for recovering geodesic distances, using the sign function in (2) is an instance of MSQ so the estimation error α in (1) decays slowly as the number of bits m increases (Yi et al., 2015).

Recently Huynh & Saab (2020) resolved these issues by replacing the simple sign function with a Sigma-Delta ($\Sigma\Delta$) quantization scheme, or alternatively other noise-shaping schemes (see (Chou & Güntürk, 2016)) whose properties will be discussed in Section 3. They use the binary embedding

$$\boldsymbol{q_x} := Q(\mathbf{DB}\boldsymbol{x}) \tag{3}$$

where Q is now a stable $\Sigma\Delta$ quantization scheme, $\mathbf{D} \in \mathbb{R}^{m\times m}$ is a diagonal matrix with random signs, and $\mathbf{B} \in \mathbb{R}^{m\times n}$ are specific structured random matrices. Their choices of $\mathbf{B}$ allow matrix vector multiplication to be implemented using the fast Fourier transform. Then the original Euclidean distance $\|\boldsymbol{x}-\boldsymbol{y}\|_2$ can be recovered via a pseudo-metric on the quantized vectors given by

$$d_{\widetilde{\boldsymbol{V}}}(\boldsymbol{q_x},\boldsymbol{q_y}) := \|\widetilde{\boldsymbol{V}}(\boldsymbol{q_x}-\boldsymbol{q_y})\|_2 \tag{4}$$

where $\widetilde{\boldsymbol{V}} \in \mathbb{R}^{p\times m}$ is a "normalized condensation operator", a sparse matrix that can be applied fast (see Section 3). Regarding the complexity of applying (3) to a single $\boldsymbol{x} \in \mathbb{R}^n$, note that $\boldsymbol{x} \mapsto \mathbf{DB}\boldsymbol{x}$ has time complexity $O(n\log n)$ while the quantization map needs $O(m)$ time and results in an m bit representation. So when $m \le n$, the total time complexity for (3) is around $O(n\log n)$.

1.2 Methods and Contributions

In this paper, we extend these results by replacing $\mathbf{DB}$ in (3) by a sparse Gaussian matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ so that now

$$\boldsymbol{q_x} := Q(\mathbf{A}\boldsymbol{x}). \tag{5}$$

Given scaled high-dimensional data $\mathcal{T} \subset \mathbb{R}^n$ contained in the ℓ_2 ball $B_2^n(\kappa)$ with radius κ, we put forward Algorithm 1 to generate binary sequences and Algorithm 2 to compute estimates of the Euclidean distances between elements of $\mathcal{T}$. The contribution of this work is threefold. First, we prove Theorem 1.1 quantifying the performance of our algorithms.

Algorithm 1: Fast Binary Embedding for Finite $\mathcal{T}$

Input: $\mathcal{T} = \{\boldsymbol{x}^{(j)}\}_{j=1}^k \subseteq B_2^n(\kappa)$ ▷ Scaled data points in ℓ_2 ball
Generate $\mathbf{A} \in \mathbb{R}^{m\times n}$ as in Definition 2.2 ▷ Sparse Gaussian matrix $\mathbf{A}$
for $j \leftarrow 1$ **to** k **do**
  $\boldsymbol{z}^{(j)} \leftarrow \mathbf{A}\boldsymbol{x}^{(j)}$
  $\boldsymbol{q}^{(j)} = Q(\boldsymbol{z}^{(j)})$ ▷ Stable $\Sigma\Delta$ quantizer Q as in (20)
Output: Binary sequences $\mathcal{B} = \{\boldsymbol{q}^{(j)}\}_{j=1}^k \subseteq \{-1,1\}^m$

Algorithm 2: ℓ_2 Norm Distance Recovery

Input: $\boldsymbol{q}^{(i)}, \boldsymbol{q}^{(j)} \in \mathcal{B}$ ▷ Binary sequences produced by Algorithm 1
$\boldsymbol{y}^{(i)} \leftarrow \widetilde{\boldsymbol{V}}\boldsymbol{q}^{(i)}$ ▷ Condense the components of $\boldsymbol{q}$
$\boldsymbol{y}^{(j)} \leftarrow \widetilde{\boldsymbol{V}}\boldsymbol{q}^{(j)}$
Output: $\|\boldsymbol{y}^{(i)} - \boldsymbol{y}^{(j)}\|_1$ ▷ Approximation of $\|\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(j)}\|_2$

Theorem 1.1 (Main result)**.** *Let $\mathcal{T} \subseteq \mathbb{R}^n$ be a finite set with elements satisfying $\|\boldsymbol{x}\|_\infty = O(n^{-1/2}\|\boldsymbol{x}\|_2)$. If $m \gtrsim p := \epsilon^{-2}\log(|\mathcal{T}|)$ where $c > 0$ is a constant and $r \geq 1$ is the integer order of Q, then with high probability on the draw of the sparse Gaussian matrix $\mathbf{A}$, the following holds uniformly over all $\boldsymbol{x}, \boldsymbol{y}$ in $\mathcal{T}$: Embedding $\boldsymbol{x}, \boldsymbol{y}$ into $\{-1, 1\}^m$ using Algorithm 1, and estimating the associated distance between them using Algorithm 2 yields the error bound*

$$\left| d_{\widetilde{\boldsymbol{V}}}(\boldsymbol{q}_{\boldsymbol{x}}, \boldsymbol{q}_y) - \|\boldsymbol{x} - \boldsymbol{y}\|_2 \right| \leq c \left(\frac{m}{p}\right)^{-r+1/2} + \epsilon\|\boldsymbol{x} - \boldsymbol{y}\|_2.$$

Theorem 1.1 yields an approximation error bounded by two components, one due to quantization and another that resembles the error from a *linear* JL embedding into a p-dimensional space. The latter part is essentially proportional to $p^{-1/2}$, while the quantization component decays polynomially fast in m, and can be made harmless by increasing m. Moreover, the number of bits $m \gtrsim \epsilon^{-2}\log(|\mathcal{T}|)$ achieves the optimal bit complexity required by any oblivious random embedding that preserves Euclidean or squared Euclidean distance, see Theorem 4.1 in (Dirksen & Stollenwerk, 2020). We will give a more precise version of Theorem 1.1 in Section 4, with all quantifiers specified explicitly, and a modification to the matrix $\mathbf{A}$ that enables the result to hold for arbitrary finite $\mathcal{T}$.

Second, due to the sparsity of $\mathbf{A}$, (5) can be computed much faster than (3), at the expense of restricting our results to "well-spread" vectors $\boldsymbol{x}$, i.e., those that are not sparse. On the other hand, in Section 5, we show that Algorithm 1 achieves $O(m)$ time and space complexity in contrast with the common $O(n \log n)$ runtime of fast binary embeddings, e.g., (Gong et al., 2013; Yi et al., 2015; Yu et al., 2014; Dirksen & Stollenwerk, 2018; 2017; Huynh & Saab, 2020) that rely on fast JL transforms or circulant matrices. Meanwhile, Algorithm 2 requires only $O(m)$ runtime.

Finally, Definition 2.3 shows that $\widetilde{\boldsymbol{V}}$ is sparse and essentially populated by integers bounded by $(m/p)^r$ where r, m, p are as in Theorem 1.1. In section 5, we note that each $\boldsymbol{y}^{(i)} = \widetilde{\boldsymbol{V}}\boldsymbol{q}^{(i)}$ can be represented by $O(p\log_2(m/p))$ bits, instead of m bits, without affecting the reconstruction accuracy.

2 Preliminaries

2.1 Notation and definitions

Throughout this paper, we use $f(n) = O(g(n))$ to mean that $|f(n)|$ is bounded above by a positive function $g(n)$ up to constants asymptotically; that is, $\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty$. Similarly, we use $f(n) = \Theta(g(n))$ to denote that $f(n)$ is bounded both above and below by a positive function $g(n)$ up to constants asymptotically. We next define operator norms.

Definition 2.1. Let $\alpha, \beta \in [1, \infty]$ be integers. The (α, β) operator norm of $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ is

$$\|\boldsymbol{K}\|_{\alpha,\beta} = \max_{x\neq 0} \frac{\|\boldsymbol{K}\boldsymbol{x}\|_\beta}{\|\boldsymbol{x}\|_\alpha}.$$

We now introduce some notation and definitions that are relevant to our construction.

Definition 2.2 (Sparse Gaussian random matrix)**.** Let $\mathbf{A} = (\mathrm{a}_{ij}) \in \mathbb{R}^{m\times n}$ be a random matrix with i.i.d. entries such that a_{ij} is 0 with probability $1 - s$ and is drawn from $\mathcal{N}(0, \frac{1}{s})$ with probability s.

We can construct a_{ij} by $\mathrm{a}_{ij} = \mathrm{b}_{ij}\mathrm{c}_{ij}$ where $\mathrm{b}_{ij} \sim \text{Bernoulli}(s)$, $\mathrm{c}_{ij} \sim \mathcal{N}(0, \frac{1}{s})$ and all b_{ij} and c_{ij} are independent. Moreover, $\mathbb{E}(\mathrm{a}_{ij}) = 0$ and $\mathrm{Var}(\mathrm{a}_{ij}) = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We adopt the definition of a condensation operator in (Huynh & Saab, 2020), see (Chou & Güntürk, 2016).

Definition 2.3 (Condensation operator)**.** Let p, r, λ be fixed positive integers such that $\lambda = r\widetilde{\lambda} - r + 1$ for some integer $\widetilde{\lambda}$. Let $m = \lambda p$ and $\boldsymbol{v}$ be a row vector in $\mathbb{R}^{\lambda}$ whose entry v_j is the j-th coefficient of the polynomial $(1 + z + \ldots + z^{\widetilde{\lambda}-1})^r$. Define the condensation operator $\boldsymbol{V} \in \mathbb{R}^{p \times m}$ by

$$\boldsymbol{V} = \boldsymbol{I}_p \otimes \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v} & & \\ & \ddots & \\ & & \boldsymbol{v} \end{bmatrix}.$$

For example, when $r = 1$, $\lambda = \widetilde{\lambda}$, and $\boldsymbol{v} \in \mathbb{R}^{\lambda}$ is simply the vector of all ones. The normalized condensation operator is given by

$$\widetilde{\boldsymbol{V}} = \frac{\sqrt{\pi/2}}{p\|\boldsymbol{v}\|_2}\boldsymbol{V}.$$

The fast JL transform was first studied by Ailon & Chazelle (2009). It admits many variants and improvements, e.g. (Krahmer & Ward, 2011; Matoušek, 2008). The idea is that given any $\boldsymbol{x} \in \mathbb{R}^n$ we use a fast "Fourier-like" transform, like the Walsh-Hadamard transform, to distribute the total mass (i.e. $||\boldsymbol{x}||_2$) of $\boldsymbol{x}$ relatively evenly to its coordinates.

Definition 2.4 (FJLT)**.** The fast JL transform can be obtained by

$$\boldsymbol{\Phi} := \mathbf{A}\boldsymbol{H}\mathbf{D} \in \mathbb{R}^{m \times n}. \tag{6}$$

Here, $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a sparse Gaussian random matrix, as in Definition 2.2, while $\boldsymbol{H} \in \mathbb{R}^{n \times n}$ is a normalized Walsh-Hadamard matrix defined by $H_{ij} = n^{-1/2}(-1)^{\langle i-1, j-1 \rangle}$ where $\langle i, j \rangle$ is the bitwise dot product of the binary representations of the numbers i and j. Finally, $\mathbf{D} \in \mathbb{R}^{n \times n}$ is diagonal with diagonal entries drawn independently from $\{-1, 1\}$ with probability $1/2$ for each.

2.2 CONDENSED JOHNSON-LINDENSTRAUSS TRANSFORMS

Definition 2.5. When $\widetilde{\boldsymbol{V}}$ is a condensation operator, and $\mathbf{A}$ is a sparse Gaussian, we refer to $\widetilde{\boldsymbol{V}}\mathbf{A}$ as a condensed sparse JL transform (CSJLT). When $\mathbf{A}$ is replaced by $\boldsymbol{\Phi}$ as in Definition 2.4 we refer to $\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}$ as a condensed fast JL transform (CFJLT).

The definition above is justified by the following lemma (see Appendix A for the proof).

Lemma 2.6 (CJLT lemma)**.** *Let $\mathcal{T}$ be a finite subset of $\mathbb{R}^n$, $\lambda \in \mathbb{N}$, $\epsilon \in (0, \frac{1}{2})$, $\delta \in (0, 1)$, $p = O(\epsilon^{-2}\log(|\mathcal{T}|^2/\delta)) \in \mathbb{N}$ and $m = \lambda p$. Let $\widetilde{\boldsymbol{V}} \in \mathbb{R}^{p \times m}$ and $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{H}\mathbf{D} \in \mathbb{R}^{m \times n}$ be as in Definition 2.3 and Definition 2.4 respectively with $s = \Theta(\epsilon^{-1}n^{-1}(\|\boldsymbol{v}\|_\infty/\|\boldsymbol{v}\|_2)^2) \leq 1$. If $\mathcal{T}$ consists of well-spread vectors, that is, $\|\boldsymbol{x}\|_\infty = O(n^{-1/2}\|\boldsymbol{x}\|_2)$ for all $\boldsymbol{x} \in \mathcal{T}$, then*

$$\left| \|\widetilde{\boldsymbol{V}}\mathbf{A}(\boldsymbol{x} - \boldsymbol{y})\|_1 - \|\boldsymbol{x} - \boldsymbol{y}\|_2 \right| \leq \epsilon\|\boldsymbol{x} - \boldsymbol{y}\|_2 \tag{7}$$

holds uniformly for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ with probability at least $1 - \delta$. If $\mathcal{T}$ is finite but arbitrary, then

$$\left| \|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x} - \boldsymbol{y})\|_1 - \|\boldsymbol{x} - \boldsymbol{y}\|_2 \right| \leq \epsilon\|\boldsymbol{x} - \boldsymbol{y}\|_2 \tag{8}$$

holds uniformly for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ with probability at least $1 - \delta$.

So $\mathcal{T} \subseteq \mathbb{R}^n$ is embedded into $\mathbb{R}^p$ with pairwise distances distorted at most ϵ, where $p = O(\epsilon^{-2}\log|\mathcal{T}|)$ as one would expect from a JL embedding. This will be needed to guarantee the accuracy associated with our embeddings algorithms. Note that the bound on p does not require extra logarithmic factors, in contrast to the bound $O(\epsilon^{-2}\log|\mathcal{T}|\log^4 n)$ in (Huynh & Saab, 2020).

3 SIGMA-DELTA QUANTIZATION

An r-th order $\Sigma\Delta$ quantizer $Q^{(r)} : \mathbb{R}^m \to \mathcal{A}^m$ maps an input signal $\boldsymbol{y} = (y_i)_{i=1}^m \in \mathbb{R}^m$ to a quantized sequence $\boldsymbol{q} = (q_i)_{i=1}^m \in \mathcal{A}^m$ via a quantization rule ρ and the following iterations

$$\begin{cases} u_0 = u_{-1} = \ldots = u_{1-r} = 0, \\ q_i = Q(\rho(y_i, u_{i-1}, \ldots, u_{i-r})) \quad \text{for } i = 1, 2, \ldots, m, \\ \boldsymbol{P}^r\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{q} \end{cases} \tag{9}$$

where $Q(y) = \arg\min_{v \in \mathcal{A}} |y - v|$ is the scalar quantizer related to alphabet $\mathcal{A}$ and $\boldsymbol{P} \in \mathbb{R}^{m \times m}$ is the first order difference matrix defined by

$$P_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that (9) is amenable to an iterative update of the state variables u_i as

$$\boldsymbol{P}^r \boldsymbol{u} = \boldsymbol{y} - \boldsymbol{q} \iff u_i = \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} u_{i-j} + y_i - q_i, \quad i = 1, 2, \ldots, m. \tag{10}$$

Definition 3.1. A quantization scheme is *stable* if there exists $\mu > 0$ such that for each input with $\|\boldsymbol{y}\|_\infty \leq \mu$, the state vector $\boldsymbol{u} \in \mathbb{R}^m$ satisfies $\|\boldsymbol{u}\|_\infty \leq C$. Crucially, μ and C do not depend on m.

Stability heavily depends on the choice of quantization rule and is difficult to guarantee for arbitrary ρ in (9) when the alphabet is small, as is the case of 1-bit quantization where $\mathcal{A} = \{\pm 1\}$. When $r = 1$ and $\mathcal{A} = \{\pm 1\}$, the simplest stable $\Sigma\Delta$ scheme $Q^{(1)} : \mathbb{R}^m \to \mathcal{A}^m$ is equipped with the greedy quantization rule $\rho(y_i, u_{i-1}) := u_{i-1} + y_i$ giving the simple iteration

$$\begin{cases} u_0 = 0, \\ q_i = \operatorname{sign}\big(u_{i-1} + y_i\big), \\ u_i = u_{i-1} + y_i - q_i. \end{cases}$$

The design and properties of stable $Q^{(r)}$ with $r \geq 2$ can be found in Appendix B.

4 Main Results

The ingredients that make our construction work are a JL embedding followed by $\Sigma\Delta$ quantization. Together these embed points into $\{\pm 1\}^m$, but it remains to define a pseudometric so that we may approximate Euclidean distances by distances on the cube. We now define this pseudometric.

Definition 4.1. Let $\mathcal{A}^m = \{\pm 1\}^m$ and let $\boldsymbol{V} \in \mathbb{R}^{p \times m}$ with $p \leq m$. We define $d_{\boldsymbol{V}}$ on $\mathcal{A}^m \times \mathcal{A}^m$ as

$$d_{\boldsymbol{V}}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \|\boldsymbol{V}(\boldsymbol{q}_1 - \boldsymbol{q}_2)\|_1 \quad \forall\, \boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathcal{A}^m.$$

We now present our main result, a more technical version of Theorem 1.1, proved in Appendix C.

Theorem 4.2 (Main result)**.** *Let λ, $r \in \mathbb{N}$, $\epsilon \in (0, \frac{1}{2})$, $\delta \in (0,1)$, $\mu \in (0,1)$, $\beta > 0$, $p = O(\epsilon^{-2} \log(|\mathcal{T}|^2/\delta)) \in \mathbb{N}$ and $m = \lambda p$. Moreover, let $\widetilde{\boldsymbol{V}} \in \mathbb{R}^{p \times m}$ and $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{H}\mathbf{D} \in \mathbb{R}^{m \times n}$ be as in Definition 2.3 and Definition 2.4 respectively with $s = \Theta(\epsilon^{-1} n^{-1} (\|\boldsymbol{v}\|_\infty / \|\boldsymbol{v}\|_2)^2) \leq 1$.*

Let $\mathcal{T}$ be a finite subset of $B_2^n(\kappa) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq \kappa\}$ and suppose that

$$\kappa \leq \frac{\mu}{2\sqrt{\beta + \log(2m)}}.$$

Defining the embedding maps $f_1 : \mathcal{T} \to \{\pm 1\}^m$ by $f_1 = Q^{(r)} \circ \mathbf{A}$ and $f_2 : \mathcal{T} \to \{\pm 1\}^m$ by $f_2 = Q^{(r)} \circ \boldsymbol{\Phi}$, there exists a constant $C(\mu, r)$ such that the following are true:

(i) If the elements of $\mathcal{T}$ satisfy $\|\boldsymbol{x}\|_\infty = O(n^{-1/2}\|\boldsymbol{x}\|_2)$, then

$$\Big| d_{\widetilde{\boldsymbol{V}}}(f_1(\boldsymbol{x}), f_1(\boldsymbol{y})) - \|\boldsymbol{x} - \boldsymbol{y}\|_2 \Big| \leq C(\mu, r)\lambda^{-r+1/2} + \epsilon \|\boldsymbol{x} - \boldsymbol{y}\|_2 \tag{11}$$

holds uniformly for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ with probability exceeding $1 - \delta - |\mathcal{T}| e^{-\beta}$.

(ii) On the other hand, for arbitrary $\mathcal{T} \subset B_2^n(\kappa)$

$$\Big| d_{\widetilde{\boldsymbol{V}}}(f_2(\boldsymbol{x}), f_2(\boldsymbol{y})) - \|\boldsymbol{x} - \boldsymbol{y}\|_2 \Big| \leq C(\mu, r)\lambda^{-r+1/2} + \epsilon \|\boldsymbol{x} - \boldsymbol{y}\|_2 \tag{12}$$

holds uniformly for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ with probability exceeding $1 - \delta - 2|\mathcal{T}| e^{-\beta}$.

Under the assumptions of Theorem 4.2, we have

$$\epsilon = O\left(\sqrt{\frac{\log(|\mathcal{T}|^2/\delta)}{p}}\right) \lesssim \frac{1}{\sqrt{p}}. \tag{13}$$

By (11), (12) and (13), we have that with high probability the inequality

$$\begin{aligned}\left|d_{\widetilde{\boldsymbol{V}}}(f_i(\boldsymbol{x}), f_i(\boldsymbol{y})) - \|\boldsymbol{x}-\boldsymbol{y}\|_2\right| &\le C(\mu, r)\left(\frac{m}{p}\right)^{-r+1/2} + \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|_2 \\ &\le C(\mu, r)\left(\frac{m}{p}\right)^{-r+1/2} + 2\kappa\epsilon \\ &\le C(\mu, r)\left(\frac{m}{p}\right)^{-r+1/2} + \frac{\mu}{\sqrt{\beta+\log(2m)}}\cdot\frac{C_2}{\sqrt{p}} \end{aligned} \tag{14}$$

holds uniformly for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$. The first error term in (14) results from $\Sigma\Delta$ quantization while the second error term is caused by the CJLT. So the term $O((m/p)^{-r+1/2})$ dominates when $\lambda = m/p$ is small. If m/p is sufficiently large, the second term $O(1/\sqrt{p})$ becomes dominant.

5 Computational and Space Complexity

Let $\mathcal{T} = \{\boldsymbol{x}^{(j)}\}_{j=1}^k \subseteq \mathbb{R}^n$. By using $\boldsymbol{H}\mathbf{D}$ to flatten each $\boldsymbol{x}^{(j)}$ if necessary, without loss of generality we may assume that $\mathcal{T}$ consists of well-spread vectors. Moreover, we will focus on stable r-th order $\Sigma\Delta$ schemes $Q^{(r)} : \mathbb{R}^m \to \mathcal{A}^m$ with $\mathcal{A} = \{-1, 1\}$. By Definition 2.3, when $r = 1$ we have $\boldsymbol{v} = (1, 1, \ldots, 1) \in \mathbb{R}^\lambda$, while when $r = 2$, $\boldsymbol{v} = (1, 2, \ldots, \widetilde{\lambda}-1, \widetilde{\lambda}, \widetilde{\lambda}-1, \ldots, 2, 1) \in \mathbb{R}^\lambda$. In general, $\|\boldsymbol{v}\|_\infty/\|\boldsymbol{v}\|_2 = O(\lambda^{-1/2})$ holds for all $r \in \mathbb{N}$. We also assume that $s = \Theta(\epsilon^{-1}n^{-1}(\|\boldsymbol{v}\|_\infty/\|\boldsymbol{v}\|_2)^2) = \Theta(\epsilon^{-1}n^{-1}\lambda^{-1}) \le 1$ as in Theorem 4.2.

We consider b-bit floating-point or fixed-point representations for numbers. Both entail the same computational complexity for computing sums and products of two numbers. Addition and subtraction require $O(b)$ operations while multiplication and division can be done in $\mathcal{M}(b) = O(b^2)$ operations using "standard" long multiplication and division. Multiplication and division can be done more efficiently, particularly for large integers and the best known methods (and best possible up to constants) have complexity $\mathcal{M}(b) = O(b\log b)$ (Harvey & Van Der Hoeven, 2019).

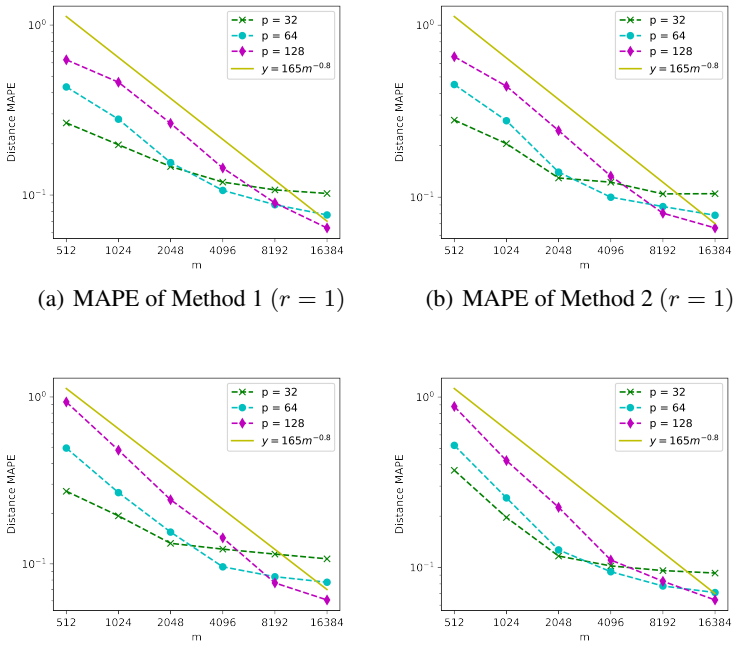
Embedding complexity. For each data point $\boldsymbol{x}^{(j)} \in \mathcal{T}$, one can use Algorithm 1 to quantize it. Since $\mathbf{A}$ has sparsity constant $s = \Theta(\epsilon^{-1}n^{-1}\lambda^{-1})$ and $\epsilon^{-1} = O(p^{1/2})$ by (13), and since $\lambda = m/p$, computing $\mathbf{A}\boldsymbol{x}^{(j)}$ needs $O(snm) = O(\lambda^{-1}\epsilon^{-1}m) = O(p^{3/2})$ time. Additionally, it takes $O(m)$ time to quantize $\mathbf{A}\boldsymbol{x}^{(j)}$ based on (20). When $p^{3/2} \le m$, Algorithm 1 can be executed in $O(m)$ for each $\boldsymbol{x}^{(j)}$. Because $\mathbf{A}$ has $O(snm) = O(m)$ nonzero entries, the space complexity is $O(m)$ bits per data point. Note that the big O notation here hides the space complexity dependence on the bit-depth b of the fixed or floating point representation of the entries of $\mathbf{A}$ and $\boldsymbol{x}^{(j)}$. This clearly has no effect on the storage space needed for each $\boldsymbol{q}^{(j)}$, which is exactly m bits.

Complexity of distance estimation. If one does not use embedding methods, storing $\mathcal{T}$ directly, i.e., by representing the coefficients of each $\boldsymbol{x}^{(j)}$ by b bits requires knb bits. Moreover, the resulting computational complexity of estimating $\|\boldsymbol{x}-\boldsymbol{y}\|_2^2$ where $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ is $O(n\mathcal{M}(b))$. On the other hand, suppose we obtain binary sequences $\mathcal{B} = \{\boldsymbol{q}^{(j)}\}_{j=1}^k \subseteq \mathcal{A}^m$ by performing Algorithm 1 on $\mathcal{T}$. Using our method with accuracy guaranteed by Theorem 4.2, high-dimensional data points $\mathcal{T} \subseteq \mathbb{R}^n$ are now transformed into short binary sequences, which only require km bits of storage instead of knb bits. Algorithm 2 can be applied to recover the pairwise ℓ_2 distances. Note that $\widetilde{\boldsymbol{V}}$ is the normalization of an integer valued matrix $\boldsymbol{V} = \boldsymbol{I}_p \otimes \boldsymbol{v}$ (by Definition 2.3) and $\boldsymbol{q}^{(i)} \in \mathcal{A}^m$ is a binary vector. So, by storing the normalization factor separately, we can ignore it when considering runtime and space complexity. Thus we observe:

1. The number of bits needed to represent each entry of $\boldsymbol{v}$ is at most $\log_2(\|\boldsymbol{v}\|_\infty) \approx (r-1)\log_2\lambda = O(\log_2\lambda)$ when $r > 1$ and $O(1)$ when $r = 1$. So the computation of $\boldsymbol{y}^{(i)} = \widetilde{\boldsymbol{V}}\boldsymbol{q}^{(i)} \in \mathbb{R}^p$ only involves m additions or subtractions of integers represented by $O(\log_2\lambda)$ bits and thus the time complexity in computing $\boldsymbol{y}^{(i)}$ is $O(m\log_2\lambda)$.

2. Each of the p entries of $\boldsymbol{y}^{(i)}$ is the sum of λ terms each bounded by λ^{r-1}. We can store $\boldsymbol{y}^{(i)}$ in $O(p \log_2 \lambda)$ bits.
3. Computing $\|\boldsymbol{y}^{(i)} - \boldsymbol{y}^{(j)}\|_1$ needs $O(p \log_2 \lambda)$ time and $O(p \log_2 \lambda)$ bits.

So we use $O(p \log_2 \lambda)$ bits to recover each pairwise distance $\|\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(j)}\|_2$ in $O(m \log_2 \lambda)$ time.



(a) MAPE of Method 1 ($r = 1$) (b) MAPE of Method 2 ($r = 1$)

(c) MAPE of Method 1 ($r = 2$) (d) MAPE of Method 2 ($r = 2$)

Figure 1: Plots of ℓ_2 distance reconstruction error when $r = 1, 2$

6 Numerical Experiments

To illustrate the performance of our fast binary embedding (Algorithm 1) and ℓ_2 distance recovery (Algorithm 2), we apply these algorithms to a real-world dataset. Our dataset comprises $20k$ images provided by Yelp[1], namely a subset of photos of businesses uploaded by users. Images are converted to grayscale and resampled using bicubic interpolation to size 128×128, so each of them can be represented by a 16384-dimensional vector. We randomly sample $k = 500$ images and scale them by the same constant so all data points are contained in the ℓ_2 unit ball. The scaled dataset is denoted by $\mathcal{T}$. We consider the two versions of our fast binary embedding algorithm from Theorem 4.2:

Method 1. We quantize FJLT embeddings, i.e. $\boldsymbol{\Phi x}$ for $\boldsymbol{x} \in \mathcal{T}$, and reconstruct pairwise distances based on Algorithm 2.

Method 2. We quantize sparse JL embeddings $\mathbf{A}\boldsymbol{x}$ and recover distances by Algorithm 2.

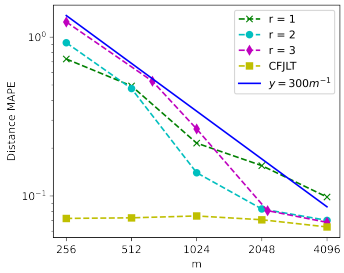
In order to test the performance of our algorithm, we compute the mean absolute percentage error (MAPE) of reconstructed ℓ_2 distances averaged over all pairwise data points, that is,

$$\frac{2}{k(k-1)} \sum_{\boldsymbol{x},\boldsymbol{y} \in \mathcal{T}} \left| \frac{\|\widetilde{\boldsymbol{V}}(\boldsymbol{q_x} - \boldsymbol{q_y})\|_1 - \|\boldsymbol{x} - \boldsymbol{y}\|_2}{\|\boldsymbol{x} - \boldsymbol{y}\|_2} \right|.$$

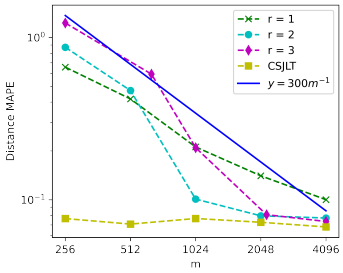
Based on Theorem 4.2, we set $n = 16384$ and $s = 1650/n \approx 0.1$. For each fixed p, we apply Algorithm 1 and Algorithm 2 for various m. We present our experimental results for stable $\Sigma\Delta$

[1] Yelp open dataset: https://www.yelp.com/dataset

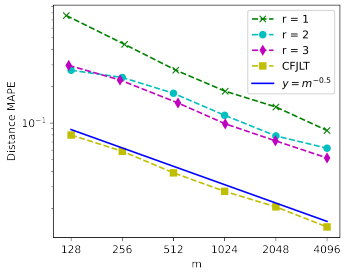
quantization schemes, given by (20), with $r = 1$ and $r = 2$ in Figure 1. For $r = 1$, we observe that the curve with small p quickly reaches an error floor while with high p the error decays like $m^{-1/2}$ and eventually reach a lower floor. The reason is that the first error term in (14) is dominant when m/p is relatively small but the second error term eventually dominates as m becomes larger and larger. When $r = 2$ the error curves decay faster and eventually achieve the same flat error because now the first term in (14) has power $-3/2$ while the second flat error term is independent of r. Moreover, the performance of Method 2 is very similar to that of Method 1.



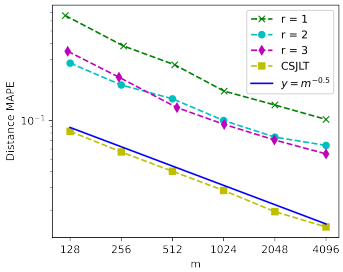
(a) MAPE of Method 1 ($p = 64$)



(b) MAPE of Method 2 ($p = 64$)



(c) MAPE of Method 1 ($p = p(m)$)



(d) MAPE of Method 2 ($p = p(m)$)

Figure 2: Plots of ℓ_2 distance reconstruction error with fixed $p = 64$ and optimal $p = p(m)$

Next, we illustrate the relationship between the quantization order r and the number of measurements m in Figure 2. The curves obtained directly from an unquantized CFJLT (resp. CSJLT) as in Lemma 2.6, with $m = 256, 512, 1024, 2048, 4096$, and $p = 64$ are used for comparison against the quantization methods. The first row of Figure 2 depicts the mean squared relative error when $p = 64$ is fixed for all distinct methods. It shows that stable quantization schemes with order $r > 1$ outperform the first order greedy quantization method, particularly when m is large. Moreover, both the $r = 2$ and $r = 3$ curves converge to the CFJLT/CSJLT result as m goes to 4096. Note that by using a quarter of the original dimension, i.e. $m = 4096$, our construction achieves less than 10% error. Furthermore, if we encode $\widetilde{\boldsymbol{V}}\boldsymbol{q}$ as discussed in Section 5, then we need at most $rp \log_2 \lambda = 64r \log_2(4096/64) = 384r$ bits per image, which is $\lesssim 0.023$ bits per pixel.

For our final experiment, we illustrate that the performance of the proposed approach can be further improved. Note that the choice of p only affects the distance computation in Algorithm 2 and does not appear in the embedding algorithm. In other words, one can vary p in Algorithm 2 to improve performance. This can be done either analytically by viewing the right hand side of (14) as a function of p and optimizing for p (up to constants). It can also be done empirically, as we do here. Following this intuition, if we vary p as a function of m, and use the empirically optimal $p := p(m)$ in the construction of $\widetilde{\boldsymbol{V}}$, then we obtain the second row of Figure 2 where the choice $r = 3$ exhibits lower error than other quantization methods. Note that the decay rate, as a function of m, very closely resembles that of the unquantized JL embedding particularly for higher orders r (as one can verify by optimizing the right hand side of (14)).

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A Proof of Lemma 2.6

We will require the following lemmas, adapted from the literature, to prove the distance-preserving properties of our condensed sparse Johnson-Lindenstrauss transform (CSJLT) and condensed fast Johnson-Lindenstrauss transform (CFJLT) in Lemma 2.6.

Lemma A.1 (Theorem 5.1 in Matoušek (2008))**.** *Let $n \in \mathbb{N}$, $\epsilon \in (0, \frac{1}{2})$, $\delta \in (0,1), \alpha \in [\frac{1}{\sqrt{n}}, 1]$ be parameters and set $m = C\epsilon^{-2}\log(\delta^{-1}) \in \mathbb{N}$ where C is a sufficiently large constant. Let $s = 2\alpha^2/\epsilon \leq 1$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ be as in Definition 2.2. Then*

$$P\Big((1-\epsilon)\|\boldsymbol{x}\|_2 \leq \frac{\sqrt{\pi/2}}{m}\|\mathbf{A}\boldsymbol{x}\|_1 \leq (1+\epsilon)\|\boldsymbol{x}\|_2\Big) \geq 1-\delta \tag{15}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_\infty \leq \alpha\|\boldsymbol{x}\|_2$.

Lemma A.2 below is adapted from (Ailon & Chazelle, 2009, Lemma 1), and we present its proof for completeness.

Lemma A.2. *Let $\boldsymbol{H} \in \mathbb{R}^{n\times n}$ and $\mathbf{D} \in \mathbb{R}^{n\times n}$ be as in Definition 2.4. For any $\lambda > 0$ and $\boldsymbol{x} \in \mathbb{R}^n$ we have*

$$P\Big(\|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \leq \lambda\|\boldsymbol{x}\|_2\Big) \geq 1 - 2ne^{-n\lambda^2/2}. \tag{16}$$

Proof. Without loss of generality, we can assume $\|\boldsymbol{x}\|_2 = 1$. Let $\boldsymbol{u} = \boldsymbol{H}\mathbf{D}\boldsymbol{x} = (u_1, \dots, u_n)$. Fix $i \in \{1, \dots, n\}$. Then $u_i = \sum_{j=1}^n a_j x_j$ with $P\Big(a_j = \frac{1}{\sqrt{n}}\Big) = P\Big(a_j = -\frac{1}{\sqrt{n}}\Big) = \frac{1}{2}$ for all j. Moreover, $a_1, a_2, \dots, a_n$ are independent and symmetric. So u_i is also symmetric, that is, u_i and $-u_i$ share the same distribution. For any $t \in \mathbb{R}$ we have

$$\mathbb{E}(e^{tnu_i}) = \prod_{j=1}^n \mathbb{E}[\exp(tna_j x_j)] = \prod_{j=1}^n \frac{\exp(t\sqrt{n}x_j) + \exp(-t\sqrt{n}x_j)}{2}$$
$$\leq \prod_{j=1}^n \exp(nt^2 x_j^2/2) = \exp(nt^2/2).$$

Since u_i is symmetric, by Markov's inequality and the above result, we get

$$P(|u_i| \geq \lambda) = 2P(e^{\lambda n u_i} \geq e^{\lambda^2 n}) \leq 2e^{-\lambda^2 n}\mathbb{E}(e^{\lambda n u_i}) = 2e^{-\lambda^2 n/2}.$$

Inequality (16) follows by the union bound over all $i \in \{1, \dots, n\}$. □

Lemma A.3. *Let $n, \lambda \in \mathbb{N}$, $\epsilon \in (0, \frac{1}{2})$, $\delta \in (0, 1)$, $p = O(\epsilon^{-2}\log(\delta^{-1})) \in \mathbb{N}$ and $m = \lambda p$. Let $\widetilde{\boldsymbol{V}} \in \mathbb{R}^{p\times m}$ and $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{H}\mathbf{D} \in \mathbb{R}^{m\times n}$ be as in Definition 2.3 and Definition 2.4 respectively with $s = \Theta(\epsilon^{-1}n^{-1}(\|\boldsymbol{v}\|_\infty/\|\boldsymbol{v}\|_2)^2) \leq 1$. Then for $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_\infty = O(n^{-1/2}\|\boldsymbol{x}\|_2)$, we have*

$$\mathrm{P}\Big((1-\epsilon)\|\boldsymbol{x}\|_2 \leq \|\widetilde{\boldsymbol{V}}\mathbf{A}\boldsymbol{x}\|_1 \leq (1+\epsilon)\|\boldsymbol{x}\|_2\Big) \geq 1 - \delta, \tag{17}$$

and for arbitrary $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\mathrm{P}\Big((1-\epsilon)\|\boldsymbol{x}\|_2 \leq \|\widetilde{\boldsymbol{V}}\Phi\boldsymbol{x}\|_1 \leq (1+\epsilon)\|\boldsymbol{x}\|_2\Big) \geq 1 - \delta. \tag{18}$$

Proof. Recall that $\boldsymbol{V} = \boldsymbol{I}_p \otimes \boldsymbol{v}$ and $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{H}\mathbf{D}$. Let $\boldsymbol{y} \in \mathbb{R}^n$ and $\mathbf{K} := \boldsymbol{V}\mathbf{A} = (\boldsymbol{I}_p \otimes \boldsymbol{v})\mathbf{A} \in \mathbb{R}^{p\times n}$. For $1 \leq i \leq p$ and $1 \leq j \leq n$, we have

$$K_{ij} = \sum_{k=1}^{\lambda} v_k a_{(i-1)\lambda+k,j}.$$

Denote the row vectors of $\mathbf{A}$ by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$. It follows that

$$(\mathbf{K}\boldsymbol{y})_i = \sum_{j=1}^n K_{ij}y_j = \sum_{j=1}^n\sum_{k=1}^{\lambda} y_j v_k a_{(i-1)\lambda+k,j} = \sum_{k=1}^{\lambda} v_k\langle \boldsymbol{y}, \mathbf{a}_{(i-1)\lambda+k}\rangle = [\mathbf{B}(\boldsymbol{v}^\top \otimes \boldsymbol{y})]_i$$

where

$$\mathbf{B} := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_\lambda \\ \mathbf{a}_{\lambda+1} & \mathbf{a}_{\lambda+2} & \dots & \mathbf{a}_{2\lambda} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{(p-1)\lambda+1} & \mathbf{a}_{(p-1)\lambda+2} & \dots & \mathbf{a}_{p\lambda} \end{bmatrix} \in \mathbb{R}^{p\times\lambda n} \quad \text{and} \quad \boldsymbol{v}^\top \otimes \boldsymbol{y} = \begin{bmatrix} v_1\boldsymbol{y} \\ v_2\boldsymbol{y} \\ \vdots \\ v_\lambda\boldsymbol{y} \end{bmatrix} \in \mathbb{R}^{\lambda n}.$$

Hence $\boldsymbol{V}\mathbf{A}\boldsymbol{y} = \mathbf{K}\boldsymbol{y} = \mathbf{B}(\boldsymbol{v}^\top \otimes \boldsymbol{y})$ holds for all $\boldsymbol{y} \in \mathbb{R}^n$. Additionally, we get a reshaped sparse Gaussian random matrix $\mathbf{B}$ by rearranging the rows of $\mathbf{A}$.

For the first assertion in the theorem, not that $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $\|\boldsymbol{x}\|_\infty = O(\|\boldsymbol{x}\|_2/\sqrt{n})$. So, we have $\boldsymbol{V}\mathbf{A}\boldsymbol{x} = \mathbf{B}(\boldsymbol{v}^\top \otimes \boldsymbol{x})$, $\|\boldsymbol{v}^\top \otimes \boldsymbol{x}\|_2 = \|\boldsymbol{v}\|_2\|\boldsymbol{x}\|_2$ and $\|\boldsymbol{v}^\top \otimes \boldsymbol{x}\|_\infty = \|\boldsymbol{v}\|_\infty\|\boldsymbol{x}\|_\infty$. Then (17) holds by applying Lemma A.1 to random matrix $\mathbf{B}$ and vector $\boldsymbol{v}^\top \otimes \boldsymbol{x}$ with $\alpha = \Theta(n^{-1/2}\|\boldsymbol{v}\|_\infty/\|\boldsymbol{v}\|_2)$.

For the second assertion, if $\boldsymbol{x} \in \mathbb{R}^n$ is arbitrary, then by substituting $\boldsymbol{H}\mathbf{D}\boldsymbol{x}$ for $\boldsymbol{y}$ one can get

$$\boldsymbol{V}\boldsymbol{\Phi}\boldsymbol{x} = \mathbf{B}(\boldsymbol{v}^\top \otimes (\boldsymbol{H}\mathbf{D}\boldsymbol{x})).$$

Note that

$$\|\boldsymbol{v}^\top \otimes (\boldsymbol{H}\mathbf{D}\boldsymbol{x})\|_2 = \|\boldsymbol{v}\|_2\|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_2 = \|\boldsymbol{v}\|_2\|\boldsymbol{x}\|_2 \quad \text{and} \quad \|\boldsymbol{v}^\top \otimes (\boldsymbol{H}\mathbf{D}\boldsymbol{x})\|_\infty = \|\boldsymbol{v}\|_\infty\|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty.$$

Inequality (18) follows immediately by using above fact and applying Lemma A.1 and Lemma A.2 to the random operator $\mathbf{B}$ and vector $\boldsymbol{v}^\top \otimes (\boldsymbol{H}\mathbf{D}\boldsymbol{x})$ with $\alpha = \Theta(n^{-1/2}\|\boldsymbol{v}\|_\infty/\|\boldsymbol{v}\|_2)$. □

Now we can embed a set of points in a high dimensional space into a space of much lower dimension in such a way that distances between the points are nearly preserved. By substituting δ with $2\delta/|\mathcal{T}|^2$ in Lemma A.3 and using the fact $1 - \binom{|\mathcal{T}|}{2}\frac{2\delta}{|\mathcal{T}|^2} = 1 - \frac{|\mathcal{T}|(|\mathcal{T}|-1)}{2} \cdot \frac{2\delta}{|\mathcal{T}|^2} > 1 - \delta$, Lemma 2.6 follows from the union bound over all pairwise data points in $\mathcal{T}$.

B STABLE SIGMA-DELTA QUANTIZATION AND ITS PROPERTIES

Although it is a non-trivial task to design a stable quantization rule ρ when $r > 1$, families of one-bit $\Sigma\Delta$ quantization schemes that achieve this goal have been designed Daubechies & DeVore (2003); Güntürk (2003); Deift et al. (2011), and we now describe one such family. To start, note that an r-th order $\Sigma\Delta$ quantization scheme may also arise from a more general difference equation of the form

$$\mathrm{y} - \mathrm{q} = \boldsymbol{f} * \boldsymbol{v} \tag{19}$$

where $*$ denotes convolution and the sequence $\boldsymbol{f} = \boldsymbol{P}^r \boldsymbol{g}$ with $\boldsymbol{g} \in \ell^1$. Then any (bounded) solution $\boldsymbol{v}$ of (19) generates a (bounded) solution $\boldsymbol{u}$ of (10) via $\boldsymbol{u} = \boldsymbol{g} * \boldsymbol{v}$. Thus (10) can be rewritten in the form (19) by a change of variables. Defining $\boldsymbol{h} := \boldsymbol{\delta}^{(0)} - \boldsymbol{f}$, where $\boldsymbol{\delta}^{(0)}$ denotes the Kronecker delta sequence supported at 0, and choosing the quantization rule ρ in terms of the new variable as $(\boldsymbol{h} * \boldsymbol{v})_i + y_i$. Then (9) reads as

$$\begin{cases} q_i = Q((\boldsymbol{h} * \boldsymbol{v})_i + y_i), \\ v_i = (\boldsymbol{h} * \boldsymbol{v})_i + y_i - q_i. \end{cases} \tag{20}$$

By designing a proper filter $\boldsymbol{h}$ one can get a stable r-th order $\Sigma\Delta$ quantizer, as was done in Deift et al. (2011); Güntürk (2003), leading to the following result from Güntürk (2003), which exploits the above relationship between $\boldsymbol{v}$ and $\boldsymbol{u}$ to bound $\|\boldsymbol{u}\|_\infty$.

Proposition B.1. *Fix an integer r, an integer $\sigma \geq 6$ and let $n_j = \sigma(j-1)^2 + 1$ for $j = 1, 2, \ldots, r$. Let the filter h be of the form*

$$\boldsymbol{h} = \sum_{j=1}^{r} d_j \boldsymbol{\delta}^{n_j}$$

where $\boldsymbol{\delta}^{n_j}$ is the Kronecker delta supported at n_j and $d_j = \prod_{i \neq j} \frac{n_i}{n_i - n_j}$ for $j = 1, 2, \ldots, r$. There exists a universal constant $C > 0$ such that the rth order $\Sigma\Delta$ scheme (20) *with* 1*-bit alphabet $\mathcal{A} = \{-1, 1\}$, is stable, and*

$$\|\boldsymbol{y}\|_\infty \leq \mu < 1 \Longrightarrow \|\boldsymbol{u}\|_\infty \leq C c(\mu)^r r^r, \tag{21}$$

where $c(\mu) > 0$ is a constant only depends on μ.

Having introduced stable $\Sigma\Delta$ quantization, we now present a lemma controlling an operator norm of $\widetilde{\boldsymbol{V}}\boldsymbol{P}^r$. We will need this result in controlling the error in approximating distances associated with our binary embedding.

Lemma B.2. *For a stable r-th order $\Sigma\Delta$ quantization scheme,*

$$\|\widetilde{\boldsymbol{V}}\boldsymbol{P}^r\|_{\infty,1} \leq \sqrt{\pi/2}(8r)^{r+1}\lambda^{-r+1/2}.$$

Proof. By the same method used in the proof of Lemma 4.6 in Huynh & Saab (2020), one can get

$$\|\boldsymbol{V}\boldsymbol{P}^r\|_{\infty,\infty} \leq r2^{3r-1} \quad \text{and} \quad \|\boldsymbol{v}\|_2 \geq \lambda^{r-1/2} r^{-r}.$$

It follows that

$$\|\widetilde{\boldsymbol{V}}\boldsymbol{P}^r\|_{\infty,1} = \frac{\sqrt{\pi/2}}{p\|\boldsymbol{v}\|_2}\|\boldsymbol{V}\boldsymbol{P}^r\|_{\infty,1} \leq \frac{\sqrt{\pi/2}}{\|\boldsymbol{v}\|_2}\|\boldsymbol{V}\boldsymbol{P}^r\|_{\infty,\infty} \leq \sqrt{\pi/2}(8r)^{r+1}\lambda^{-r+1/2}.$$

□

The following result guarantees that the linear part of our embedding generates a bounded vector, and therefore allows us to later appeal to the stability property of $\Sigma\Delta$ quantizers. In other words, it will allow us to use (21) to control the infinity norm of state vectors generated by $\Sigma\Delta$ quantization.

Lemma B.3 (Concentration inequality for $\|\cdot\|_\infty$)**.** *Let $\beta > 0$, $\epsilon \in (0, 1)$ and $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{H}\mathbf{D} \in \mathbb{R}^{m \times n}$ be the FJLT in Definition 2.4 with $s = \Theta(\epsilon^{-1}n^{-1}) \leq 1$. If*

$$2\sqrt{\beta + \log(2m)} \leq \mu \leq \frac{4}{\sqrt{\epsilon}}, \tag{22}$$

then

$$P\big(\|\mathbf{A}\boldsymbol{x}\|_\infty \le \mu\|\boldsymbol{x}\|_2\big) \ge 1 - e^{-\beta} \tag{23}$$

holds for $\boldsymbol{x} \in \mathbb{R}^n$ *with* $\|\boldsymbol{x}\|_\infty = O(n^{-1/2}\|\boldsymbol{x}\|_2)$ *and*

$$P\big(\|\boldsymbol{\Phi}\boldsymbol{x}\|_\infty \le \mu\|\boldsymbol{x}\|_2\big) \ge 1 - 2e^{-\beta} \tag{24}$$

holds for $\boldsymbol{x} \in \mathbb{R}^n$.

Proof. Without loss of generality, we can assume that $\boldsymbol{x}$ is a unit vector with $\|\boldsymbol{x}\|_2 = 1$. By applying Lemma A.2 to $\boldsymbol{x}$ with $\lambda = \Theta(n^{-1/2})$, we have

$$P\Big(\|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \lambda\Big) \ge 1 - e^{-\beta}. \tag{25}$$

Let A be as in Definition 2.2 with $s = 2\lambda^2/\epsilon = \Theta(\epsilon^{-1}n^{-1}) \le 1$ and recall that $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{H}\mathbf{D}$.

Suppose that $\boldsymbol{y} \in \mathbb{R}^n$ with $\|\boldsymbol{y}\|_2 = 1$ and $\|\boldsymbol{y}\|_\infty \le \lambda$. Let $\mathbf{Y} = \mathbf{A}\boldsymbol{y}$. Then $Y_i := (\mathbf{A}\boldsymbol{y})_i = \sum_{j=1}^n a_{ij}y_j$ for $1 \le i \le m$. For $t \le t_0 := \sqrt{2s}/\lambda = 2/\sqrt{\epsilon}$, we get $t^2y_j^2/2s \le 1$ for all j. Since $e^x \le 1+2x$ for all $x \in [0,1]$ and $1+x \le e^x$ for all $x \in \mathbb{R}$, $se^{t^2y_j^2/2s}+1-s \le s(1+t^2y_j^2/s)+1-s = 1+t^2y_j^2 \le e^{t^2y_j^2}$. It follows that

$$\mathbb{E}(e^{tY_i}) = \prod_{j=1}^n \mathbb{E}(e^{ta_{ij}y_j}) = \prod_{j=1}^n \big(se^{t^2y_j^2/2s} + 1 - s\big) \le \prod_{j=1}^n e^{t^2y_j^2} = e^{t^2}$$

holds for all $1 \le i \le m$ and $t \in [0, t_0]$. So for $t \in [0, t_0]$, by Markov inequality and above inequality we have

$$P\Big(Y_i \ge \mu\Big) = P\Big(e^{tY_i} \ge e^{t\mu}\Big) \le e^{-t\mu}\mathbb{E}(e^{tY_i}) \le e^{-t\mu+t^2}.$$

According to (22) we can set $t = \mu/2 \le t_0 = 2/\sqrt{\epsilon}$, then $P\Big(Y_i \ge \mu\Big) \le e^{-\mu^2/4}$. By symmetry we have $P\Big(-Y_i \ge \mu\Big) \le e^{-\mu^2/4}$. Consequently, for all $1 \le i \le m$ we have

$$P\Big(|Y_i| \ge \mu\Big) \le 2e^{-\mu^2/4}. \tag{26}$$

By a union bound, (22), and (26)

$$\begin{aligned} P\big(\|\mathbf{A}\boldsymbol{y}\|_\infty \ge \mu\big) &= P\Big(\max_{1\le i\le m} |Y_i| \ge \mu\Big) \le mP\Big(|Y_i| \ge \mu\Big) \\ &= 2me^{-\mu^2/4} \le e^{-\beta}. \end{aligned} \tag{27}$$

It follows immediately from (25) and (27) with $\boldsymbol{y} = \boldsymbol{H}\mathbf{D}\boldsymbol{x}$ that

$$\begin{aligned} P(\|\boldsymbol{\Phi}\boldsymbol{x}\|_\infty \le \mu) &= P(\|\mathbf{A}\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \mu) \\ &\ge P(\|\mathbf{A}\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \mu, \|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \lambda) \\ &= P(\|\mathbf{A}\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \mu \mid \|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \lambda)P(\|\boldsymbol{H}\mathbf{D}\boldsymbol{x}\|_\infty \le \lambda) \\ &\ge (1 - e^{-\beta})^2 \\ &\ge 1 - 2e^{-\beta}. \end{aligned}$$

Furthermore, if we replace $\boldsymbol{y}$ by $\boldsymbol{x}$ in (27), then inequality (23) follows. □

C Proof of Theorem 4.2

Proof. Since the proofs of (11) and (12) are almost identical except for using different random projections $\mathbf{A}$ and $\boldsymbol{\Phi}$, we shall only establish the result for (12) in detail. For any $\boldsymbol{x} \in \mathcal{T} \subseteq B_2^n(\kappa)$ we have $\|\boldsymbol{x}\|_2 \le \kappa$. By applying Lemma B.3 we get

$$\begin{aligned} P\big(\|\boldsymbol{\Phi}\boldsymbol{x}\|_\infty < \mu\big) &\ge P\big(\|\boldsymbol{\Phi}\boldsymbol{x}\|_\infty < \mu\|\boldsymbol{x}\|_2/\kappa\big) \\ &\ge P\big(\|\boldsymbol{\Phi}\boldsymbol{x}\|_\infty < 2\sqrt{\beta + \log(2m)}\|\boldsymbol{x}\|_2\big) \\ &\ge 1 - 2e^{-\beta}. \end{aligned}$$

Since above inequality holds for arbitrary $\boldsymbol{x} \in \mathcal{T}$, by union bound one can get

$$P\Big(\max_{\boldsymbol{x}\in\mathcal{T}} \|\boldsymbol{\Phi}\boldsymbol{x}\|_\infty < \mu\Big) \geq 1 - 2|\mathcal{T}|e^{-\beta}.$$

Suppose that $\boldsymbol{u}_{\boldsymbol{x}}$ is the state vector of input signal $\boldsymbol{\Phi}\boldsymbol{x}$ which is produced by stable r-th order $\Sigma\Delta$ scheme. Using Lemma B.2 and formula (21) to get

$$\|\widetilde{\boldsymbol{V}}\boldsymbol{P}^r\|_{\infty,1}\|\boldsymbol{u}_{\boldsymbol{x}}\|_\infty \leq Cc(\mu)^r r^r (8r)^{r+1}\sqrt{\pi/2}\lambda^{-r+1/2}, \tag{28}$$

which holds uniformly for all $\boldsymbol{x} \in \mathcal{T}$ with probability exceeding $1 - 2|\mathcal{T}|e^{-\beta}$.

Furthermore, by Lemma 2.6 the probability that

$$\Big|\|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{y})\|_1 - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \leq \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|_2. \tag{29}$$

holds simultaneously for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ is at least $1 - \delta$.

We deduce from triangle inequality and equations (28), (29) that

$$\begin{aligned}
&\Big|d_{\widetilde{\boldsymbol{V}}}(f_2(\boldsymbol{x}), f_2(\boldsymbol{y})) - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \\
&= \Big|\|\widetilde{\boldsymbol{V}}Q^{(r)}(\boldsymbol{\Phi}\boldsymbol{x}) - \widetilde{\boldsymbol{V}}Q^{(r)}(\boldsymbol{\Phi}\boldsymbol{y})\|_1 - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \\
&\leq \Big|\|\widetilde{\boldsymbol{V}}Q^{(r)}(\boldsymbol{\Phi}\boldsymbol{x}) - \widetilde{\boldsymbol{V}}Q^{(r)}(\boldsymbol{\Phi}\boldsymbol{y})\|_1 - \|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{y})\|_1\Big| + \Big|\|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{y})\|_1 - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \\
&\leq \|\widetilde{\boldsymbol{V}}(Q^{(r)}(\boldsymbol{\Phi}\boldsymbol{x}) - \boldsymbol{\Phi}\boldsymbol{x}) - \widetilde{\boldsymbol{V}}(Q^{(r)}(\boldsymbol{\Phi}\boldsymbol{y}) - \boldsymbol{\Phi}\boldsymbol{y})\|_1 + \Big|\|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{y})\|_1 - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \\
&\leq \|\widetilde{\boldsymbol{V}}\boldsymbol{P}^r\boldsymbol{u}_{\boldsymbol{x}}\|_1 + \|\widetilde{\boldsymbol{V}}\boldsymbol{P}^r\boldsymbol{u}_{\boldsymbol{y}}\|_1 + \Big|\|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{y})\|_1 - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \\
&\leq \|\widetilde{\boldsymbol{V}}\boldsymbol{P}^r\|_{\infty,1}(\|\boldsymbol{u}_{\boldsymbol{x}}\|_\infty + \|\boldsymbol{u}_{\boldsymbol{y}}\|_\infty) + \Big|\|\widetilde{\boldsymbol{V}}\boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{y})\|_1 - \|\boldsymbol{x}-\boldsymbol{y}\|_2\Big| \\
&\leq 2Cc(\mu)^r r^r (8r)^{r+1}\sqrt{\pi/2}\lambda^{-r+1/2} + \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|_2 \\
&= \sqrt{2\pi}Cc(\mu)^r r^r (8r)^{r+1}\lambda^{-r+1/2} + \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|_2 \\
&= C(\mu, r)\lambda^{-r+1/2} + \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|_2
\end{aligned}$$

holds uniformly for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}$ with probability at least $1 - \delta - 2|\mathcal{T}|e^{-\beta}$. □