Predicting generalization with degrees of freedom in neural networks

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Abstract

Model complexity is fundamentally tied to predictive power in the sciences as well as in applications. However, there is a divergence between naive measures of complexity such as parameter count and the generalization performance of over-parameterized machine learning models. Prior empirical approaches to capturing intrinsic complexity in a more sophisticated manner than parameter count are computationally intractable, do not capture the implicitly regularizing effects of the entire machine-learning pipeline, or do not provide a quantitative fit to the double descent behavior of overparameterized models. In this work, we introduce an empirical complexity measure inspired by the classical notion of generalized degrees of freedom in statistics. This measure can be approximated efficiently and is a function of the entire machine learning training pipeline. We demonstrate that this measure correlates with generalization performance in the double-descent regime.

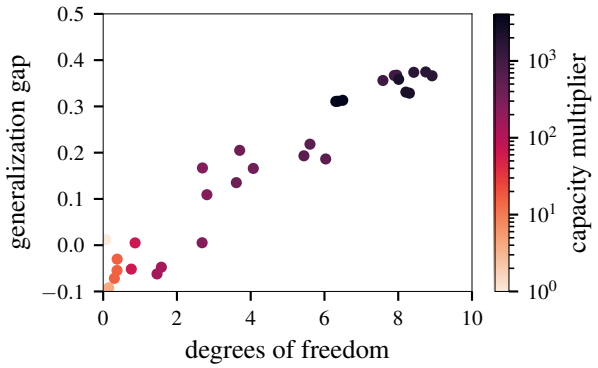

Figure 1. The generalization gap among neural network models that vary by a capacity multiplier (here, kernel parameter multiplier for convolutional neural networks) is monotonically increasing in the complexity measure *generalized degrees of freedom*

1. Introduction

What makes a machine learning model a good scientific explanation? Classical perspectives call for trading off low *complexity* and high *predictive accuracy* (Forster & Sober, 1994). However, deep learning approaches to scientific modeling (Baraniuk et al., 2020; Bianchini et al., 2020; Raghu & Schmidt, 2020; Hope et al., 2022) come with new challenges in navigating this trade-off due to the difficulty in assessing *both* complexity and predictive accuracy. First, many established measures of complexity do not apply to deep learning models because they are vacuous (Dziugaite & Roy, 2017), or they do not capture the non-monotonic behavior of the evaluation error to which complexity is fundamentally tied (Belkin et al., 2019; Nakkiran et al., 2021). Secondly, over-parameterized deep learning models often obtain zero training error (Zhang et al., 2017), meaning that model selection by predictive accuracy on the training set alone is not possible, while maintaining a test set representative of downstream applications is challenging in practice (Nagarajan et al., 2021). These challenges mean that practitioners lack foundational methods for model selection, hindering the reliable deployment of many machine learning techniques (Amodei et al., 2016; D'Amour et al., 2020; Geirhos et al., 2020) and giving rise to poorly understood pathologies such as susceptibility to adversarial examples (Szegedy et al., 2014; Goodfellow et al., 2015).

In this work, we explore the extent to which the *generalized degrees of freedom (GDoF)* framework introduced by Efron (1983) and Ye (1998) can respond to these challenges. GDoF, as its name suggests, generalizes the basic notion of degrees of freedom for non-linear models, and is equivalent to the number of features for linear models. This correspondence provides a natural measurement of the complexity of machine learning models, including deep neural networks. Given the fundamental connection between complexity and generalization, we hypothesize that the GDoF predicts a model's generalization performance. GDoF has been ex-

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plored in various contexts, though Gao & Jojic (2016) was the first to study this measurement for deep neural networks. In this work, we make the following additional contributions:

- We introduce approximations that enable scalable computation of the GDoF during training of deep neural networks (Section (**3**)).
- We reconcile the double-descent phenomenon with the classical trade-off between complexity and predictive accuracy by demonstrating the monotonicity of the generalization gap in GDoF (Fig. (**1**)).
- We use the online nature of the approximation to reveal the evolution of GDoF over training (Fig. (**3**) and Fig. (**5**)).
- We demonstrate that controlling GDoF via subspace training (Li et al., 2018) can improve generalization (Fig. (**4**) and Fig. (**5**)).

2. Background

When reasoning about model complexity, we contrast changes in the *parameter dimension* of a machine learning model—measures proportional to the parameter count of a model—with changes in the *effective dimensionality*, which is indirectly controlled by various components such as the architecture and training procedure. While trivially equivalent re-parametrizations—for instance, replacing a connection weight by the product of two weights, $w = w_1\, w_2$—may result in vastly different parameter dimensions, effective dimensionality captures the intrinsic complexity as a result of the complete system specification. According to classic notions of complexity-fit trade-off, models with low effective dimensionality have low complexity, are less likely to overfit and thus have lower generalization error (Schmidhuber, 1997). The preference for simplicity in model selection is also known as *Occam's Razor* (MacKay, 1991).

However, the classical trade-off between complexity and predictive power, which focuses on the bias and variance trade-off (Geman et al., 1992), has been challenged by the recently re-discovered double descent phenomenon (Belkin et al., 2019): While large models are traditionally understood to be inherently more complex and thus subject to poor generalization as a result of overfitting, modern over-parameterized machine learning models exhibit surprisingly good generalization performance (Zhang et al., 2017). While the origin of the discrepancy between parameter dimension and effective dimensionality is outside of the scope of this work, previous work has shown that it may arise from implicit regularization due to various components of the machine-learning pipeline, including the model architecture (Golubeva et al., 2020), the specific initialization procedure (Kubo et al., 2019; Mehta et al., 2021), gradient-based optimization (Barrett & Dherin, 2021), and overparametrization itself (Neyshabur et al., 2018; Advani et al., 2020).

Various attempts have been made to capture effective dimensionality via model complexity from the perspective of a model's loss landscape (Larsen et al., 2021) and, in particular, its local curvature (Maddox et al., 2020; Loukas et al., 2021). The statistical concept of *degrees of freedom* can also be related to the loss landscape. For linear models, the degree of freedom is equivalent to parameter count, but Efron (1983) and Ye (1998) generalized this concept to non-linear models from the perspectives of *expected optimism* and *average sensitivity* as follows: For a classifier f parameterized by θ, the GDoF is defined from the derivatives of the *model-predicted label* $f_{n,c}$ with respect to the *data label* $y_{n,c}$:

$$\Phi \coloneqq \sum_{n=1}^{N} \sum_{c=1}^{C} \frac{\partial f_{n,c}(\theta)}{\partial y_{n,c}} \,, \tag{1}$$

where $n \in \{1, \ldots, N\}$ is the index of the sample in the training set, $c \in \{1, \ldots, C\}$ is the index of the class. When explicit dependence on the training procedure is required, we further expand Eq. (1) and specify the algorithm $\mathcal{A}$, the initial parameter θ_0, and the training step t such that the GDoF can reflect the model complexity at any step t before convergence as

$$\Phi(t) \coloneqq \sum_{n=1}^{N} \sum_{c=1}^{C} \frac{\partial f_{n,c}(\theta_t = \mathcal{A}(\theta_0, t))}{\partial y_{n,c}} \,. \tag{2}$$

Naively, computing these derivatives would require back-propagating through the whole training procedure specified by $\mathcal{A}$.

The GDoF measurement Φ in Eq. (2) can be interpreted as the sensitivity of the model f to perturbation in the training label $y_{n,c}$. Gao & Jojic (2016) introduced GDoF to neural networks by approximating the derivatives in Eq. (2) by a specific random perturbation and demonstrated that the GDoF of such over-parameterized models are only a small fraction of the total number of parameters. However, (Gao & Jojic, 2016) computed GDoF by training a perturbed model, as well as the original model, on the complete training dataset until convergence, thus requiring dual training runs for each iteration of computing the GDoF. Here we show that the approximation of Eq. (2) can be broken-up in both space—with mini-batches—and in time—by incremental computation. In addition, the latter incremental approximation enables us to reveal the evolution of GDoF during training.

3. Method

In this work, we introduce additional approximations to GDoF, defined as Φ in Eq. (2) for a more scalable estimate

of model complexity. The first straightforward step of approximation is using a Monte-Carlo estimator based on a subset $B < N$ of samples from the training set:

$$\Phi(t) \approx \Phi^{\text{MC}}(t) := \frac{N}{B} \sum_{n=1}^{B} \sum_{c=1}^{C} \frac{\partial f_{n,c}\left(\theta_t = \mathcal{A}(\theta_0, t)\right)}{\partial y_{n,c}} \,. \tag{3}$$

The second is to remove the requirement of training the perturbed model to convergence by exploiting the sequential updating of model parameters θ during gradient-based training. The following proposition shows that, given the GDoF of a model trained from initialization to time t, the GDoF at step $t' > t$ can be approximated by simply adding the additional sensitivity brought by the increment $\Delta\theta = \theta_{t'} - \theta_t$ as long as $\Delta\theta$ is small.

Proposition 1. *The GDoF at step t' is related to the GDoF at $t < t'$ via:*

$$\Phi(t') \approx \Phi(t) + \Delta\Phi(t, t') \,, \tag{4}$$

where

$$\Delta\Phi(t, t') = \sum_{n=1}^{N} \sum_{c=1}^{C} \frac{\partial f_{n,c}(\theta_{t'} = \mathcal{A}(\theta_t, t' - t))}{\partial y_{n,c}} \tag{5}$$

measures the increment of sensitivity after training f with $\mathcal{A}$ to step t' from θ_t at step t.

Proof. We first use the following shorthand to represent the instance- and class-wise contributions to Φ:

$$\phi_{n,c}(t, t') := \frac{\partial f_{n,c}(\theta_{t'} = \mathcal{A}(\theta_t, t' - t))}{\partial y_{n,c}} \,. \tag{6}$$

With this, Eq. (2) can be written as

$$\Phi(t) = \sum_{n=1}^{N} \sum_{c=1}^{C} \phi_{n,c}(0, t) \,. \tag{7}$$

For additional brevity, we drop the example subscript i and the class subscript c when they are irrelevant to the derivation, giving

$$\phi(0, t') = \frac{\partial f(\theta_{t'})}{\partial y} \,. \tag{8}$$

We then construct a Taylor expansion of $f(\theta)$ near $\theta = \theta_t$:

$$f(\theta) = f(\theta_t) + (\theta - \theta_t)^T \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=\theta_t} + \mathcal{O}(\|\theta - \theta_t\|^2) \,, \tag{9}$$

which gives, when $\Delta\theta = \theta_{t'} - \theta_t$ is small,

$$f(\theta_{t'}) \approx f(\theta_t) + (\Delta\theta)^T \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=\theta_t} \,. \tag{10}$$

Plugging in the approximation Eq. (10) into Eq. (8) gives

$$\begin{aligned} \phi(0, t') &\approx \frac{\partial f(\theta_t)}{\partial y} + \left(\frac{\partial \Delta\theta}{\partial y}\right)^T \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=\theta_t} \\ &\quad + (\Delta\theta)^T \left.\frac{\partial^2 f(\theta)}{\partial y\, \partial \theta}\right|_{\theta=\theta_t} \\ &\approx \phi(0, t) + \left(\frac{\partial \Delta\theta}{\partial y}\right)^T \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=\theta_t} \,. \end{aligned} \tag{11}$$

Here the approximation comes from dropping the second order term, which is negligible when the update $\Delta\theta$ or the second derivative $\frac{\partial^2 f(\theta)}{\partial y\, \partial \theta}\big|_{\theta=\theta_t}$ is small.

Moreover, since $\phi(t, t')$ depends on y only through the update $\Delta\theta$, we can employ the chain rule to give:

$$\begin{aligned} \phi(t, t') &= \frac{\partial f(\theta_{t'} = \mathcal{A}(\theta_t, t' - t))}{\partial y} \\ &= \left(\frac{\partial \Delta\theta}{\partial y}\right)^T \left.\frac{\partial f(\theta_t + \Delta)}{\partial \Delta}\right|_{\Delta=\Delta\theta} \\ &= \left(\frac{\partial \Delta\theta}{\partial y}\right)^T \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=\theta_{t'}} \qquad (12) \\ &\approx \left(\frac{\partial \Delta\theta}{\partial y}\right)^T \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=\theta_t} \,. \qquad (13) \end{aligned}$$

where the last line of approximation comes from using the derivatives of $\frac{\partial f(\theta)}{\partial \theta}$ at θ_t for $\theta_{t'}$. In Eq. (12), we used the definition of the derivative, while changing the variable $\theta = \theta_t + \Delta$:

$$\begin{aligned} \frac{\partial f(\theta_t + \Delta)}{\partial \Delta} &= \lim_{\delta \to 0} \frac{f(\theta_t + \Delta + \delta) - f(\theta_t + \Delta)}{\delta} \\ &= \lim_{\delta \to 0} \frac{f(\theta + \delta) - f(\theta)}{\delta} \\ &= \frac{\partial f(\theta)}{\partial \theta} \,. \end{aligned} \tag{14}$$

Combining Eq. (13) with Eq. (11), we have

$$\phi(0, t') \approx \phi(0, t) + \phi(t, t') \,. \tag{15}$$

We can now obtain Proposition (1) from the definitions of ϕ and Φ. □

Although several steps of the approximation require $\|\Delta\theta\|^2$ to be small, in our experiments we found the resulting estimator of GDoF predictive of generalization even when $\Delta\theta$ represents a change of up to 1000 update steps. In addition, we found performing multiple update steps while re-sampling mini-batches (K in Algorithm (1)) improves the estimate of GDoF, as in our experiments in Section (4). We leave detailed analysis of the approximations, such as the error accumulated through the trajectory, for future work.

The next section empirically demonstrates the predictive power of the measurement Eq. (2) and the two approximations we make in Eq. (3) and Eq. (4) for neural networks. Algorithm (**1**) summarizes the algorithm for computing the GDoF from a series of checkpoints saved during training. Notice that taking $T = 0$, and $K = \infty$ (*i.e.,* training the original and perturbed model from initialization until convergence) recovers the algorithm of Gao & Jojic (2016).

Algorithm 1 Algorithm for computing GDoF offline

Require: saved checkpoints $\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_T$, the number of update steps K, mini-batch size B, perturbation step ϵ
$\Phi \leftarrow 0$
for $t = 1, \ldots T$ **do**
  Load model $f(\theta_t)$ from checkpoint $\mathcal{C}_t$
  for $k = 1, \ldots K$ **do**
    Sample a mini-batch of data $\{x_b, y_b\}_{b=1}^{B}$
    Sample perturbation $\{z_b\}_{b=1}^{B} \sim \mathcal{N}(0, 1)$
    Update $f(\theta_{t+1})$ using $\{x_b, y_b\}_{b=1}^{B}$
    Update $f'(\theta_{t+1})$ using $\{x_b, y_b + \epsilon \cdot z_b\}_{b=1}^{B}$
  end for
  Approximate
    $\Delta\Phi(t, t+1) \leftarrow$
      $\frac{N}{B} \sum_{b=1}^{B} \sum_{c=1}^{C} z_b(c) \frac{f'_{i,c}(\theta_{t+1}) - f_{i,c}(\theta_{t+1})}{\epsilon}$
  Accumulate $\Phi(t+1) \leftarrow \Phi(t) + \Delta\Phi(t, t+1)$
end for

4. Experiments

We demonstrate results measuring the GDoF of a convolutional neural network (CNN) trained on the CIFAR-10 dataset. The CNNs follow the architecture from Nakkiran et al. (2021) and have 3 convolutional layers. We vary their capacities via a *capacity multiplier*, the multiple of the number of filter channels. We thus have a batch of models whose first layer has the filter sizes of $\{1, 2, 4, 8, 12, 16, 20, 24, 32, 40, 48, 64\}$. All models are trained with a batch size of 128 for 3×10^5 training iterations. In Fig. (**2**) we plot the training error, evaluation error, alongside their difference—the generalization gap—and the approximated GDoF for models with different capacities. The double descent curve (Belkin et al., 2019) can be observed in the evaluation error curve, and the measured GDoF closely tracks the generalization gap.

To further investigate the correlation between the GDoF and the generalization gap, we represent the same data in a scatter plot in Fig. (**1**), which reveals that the generalization gap increased monotonically with the GDoF, as predicted by the classical theory of computational complexity (Schmidhuber, 1997). To quantify the correlation and verify our observation, we compute Spearman's rank correlation as $r(36) = .99$, $p = 2.5 \times 10^{-11}$ and Pearson's correlation

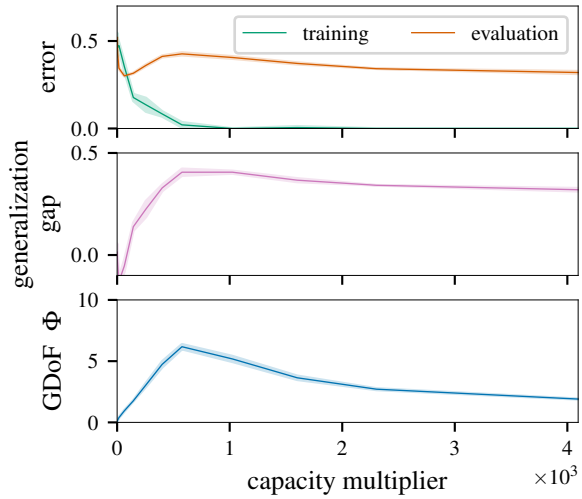

Figure 2. Training and evaluation error **(top)**, the generalization gap, **(middle)**, and GDoF **(bottom)** vs. capacity multiplier (here, kernel parameter multiplier) for 3 CNNs trained on the CIFAR-10 dataset for 300 thousand iterations. The evaluation error exhibits the double-descent phenomenon, while the GDoF tracks the generalization gap.

coefficient as $r(36) = .88, p = 9.9 \times 10^{-11}$.

For each capacity multiplier, we expand the whole training curves in Fig. (**3**) to show the evolution of all the quantities measured. The early plateau of the generalization gap for larger models (darker curves) suggests the improved generalization for model sizes past the interpolation threshold as occurred towards the tail of the "double descent" (Belkin et al., 2019). This behavior is reflected in the low-down (though not plateau) of the GDoF for larger models.

We also explore how the GDoF reacts when model complexity is controlled. We measure GDoF of the same set of models as in Fig. (**2**), here trained with the subspace method of Li et al. (2018), which was originally proposed as an ingredient of another method to approximate the intrinsic dimensionality of a model. In particular, following Li et al. (2018), we randomly project model parameters to a random, lower-dimensional subspace $\theta_S = R\,\theta$, where R is a random matrix fixed at the start of training, and optimize the lower-dimensional vector θ_S instead of θ to train the model. Intuitively, the model complexity is constrained by the coupling introduced by the random projection.

We choose as the base model the CNN with the highest capacity (first layer filter size 64, ≈ 1.55 million parameters). We then train this base mode using subspace dimensions of $\{1, 2, 3, 5, 7, 10, 20, 30, 50, 70, 90\} \times 10^4$. Fig. (**4**) demonstrates that the GDoF still tracks the generalization gap when complexity is controlled with subspace optimization, and Fig. (**5**) and Fig. (**6**) further show that a suitable space

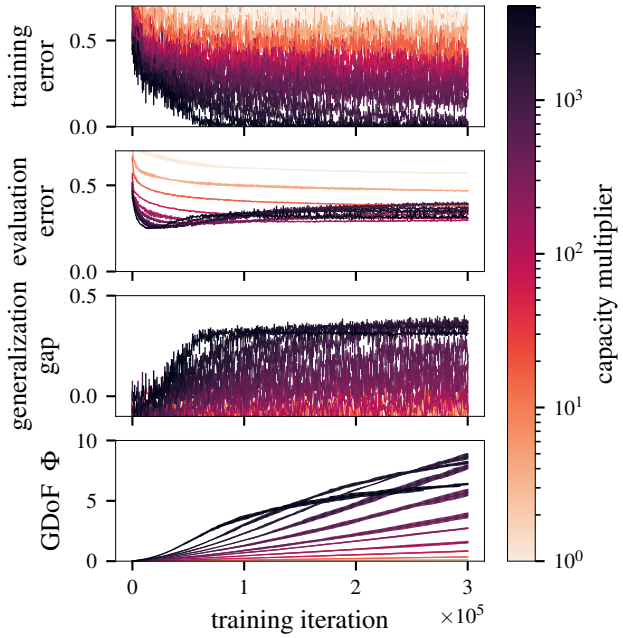

Figure 3. Training and evaluation error **(top two)**, the generalization gap, **(middle)**, and GDoF **(bottom)** vs. training iteration for different values of the capacity multiplier (here, kernel parameter multiplier) for 3 CNNs trained on the CIFAR-10 dataset for 300 thousand iterations. The GDoF captures the evolution of the generalization gap over training for the various capacity settings.

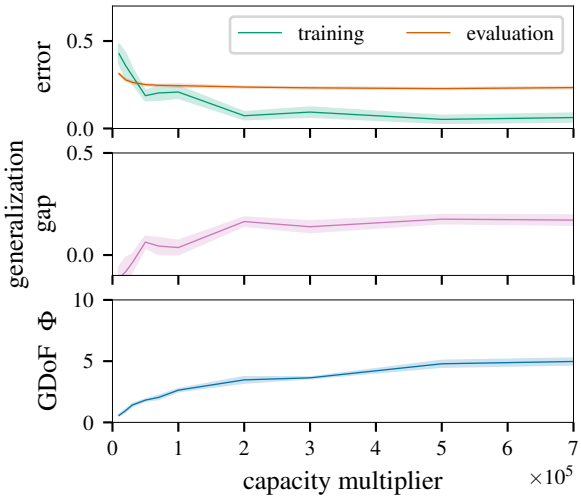

Figure 4. Training and evaluation error **(top)**, the generalization gap, **(middle)**, and GDoF **(bottom)** vs capacity multiplier (here, subspace dimension in the algorithm of Li et al. (2018)) for 3 CNNs trained with subspace optimization on the CIFAR-10 dataset for 300 thousand iterations. In contrast to Fig. (**2**), the evaluation error is monotonically decreasing. The GDoF still tracks the generalization gap.

dimension can *improve* performance by reducing the generalization gap; that is, intermediate subspace dimensions can lower the evaluation error below that of the full model.

5. Related Work

Whence generalization? Generalization properties of deep neural networks has received much renewed interest from both theoretical and empirical perspectives (Jiang et al., 2020a). However, it seems that many properties that correlate with generalization performance are neither necessary nor sufficient to ensure it; for example, sharpness of the loss surface around a minimum (Dinh et al., 2017), or monotonic linear interpolation (Lucas et al., 2021). Loukas et al. (2021) explore the Benevolent Training Hypothesis and provides evidence that model complexity, as measured by Lipschitz constant in their work, can be deduced from training dynamics. They further provided a generalization bound depends on both data and training. Jiang et al. (2021) discuss the results of a recent community competition (Jiang et al., 2020a) that targeted complexity measures that reliably capture generalization performance; however, they do not investigate the trend of the these complexity measures as the parameter dimension is manipulated. Maddox et al. (2020) resolve the double descent paradox with the introduction of a capacity estimate inspired by posterior contraction in Bayesian learning that renders evaluation performance monotonically increasing.

Capturing complexity. Proposals to measure effective dimensionality have taken many forms, from more classical parameter and representation norms (Bartlett et al., 2017) and sharpness measures (Keskar et al., 2017), to measures motivated specifically by modern deep learning systems, such as measures of gradient noise (Smith & Le, 2018) and optimization speed (Hardt et al., 2016); see Jiang et al. (2020b) for a thorough overview and an empirical comparison. The common idea behind all such approaches is to quantify variability and thus complexity in the function represented by a neural network. This functional complexity may result from the interaction of the particular parameterization or architecture in addition to all other aspects of the machine learning pipeline, including the optimization algorithm and other hyper-parameters. Crucially, however, the exact relationship between neural network design decisions and functional complexity is unknown, prompting the study of *implicit regularization* in deep learning (Neyshabur et al., 2015; Neyshabur, 2017; Dherin et al., 2021).

Finding neural networks with minimal complexity One class of methods directly search models as programs with low Kolmogorov complexity. For example, Schmidhuber (1997) proposes a probabilistic search algorithm to

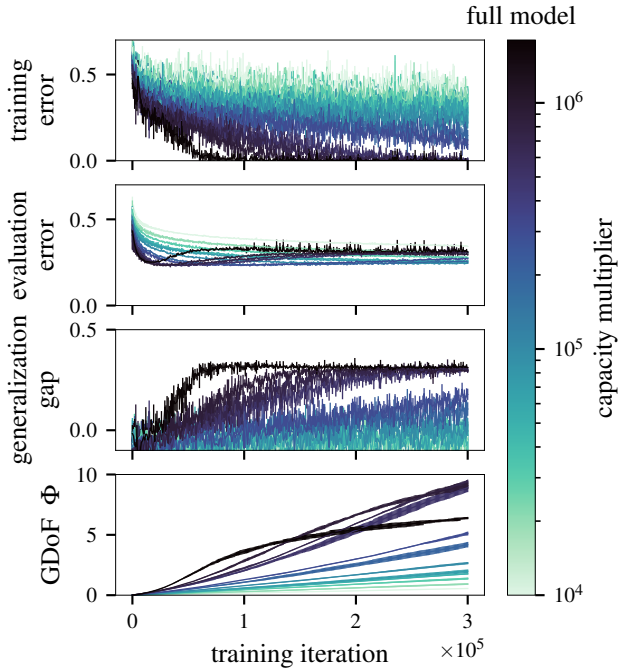

Figure 5. Training and evaluation error **(top two)**, the generalization gap, **(middle)**, and GDoF **(bottom)** vs. training iteration for different values of the capacity multiplier (here, subspace dimension in the algorithm of Li et al. (2018)) for 3 CNNs trained on the CIFAR-10 dataset for 300 thousand iterations. The GDoF captures the evolution of the generalization gap over training for the various capacity settings.

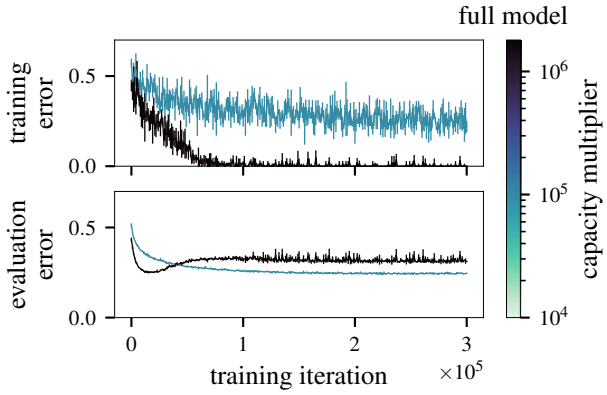

Figure 6. Partial reproduction of Fig. (5), for emphasis. Training **(top)** and validation error **(bottom)** vs. training iteration for the full model as compared to a subspace model (here, 10^5) from Fig. (5). Controlling the complexity via subspace optimization in an intermediate subspace dimension results in a lower validation error.

discover neural networks with low Kolmogorov complexity, although it has only been demonstrated with toy problems that can be solved with relatively short programs consist of limited primitives. Another class of methods encourages optimizers to find simpler neural networks by penalize deviation from a specified *prior* of the weights. The simplest example is "weight-decay" (Hinton, 1987), which can be interpreted as finding the *maximum a posteriori* (MAP) solution given an isotropic Gaussian prior over weights. Nowlan & Hinton (1992) elaborates this idea by introducing mixture of Gaussian priors over weights, but still assumes that all weights are quantized with the same tolerance of noise. This assumption has been drop in (Hinton & Van Camp, 1993), which employs the variational method to approximate the posterior distribution of *noisy weights* with different precisions.

Bayesian model selection and Occam's Razor. MacKay (1992a) reviews how *model selection* or *comparison* can be achieved in a Bayesian context by comparing the evidence for each model. In particular, given a parameterized model family $\mathcal{H}$ that specifies a *prior* distribution over parameters $\mathbf{w}$, $P(\mathbf{w} \mid \mathcal{H})$ and a *predictive* distribution over data $\mathcal{D}$, $P(\mathcal{D} \mid \mathbf{w}, \mathcal{H})$, the evidence is $P(\mathcal{D} \mid \mathcal{H}) = \int P(\mathcal{D} \mid \mathbf{w}, \mathcal{H}) P(\mathbf{w} \mid \mathcal{H}) \, \mathrm{d}\mathbf{w}$. As a rule of thumb, greater entropy in the prior term $P(\mathbf{w} \mid \mathcal{H})$, corresponding to a more flexible hypothesis class, results in a lesser value for the evidence, a regularizing effect known as *Occam's Razor*. It should be noted that the behavior of Occam's Razor—in particular what notion of flexibility is penalized—implicitly relies on the model parameterization and is therefore not an *a priori* effect (Wolpert, 1995); for instance, MacKay (1992a) discuss Occam's Razor and Bayesian model comparison for models that admit Gaussian approximation and quadratic regularizers, and MacKay (1992b) study such approximations in the context of neural networks. Nevertheless, Bayesian model selection allows coherent comparison of appropriately related prior distributions (Jefferys & Berger, 1992), and robustness with respect to the selection of the prior distribution can be ensured in some situations (Berger et al., 1994).

Implicit biases in model design. Lineages of machine learning models are often the result of selection on the basis of performance on standardized benchmarks (Dehghani et al., 2021). As a result, such lineages implicitly depend on the types of data to which they have been applied—in particular, such models are likely to have implicit biases that aid performance on such datasets. Analogously, prior selection in Bayesian analysis often implicitly relies on data (Gelman et al., 2017). From a Bayesian perspective, the impact of prior information can be quantified using sensitivity analysis (e.g., Müller 2012) or with measures of prior-data and prior-likelihood conflict (Bousquet, 2008; Nott et al., 2020; Reimherr et al., 2021).

6. Conclusion

In this work, we investigated a complexity measure inspired by the classical notion of degrees of freedom in statistics. This measure is scalable and is a function of the entire machine learning training pipeline. We demonstrated that this measure strongly correlates with generalization performance in the double-descent regime, and that directly controlling model complexity via subspace training improves can improve generalization.

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