A Simple Convergence Proof of Adam and Adagrad

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Abstract

We provide a simple proof of convergence covering both the Adam and Adagrad adaptive optimization algorithms when applied to smooth (possibly non-convex) objective functions with bounded gradients. We show that in expectation, the squared norm of the objective gradient averaged over the trajectory has an upper-bound which is explicit in the constants of the problem, parameters of the optimizer and the total number of iterations N. This bound can be made arbitrarily small: Adam with a learning rate $\alpha = 1/\sqrt{N}$ and a momentum parameter on squared gradients $\beta_2 = 1 - 1/N$ achieves the same rate of convergence $O(\ln(N)/\sqrt{N})$ as Adagrad. Finally, we obtain the tightest dependency on the heavy ball momentum among all previous convergence bounds for non-convex Adam and Adagrad, improving from $O((1-\beta_1)^{-3})$ to $O((1-\beta_1)^{-1})$. Our technique also improves the best known dependency for standard SGD by a factor $1 - \beta_1$.

1 Introduction

First-order methods with adaptive step sizes have proved useful in many fields of machine learning, be it for sparse optimization (Duchi et al., 2013), tensor factorization (Lacroix et al., 2018) or deep learning (Goodfellow et al., 2016). Duchi et al. (2011) introduced Adagrad, which rescales each coordinate by a sum of squared past gradient values. While Adagrad proved effective for sparse optimization (Duchi et al., 2013), experiments showed that it under-performed when applied to deep learning (Wilson et al., 2017). RMSProp (Tieleman & Hinton, 2012) proposed an exponential moving average instead of a cumulative sum to solve this. Kingma & Ba (2015) developed Adam, one of the most popular adaptive methods in deep learning, built upon RMSProp and added corrective terms at the beginning of training, together with heavy-ball style momentum.

In the online convex optimization setting, Duchi et al. (2011) showed that Adagrad achieves optimal regret for online convex optimization. Kingma & Ba (2015) provided a similar proof for Adam when using a decreasing overall step size, although this proof was later shown to be incorrect by Reddi et al. (2018), who introduced AMSGrad as a convergent alternative. Ward et al. (2019) proved that Adagrad also converges to a critical point for non convex objectives with a rate $O(\ln(N)/\sqrt{N})$ when using a scalar adaptive step-size, instead of diagonal. Zou et al. (2019b) extended this proof to the vector case, while Zou et al. (2019a) displayed a bound for Adam, showing convergence when the decay of the exponential moving average scales as $1 - 1/N$ and the learning rate as $1/\sqrt{N}$.

In this paper, we present a simplified and unified proof of convergence to a critical point for Adagrad and Adam for stochastic non-convex smooth optimization. We assume that the objective function is lower bounded, smooth and the stochastic gradients are almost surely bounded. We recover the standard $O(\ln(N)/\sqrt{N})$ convergence rate for Adagrad for all step sizes, and the same rate with Adam with an appropriate choice of the step sizes and decay parameters, in particular, Adam can converge without using the AMSGrad variant. Compared to previous work, our bound significantly improves the dependency on the momentum parameter β_1. The best know bounds for Adagrad and Adam are respectively in $O((1-\beta_1)^{-3})$ and $O((1-\beta_1)^{-5})$ (see Section 3), while our result is in $O((1-\beta_1)^{-1})$ for both algorithms. Our proof technique for heavy-ball momentum can also be applied to plain SGD, and improves the dependency on $1 - \beta_1$ from a -2 to a -1 exponent (Yang et al., 2016). This improvement is a step toward understanding the practical efficiency of heavy-ball momentum.

Outline. The precise setting and assumptions are stated in the next section, and previous work is then described in Section 3. The main theorems are presented in Section 4, followed by a full proof for the case without momentum in Section 5. The proof of the convergence with momentum is deferred to the supplementary material, along with the same technique applied to SGD. Finally we compare our bounds with experimental results, both on toy and real life problems in Section 6.

2 Setup

2.1 Notation

Let $d \in \mathbb{N}$ be the dimension of the problem (i.e. the number of parameters of the function to optimize) and take $[d] = \{1, 2, \dots, d\}$. Given a function $h : \mathbb{R}^d \to \mathbb{R}$, we denote by ∇h its gradient and $\nabla_i h$ the i-th component of the gradient. We use a small constant ϵ, e.g. 10^{-8}, for numerical stability. Given a sequence $(u_n)_{n\in\mathbb{N}}$ with $\forall n \in \mathbb{N}, u_n \in \mathbb{R}^d$, we denote $u_{n,i}$ for $n \in \mathbb{N}$ and $i \in [d]$ the i-th component of the n-th element of the sequence.

We want to optimize a function $F : \mathbb{R}^d \to \mathbb{R}$. We assume there exists a random function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}\left[\nabla f(x)\right] = \nabla F(x)$ for all $x \in \mathbb{R}^d$, and that we have access to an oracle providing i.i.d. samples $(f_n)_{n\in\mathbb{N}^*}$. We note $\mathbb{E}_{n-1}\left[\cdot\right]$ the conditional expectation knowing $f_1, \dots, f_{n-1}$. In machine learning, x typically represents the weights of a linear or deep model, f represents the loss from individual training examples or minibatches, and F is the full training objective function. The goal is to find a critical point of F.

2.2 Adaptive methods

We study both Adagrad (Duchi et al., 2011) and Adam (Kingma & Ba, 2015) using a unified formulation. We assume we have $0 < \beta_2 \leq 1$, $0 \leq \beta_1 < \beta_2$, and a non negative sequence $(\alpha_n)_{n\in\mathbb{N}^*}$. We define three vectors $m_n, v_n, x_n \in \mathbb{R}^d$ iteratively. Given $x_0 \in \mathbb{R}^d$ our starting point, $m_0 = 0$, and $v_0 = 0$, we define for all iterations $n \in \mathbb{N}^*$,

$$m_{n,i} = \beta_1 m_{n-1,i} + \nabla_i f_n(x_{n-1}) \tag{1}$$

$$v_{n,i} = \beta_2 v_{n-1,i} + \left(\nabla_i f_n(x_{n-1})\right)^2 \tag{2}$$

$$x_{n,i} = x_{n-1,i} - \alpha_n \frac{m_{n,i}}{\sqrt{\epsilon + v_{n,i}}}. \tag{3}$$

The parameter β_1 is a heavy-ball style momentum parameter (Polyak, 1964), while β_2 controls the rate at which the scale of past gradients is forgotten. Taking $\beta_1 = 0$, $\beta_2 = 1$ and $\alpha_n = \alpha$ gives Adagrad. While the original Adagrad algorithm did not include a heavy-ball-like momentum, our analysis also applies to the case $\beta_1 > 0$. On the other hand, when $0 < \beta_2 < 1$, $0 \leq \beta_1 < \beta_2$, taking

$$\alpha_n = \alpha(1 - \beta_1)\sqrt{\frac{1 - \beta_2^n}{1 - \beta_2}} \tag{4}$$

leads to an algorithm close to Adam. We moved the $1 - \beta_1$ and $1 - \beta_2$ factors originally in (1) and (2) to the step size α_n, as this allows for a common treatment of Adam and Adagrad. We also integrate the corrective term $\sqrt{1 - \beta_2^n}$ into the step size. However, we chose to drop the corrective term in $1 - \beta_1^n$ in the original algorithm. Indeed, keeping both can make α_n non monotonic, which complicates the proof. The first few $1/(1 - \beta_1)$ iterations will be smaller than with the usual Adam, i.e., for a typical β_1 of 0.9 (Kingma & Ba, 2015), our algorithm differs from Adam only for the first 50 iterations.

2.3 Assumptions

We make three assumptions. We first assume F is bounded below by F_*, that is,

$$\forall x \in \mathbb{R}^d,\ F(x) \geq F_*. \tag{5}$$

We then assume *the ℓ_∞ norm of the stochastic gradients is uniformly almost surely bounded*, i.e. there is $R \geq \sqrt{\epsilon}$ ($\sqrt{\epsilon}$ is used here to simplify the final bounds) so that

$$\forall x \in \mathbb{R}^d \, \|\nabla f(x)\|_\infty \leq R - \sqrt{\epsilon} \quad \text{a.s.}, \tag{6}$$

and finally, the *smoothness of the objective function*, e.g., its gradient is L-Liptchitz-continuous with respect to the ℓ_2-norm:

$$\forall x, y \in \mathbb{R}^d, \|\nabla F(x) - \nabla F(y)\|_2 \leq L \|x - y\|_2 . \tag{7}$$

3 Related work

Early work on adaptive methods (McMahan & Streeter, 2010; Duchi et al., 2011) showed that Adagrad achieves an optimal rate of convergence of $O(1/\sqrt{N})$ for convex optimization (Agarwal et al., 2009). Later, RMSProp (Tieleman & Hinton, 2012) and Adam (Kingma & Ba, 2015) were developed for training deep neural networks, using an exponential moving average of the past squared gradients.

Kingma & Ba (2015) offered a proof that Adam with a decreasing step size converges for convex objectives. However, the proof contained a mistake spotted by Reddi et al. (2018), who also gave examples of convex problems where Adam does not converge to an optimal solution. They proposed AMSGrad as a convergent variant, which consisted in retaining the maximum value of the exponential moving average. When α goes to zero, AMSGrad is shown to converge in the convex and non-convex setting (Fang & Klabjan, 2019; Zhou et al., 2018). Despite this apparent flaw in the Adam algorithm, it remains a widely popular optimizer, be it for image generation (Karras et al., 2019), music synthesis (Dhariwal et al., 2020), or language modeling (Devlin et al., 2019), raising the question, does Adam really not converge? When β_2 goes to 1 and α to zero, our results and previous work (Zou et al., 2019a) show that Adam does converge with the same rate as Adagrad. This is coherent with the counter examples of Reddi et al. (2018), because they uses a small exponential decay parameter $\beta_2 < 1/5$.

The convergence of Adagrad for non-convex objectives was first tackled by Li & Orabona (2019), who proved the convergence of Adagrad, but under restrictive conditions (e.g., $\alpha \leq \sqrt{\epsilon}/L$). The proof technique was improved by Ward et al. (2019), who showed the convergence of "scalar" Adagrad, i.e., with a single learning rate, for any value of α with a rate of $O(\ln(N)/\sqrt{N})$. Our approach builds on this work but we extend it to apply to both Adagrad and Adam, in their coordinate-wise version, as used in practice, while also supporting heavy-ball momentum.

The coordinate-wise version of Adagrad was also tackled by Zou et al. (2019b), offering a convergence result for Adagrad with either heavy-ball or Nesterov style momentum. We obtain the same rate for heavy-ball momentum with respect to N (i.e., $O(\ln(N)/\sqrt{N})$), but we improve the dependence on the momentum parameter β_1 from $O((1-\beta_1)^{-3})$ to $O((1-\beta_1)^{-1})$. Chen et al. (2019) also provided a bound for Adagrad and Adam, but without convergence guarantees for Adam for any hyper-parameter choice, and with a worse dependency on β_1. Zhou et al. (2018) also cover Adagrad in the stochastic setting, however their proof technique rely on ϵ being quite large, as shown by the $\sqrt{1/\epsilon}$ term in their bound. Finally, a convergence bound for Adam was introduced by Zou et al. (2019a). We recover the same scaling of the bound with respect to α and β_2. However their bound has a dependency of $O((1-\beta_1)^{-5})$ with respect to β_1, while we get $O((1-\beta_1)^{-1})$, a significant improvement. Shi et al. (2020) obtain similar convergence results for RMSProp and Adam when considering the random shuffling setup. They use a strong growth condition (i.e. norm of the stochastic gradient is bounded by an affine function of the norm of the deterministic gradient) instead of the boundness of the gradient, but their bound decays with the number of total epochs, not stochastic updates leading to an overall $\sqrt{s}$ extra term with s the size of the dataset. Finally, Faw et al. (2022) use the same affine growth assumption to derive high probability bounds for scalar Adagrad.

Non adaptive methods like SGD are also well studied in the non convex setting (Ghadimi & Lan, 2013), with a convergence rate of $O(1/\sqrt{N})$ for a smooth objective with bounded variance of the gradients. Unlike adaptive methods, SGD requires knowing the smoothness constant. When adding heavy-ball momentum, Yang et al. (2016) showed that the convergence bound degrades as $O((1-\beta_1)^{-2})$, assuming that the gradients are bounded. We apply our proof technique for momentum to SGD in the Appendix, Section B and improve this dependency to $O((1-\beta_1)^{-1})$.

4 Main results

For a number of iterations $N \in \mathbb{N}^*$, we note τ_N a random index with value in $\{0, \dots, N-1\}$, so that

$$\forall j \in \mathbb{N}, j < N, \mathbb{P}\left[\tau = j\right] \propto 1 - \beta_1^{N-j}. \tag{8}$$

If $\beta_1 = 0$, this is equivalent to sampling τ uniformly in $\{0, \dots, N-1\}$. If $\beta_1 > 0$, the last few $\frac{1}{1-\beta_1}$ iterations are sampled rarely, and iterations older than a few times that number are sampled almost uniformly. Our results bound the expected squared norm of the gradient at iteration τ, which is standard for non convex stochastic optimization (Ghadimi & Lan, 2013).

4.1 Convergence bounds

For simplicity, we first give convergence results for $\beta_1 = 0$, along with a complete proof in Section 5. We then provide the results with momentum, with their proofs in the Appendix, Section A.6. We also provide a bound on the convergence of SGD with an improved dependency on β_1 in the Appendix, Section B.2, along with its proof in Section B.4.

No heavy-ball momentum

Theorem 1 (Convergence of Adagrad without momentum)**.** *Given the assumptions from Section 2.3, the iterates x_n defined in Section 2.2 with hyper-parameters verifying $\beta_2 = 1$, $\alpha_n = \alpha$ with $\alpha > 0$ and $\beta_1 = 0$, and τ defined by (8), we have for any $N \in \mathbb{N}^*$,*

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \leq 2R\frac{F(x_0) - F_*}{\alpha\sqrt{N}} + \frac{1}{\sqrt{N}}\left(4dR^2 + \alpha dRL\right)\ln\left(1 + \frac{NR^2}{\epsilon}\right). \tag{9}$$

Theorem 2 (Convergence of Adam without momentum)**.** *Given the assumptions from Section 2.3, the iterates x_n defined in Section 2.2 with hyper-parameters verifying $0 < \beta_2 < 1$, $\alpha_n = \alpha\sqrt{\frac{1-\beta_2^n}{1-\beta_2}}$ with $\alpha > 0$ and $\beta_1 = 0$, and τ defined by (8), we have for any $N \in \mathbb{N}^*$,*

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \leq 2R\frac{F(x_0) - F_*}{\alpha N} + C\left(\frac{1}{N}\ln\left(1 + \frac{R^2}{(1-\beta_2)\epsilon}\right) - \ln(\beta_2)\right), \tag{10}$$

with

$$C = \frac{4dR^2}{\sqrt{1-\beta_2}} + \frac{\alpha dRL}{1-\beta_2}.$$

With heavy-ball momentum

Theorem 3 (Convergence of Adagrad with momentum)**.** *Given the assumptions from Section 2.3, the iterates x_n defined in Section 2.2 with hyper-parameters verifying $\beta_2 = 1$, $\alpha_n = \alpha$ with $\alpha > 0$ and $0 \leq \beta_1 < 1$, and τ defined by (8), we have for any $N \in \mathbb{N}^*$ such that $N > \frac{\beta_1}{1-\beta_1}$,*

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \leq 2R\sqrt{N}\frac{F(x_0) - F_*}{\alpha\tilde{N}} + \frac{\sqrt{N}}{\tilde{N}}C\ln\left(1 + \frac{NR^2}{\epsilon}\right), \tag{11}$$

with $\tilde{N} = N - \frac{\beta_1}{1-\beta_1}$, and,

$$C = \alpha dRL + \frac{12dR^2}{1-\beta_1} + \frac{2\alpha^2 dL^2\beta_1}{1-\beta_1}.$$

Theorem 4 (Convergence of Adam with momentum)**.** *Given the assumptions from Section 2.3, the iterates x_n defined in Section 2.2 with hyper-parameters verifying $0 < \beta_2 < 1$, $0 \leq \beta_1 < \beta_2$, and, $\alpha_n = \alpha(1-\beta_1)\sqrt{\frac{1-\beta_2^n}{1-\beta_2}}$ with $\alpha > 0$, and τ defined by (8), we have for any $N \in \mathbb{N}^*$ such that $N > \frac{\beta_1}{1-\beta_1}$,*

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \leq 2R\frac{F(x_0) - F_*}{\alpha\tilde{N}} + C\left(\frac{1}{\tilde{N}}\ln\left(1 + \frac{R^2}{(1-\beta_2)\epsilon}\right) - \frac{N}{\tilde{N}}\ln(\beta_2)\right), \tag{12}$$

with $\tilde{N} = N - \frac{\beta_1}{1-\beta_1}$, *and*

$$C = \frac{\alpha d R L (1-\beta_1)}{(1-\beta_1/\beta_2)(1-\beta_2)} + \frac{12 d R^2 \sqrt{1-\beta_1}}{(1-\beta_1/\beta_2)^{3/2}\sqrt{1-\beta_2}} + \frac{2\alpha^2 d L^2 \beta_1}{(1-\beta_1/\beta_2)(1-\beta_2)^{3/2}}.$$

4.2 Analysis of the bounds

Dependency on d. The dependency in d is present in previous works on coordinate wise adaptive methods (Zou et al., 2019a;b). Indeed, for the diagonal version of Adagrad and Adam, we will see in Section 5 that we apply Lemma 5.2 once per dimension. The contribution from each coordinate is mostly independent of the actual scale of its gradients (as it only appears in the log), so that the right hand side of the convergence bound will grow as d. In contrast, the scalar version of Adagrad (Ward et al., 2019) has a single learning rate, so that Lemma 5.2 is only applied once, removing the dependency on d. However, this variant is rarely used in practice.

Almost sure bound on the gradient. We chose to assume the existence of an almost sure uniform ℓ_∞-bound on the gradients given by (6). It is possible instead to assume a uniform bound on the gradients in expectation. We use (6) in Lemma 5.1, to obtain (23) and (26), however in that case, a bound on the expected squared norm of the gradients is sufficient. We then use (6) to derive (31) and (33) in Section 5.2. For those, one can assume only a bound in expectation and use Hölder inequality, as done by Ward et al. (2019) and Zou et al. (2019b). This however deteriorates the bound, as instead of a bound on $\mathbb{E}\left[\|\nabla F(x_\tau)\|_2^2\right]$, one would obtain a bound on $\mathbb{E}\left[\|\nabla F(x_\tau)\|_2^{4/3}\right]^{2/3}$.

Impact of heavy-ball momentum. Looking at Theorems 3 and 4, we see that increasing β_1 always deteriorates the bounds. Taking $\beta_1 = 0$ in those theorems gives us almost exactly the bound without heavy-ball momentum from Theorems 1 and 2, up to a factor 3 in the terms of the form dR^2.

As discussed in Section 3, previous bounds for Adagrad in the non-convex setting deteriorates as $O((1-\beta_1)^{-3})$ (Zou et al., 2019b), while bounds for Adam deteriorates as $O((1-\beta_1)^{-5})$ (Zou et al., 2019a). Instead, our unified proof for Adam and Adagrad achieves a dependency of $O((1-\beta_1)^{-1})$, a significant improvement. We refer the reader to the Appendix, Section A.3, for a detailed analysis. Note that our proof technique can also be applied to SGD and achieve a dependency of $O((1-\beta_1)^{-1})$, compared to $O((1-\beta_1)^{-2})$ for the best existing result Yang et al. (2016). We provide a complete proof in the Appendix, Section B.

While our dependency still contradicts the benefits of using momentum observed in practice, see Section 6, our tighter analysis is a step in the right direction.

4.3 Optimal finite horizon Adam is Adagrad

Let us take a closer look at the result from Theorem 2. It could seem like some quantities can explode but actually not for any reasonable values of α, β_2 and N. Let us assume $\epsilon \ll R^2$, $\alpha = N^{-a}$ and $\beta_2 = 1 - N^{-b}$. Then we immediately have

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \leq 2R\frac{F(x_0) - F_*}{N^{1-a}} + C\left(\frac{1}{N}\ln\left(\frac{R^2 N^b}{\epsilon}\right) + N^{-b}\right), \tag{13}$$

with $C = 4dR^2N^{b/2} + dRLN^{b-a}$. Putting those together and ignoring the log terms for now,

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \lessapprox 2R\frac{F(x_0) - F_*}{N^{1-a}} + 4dR^2N^{b/2-1} + 4dR^2N^{-b/2} + RLN^{b-a-1} + \frac{L}{2}N^{-a}.$$

The best overall rate we can obtain is $O(1/\sqrt{N})$, and it is only achieved for $a = 1/2$ and $b = 1$, i.e., $\alpha = \alpha_1/\sqrt{N}$ and $\beta_2 = 1 - 1/N$. We can see the resemblance between Adagrad on one side and Adam with a finite horizon and such parameters on the other. Indeed, an exponential moving average with a parameter

$\beta_2 = 1 - 1/N$ as a typical averaging window length of size N, while Adagrad would be an exact average of the past N terms. In particular, the bound for Adam now becomes

$$\mathbb{E}\left[\|\nabla F(x_\tau)\|^2\right] \leq \frac{F(x_0) - F_*}{\alpha_1 \sqrt{N}} + \frac{1}{\sqrt{N}} \left(dR + \frac{\alpha_1 dL}{2}\right) \left(\ln\left(1 + \frac{RN}{\epsilon}\right) + 1\right), \tag{14}$$

which differ from (9) only by a +1 next to the log term.

Adam and Adagrad are twins. Our analysis highlights an important fact: Adam is to Adagrad like constant step size SGD is to decaying step size SGD. While Adagrad is asymptotically optimal, it has a slower forgetting of the initial condition $F(x_0) - F_*$, as $1/\sqrt{N}$ instead of $1/N$ for Adam. The fast forgetting of the initial condition of Adam comes at a cost as it does not converge. It is however possible to choose α and β_2 to achieve an ϵ critical point for ϵ arbitrarily small and, for a known time horizon, they can be chosen to obtain the exact same bound as Adagrad.

5 Proofs for $\beta_1 = 0$ (no momentum)

We assume here for simplicity that $\beta_1 = 0$, i.e., there is no heavy-ball style momentum. Taking $n \in \mathbb{N}^*$, the recursions introduced in Section 2.2 can be simplified into

$$\begin{cases} v_{n,i} &= \beta_2 v_{n-1,i} + \left(\nabla_i f_n(x_{n-1})\right)^2, \\ x_{n,i} &= x_{n-1,i} - \alpha_n \frac{\nabla_i f_n(x_{n-1})}{\sqrt{\epsilon + v_{n,i}}}. \end{cases} \tag{15}$$

Remember that we recover Adagrad when $\alpha_n = \alpha$ for $\alpha > 0$ and $\beta_2 = 1$, while Adam can be obtained taking $0 < \beta_2 < 1$ and

$$\alpha_n = \alpha \sqrt{\frac{1 - \beta_2^n}{1 - \beta_2}}, \tag{16}$$

for $\alpha > 0$.

Throughout the proof we denote by $\mathbb{E}_{n-1}[\cdot]$ the conditional expectation with respect to $f_1, \ldots, f_{n-1}$. In particular, x_{n-1} and v_{n-1} are deterministic knowing $f_1, \ldots, f_{n-1}$. For all $n \in \mathbb{N}^*$, we also define $\tilde{v}_n \in \mathbb{R}^d$ so that for all $i \in [d]$,

$$\tilde{v}_{n,i} = \beta_2 v_{n-1,i} + \mathbb{E}_{n-1}\left[\left(\nabla_i f_n(x_{n-1})\right)^2\right], \tag{17}$$

i.e., we replace the last gradient contribution by its expected value conditioned on $f_1, \ldots, f_{n-1}$.

5.1 Technical lemmas

A problem posed by the update (15) is the correlation between the numerator and denominator. This prevents us from easily computing the conditional expectation and as noted by Reddi et al. (2018), the expected direction of update can have a positive dot product with the objective gradient. It is however possible to control the deviation from the descent direction, following Ward et al. (2019) with this first lemma.

Lemma 5.1 (adaptive update approximately follow a descent direction)**.** *For all $n \in \mathbb{N}^*$ and $i \in [d]$, we have:*

$$\mathbb{E}_{n-1}\left[\nabla_i F(x_{n-1}) \frac{\nabla_i f_n(x_{n-1})}{\sqrt{\epsilon + v_{n,i}}}\right] \geq \frac{\left(\nabla_i F(x_{n-1})\right)^2}{2\sqrt{\epsilon + \tilde{v}_{n,i}}} - 2R\mathbb{E}_{n-1}\left[\frac{\left(\nabla_i f_n(x_{n-1})\right)^2}{\epsilon + v_{n,i}}\right]. \tag{18}$$

Proof. We take $i \in [d]$ and note $G = \nabla_i F(x_{n-1})$, $g = \nabla_i f_n(x_{n-1})$, $v = v_{n,i}$ and $\tilde{v} = \tilde{v}_{n,i}$.

$$\mathbb{E}_{n-1}\left[\frac{Gg}{\sqrt{\epsilon + v}}\right] = \mathbb{E}_{n-1}\left[\frac{Gg}{\sqrt{\epsilon + \tilde{v}}}\right] \mathbb{E}_{n-1}\Bigg[\underbrace{Gg\left(\frac{1}{\sqrt{\epsilon + v}} - \frac{1}{\sqrt{\epsilon + \tilde{v}}}\right)}_{A}\Bigg]. \tag{19}$$

Given that g and $\tilde{v}$ are independent knowing $f_1, \ldots, f_{n-1}$, we immediately have

$$\mathbb{E}_{n-1}\left[\frac{Gg}{\sqrt{\epsilon+\tilde{v}}}\right] = \frac{G^2}{\sqrt{\epsilon+\tilde{v}}}. \tag{20}$$

Now we need to control the size of the second term A,

$$\begin{aligned} A &= Gg\frac{\tilde{v}-v}{\sqrt{\epsilon+v}\sqrt{\epsilon+\tilde{v}}(\sqrt{\epsilon+v}+\sqrt{\epsilon+\tilde{v}})} \\ &= Gg\frac{\mathbb{E}_{n-1}\left[g^2\right]-g^2}{\sqrt{\epsilon+v}\sqrt{\epsilon+\tilde{v}}(\sqrt{\epsilon+v}+\sqrt{\epsilon+\tilde{v}})} \\ |A| &\leq \underbrace{|Gg|\frac{\mathbb{E}_{n-1}\left[g^2\right]}{\sqrt{\epsilon+v}(\epsilon+\tilde{v})}}_{\kappa} + \underbrace{|Gg|\frac{g^2}{(\epsilon+v)\sqrt{\epsilon+\tilde{v}}}}_{\rho}. \end{aligned}$$

The last inequality comes from the fact that $\sqrt{\epsilon+v}+\sqrt{\epsilon+\tilde{v}} \geq \max(\sqrt{\epsilon+v}, \sqrt{\epsilon+\tilde{v}})$ and $\left|\mathbb{E}_{n-1}\left[g^2\right]-g^2\right| \leq \mathbb{E}_{n-1}\left[g^2\right]+g^2$. Following Ward et al. (2019), we can use the following inequality to bound κ and ρ,

$$\forall \lambda > 0,\, x, y \in \mathbb{R}, xy \leq \frac{\lambda}{2}x^2 + \frac{y^2}{2\lambda}. \tag{21}$$

First applying (21) to κ with

$$\lambda = \frac{\sqrt{\epsilon+\tilde{v}}}{2},\; x = \frac{|G|}{\sqrt{\epsilon+\tilde{v}}},\; y = \frac{|g|\,\mathbb{E}_{n-1}\left[g^2\right]}{\sqrt{\epsilon+\tilde{v}}\sqrt{\epsilon+v}},$$

we obtain

$$\kappa \leq \frac{G^2}{4\sqrt{\epsilon+\tilde{v}}} + \frac{g^2\mathbb{E}_{n-1}\left[g^2\right]^2}{(\epsilon+\tilde{v})^{3/2}(\epsilon+v)}.$$

Given that $\epsilon+\tilde{v} \geq \mathbb{E}_{n-1}\left[g^2\right]$ and taking the conditional expectation, we can simplify as

$$\mathbb{E}_{n-1}\left[\kappa\right] \leq \frac{G^2}{4\sqrt{\epsilon+\tilde{v}}} + \frac{\mathbb{E}_{n-1}\left[g^2\right]}{\sqrt{\epsilon+\tilde{v}}}\mathbb{E}_{n-1}\left[\frac{g^2}{\epsilon+v}\right]. \tag{22}$$

Given that $\sqrt{\mathbb{E}_{n-1}\left[g^2\right]} \leq \sqrt{\epsilon+\tilde{v}}$ and $\sqrt{\mathbb{E}_{n-1}\left[g^2\right]} \leq R$, we can simplify (22) as

$$\mathbb{E}_{n-1}\left[\kappa\right] \leq \frac{G^2}{4\sqrt{\epsilon+\tilde{v}}} + R\mathbb{E}_{n-1}\left[\frac{g^2}{\epsilon+v}\right]. \tag{23}$$

Now turning to ρ, we use (21) with

$$\lambda = \frac{\sqrt{\epsilon+\tilde{v}}}{2\mathbb{E}_{n-1}\left[g^2\right]},\; x = \frac{|Gg|}{\sqrt{\epsilon+\tilde{v}}},\; y = \frac{g^2}{\epsilon+v},$$

we obtain

$$\rho \leq \frac{G^2}{4\sqrt{\epsilon+\tilde{v}}}\frac{g^2}{\mathbb{E}_{n-1}\left[g^2\right]} + \frac{\mathbb{E}_{n-1}\left[g^2\right]}{\sqrt{\epsilon+\tilde{v}}}\frac{g^4}{(\epsilon+v)^2}, \tag{24}$$

Given that $\epsilon+v \geq g^2$ and taking the conditional expectation we obtain

$$\mathbb{E}_{n-1}\left[\rho\right] \leq \frac{G^2}{4\sqrt{\epsilon+\tilde{v}}} + \frac{\mathbb{E}_{n-1}\left[g^2\right]}{\sqrt{\epsilon+\tilde{v}}}\mathbb{E}_{n-1}\left[\frac{g^2}{\epsilon+v}\right], \tag{25}$$

which we simplify using the same argument as for (23) into

$$\mathbb{E}_{n-1}\left[\rho\right] \leq \frac{G^2}{4\sqrt{\epsilon+\tilde{v}}} + R\mathbb{E}_{n-1}\left[\frac{g^2}{\epsilon+v}\right]. \tag{26}$$

Notice that in (24), we possibly divide by zero. It suffice to notice that if $\mathbb{E}_{n-1}\left[g^2\right] = 0$ then $g^2 = 0$ a.s. so that $\rho = 0$ and (26) is still verified. Summing (23) and (26) we can bound

$$\mathbb{E}_{n-1}\left[|A|\right] \leq \frac{G^2}{2\sqrt{\epsilon+\tilde{v}}} + 2R\mathbb{E}_{n-1}\left[\frac{g^2}{\epsilon+v}\right]. \tag{27}$$

Injecting (27) and (20) into (19) finishes the proof. □

Anticipating on Section 5.2, the previous Lemma gives us a bound on the deviation from a descent direction. While for a specific iteration, this deviation can take us away from a descent direction, the next lemma tells us that the sum of those deviations cannot grow larger than a logarithmic term. This key insight introduced in Ward et al. (2019) is what makes the proof work.

Lemma 5.2 (sum of ratios with the denominator being the sum of past numerators)**.** *We assume we have* $0 < \beta_2 \leq 1$ *and a non-negative sequence* $(a_n)_{n\in\mathbb{N}^*}$*. We define for all* $n \in \mathbb{N}^*$*,* $b_n = \sum_{j=1}^{n} \beta_2^{n-j} a_j$*. We have*

$$\sum_{j=1}^{N} \frac{a_j}{\epsilon + b_j} \leq \ln\left(1 + \frac{b_N}{\epsilon}\right) - N\ln(\beta_2). \tag{28}$$

Proof. Given that concavity of ln, and the fact that $b_j > a_j \geq 0$, we have for all $j \in \mathbb{N}^*$,

$$\begin{aligned}
\frac{a_j}{\epsilon + b_j} &\leq \ln(\epsilon + b_j) - \ln(\epsilon + b_j - a_j) \\
&= \ln(\epsilon + b_j) - \ln(\epsilon + \beta_2 b_{j-1}) \\
&= \ln\left(\frac{\epsilon + b_j}{\epsilon + b_{j-1}}\right) + \ln\left(\frac{\epsilon + b_{j-1}}{\epsilon + \beta_2 b_{j-1}}\right).
\end{aligned}$$

The first term forms a telescoping series, while the second one is bounded by $-\ln(\beta_2)$. Summing over all $j \in [N]$ gives the desired result. □

5.2 Proof of Adam and Adagrad without momentum

Let us take an iteration $n \in \mathbb{N}^*$, we define the update $u_n \in \mathbb{R}^d$:

$$\forall i \in [d], u_{n,i} = \frac{\nabla_i f_n(x_{n-1})}{\sqrt{\epsilon + v_{n,i}}}. \tag{29}$$

Adagrad. As explained in Section 2.2, we have $\alpha_n = \alpha$ for $\alpha > 0$. Using the smoothness of F (7), we have

$$F(x_{n+1}) \leq F(x_n) - \alpha\nabla F(x_n)^T u_n. + \frac{\alpha^2 L}{2}\left\|u_n\right\|_2^2. \tag{30}$$

Taking the conditional expectation with respect to $f_0, \ldots, f_{n-1}$ we can apply the descent Lemma 5.1. Notice that due to the a.s. ℓ_∞ bound on the gradients (6), we have for any $i \in [d]$, $\sqrt{\epsilon + \tilde{v}_{n,i}} \leq R\sqrt{n}$, so that,

$$\frac{\alpha\left(\nabla_i F(x_{n-1})\right)^2}{2\sqrt{\epsilon + \tilde{v}_{n,i}}} \geq \frac{\alpha\left(\nabla_i F(x_{n-1})\right)^2}{2R\sqrt{n}}. \tag{31}$$

This gives us

$$\mathbb{E}_{n-1}\left[F(x_n)\right] \leq F(x_{n-1}) - \frac{\alpha}{2R\sqrt{n}}\left\|\nabla F(x_{n-1})\right\|_2^2 + \left(2\alpha R + \frac{\alpha^2 L}{2}\right)\mathbb{E}_{n-1}\left[\left\|u_n\right\|_2^2\right].$$

Summing the previous inequality for all $n \in [N]$, taking the complete expectation, and using that $\sqrt{n} \leq \sqrt{N}$ gives us,

$$\mathbb{E}\left[F(x_N)\right] \leq F(x_0) - \frac{\alpha}{2R\sqrt{N}} \sum_{n=0}^{N-1} \mathbb{E}\left[\|\nabla F(x_n)\|_2^2\right] + \left(2\alpha R + \frac{\alpha^2 L}{2}\right) \sum_{n=0}^{N-1} \mathbb{E}\left[\|u_n\|_2^2\right].$$

From there, we can bound the last sum on the right hand side using Lemma 5.2 once for each dimension. Rearranging the terms, we obtain the result of Theorem 1.

Adam. As given by (4) in Section 2.2, we have $\alpha_n = \alpha\sqrt{\frac{1-\beta_2^n}{1-\beta_2}}$ for $\alpha > 0$. Using the smoothness of F defined in (7), we have

$$F(x_n) \leq F(x_{n-1}) - \alpha_n \nabla F(x_{n-1})^T u_n + \frac{\alpha_n^2 L}{2} \|u_n\|_2^2. \tag{32}$$

We have for any $i \in [d]$, $\sqrt{\epsilon + \tilde{v}_{n,i}} \leq R\sqrt{\sum_{j=0}^{n-1} \beta_2^j} = R\sqrt{\frac{1-\beta_2^n}{1-\beta_2}}$, thanks to the a.s. ℓ_∞ bound on the gradients (6), so that,

$$\alpha_n \frac{\left(\nabla_i F(x_{n-1})\right)^2}{2\sqrt{\epsilon + \tilde{v}_{n,i}}} \geq \frac{\alpha \left(\nabla_i F(x_{n-1})\right)^2}{2R}. \tag{33}$$

Taking the conditional expectation with respect to $f_1, \ldots, f_{n-1}$ we can apply the descent Lemma 5.1 and use (33) to obtain from (32),

$$\mathbb{E}_{n-1}\left[F(x_n)\right] \leq F(x_{n-1}) - \frac{\alpha}{2R} \|\nabla F(x_{n-1})\|_2^2 + \left(2\alpha_n R + \frac{\alpha_n^2 L}{2}\right) \mathbb{E}_{n-1}\left[\|u_n\|_2^2\right].$$

Given that $\beta_2 < 1$, we have $\alpha_n \leq \frac{\alpha}{\sqrt{1-\beta_2}}$. Summing the previous inequality for all $n \in [N]$ and taking the complete expectation yields

$$\mathbb{E}\left[F(x_N)\right] \leq F(x_0) - \frac{\alpha}{2R} \sum_{n=0}^{N-1} \mathbb{E}\left[\|\nabla F(x_n)\|_2^2\right] + \left(\frac{2\alpha R}{\sqrt{1-\beta_2}} + \frac{\alpha^2 L}{2(1-\beta_2)}\right) \sum_{n=0}^{N-1} \mathbb{E}\left[\|u_n\|_2^2\right].$$

Applying Lemma 5.2 for each dimension and rearranging the terms finishes the proof of Theorem 2.

6 Experiments

On Figure 1, we compare the effective dependency of the average squared norm of the gradient in the parameters α, β_1 and β_2 for Adam, when used on a toy task and CIFAR-10.

6.1 Setup

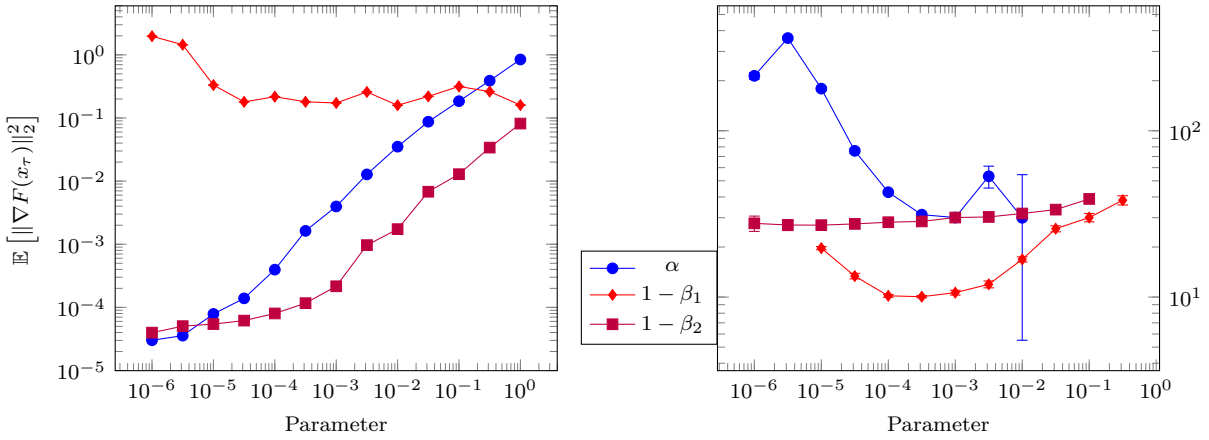
Toy problem. In order to support the bounds presented in Section 4, in particular the dependency in β_2, we test Adam on a specifically crafted toy problem. We take $x \in \mathbb{R}^6$ and define for all $i \in [6]$, $p_i = 10^{-i}$. We take $(Q_i)_{i\in[6]}$, Bernoulli variables with $\mathbb{P}\left[Q_i = 1\right] = p_i$. We then define f for all $x \in \mathbb{R}^d$ as

$$f(x) = \sum_{i\in[6]} (1 - Q_i)\,\mathrm{Huber}(x_i - 1) + \frac{Q_i}{\sqrt{p_i}}\,\mathrm{Huber}(x_i + 1), \tag{34}$$

with for all $y \in \mathbb{R}$,

$$\mathrm{Huber}(y) = \begin{cases} \frac{y^2}{2} & \text{when } |y| \leq 1 \\ |y| - \frac{1}{2} & \text{otherwise.} \end{cases}$$

Intuitively, each coordinate is pointing most of the time towards 1, but exceptionally towards -1 with a weight of $1/\sqrt{p_i}$. Those rare events happens less and less often as i increase, but with an increasing weight. Those



(a) Average squared norm of the gradient on a toy task, see Section 6, for more details. For the α and $1-\beta_2$ curves, we initialize close to the optimum to make the $F_0 - F_*$ term negligible.

(b) Average squared norm of the gradient of a small convolutional model Gitman & Ginsburg (2017) trained on CIFAR-10, with a random initialization. The full gradient is evaluated every epoch.

Figure 1: Observed average squared norm of the objective gradients after a fixed number of iterations when varying a single parameter out of α, $1-\beta_1$ and $1-\beta_2$, on a toy task (left, 10^6 iterations) and on CIFAR-10 (right, 600 epochs with a batch size 128). All curves are averaged over 3 runs, error bars are negligible except for small values of α on CIFAR-10. See Section 6 for details.

weights are chosen so that the variances of all the coordinates of the gradient are equals[1] It is necessary to take different probabilities for each coordinate. If we use the same p for all, we observe a phase transition when $1-\beta_2 \approx p$, but not the continuous improvement we obtain on Figure 1a.

We plot the variation of $\mathbb{E}\left[\|F(x_\tau)\|_2^2\right]$ after 10^6 iterations with batch size 1 when varying either α, $1-\beta_1$ or $1-\beta_2$ through a range of 13 values uniformly spaced in log-scale between 10^{-6} and 1. When varying α, we take $\beta_1 = 0$ and $\beta_2 = 1 - 10^{-6}$. When varying β_1, we take $\alpha = 10^{-5}$ and $\beta_2 = 1 - 10^{-6}$ (i.e. β_2 is so that we are in the Adagrad-like regime). Finally, when varying β_2, we take $\beta_1 = 0$ and $\alpha = 10^{-6}$. When varying α and β_2, we start from x_0 close to the optimum by running first 10^6 iterations with $\alpha = 10^{-4}$, then 10^{-6} iterations with $\alpha = 10^{-5}$, always with $\beta_2 = 1 - 10^{-6}$. This allows to have $F(x_0) - F_* \approx 0$ in (10) and (12) and focus on the second part of both bounds. All curves are averaged over three runs. Error bars are plotted but not visible in log-log scale.

CIFAR-10. We train a simple convolutional network (Gitman & Ginsburg, 2017) on the CIFAR-10[2] image classification dataset. Starting from a random initialization, we train the model on a single V100 for 600 epochs with a batch size of 128, evaluating the full training gradient after each epoch. This is a proxy for $\mathbb{E}\left[\|F(x_\tau)\|_2^2\right]$, which would be to costly to evaluate exactly. All runs use the default config $\alpha = 10^{-3}$, $\beta_2 = 0.999$ and $\beta_1 = 0.9$, and we then change one of the parameter.

We take α from a uniform range in log-space between 10^{-6} and 10^{-2} with 9 values, for $1-\beta_1$ the range is from 10^{-5} to 0.3 with 9 values, and for $1-\beta_2$, from 10^{-6} to 10^{-1} with 11 values. Unlike for the toy problem, we do not initialize close to the optimum, as even after 600 epochs, the norm of the gradients indicates that we are not at a critical point. All curves are averaged over three runs. Error bars are plotted but not visible in log-log scale, except for large values of α.

[1] We deviate from the a.s. bounded gradient assumption for this experiment, see Section 4.2 for a discussion on a.s. bound vs bound in expectation.

[2] `https://www.cs.toronto.edu/~kriz/cifar.html`

6.2 Analysis

Toy problem. Looking at Figure 1a, we observe a continual improvement as β_2 increases. Fitting a linear regression in log-log scale of $\mathbb{E}[\|\nabla F(x_\tau)\|_2^2]$ with respect to $1-\beta_2$ gives a slope of 0.56 which is compatible with our bound (10), in particular the dependency in $O(1/\sqrt{1-\beta_2})$. As we initialize close to the optimum, a small step size α yields as expected the best performance. Doing the same regression in log-log scale, we find a slope of 0.87, which is again compatible with the $O(\alpha)$ dependency of the second term in (10). Finally, we observe a limited impact of β_1, except when $1-\beta_1$ is small. The regression in log-log scale gives a slope of -0.16, while our bound predicts a slope of -1.

CIFAR 10. Let us now turn to Figure 1b. As we start from random weights for this problem, we observe that a large step size gives the best performance, although we observe a high variance for the largest α. This indicates that training becomes unstable for large α, which is not predicted by the theory. This is likely a consequence of the bounded gradient assumption (6) not being verified for deep neural networks. We observe a small improvement as $1-\beta_2$ decreases, although nowhere near what we observed on our toy problem. Finally, we observe a sweet spot for the momentum β_1, not predicted by our theory. We conjecture that this is due to the variance reduction effect of momentum (averaging of the gradients over multiple mini-batches, while the weights have not moved so much as to invalidate past information).

7 Conclusion

We provide a simple proof on the convergence of Adam and Adagrad without heavy-ball style momentum. Our analysis highlights a link between the two algorithms: with right the hyper-parameters, Adam converges like Adagrad. The extension to heavy-ball momentum is more complex, but we significantly improve the dependence on the momentum parameter for Adam, Adagrad, as well as SGD. We exhibit a toy problem where the dependency on α and β_2 experimentally matches our prediction. However, we do not predict the practical interest of momentum, so that improvements to the proof are needed for future work.

Broader Impact Statement

The present theoretical results on the optimization of non convex losses in a stochastic settings impact our understanding of the training of deep neural network. It might allow a deeper understanding of neural network training dynamics and thus reinforce any existing deep learning applications. There would be however no direct possible negative impact to society.

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