Fast Bayesian Inference of Point Process Intensity as Function of Covariates

Anonymous Author(s)
Affiliation
Address
email

Abstract

In this paper, we tackle the Bayesian inferencing of point process intensity as a function of covariates. We propose a novel augmentation of permanental process called *augmented permanental process*, a doubly-stochastic point process that uses a Gaussian process on covariate space to describe the Bayesian a priori uncertainty present in the square root of intensity, and derive a fast Bayesian inference algorithm that scales linearly with data size without relying on either domain discretization or Markov Chain Monte Carlo computation. The proposed algorithm is based on our finding that the representer theorem, which is related to RKHS theory, holds for augmented permanental process, which provides us with many significant computational advantages. We evaluate our algorithm on synthetic and real-world data, and show that it outperforms state-of-the-art methods in terms of predictive accuracy while being substantially faster than a conventional Bayesian method.

1 Introduction

Point processes are widely-used statistical tools to analyze point patterns observed on temporal, spatial, and spatio-temporal domains. They have a wide ranging list of applications in neuroscience [10], finance [7], epidemiology [15], ecology [40], and others.

The essential task in point processes is to infer from event data an *intensity* function, that is, an instantaneous probability of events occurring at each point in the observation domain. Generally speaking, there are two kinds of scenarios in the inference task: one is to infer intensities as functions of every point in the observation domain; the other is to infer intensities as functions of covariates with the assumption that covariates are given at every point in the observation domain. For example, in ecological applications where the event points are the locations of trees, the former estimates the value of intensity at each geographical point (longitude and latitude), while the latter estimates the intensity as function of spatial covariates such as slope, elevation, and humidity, each of which varies across the geographical domain. The latter scenario has greater practical importance because it enables us to investigate factors that may be causing events. Unfortunately, data-driven approaches have not been actively exploited in this scenario until recently. This paper focuses on using the function of covariates to solve the inference problem of intensity.

The simplest and most classical model used to estimate intensities in the covariate domain is the Cox proportional hazards model [9, 42, 11] (or the log-linear model), which forms the logarithm of the intensity function as a linear combination of covariates. Although this parametric model is very easy to handle, it fails to address the nonlinear dependence of point patterns on covariates, which is often the case in real-world data. To rectify this limitation, two nonparametric approaches have been proposed: Kernel-based intensity estimator (KIE) [17, 18, 6], which is a rescaled version of kernel density estimator, is the gold standard in estimating intensity as a function of covariates in a non-

Submitted to 36th Conference on Neural Information Processing Systems (NeurIPS 2022). Do not distribute.

Bayesian way; it is easy to implement and rapid in computation. A Bayesian alternative is Gaussian Cox process (GCP) [29, 36, 22, 21, 28, 49], in which a flexible prior over a latent intensity is established by a Gaussian process/Gaussian random field [34] through a positive link function. GCPs are promising approaches as they allow the estimation of the uncertainty of intensity and the hyper-parameter optimization in a statistically formal way. However, the Bayesian inference for GCPs is highly challenging as the likelihood function depends on the functional form of the latent intensity over a compact domain, which imposes the intractable computation of infinite-dimensional distributions. Most algorithms published to date have dealt with the difficulty by using domain discretization or/and Markov Chain Monte Carlo (MCMC). But domain discretization suffers from poor scaling in terms of domain dimension, as well as being sensitive to discretization size. Moreover, MCMC has significant computation demands, making GCPs less attractive than KIEs.

In this paper, for the problem of using functions of covariates for intensity inferencing, we propose a novel Bayesian inference scheme for GCPs with quadratic link functions without recourse to either domain discretization or MCMC. GCPs with quadratic link functions are usually called *permanental processes* [27], in which intensity and data points are assumed to be generated on the same observation domain. But intensity is generated from a Gaussian process on covariate domain in our proposal, which we call *augmented permanental process* (APP). APP is based on the validity of the representer theorem [45], which states that the maximum *a posteriori* estimator of the latent intensity can be represented as a finite linear combination of a transformed kernel product evaluated on the training data points. The representer theorem provides us with many significant computational advantages, resulting in a fast Bayesian inference scheme that scales linearly with the number of data points. Furthermore, based on a Laplace approximation in the functional space, we derive the predictive distribution and the marginal likelihood in a feasible form, which enables us to implement the uncertainty evaluation and the hyper-parameter optimization in a Bayesian manner.

In Section 2, we introduce APP and construct a scalable Bayesian inference scheme for it. In Section 3, we outline related work. In Section 4, we compare APP against KIE- and GCP-based algorithms on synthetic and real-world data, and confirm that our scheme achieves higher predictive accuracy than the state-of-the-art algorithms as well as substantial speed improvements over the conventional GCP-based algorithm. Finally, Section 5 states our conclusions.

2 Methods

2.1 Augmented Permanental Processes

Suppose that a dataset of N point events and a map of D_y-dimensional covariate, denoted by $\mathcal{D} = \{\boldsymbol{t}_n \in \mathcal{T}\}_{n=1}^N$ and $\boldsymbol{y}(\boldsymbol{t}) : \mathcal{T} \to \mathcal{Y} \subset \mathbb{R}^{D_y}$, respectively, are observed for a compact D_t-dimensional domain, $\mathcal{T} \subset \mathbb{R}^{D_t}$. We consider a variant of Gaussian Cox processes which assumes that latent function, $x(\boldsymbol{y}) : \mathcal{Y} \to \mathbb{R}$, is generated from a Gaussian process (GP) on $\mathcal{Y}$, and point events are generated on $\mathcal{T}$ from a point process with *intensity* function, $\lambda(\boldsymbol{t})$, so that

$$p(x(\boldsymbol{y})|\mathcal{D}) = \frac{p(\mathcal{D}|x(\boldsymbol{y}))\, \mathcal{GP}(x(\boldsymbol{y})|k)}{\int \mathscr{D}x(\boldsymbol{y})\, p(\mathcal{D}|x(\boldsymbol{y}))\, \mathcal{GP}(x(\boldsymbol{y})|k)}, \tag{1}$$

$$\log p(\mathcal{D}|x(\boldsymbol{y})) = \sum_{n=1}^{N} \log \lambda(\boldsymbol{t}_n) - \int_{\mathcal{T}} \lambda(\boldsymbol{t}) d\boldsymbol{t}, \quad \lambda(\boldsymbol{t}) = \kappa\big(x(\boldsymbol{y}(\boldsymbol{t}))\big), \tag{2}$$

where $\mathcal{GP}(x(\boldsymbol{y})|k)$ represents a GP with kernel function $k(\boldsymbol{y}, \boldsymbol{y}')$, $p(\mathcal{D}|x(\boldsymbol{y}))$ is the likelihood function of the point process, $\kappa(x) : \mathbb{R} \to \mathbb{R}_+$ is a non-negative function called the *link function*, while $\int \mathscr{D}x(\boldsymbol{y})$ in the denominator represents the integral over the function or the infinite-dimensional variable $x(\boldsymbol{y})$. Among popular link functions such as exponential and sigmoidal, this paper adopts the quadratic form of the link function, $\kappa(x) = x^2$, for its computing benefits. We call the model defined by (1-2) with the quadratic link function the *augmented permanental process* (APP), because it reduces to the ordinary permanental process when the covariate map is the identity mapping $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{t}$. Intensity $\lambda(\boldsymbol{t})$ represents an instantaneous probability of events occurring at each point on $\mathcal{T}$, and our goal is to infer, based on APP, the functional form of the intensity over covariate domain, $\kappa(x(\boldsymbol{y})) = x^2(\boldsymbol{y})$, from the observed data, $\big\{\mathcal{T}, \{\boldsymbol{t}_n\}_{n=1}^N, \boldsymbol{y}(\boldsymbol{t})\big\}$.

In general, Bayesian inference on normal Gaussian Cox processes, equivalent to (1-2) with $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{t}$, is often referred as "*doubly-intractable*", because it requires solving the integral of the stochastic

function $x(\boldsymbol{y})$ as well as computing the intractable posterior probability of an infinite-dimensional variable. Furthermore, APP is challenged by one more difficulty: the domain on which the GP is defined ($\mathcal{Y}$) differs from that over which the intensity integral is performed ($\mathcal{T}$), which degrades mathematical tractability. Below, we address this "*triply-intractable*" problem with the help of *path integral* representation, a mathematical tool that has been proposed most recently [24].

2.2 Maximum *A Posteriori* Estimator

We consider the problem of obtaining the maximum *a posteriori* (MAP) estimator of $x(\boldsymbol{y})$ that maximizes the posterior probability (1). First, we solve the discrepancy between the domains for the GP in (1) and the intensity integral in the likelihood function (2), as it makes functional analyses problematic. If the covariate map $\boldsymbol{y}(\boldsymbol{t})$ is bijective, it is trivial that the intensity integral over $\mathcal{T}$ in (2) can be transformed into the integral over the covariate domain as, $\int_{\mathcal{T}} \cdot d\boldsymbol{t} = \int_{\mathcal{Y}} \cdot |\boldsymbol{J}_{\boldsymbol{y}}^{-1}| d\boldsymbol{y}$, where $\boldsymbol{J}_{\boldsymbol{y}}$ is the Jacobian matrix of the map $\boldsymbol{y}(\boldsymbol{t})$. We can generalize the transformation for a non-bijective map, which often happens in practice (e.g. the value of a covariate varies periodically over time), as follows:

$$\int_{\mathcal{T}} \lambda(\boldsymbol{t}) d\boldsymbol{t} = \int_{\mathcal{T}} x^2(\boldsymbol{y}(\boldsymbol{t})) d\boldsymbol{t} = \int_{\mathcal{T}} d\boldsymbol{t} \int_{\mathcal{Y}} x^2(\boldsymbol{y}) \delta(\boldsymbol{y}(\boldsymbol{t}) - \boldsymbol{y}) d\boldsymbol{y} = \int_{\mathcal{Y}} \rho(\boldsymbol{y}) x^2(\boldsymbol{y}) d\boldsymbol{y}, \tag{3}$$

where $\delta(\cdot)$ is the Dirac delta function, and $\rho(\boldsymbol{y}) = \int_{\mathcal{T}} \delta(\boldsymbol{y}(\boldsymbol{t}) - \boldsymbol{y}) d\boldsymbol{t}$. We call $\rho(\boldsymbol{y})$ the *weight distribution of the covariate*. Actually, our proposed algorithm does not need to evaluate $\rho(\boldsymbol{y})$ directly, and thus we do not go into detail about how to compute $\rho(\boldsymbol{y})$. Rather, we emphasize that the following transformation holds for arbitrary functions of covariates, $f(\boldsymbol{y})$:

$$\int_{\mathcal{Y}} \rho(\boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y} = \int_{\mathcal{T}} f(\boldsymbol{y}(t)) d\boldsymbol{t}. \tag{4}$$

Next, we derive the MAP estimator of $x(\boldsymbol{y})$, denoted by $\hat{x}(\boldsymbol{y})$, by using the path integral representation of GP [24],

$$\mathcal{GP}(x(\boldsymbol{y})|k)\, \mathscr{D}x(\boldsymbol{y}) = \sqrt{\frac{1}{|\mathcal{K}|}} \exp\Big[-\frac{1}{2} \iint_{\mathcal{Y}\times\mathcal{Y}} k^*(\boldsymbol{y}, \boldsymbol{y}') x(\boldsymbol{y}) x(\boldsymbol{y}') d\boldsymbol{y} d\boldsymbol{y}'\Big]\, \mathscr{D}x(\boldsymbol{y}), \tag{5}$$

where $\mathcal{K}$ and $\int_{\mathcal{Y}} k^*(\boldsymbol{y}, \boldsymbol{y}') \cdot d\boldsymbol{y}'$ are the integral operator with kernel function $k(\boldsymbol{y}, \boldsymbol{y}')$ and its inverse operator, respectively: $\mathcal{K}^{(*)} x(\boldsymbol{y}) = \int_{\mathcal{Y}} k^{(*)}(\boldsymbol{y}, \boldsymbol{y}') x(\boldsymbol{y}') d\boldsymbol{y}'$, $\mathcal{K} k^*(\boldsymbol{y}, \boldsymbol{y}') = \delta(\boldsymbol{y} - \boldsymbol{y}')$. Here $|\mathcal{K}|$ represents the function determinant [16] of $\mathcal{K}$, defined by the product of its eigenvalues. Using the relation (3) and the representation (5), we write the posterior of APP (1-2) in the following functional form,

$$p(x(\boldsymbol{y})|\mathcal{D}) \mathscr{D}x = \frac{1}{p(\mathcal{D})} \exp\Big[-S\big(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y})\big)\Big] \mathscr{D}x, \tag{6}$$

where $S\big(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y})\big)$ is the *action integral*, defined by

$$S\big(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y})\big) = \int_{\mathcal{Y}} \Big[\frac{1}{2} x(\boldsymbol{y}) \underline{x}(\boldsymbol{y}) + \rho(\boldsymbol{y}) x^2(\boldsymbol{y}) - 2 \sum_{n=1}^{N} \log x(\boldsymbol{y}) \delta(\boldsymbol{y} - \boldsymbol{y}_n)\Big] d\boldsymbol{y} + \frac{1}{2} \log |\mathcal{K}|, \tag{7}$$

$\boldsymbol{y}_n = \boldsymbol{y}(\boldsymbol{t}_n)$, and $\underline{x}(\boldsymbol{y}) = \int_{\mathcal{Y}} k^*(\boldsymbol{y}, \boldsymbol{y}') x(\boldsymbol{y}') d\boldsymbol{y}'$. Then we apply calculus of variations to the action integral, where the functional derivative of $S(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y}))$ on the MAP estimator $\hat{x}(\boldsymbol{y})$ should be equal to zero: $\frac{\delta S}{\delta \hat{x}(\boldsymbol{y})} \delta \hat{x}(\boldsymbol{y}) + \frac{\delta S}{\delta \underline{\hat{x}}(\boldsymbol{y})} \delta \underline{\hat{x}}(\boldsymbol{y}) = 0$, which results in the exact MAP estimator in a feasible form,

$$\hat{x}(\boldsymbol{y}) = 2 \sum_{n=1}^{N} h(\boldsymbol{y}, \boldsymbol{y}_n) v_n, \quad v_n = \hat{x}(\boldsymbol{y}_n)^{-1}, \tag{8}$$

where $h(\boldsymbol{y}, \boldsymbol{y}')$ is a transformed kernel function defined by a Fredholm integral equation of the second kind [32],

$$h(\boldsymbol{y}, \boldsymbol{y}') + 2 \int_{\mathcal{Y}} k(\boldsymbol{y}, \boldsymbol{s}) \rho(\boldsymbol{s}) h(\boldsymbol{s}, \boldsymbol{y}') d\boldsymbol{s} = k(\boldsymbol{y}, \boldsymbol{y}'). \tag{9}$$

See the supplementary material (§1) for the detailed derivations. Equation (8) shows that the MAP estimator of augmented permanental process involves the representer theorem under the transformed kernel function $h(\boldsymbol{y}, \boldsymbol{y}')$, and thus the Bayesian inference reduces to a finite-dimensional optimization problem. Here, we call $h(\boldsymbol{y}, \boldsymbol{y}')$ the *equivalent kernel* following studies by Flaxman et al. [14] and Walder & Bishop [46]. Given the equivalent kernel, the unknown coefficients v_n in (8) solve the simultaneous quadratic equations derived from (8),

$$\Delta_n \triangleq 2\, v_n \sum_{n'=1}^{N} h(\boldsymbol{y}_n, \boldsymbol{y}_{n'}) v_{n'} - 1 = 0, \quad n = 1, 2, \ldots, N. \tag{10}$$

In this paper, we estimate a set of coefficients, $\{v_n\}_{n=1}^N$, by solving a minimization problem of the sum of the squared residuals, $\sum_{n=1}^N |\Delta_n|^2$, with a popular gradient descent algorithm, *Adam* [25]. We discuss how to obtain $h(\boldsymbol{y}, \boldsymbol{y}')$ in Section 2.4.

Computational Complexity

The objective function to be minimized in MAP estimation is $\sum_{n=1}^N |\Delta_n|^2$, which, in gradient descent algorithms, naively demands the computation of $\mathcal{O}(N^2)$ for each evaluation. However, when the equivalent kernel is given in degenerate form with rank M ($< N$) such that $h(\boldsymbol{y}, \boldsymbol{y}') = \sum_{m=1}^M \varphi_m(\boldsymbol{y})\varphi_m(\boldsymbol{y}')$, the computational complexity reduces to $\mathcal{O}(NM)$, a linear computation against data size N. See Section 2.4 for the construction of the equivalent kernel $h(\boldsymbol{y}, \boldsymbol{y}')$.

2.3 Predictive Distribution and Marginal Likelihood

One of the advantages of GP models over non-Bayesian approaches is that they can provide predictive distributions and marginal likelihoods, which enable us to perform uncertainty evaluations, hyper-parameter optimization, and model selection in a Bayesian manner. We apply a Laplace approximation to APP (1-2), and find the approximate form of the predictive distribution and the marginal likelihood.

We now know the mode of the posterior, $\hat{x}(\boldsymbol{y})$, and consider a Taylor expansion of functional action potential $S\big(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y})\big)$ centered on the mode such that

$$S\big(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y})\big) \simeq S\big(\hat{x}(\boldsymbol{y}), \underline{\hat{x}}(\boldsymbol{y})\big) + \frac{1}{2}\iint_{\mathcal{Y}\times\mathcal{Y}} \sigma^*(\boldsymbol{y}, \boldsymbol{y}')(x(\boldsymbol{y}) - \hat{x}(\boldsymbol{y}))(x(\boldsymbol{y}') - \hat{x}(\boldsymbol{y}'))d\boldsymbol{y}d\boldsymbol{y}', \tag{11}$$

where $\sigma^*(\boldsymbol{y}, \boldsymbol{y}') = \frac{\delta^2 S(x, \underline{x})}{\delta x(\boldsymbol{y})\delta x(\boldsymbol{y}')}\Big|_{x=\hat{x}}$ is the second derivative of S. The first term in the Taylor expansion vanishes due to the stationary condition. The quadratic approximation of the action integral corresponds to the approximation of the posterior process by a GP, and the predictive covariance or the kernel function for the posterior GP, denoted by $\sigma(\boldsymbol{y}, \boldsymbol{y}')$, can be obtained by the functional inversion of $\sigma^*(\boldsymbol{y}, \boldsymbol{y}')$, which results in

$$\sigma(\boldsymbol{y}, \boldsymbol{y}') = h(\boldsymbol{y}, \boldsymbol{y}') - \boldsymbol{h}(\boldsymbol{y})^\top (\boldsymbol{Z} + \boldsymbol{H})^{-1} \boldsymbol{h}(\boldsymbol{y}'), \tag{12}$$

where

$$\boldsymbol{Z}_{nn'} = (2v_n^2)^{-1}\delta_{nn'}, \;\; \boldsymbol{H}_{nn'} = h(\boldsymbol{y}_n, \boldsymbol{y}_{n'}), \;\; \boldsymbol{h}(\boldsymbol{y}) = (h(\boldsymbol{y}, \boldsymbol{y}_1), \ldots, h(\boldsymbol{y}, \boldsymbol{y}_N))^\top. \tag{13}$$

The full derivation of (12) is given in the supplementary material (§2). When the latent function $x(\boldsymbol{y})$ follows a posterior GP with mean of $\hat{x}(\boldsymbol{y})$ and the kernel $\sigma(\boldsymbol{y}, \boldsymbol{y}')$, it is easily verified that values of the squared function, $\lambda = x^2(\boldsymbol{y})$, on each point of the covariate domain $\boldsymbol{y} \in \mathcal{Y}$ follows a Gamma distribution,

$$p_{\mu,\nu}(\lambda) = \frac{1}{\Gamma(\nu)\mu^\nu}\lambda^{\nu-1}\exp(-\lambda/\mu), \quad \mu = 2\sigma\frac{2\hat{x}^2 + \sigma}{\hat{x}^2 + \sigma}, \; \nu = \frac{(\hat{x}^2 + \sigma)^2}{2\sigma(2\hat{x}^2 + \sigma)}, \tag{14}$$

where $\hat{x}$ and σ are abbreviations of $\hat{x}(\boldsymbol{y})$ and $\sigma(\boldsymbol{y}, \boldsymbol{y})$, respectively. (14) represents the predictive distribution of the intensity as function of covariates.

Furthermore, under Laplace approximation (11), we can obtain the marginal likelihood, $p(\mathcal{D})$, in (6) by performing the path integral as,

$$\log p(\mathcal{D}) = \log \int \exp\big[-S\big(x(\boldsymbol{y}), \underline{x}(\boldsymbol{y})\big)\big]\mathscr{D}x \;\simeq\; -S\big(\hat{x}(\boldsymbol{y}), \underline{\hat{x}}(\boldsymbol{y})\big) + \frac{1}{2}\log|\Sigma|, \tag{15}$$

where $|\Sigma|$ is the functional determinant of integral operator $\Sigma = \int_{\mathcal{Y}} \cdot\, \sigma(\boldsymbol{y}, \boldsymbol{y}')d\boldsymbol{y}'$. We can rewrite the result in a more tractable form by substituting (7, 12) into (15):

$$\log p(\mathcal{D}) = \log|\boldsymbol{Z}| - \frac{1}{2}\log|\boldsymbol{I}_N + \boldsymbol{Z}^{-1}\boldsymbol{H}| + \frac{1}{2}\big(\log|\mathcal{H}| - \log|\mathcal{K}|\big) + (2\log 2 - 1)N, \tag{16}$$

where $\boldsymbol{I}_N$ is the identity matrix with size N, and $|\mathcal{H}|$ is the functional determinant of integral operator $\mathcal{H} = \int_{\mathcal{Y}} \cdot\, h(\boldsymbol{y}, \boldsymbol{y}')d\boldsymbol{y}'$. The full derivation of (16) is given in the supplementary material (§3), and the method of calculating the functional determinant term is discussed in Section 2.4.

Computational Complexity

Given the equivalent kernel $h(\boldsymbol{y}, \boldsymbol{y}')$, the evaluation of the predictive covariance (12) for a covariate pair, $(\boldsymbol{y}, \boldsymbol{y}')$, naively needs the computation of $\mathcal{O}(N^3)$ due to the matrix inversion in the second term. Fortunately, the complexity reduces to $\mathcal{O}(NM^3)$ given a kernel with degenerate form, $h(\boldsymbol{y}, \boldsymbol{y}') = \sum_{m=1}^{M} \varphi_m(\boldsymbol{y})\varphi_m(\boldsymbol{y}')$, through the Woodbury matrix identity. The $N \times N$ matrix $\boldsymbol{H}$ can be decomposed into a product of $N \times M$ matrix $\boldsymbol{R}$, defined by $\boldsymbol{R}_{nm} = \varphi_m(\boldsymbol{y}_n)$, and its transpose as $\boldsymbol{H} = \boldsymbol{R}\boldsymbol{R}^\top$; the matrix inversion is transformed as $(\boldsymbol{Z} + \boldsymbol{R}\boldsymbol{R}^\top)^{-1} = \boldsymbol{Z}^{-1} - \boldsymbol{Z}^{-1}\boldsymbol{R}(\boldsymbol{I}_M + \boldsymbol{R}^\top\boldsymbol{Z}^{-1}\boldsymbol{R})^{-1}\boldsymbol{R}^\top\boldsymbol{Z}^{-1}$. Note that $\boldsymbol{Z}^{-1}\boldsymbol{R}$ costs $\mathcal{O}(NM)$ because $\boldsymbol{Z}$ is a diagonal matrix. Therefore, the computational complexity of the predictive distribution (14) is $\mathcal{O}(N^3)$ naively and $\mathcal{O}(NM^3)$ with degenerate equivalent kernel.

In computing the marginal likelihood (16), the complexity of the first and the second terms is $\mathcal{O}(N)$ and $\mathcal{O}(N^3)$ with a naive computation, respectively, but the second one reduces to $\mathcal{O}(NM^3)$ with degenerate equivalent kernel. The complexity of the third term, which is related to the functional determinants, depends on how the equivalent kernel $h(\boldsymbol{y}, \boldsymbol{y}')$ is constructed, as discussed below.

2.4 Construction of Equivalent Kernels

2.4.1 Naive Approach

The equivalent kernel $h(\boldsymbol{y}, \boldsymbol{y}')$ solves the Fredholm integral equation (9), which includes the weight distribution of covariate, $\rho(\boldsymbol{y})$. Although it is possible to evaluate $\rho(\boldsymbol{y})$ numerically, we take an approach that avoids to treat $\rho(\boldsymbol{y})$ directly. Through the relation (4), we rewrite the integral term in (9) as $\int_{\mathcal{Y}} k(\boldsymbol{y}, \boldsymbol{s})\rho(\boldsymbol{y})h(\boldsymbol{s}, \boldsymbol{y}')d\boldsymbol{s} = \int_{\mathcal{T}} k(\boldsymbol{y}, \boldsymbol{y}(\boldsymbol{t}))h(\boldsymbol{y}(\boldsymbol{t}), \boldsymbol{y}')d\boldsymbol{t}$, and apply to (9) the *Nyström method* [30], a popular method to solve integral equations that approximates the integral operator by J-point numerical integration:

$$h(\boldsymbol{y}, \boldsymbol{y}') + 2\sum_{j=1}^{J} w_j k(\boldsymbol{y}, \boldsymbol{y}_j)h(\boldsymbol{y}_j, \boldsymbol{y}') = k(\boldsymbol{y}, \boldsymbol{y}'), \quad \boldsymbol{y}_j = \boldsymbol{y}(\boldsymbol{t}_j), \tag{17}$$

where $\boldsymbol{t}_j \in \mathcal{T}$ and $w_j \in \mathbb{R}$ represent the integration points and weights, respectively. It is easily shown that the equation (17) can be solved in an analytical form as follows,

$$h(\boldsymbol{y}, \boldsymbol{y}') = k(\boldsymbol{y}, \boldsymbol{y}') - \boldsymbol{k}(\boldsymbol{y})^\top\big(\boldsymbol{W}^{-1} + \boldsymbol{K}\big)^{-1}\boldsymbol{k}(\boldsymbol{y}'), \tag{18}$$

where $\boldsymbol{k}(\boldsymbol{y}) = (k(\boldsymbol{y}, \boldsymbol{y}_1), \ldots, k(\boldsymbol{y}, \boldsymbol{y}_J))^\top$, $\boldsymbol{W}_{jj'} = 2w_j\delta_{jj'}$, and $\boldsymbol{K}_{jj'} = k(\boldsymbol{y}_j, \boldsymbol{y}_{j'})$ for $1 \le j, j' \le J$. We call (18) the *naive approach* to constructing an equivalent kernel. In this paper, we use the quasi-Monte Carlo method [8], where $\{\boldsymbol{t}_j\}_{j=1}^{J}$ is a low-discrepancy sequence (Sobol sequence is used here) and $w_j = 1/J$. However, other integration techniques like Monte Carlo methods are also possible.

The functional determinant of operator, $\mathcal{H} = \int_{\mathcal{Y}} \cdot\, h(\boldsymbol{y}, \boldsymbol{y}')d\boldsymbol{y}'$, under (18) can be obtained as a product of the functional determinant of $\mathcal{K}$ and a $J \times J$ matrix determinant: $|\mathcal{H}| = |\mathcal{K}||\boldsymbol{I}_J + \boldsymbol{W}\boldsymbol{K}|^{-1}$ (see the supplementary material (§4)). Thus the determinant term in (16) is given by the tractable determinant: $(\log|\mathcal{H}| - \log|\mathcal{K}|)/2 = -1/2\log|\boldsymbol{I}_J + \boldsymbol{W}\boldsymbol{K}|$. The computational complexity of evaluating $h(\boldsymbol{y}, \boldsymbol{y}')$ or $(\log|\mathcal{H}| - \log|\mathcal{K}|)/2$is, under the naive approach, $\mathcal{O}(J^3)$.

2.4.2 Degenerate Approach

When the kernel function of GP has degenerate form with rank $M(< \infty)$,

$$k(\boldsymbol{y}, \boldsymbol{y}') = \sum_{m=1}^{M} \phi_m(\boldsymbol{y})\phi_m(\boldsymbol{y}') = \boldsymbol{\phi}(\boldsymbol{y})^\top\boldsymbol{\phi}(\boldsymbol{y}'), \tag{19}$$

it is well known that the Fredholm integral equation (9) can be solved analytically [3] as,

$$h(\boldsymbol{y}, \boldsymbol{y}') = \boldsymbol{\phi}(\boldsymbol{y})^\top (\boldsymbol{I}_M + 2\boldsymbol{A})^{-1} \boldsymbol{\phi}(\boldsymbol{y}'), \tag{20}$$

where $\boldsymbol{\phi}(\boldsymbol{y}) = (\phi_1(\boldsymbol{y}), \phi_2(\boldsymbol{y}), \ldots, \phi_M(\boldsymbol{y}))^\top$, and $\boldsymbol{A}$ is the $M \times M$ matrix defined by $\boldsymbol{A} = \int_{\mathcal{Y}} \rho(\boldsymbol{y}) \boldsymbol{\phi}(\boldsymbol{y}) \boldsymbol{\phi}(\boldsymbol{y})^\top d\boldsymbol{y}$. As in the naive approach, we rewrite and approximate the integral operator in $\boldsymbol{A}$ by using relation (4) and a J-point numerical integration, resulting in

$$\boldsymbol{A} = \sum_{j=1}^{J} w_j \boldsymbol{\phi}(\boldsymbol{y}_j) \boldsymbol{\phi}(\boldsymbol{y}_j)^\top, \quad \boldsymbol{y}_j = \boldsymbol{y}(\boldsymbol{t}_j), \tag{21}$$

where $\boldsymbol{t}_j \in \mathcal{T}$ and $w_j \in \mathbb{R}$ represent the integration points and weights, respectively. We call (20-21) the *degenerate approach* for constructing an equivalent kernel.

The functional determinant of operator $\mathcal{H}$ under (20) can be obtained as a product of the functional determinant of $\mathcal{K}$ and a $M \times M$ matrix determinant: $|\mathcal{H}| = |\mathcal{K}||\boldsymbol{I}_M + 2\boldsymbol{A}|^{-1}$ (see the supplementary material (§4)). Thus the determinant term in (16) is given by the tractable determinant: $(\log|\mathcal{H}| - \log|\mathcal{K}|)/2 = -1/2 \log|\boldsymbol{I}_M + 2\boldsymbol{A}|$. The computational complexity of evaluating $h(\boldsymbol{y}, \boldsymbol{y}')$ or $(\log|\mathcal{H}| - \log|\mathcal{K}|)/2$ is, under the degenerate approach, $\mathcal{O}(M^3 + JM^2)$, where $\mathcal{O}(JM^2)$ is given by numerical integration (21).

It should be emphasized here that when $k(\boldsymbol{y}, \boldsymbol{y}')$ has degenerate form with rank M, the relation (20) shows that the equivalent kernel $h(\boldsymbol{y}, \boldsymbol{y}')$ also has degenerate form obtained through Cholesky decomposition:

$$h(\boldsymbol{y}, \boldsymbol{y}') = (\boldsymbol{L}\boldsymbol{\phi}(\boldsymbol{y}))^\top (\boldsymbol{L}\boldsymbol{\phi}(\boldsymbol{y}')), \quad \boldsymbol{L}^\top \boldsymbol{L} = (\boldsymbol{I}_M + 2\boldsymbol{A})^{-1}. \tag{22}$$

The degenerate equivalent kernel (22) offers fast Bayesian estimation that scales linearly with N, the number of data points (see Section 2.2-2.3).

There are several ways of preparing degenerate kernel functions (19). *Random feature map* [33, 43, 44] approximates popular kernel functions, such as shift invariant kernels and additive kernels, by the inner products of randomized feature map vectors. Also, *Nyström approximation* for kernel methods [47, 48] finds a degenerate form of the function that best approximates an arbitrary kernel function at a finite set of data points. We apply the two approximation methods to a Gaussian kernel in our experiments.

3 Related Work

Kernel Intensity Estimators (KIEs) are the *de facto* standard nonparametric approaches for estimating point process intensities as functions of covariates in non-Bayesian manner [17, 18, 6]. They construct intensity estimators by rescaling well-studied kernel (probability) density estimators for biased sample data. KIEs offer the lowest computational complexity as well as sophisticated methods for bandwidth selection. Analogue estimators, denoted by LLEs, have been also proposed based on local likelihood density estimators [6]. KIEs and LLEs have a variety of variants, some of which are readily available from library `spatstat` in `R` [5]. It is notable that the estimators also provide approximate point-wise confidence intervals for estimated intensities, which allows for uncertainty analysis.

Gaussian Cox Processes (GCPs) are the Bayesian alternatives to KIEs, where latent intensities are modeled by Gaussian processes or Gaussian random fields through positive link functions [29]. They provide us a methodology to perform the interval estimation of intensities as well as the hyper-parameter optimization in a Bayesian manner. GCPs have been intensively developed for the problem of estimating intensities as functions of each point on the observation space [1, 2, 10, 13, 12, 19, 23, 26, 35, 41], being a simpler case ($\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{t}$) of what we consider in this paper. The estimation of intensities as functions of general covariates, which varies across observation domain, has recently been tackled by using domain discretization [21, 22, 28, 36, 49]; it partitions the observation domain into finite disjoint sub-areas and assumes the intensity to be constant over each sub-area. Although domain discretization is applicable to general GCPs regardless of link/kernel functions, it suffers from poor scaling in terms of domain dimension, as well as being sensitive to discretization size. Fitting GCPs to data are often based on time-consuming Markov chain Monte Carlo methods, but the integrated nested Laplace approximation (INLA) [21, 22, 36]

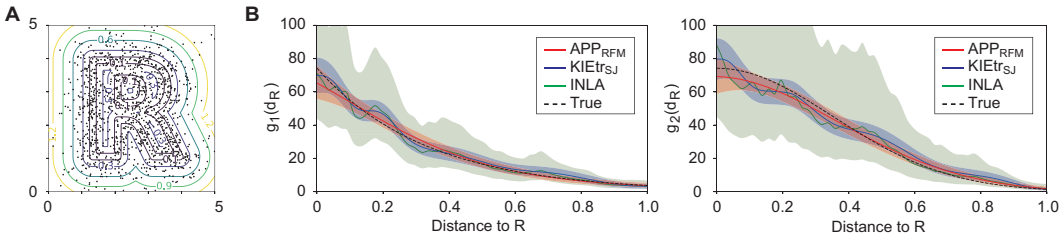

Figure 1: (A) Simulated data on $g_1(d_R)$ with $\alpha = 0.5$ and contours of distance to letter "R". (B) The estimated intensity functions. Solid lines and shaded areas represent predictive medians and [0.025, 0.975] prediction intervals, respectively.

can alleviate the fitting costs substantially. Fortunately, INLA is also accessible through the library `INLA` in `R`. Although it is out-of-scope of this paper, modeling local repulsion/clustering between point events also has been a hot topic in this research field [4], which could be a future direction of our study.

Permanental Processes (PPs) are special GCPs that adopt quadratic forms of link functions [27], of which computational advantages have been highlighted by recent researches [14, 23, 26, 46]. The fact that PPs involve a representer theorem when $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{t}$ was exploited through the RKHS theory [14], the Mercer's theorem [46], and the path integral formulation [24]. Therefore, our proposed model is a generalization of their works for $\boldsymbol{y}(\boldsymbol{t}) \neq \boldsymbol{t}$.

Neural Network-based Point Processes to explorer the dependence of point patterns on covariates have rarely been proposed. An exception is the deep mixture point process [31], which locates the finite representative points on the observation domain and models the intensity value on the point by a neural network model that eats as input the covariates on the point. This model suffers from the same difficulties as the conventional GCPs due to the domain discretization approach. Also, it cannot perform the interval estimation of intensities.

4 Experiments

We examined the validity and the potential efficiency of our proposed model by evaluating it against the state-of-the-art Bayesian and non-Bayesian approaches on synthetic and open real-world data. As non-Bayesian approaches, we adopted KIEs (KIE) and LLEs (LLE) with the ratio (ra), the reweighted (re), and the transformation (tr) forms [6], and selected the kernel bandwidths of KIEs by Silverman's rule of thumb [39] (SI) and a method by Sheather & Jones [38] (SJ), resulting in the nine non-Bayesian variants (e.g. KIEtr_{SJ}). Some of them are not implemented for multivariate covariates in `spatstat`, and they were applied only to synthetic data, where $\mathcal{Y} \subset \mathbb{R}$. As a Bayesian approach, we employed the log-Gaussian Cox process with INLA [22] (INLA) with 100×100 domain discretization, where the random walk model of order 1 was used as the prior process, and the hyper-parameters were optimized by maximizing the marginal likelihood through the 16-points grid search. For our proposal, we adopted APPs with the naive approach (APP_{NAI}) and with the degenerate approaches of the random Fourier map [33] (APP_{RFM}) and the Nyström approximation [48] (APP_{NYS}). We applied to the APPs a multiplicative Gaussian kernel, $k(\boldsymbol{y}, \boldsymbol{y}') = \theta_0 \prod_{d=1}^{D_y} e^{-(\theta_d(y_d - y'_d))^2}$, where the hyper-parameter $\theta = (\theta_0, \ldots, \theta_{D_y})$ was optimized for each data by maximizing the marginal likelihood (16) through the 25-points grid search. A MacBook Pro with 4-core CPU (2.8 GHz Intel Core i7) was used. For details of the model configurations, see the supplementary material (§5).

4.1 Synthetic Data

Following [6], we created 2D data sets ($\mathcal{T} \subset \mathbb{R}^2$, $\mathcal{Y} \subset \mathbb{R}$) generated from two types of intensity functions: $\lambda_1(\boldsymbol{t}) = g_1(d_R(\boldsymbol{t})) = \alpha \exp(5 - 3d_R(\boldsymbol{t}))$ and $\lambda_2(\boldsymbol{t}) = g_2(d_R(\boldsymbol{t})) = \alpha \exp(5 - 4d_R(\boldsymbol{t})^2)$, each of which has 100 trial sequences. Here $d_R(\boldsymbol{t})$ denotes the shortest distance from a given location $\boldsymbol{t}$ to the set of lines arranged in the shape of the letter "R" (see Figure 1A). The coefficient α was set as 0.2, 0.5, and 1.0. The predictive performance was evaluated based on the integrated *ρ-quantile loss* [37], defined as $l_\rho \triangleq \int_{\mathcal{Y}} 2\big(g(y) - \hat{g}(y)\big)\big(\rho \mathrm{I}_{g(y) > \hat{g}(y)} - (1-\rho)\mathrm{I}_{g(y) \leq \hat{g}(y)}\big)dy$, where I, $\hat{g}(y)$, and $g(y)$ denote the indicator, the predicted ρ-quantile of the intensity function on covariate

Table 1: Results on two types of synthetic data across 100 trials with standard errors in brackets. The underlines represent the best predictive performances on each metric, and the performances not significantly ($p < 10^{-2}$) different from the best one under the Mann-Whitney U test with Holm method [20] are shown in bold. $\tau(s)$ is the CPU times in second, and $\tilde{N}$ is the average data size.

$g_1(d_R)$	$\alpha = 0.2$ ($\tilde{N} = 232$)				$\alpha = 0.5$ ($\tilde{N} = 586$)				$\alpha = 1.0$ ($\tilde{N} = 1180$)			
	$l_{.025}$	$l_{.5}$	$l_{.975}$	$\tau(s)$	$l_{.025}$	$l_{.5}$	$l_{.975}$	$\tau(s)$	$l_{.025}$	$l_{.5}$	$l_{.975}$	$\tau(s)$
KIEtr_{SJ}	0.15 (0.05)	**1.16** (0.35)	0.21 (0.10)	0.05 (0.00)	0.29 (0.20)	2.26 (0.52)	0.36 (0.10)	0.06 (0.00)	0.43 (0.26)	3.45 (0.84)	0.51 (0.12)	0.06 (0.00)
LLEra	0.17 (0.09)	1.41 (0.39)	0.23 (0.05)	0.07 (0.00)	0.43 (0.34)	2.91 (0.80)	0.57 (0.29)	0.08 (0.00)	1.11 (0.85)	5.05 (1.11)	0.92 (0.45)	0.08 (0.00)
INLA	0.27 (0.03)	1.31 (0.30)	0.62 (0.13)	101 (6.01)	0.54 (0.06)	2.54 (0.51)	1.00 (0.14)	108 (5.19)	0.89 (0.07)	3.71 (0.70)	1.51 (0.20)	110 (6.10)
APP_{NAI}	**0.14** (0.06)	**1.08** (0.32)	**0.18** (0.09)	9.30 (0.52)	**0.24** (0.10)	**1.93** (0.63)	**0.34** (0.22)	14.1 (0.80)	**0.38** (0.26)	**2.74** (0.78)	**0.45** (0.33)	25.47 (1.05)
APP_{NYS}	**0.14** (0.05)	**1.08** (0.31)	**0.18** (0.09)	0.61 (0.09)	**0.24** (0.10)	**1.93** (0.62)	**0.34** (0.22)	2.62 (0.19)	**0.38** (0.27)	**2.72** (0.78)	**0.44** (0.33)	2.81 (0.35)
APP_{RFM}	**0.13** (0.07)	**1.06** (0.33)	**0.18** (0.13)	2.42 (0.17)	**0.24** (0.10)	**1.90** (0.60)	**0.34** (0.24)	2.48 (0.07)	**0.37** (0.27)	**2.65** (0.75)	**0.43** (0.34)	2.56 (0.05)
APP'_{RFM}	0.17 (0.04)	1.38 (0.32)	0.24 (0.07)	0.30 (0.01)	0.31 (0.12)	2.48 (0.64)	0.45 (0.23)	0.32 (0.01)	0.49 (0.24)	3.66 (0.82)	0.60 (0.29)	0.35 (0.01)
$g_2(d_R)$	$\alpha = 0.2$ ($\tilde{N} = 322$)				$\alpha = 0.5$ ($\tilde{N} = 801$)				$\alpha = 1.0$ ($\tilde{N} = 1610$)			
	$l_{.025}$	$l_{.5}$	$l_{.975}$	$\tau(s)$	$l_{.025}$	$l_{.5}$	$l_{.975}$	$\tau(s)$	$l_{.025}$	$l_{.5}$	$l_{.975}$	$\tau(s)$
KIEtr_{SJ}	0.18 (0.05)	1.47 (0.35)	0.24 (0.10)	0.06 (0.00)	0.38 (0.20)	2.76 (0.52)	0.40 (0.10)	0.06 (0.00)	0.83 (0.26)	4.77 (0.84)	0.64 (0.12)	0.06 (0.00)
LLEra	0.24 (0.09)	1.67 (0.39)	0.28 (0.05)	0.07 (0.00)	0.74 (0.34)	3.47 (0.80)	0.54 (0.29)	0.08 (0.00)	1.95 (0.85)	6.17 (1.11)	1.02 (0.45)	0.08 (0.00)
INLA	0.45 (0.03)	1.73 (0.30)	1.34 (0.13)	112 (6.01)	0.96 (0.06)	3.31 (0.51)	2.22 (0.14)	115 (5.19)	1.68 (0.07)	5.04 (0.70)	3.45 (0.20)	116 (6.10)
APP_{NAI}	**0.15** (0.06)	**1.06** (0.32)	**0.17** (0.09)	10.1 (0.52)	**0.25** (0.10)	**1.87** (0.63)	**0.27** (0.22)	17.7 (0.80)	**0.40** (0.26)	**2.75** (0.78)	**0.44** (0.33)	40.1 (1.05)
APP_{NYS}	**0.15** (0.05)	**1.08** (0.31)	**0.18** (0.09)	1.02 (0.09)	**0.25** (0.10)	**1.88** (0.62)	**0.27** (0.22)	2.72 (0.19)	**0.39** (0.27)	**2.79** (0.78)	**0.44** (0.33)	3.15 (0.35)
APP_{RFM}	**0.14** (0.07)	**0.99** (0.33)	**0.17** (0.13)	2.40 (0.17)	**0.22** (0.10)	**1.76** (0.60)	**0.26** (0.24)	2.57 (0.07)	**0.36** (0.27)	**2.53** (0.75)	**0.40** (0.34)	2.78 (0.05)
APP'_{RFM}	0.20 (0.04)	1.57 (0.32)	0.28 (0.07)	0.30 (0.01)	0.37 (0.12)	2.76 (0.64)	0.46 (0.23)	0.34 (0.01)	0.57 (0.24)	4.06 (0.82)	0.65 (0.29)	0.40 (0.01)

domain, and the true one, respectively. Here, we adopted $l_{.025}$, $l_{.5}$ (integrated absolute error) and $l_{.975}$.

Table 1 displays the predictive performances on various synthetic data, where only the methods with relatively good accuracy are listed due to space limitations (see the supplementary material (§5) for the full results). It shows that our approaches (APPs) outperformed the reference methods across all predictive metrics, and that APPs with the degenerate approaches (APP_{NYS} and APP_{RFM}) were performed several tens of times faster than INLA, a feasible Bayesian method based on domain discretization. Especially, APPs' good scores regarding $l_{.025}$ and $l_{.975}$ demonstrate that our Laplace approximation on the functional space (see Section 2.3) can recover the posterior distribution accurately as well as the MAP estimator. Figure 1B implies that INLA tends to overestimate the predictive errors. Also, We confirmed that APP_{RFM} without hyper-parameter optimization (APP'_{RFM}) was inferior to APP_{RFM}, showing the practical utility of the approximate marginal likelihood for determining hyper-parameters.

4.2 Real-world Data

We examined the validity of our approach based on real-world spatial ($\mathcal{T} \subset \mathbb{R}^2$) data provided by `spatstat.data` in `R` (GPL-3) [5]: `copper` consists of 67 location points of copper ore deposits and 146 line segments representing faults. The covariate of interest is the shortest distance from a given location $\boldsymbol{t}$ to the set of faults ($\mathcal{Y} \subset \mathbb{R}$); `bei` consists of locations of 3605 trees of the species *Beilschmiedia pendula* and geo-information in a tropical rain forest. The covariates of interest are

Table 2: Results on real-world data across 10 trials. Notations follow Table 1.

	copper			bei			clmfires		
	ll_{test}	cl_{test}	$\tau(s)$	ll_{test}	cl_{test}	$\tau(s)$	ll_{test}	cl_{test}	$\tau(s)$
KIEra_{SI}	**5.65** (0.37)	**1.64** (0.09)	0.32 (0.01)	2.92 (0.00)	1.31 (0.07)	0.26 (0.00)	2.39 (0.00)	1.30 (0.08)	0.24 (0.01)
KIEre_{SI}	**5.65** (0.37)	**1.64** (0.09)	0.33 (0.01)	2.92 (0.00)	1.31 (0.07)	0.25 (0.00)	2.40 (0.00)	1.38 (0.08)	0.24 (0.01)
INLA	**5.64** (0.34)	**1.66** (0.10)	82.9 (3.00)	2.90 (0.01)	1.13 (0.06)	131 (3.37)	2.39 (0.01)	**1.18** (0.06)	129 (4.61)
APP_{NYS}	**5.65** (0.37)	**1.64** (0.09)	0.60 (0.06)	**2.87** (0.01)	**1.01** (0.05)	15.8 (0.41)	**2.36** (0.01)	**1.15** (0.07)	17.9 (1.06)
APP_{RFM}	**5.65** (0.37)	**1.64** (0.09)	2.12 (0.10)	**2.88** (0.01)	**1.02** (0.05)	4.64 (0.21)	**2.36** (0.01)	**1.16** (0.07)	4.28 (0.22)

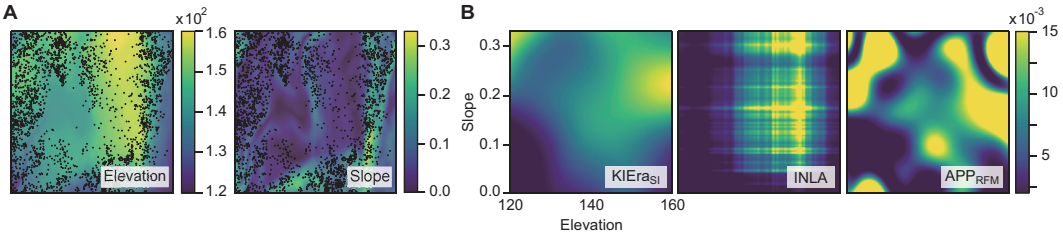

Figure 2: Estimation results on `bei`. (A) Covariate maps on 2D spatial domain with observed points (black dots). (B) The estimated intensities as functions of 2D covariates.

the terrain elevation and the terrain slope ($\mathcal{Y} \subset \mathbb{R}^2$); `clmfires` consists of locations of forest fires in the Castilla-La Mancha region of Spain and the geographical information. We restricted the analysis to events happening within a rectangular region (180, 60) – (330, 360), resulting in 4241 location points. The covariates of interest are the terrain elevation and the terrain slope ($\mathcal{Y} \subset \mathbb{R}^2$). For each data set, we randomly split the data points into 10 subsets, assigned one to test and the others to training data, and conducted 10-fold cross validations of the predictive performances based on the negative test log likelihood of point patterns (ll_{test}) and the negative test likelihood of counts (cl_{test}). Details of the metrics are given in the supplementary material (§5). Note that to eliminate the effect of data assignment bias, the evaluation multiplied estimated intensity by a factor, $N_{\text{test}}/N_{\text{train}}$.

Table 2 displays the predictive performances on the three real-world data sets, which shows that the Bayesian approaches (INLA, APP_*) outperformed the non-Bayesian alternatives (KIE_*) on the predictive performances. Our approach (APP_*) achieved higher accuracy than INLA in most cases, while the gaps on metrics were significant but not large. The results rather highlight the advantage of our approach on computation cost over the conventional Bayesian method. But, it should be noted that as Table 2 demonstrates, the CPU times of INLA are less affected by the size of data due to domain discretization. Thus INLA remains a valuable inference approach when the size of data is large and the dimension of observation domain is low. Figure 2 displays the estimation results on `bei`, which implies that KIEs tend to underestimate the intensity modulation than Bayesian methods. Here, the estimated intensity by INLA was less smooth than the others due to the additive assumption: $\lambda(\boldsymbol{t}) \propto \exp[g_1(\text{elev}(\boldsymbol{t})) + g_2(\text{slope}(\boldsymbol{t}))]$, where we could not find other ways of implementations in the `R` package `INLA`.

5 Conclusions

We have proposed a novel Bayesian inference method of point process intensity as function of covariates. We augmented a permanental process such that latent intensities can be defined on covariate space, and showed that a representer theorem holds for the augmented process. By exploiting the representer theorem, we have derived a fast inference scheme which scales linearly with data size without using either domain discretization or MCMC. Based on synthetic and open real-world data, we confirmed that our method outperforms state-of-the-art methods in terms of predictive accuracy while being substantially faster than a conventional Bayesian method.

References

[1] Ryan Prescott Adams, Iain Murray, and David J.C. MacKay. Tractable nonparametric Bayesian inference in Poisson processes with Gaussian process intensities. In *Proceedings of the 26th Annual International Conference on Machine Learning*, pages 9–16, 2009.

[2] Virginia Aglietti, Edwin V. Bonilla, Theodoros Damoulas, and Sally Cripps. Structured variational inference in continuous Cox process models. In *Advances in Neural Information Processing Systems 32*, 2019.

[3] Kendall Atkinson. A personal perspective on the history of the numerical analysis of Fredholm integral equations of the second kind. In *The Birth of Numerical Analysis*, pages 53–72. World Scientific, 2010.

[4] Adrian Baddeley and Rolf Turner. Practical maximum pseudolikelihood for spatial point patterns: (with discussion). *Australian & New Zealand Journal of Statistics*, 42(3):283–322, 2000.

[5] Adrian Baddeley and Rolf Turner. Spatstat: an R package for analyzing spatial point patterns. *Journal of Statistical Software*, 12:1–42, 2005.

[6] Adrian Baddeley, Ya-Mei Chang, Yong Song, and Rolf Turner. Nonparametric estimation of the dependence of a spatial point process on spatial covariates. *Statistics and Its Interface*, 5 (2):221–236, 2012.

[7] Sankarshan Basu and Angelos Dassios. A Cox process with log-normal intensity. *Insurance: Mathematics and Economics*, 31(2):297–302, 2002.

[8] Russel E Caflisch. Monte Carlo and quasi-Monte Carlo methods. *Acta Numerica*, 7:1–49, 1998.

[9] David R Cox. Regression models and life-tables. *Journal of the Royal Statistical Society: Series B (Methodological)*, 34(2):187–202, 1972.

[10] John P. Cunningham, M. Yu Byron, Krishna V. Shenoy, and Maneesh Sahani. Inferring neural firing rates from spike trains using Gaussian processes. In *Advances in Neural Information Processing Systems 20*, 2007.

[11] Steve Cygu, Jonathan Dushoff, and Benjamin M Bolker. pcoxtime: Penalized Cox proportional hazard model for time-dependent covariates. *arXiv preprint arXiv:2102.02297*, 2021.

[12] Peter J. Diggle, Paula Moraga, Barry Rowlingson, and Benjamin M. Taylor. Spatial and spatio-temporal log-Gaussian Cox processes: extending the geostatistical paradigm. *Statistical Science*, 28(4):542–563, 2013.

[13] Christian Donner and Manfred Opper. Efficient Bayesian inference of sigmoidal Gaussian Cox processes. *Journal of Machine Learning Research*, 19:1–34, 2018.

[14] Seth Flaxman, Yee Whye Teh, and Dino Sejdinovic. Poisson intensity estimation with reproducing kernels. In *Artificial Intelligence and Statistics*, pages 270–279. PMLR, 2017.

[15] Anthony C. Gatrell, Trevor C. Bailey, Peter J. Diggle, and Barry S. Rowlingson. Spatial point pattern analysis and its application in geographical epidemiology. *Transactions of the Institute of British Geographers*, pages 256–274, 1996.

[16] Israel Gohberg, Seymour Goldberg, and Nahum Krupnik. *Traces and Determinants of Linear Operators*, volume 116. Birkhäuser, 2012.

[17] Yongtao Guan. On consistent nonparametric intensity estimation for inhomogeneous spatial point processes. *Journal of the American Statistical Association*, 103(483):1238–1247, 2008.

[18] Yongtao Guan and Hansheng Wang. Sufficient dimension reduction for spatial point processes directed by Gaussian random fields. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 72(3):367–387, 2010.

[19] Tom Gunter, Chris Lloyd, Michael A. Osborne, and Stephen J. Roberts. Efficient Bayesian nonparametric modelling of structured point processes. In *Uncertainty in Artificial Intelligence*, 2014.

[20] Sture Holm. A simple sequentially rejective multiple test procedure. *Scandinavian Journal of Statistics*, pages 65–70, 1979.

[21] Janine B Illian, Sigrunn H Sørbye, and Håvard Rue. A toolbox for fitting complex spatial point process models using integrated nested Laplace approximation (INLA). *The Annals of Applied Statistics*, 6(4):1499–1530, 2012.

[22] Janine Barbel Illian, Sigrunn Holbek Sørbye, Håvard Rue, and Ditte Katrine Hendrichsen. Using INLA to fit a complex point process model with temporally varying effects-a case study. *Journal of Environmental Statistics*, 3(7), 2012.

[23] S. T. John and James Hensman. Large-scale Cox process inference using variational Fourier features. In *International Conference on Machine Learning*, volume 80, pages 2362–2370. PMLR, 2018.

[24] Hideaki Kim. Fast Bayesian inference for Gaussian Cox processes via path integral formulation. In *Advances in Neural Information Processing Systems 34*, 2021.

[25] Diederik P. Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.

[26] Chris Lloyd, Tom Gunter, Michael Osborne, and Stephen Roberts. Variational inference for Gaussian process modulated Poisson processes. In *International Conference on Machine Learning*, volume 37, pages 1814–1822. PMLR, 2015.

[27] Peter McCullagh and Jesper Møller. The permanental process. *Advances in applied probability*, 38(4):873–888, 2006.

[28] Jesper Møller and Rasmus Plenge Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. CRC Press, 2003.

[29] Jesper Møller, Anne Randi Syversveen, and Rasmus Plenge Waagepetersen. Log Gaussian Cox processes. *Scandinavian Journal of Statistics*, 25(3):451–482, 1998.

[30] E. J. Nyström. Über die praktische Auflösung von Integralgleichungen mit Anwendungen auf Randwertaufgaben. *Acta Mathematica*, 54:185–204, 1930.

[31] Maya Okawa, Tomoharu Iwata, Takeshi Kurashima, Yusuke Tanaka, Hiroyuki Toda, and Naonori Ueda. Deep mixture point processes: Spatio-temporal event prediction with rich contextual information. In *ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pages 373–383, 2019.

[32] Andrei D. Polyanin and Alexander V. Manzhirov. *Handbook of Integral Equations*. CRC press, 1998.

[33] Ali Rahimi and Benjamin Recht. Random features for large-scale kernel machines. In *Advances in Neural Information Processing Systems 20*, 2007.

[34] Carl Edward Rasmussen and Christopher K. I. Williams. *Gaussian Processes for Machine Learning*. MIT press, 2006.

[35] Stephen L. Rathbun and Noel Cressie. Asymptotic properties of estimators for the parameters of spatial inhomogeneous Poisson point processes. *Advances in Applied Probability*, 26(1): 122–154, 1994.

[36] Håvard Rue, Sara Martino, and Nicolas Chopin. Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 71(2):319–392, 2009.

[37] Matthias Seeger, David Salinas, and Valentin Flunkert. Bayesian intermittent demand forecasting for large inventories. In *Advances in Neural Information Processing Systems 29*, 2016.

[38] Simon J. Sheather and Michael C. Jones. A reliable data-based bandwidth selection method for kernel density estimation. *Journal of the Royal Statistical Society: Series B (Methodological)*, 53(3):683–690, 1991.

[39] Bernard W. Silverman. *Density Estimation for Statistics and Data Analysis*. Routledge, 2018.

[40] Dietrich Stoyan and Antti Penttinen. Recent applications of point process methods in forestry statistics. *Statistical Science*, pages 61–78, 2000.

[41] Ming Teng, Farouk Nathoo, and Timothy D. Johnson. Bayesian computation for log-Gaussian Cox processes: A comparative analysis of methods. *Journal of Statistical Computation and Simulation*, 87:2227–2252, 2017.

[42] Terry Therneau, Cindy Crowson, and Elizabeth Atkinson. Using time dependent covariates and time dependent coefficients in the Cox model. *Survival Vignettes*, 2:3, 2017.

[43] Andrea Vedaldi and Andrew Zisserman. Efficient additive kernels via explicit feature maps. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 34(3):480–492, 2012.

[44] Sreekanth Vempati, Andrea Vedaldi, Andrew Zisserman, and C. V. Jawahar. Generalized RBF feature maps for efficient detection. In *BMVC*, pages 1–11, 2010.

[45] Grace Wahba. *Spline Models for Observational Data*, volume 59. SIAM, 1990.

[46] Christian J. Walder and Adrian N. Bishop. Fast Bayesian intensity estimation for the permanental process. In *International Conference on Machine Learning*, volume 70, pages 3579–3588. PMLR, 2017.

[47] Christopher Williams and Matthias Seeger. The effect of the input density distribution on kernel-based classifiers. In *Proceedings of the 17th international conference on machine learning*. Citeseer, 2000.

[48] Christopher Williams and Matthias Seeger. Using the Nyström method to speed up kernel machines. In *Advances in Neural Information Processing Systems 13*, 2000.

[49] Yu Ryan Yue and Ji Meng Loh. Bayesian semiparametric intensity estimation for inhomogeneous spatial point processes. *Biometrics*, 67(3):937–946, 2011.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] See supplementary materials
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Section 2
   (b) Did you include complete proofs of all theoretical results? [Yes] Also see supplementary materials
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] Full instruction of experiments are in supplementary materials, and code is also submitted.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 4 and supplementary materials

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] See Section 4

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See Section 4

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [Yes] See Section 4

(b) Did you mention the license of the assets? [Yes] See Section 4

(c) Did you include any new assets either in the supplemental material or as a URL? [No]

(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [No]

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]