# ON EPISODES, PROTOTYPICAL NETWORKS, AND FEW-SHOT LEARNING

**Anonymous authors**
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## ABSTRACT

Episodic learning is a popular practice among researchers and practitioners interested in few-shot learning. It consists of organising training in a series of learning problems, each relying on small "support" and "query" sets to mimic the few-shot circumstances encountered during evaluation. In this paper, we investigate the usefulness of episodic learning in Prototypical Networks, one the most popular algorithms making use of this practice. Surprisingly, in our experiments we found that episodic learning is detrimental to performance, and that it is under no circumstance beneficial to differentiate between a support and query set within a training batch. This "non-episodic" version of Prototypical Networks, which corresponds to the classic Neighbourhood Component Analysis, reliably improves over its episodic counterpart in multiple datasets, achieving an accuracy that is competitive with the state-of-the-art, despite being extremely simple.

## 1 INTRODUCTION

The problem of few-shot learning (FSL) – classifying examples from previously unseen classes given only a handful of training data – has considerably grown in popularity within the machine learning community in the last few years. The reason is likely twofold. First, being able to perform well on FSL problems is important for several applications, from learning new characters (Lake et al., 2015) to drug discovery (Altae-Tran et al., 2017). Second, since the aim of researchers interested in meta-learning is to design systems that can quickly learn novel concepts by generalising from previously encountered learning tasks, FSL benchmarks are often adopted as a practical way to empirically validate meta-learning algorithms.

To the best of our knowledge, there is not a widely recognised definition of meta-learning. In a recent survey, Hospedales et al. (2020) informally describe it as *"the process of improving a learning algorithm over multiple learning episodes"*. Several popular papers in the FSL community (e.g. Vinyals et al. (2016); Ravi & Larochelle (2017); Finn et al. (2017); Snell et al. (2017)) have emphasised the importance of organising training into *episodes*, i.e. learning problems with a limited amount of training and (pseudo-)test examples that resemble the test-time scenario. This popularity has reached such a point that an "episodic" data-loader is often at the core of new FSL algorithms, a practice facilitated by frameworks such as Deleu et al. (2019) and Grefenstette et al. (2019).

Despite the considerable strides made in FSL over the past few years, several recent works (e.g. Chen et al. (2019); Wang et al. (2019); Dhillon et al. (2020); Tian et al. (2020)) showed that simple baselines can outperform established meta-learning methods by using embeddings pre-trained with standard classification losses. These results legitimately cast a doubt in the FSL community on the usefulness of meta-learning and its pervasive *episodes*. However, these papers lack a clear explanation of this phenomenon. Inspired by these results, we aim at understanding the practical usefulness of episodic learning in arguably the simplest method which makes use of it: Prototypical Networks (Snell et al., 2017). We chose to analyse Prototypical Networks not only for their simplicity, but also because they routinely appear as important building blocks of newly-proposed methods (e.g. Oreshkin et al. (2018); Cao et al. (2020); Gidaris et al. (2019); Yoon et al. (2019)).

With a set of ablative experiments, we show that episodic learning *a)* is detrimental for FSL performance, *b)* is analogous to randomly discarding examples from a batch and *c)* it introduces a set of unnecessary hyper-parameters that require careful tuning. We also show that, without episodic learning, Prototypical Networks corresponds to the classic *Neighbourhood Component Analysis*

(NCA) (Goldberger et al., 2005; Salakhutdinov & Hinton, 2007) on deep embeddings. Without bells and whistles, our implementation of the NCA loss achieves an accuracy that is comparable to the state-of-the-art on multiple FSL benchmarks: *mini*ImageNet, CIFAR-FS and *tiered*ImageNet.

## 2 RELATED WORK

Pioneered by the seminal work of Utgoff (1986), Schmidhuber (1987; 1992), Bengio et al. (1992) and Thrun (1996), the general concept of meta-learning is several decades old (for a survey see Vilalta & Drissi (2002); Hospedales et al. (2020)). However, in the last few years it has become the most popular paradigm for learning from very few examples. Between 2016 and 2017, several methods addressing the FSL problem using learning on episodes were proposed. MANN (Santoro et al., 2016) uses a Neural Turing Machine to save and access the information useful to meta-learn; Bertinetto et al. (2016) propose a deep network in which a "teacher" branch is tasked with predicting the parameters of a "student" branch; Matching (Vinyals et al., 2016) and Prototypical Networks (Snell et al., 2017) are two non-parametric methods in which the contributions of different examples in the support set are weighted by either an LSTM or a simple average, respectively; Ravi & Larochelle (2017) propose instead to use an LSTM to learn the hyperparameters of SGD, while MAML (Finn et al., 2017) learns to fine-tune an entire deep network by backpropagating through SGD. Despite these works widely differing in nature, they all stress on the importance of organising training in a series of small learning problems (*episodes*) that are similar to those encountered during inference at test time.

In contrast with this trend, a handful of papers have recently shown that simple approaches that forego episodes and meta-learning can perform well on FSL benchmarks. These methods all have in common that they pre-train a feature extractor with the cross-entropy loss on the "meta-training classes" of the dataset. Then, at test time a classifier is adapted to the support set by weight imprinting (Qi et al., 2018; Dhillon et al., 2020), fine-tuning (Chen et al., 2019), transductive fine-tuning (Dhillon et al., 2020) or logistic regression (Tian et al., 2020). Wang et al. (2019) suggest performing test-time classification by using the label of the closest centroid to the query image.

Differently from these papers, we try to shed some light on one of the possible causes behind the poor performance of episodic-based algorithms like Prototypical Networks. An analysis similar to ours in spirit is the one of Raghu et al. (2020). After showing that the efficacy of MAML in FSL is due to the adaptation of the final layer and the "reuse" of the features of previous layers, they propose a variant with the same accuracy and computational advantages. In this paper, we focus on an FSL algorithm just as popular and uncover inefficiencies that allow for a notable conceptual simplification of Prototypical Networks, which surprisingly also brings a significant boost in performance.

## 3 BACKGROUND AND METHOD

This section is divided as follows: Sec. 3.1 introduces episodic learning and the formalism used in FSL, Sec. 3.2 reviews Prototypical Networks (often referred to as PN from now on), Sec 3.3 describes the classic NCA loss and how exactly it relates to PN, and Sec. 3.4 explains the three options we explored to perform FSL classification with an NCA-trained feature embedding.

### 3.1 EPISODIC LEARNING

A common strategy to train few-shot learning algorithms is to construct a distribution of episodes $\hat{\mathcal{E}}$ as close as possible to the one encountered during evaluation $\mathcal{E}$ [1] . The distribution of episodes is based on the possible label (sub)sets $L$ that can be constructed from all available labels. Each episodic batch $B_E = \{S, Q\}$ is obtained by first sampling a set of labels $L$ from $\hat{\mathcal{E}}$, and then sampling images constituting both *support set* $S$ and *query set* $Q$ from the set of images with label $L$, where $S = \{(\mathbf{s}_1, y_1), \ldots, (\mathbf{s}_n, y_n)\}$, $Q = \{(\mathbf{q}_1, y_1), \ldots, (\mathbf{q}_m, y_m)\}$, and $S_k$ and $Q_k$ denote the set of images with label $y = k$ in the support set of query set respectively. In a Maximum Likelihood Estimation framework, training on these episodes can be written (Vinyals et al., 2016) as

[1] Note that, in FSL, the sets of classes of training and evaluation are disjoint.

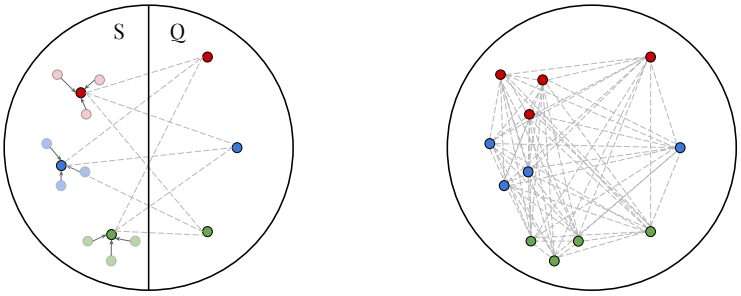


Figure 1: **Batch exploitation** during training for Prototypical Networks (Snell et al., 2017) (left) vs. Neighborhood Component Analysis (Goldberger et al., 2005) (right) on a toy few-shot problem of 3 "ways", 3 "shots" and 1 "query" per class. By dividing batches between support set $S$ and query set $Q$, Prototypical Networks disregards many of the distances between labelled examples that would constitute useful training signal. Details in Sec. 3.3.

the optimisation

$$\arg\max_{\theta} E_{L\sim\hat{\mathcal{E}}}\left[E_{S\sim L,Q\sim L}\left[\sum_{(q_i,y_i)\in Q}\log P_\theta\left(y_i|q_i,S\right)\right]\right]. \tag{1}$$

In most implementations this corresponds to training on a series of mini-batches in which each image is characterised by a flag indicating whether it corresponds to the support or the query set, besides a class label as usual. Support and query sets are constructed such that they both contain all the classes of $L$, and a set number of images per class. Therefore, episodes are characterised by three values: the number of classes $w = |L|$ (the "ways"), the number of examples per class in the support set $n = |S_k|$ (the "shots"), and the number of examples per class in the query set $m = |Q_k|$. Importantly, during evaluation the triplet $\{w, n, m\}$ defines the problem setup (and thus it must stay unchanged across methods). However, at training time it can be seen as a set of hyper-parameters controlling the batch creation, and that (as we will see in Sec. 4.2) requires careful tuning.

### 3.2 Prototypical Networks (PN)

Prototypical Networks (Snell et al., 2017) are one of the most popular and effective approaches in the few-shot learning literature, and they are at the core of many newly proposed methods (e.g. Oreshkin et al. (2018); Gidaris et al. (2019); Allen et al. (2019); Yoon et al. (2019); Cao et al. (2020)).

During *training*, episodes constituted by support set $S$ and query set $Q$ are sampled as described in Sec. 3.1. Then, a *prototype* for each class $k$ is computed as the mean embedding of the samples from the support set belonging to that class: $\mathbf{c}_k = (1/|S_k|) \cdot \sum_{(\mathbf{s}_i,y_k)\in S_k} f_\theta(\mathbf{s}_i)$, where $f_\theta$ is a deep neural networks with parameters $\theta$ learned via Eq. 1.

Let $C = \{(\mathbf{c}_1, y_1), \ldots, (\mathbf{c}_k, y_k)\}$ be the set of prototypes and corresponding labels. The loss can be written as follows:

$$\mathcal{L}_{\text{proto}}(S,Q) = -\frac{1}{|Q|+|S|}\sum_{(\mathbf{q}_i,y_i)\in Q}\log\left(\frac{\exp -\|f_\theta(\mathbf{q}_i)-\mathbf{c}_{y_i}\|^2}{\sum_{k'}\exp -\|f_\theta(\mathbf{q}_i)-\mathbf{c}_{k'}\|^2}\right). \tag{2}$$

Here, $k'$ is an index that goes over all classes. This loss is minimised over a number of training episodes. After training, given a query image $\mathbf{x}_i$ from a new test episode, classification is conducted by simply consulting the nearest-neighboring prototype computed from the support set from the same episode, i.e. $y(\mathbf{x}_i) = \arg\min_{j\in\{1,\ldots,k\}}\|f_\theta(\mathbf{x}_i)-\mathbf{c}_j\|$.

### 3.3 Neighbourhood Component Analysis (NCA)

Eq. 2 computes the likelihood that a query image is sampled from a prototype by computing the softmax over the distances to all prototypes. This formulation is similar to the *Neighbourhood*

*Component Analysis* approach by Goldberger et al. (2005) (and expanded to the non-linear case by Salakhutdinov & Hinton (2007)), except for a few important differences which we will now discuss.

Let $i \in [1, b]$ be the indices of the images within a batch $B$. The NCA loss can be written as:

$$\mathcal{L}_{\text{NCA}}(X) = -\frac{1}{b} \sum_{i \in 1,\dots,b} \log \left( \frac{\sum\limits_{\substack{j \in 1,\dots,b \\ j \neq i \\ y_i = y_j}} \exp - \|\mathbf{z}_i - \mathbf{z}_j\|^2}{\sum\limits_{\substack{k \in 1,\dots,b \\ k \neq i}} \exp - \|\mathbf{z}_i - \mathbf{z}_k\|^2} \right), \tag{3}$$

where $\mathbf{z}_i = f_\theta(\mathbf{x}_i)$ is an image embedding and $y_i$ its corresponding label. By minimising this loss, distances between embeddings from the same class will be minimised, while distances between embeddings from different classes will be maximised. This bears similarities to the (supervised) contrastive loss (Khosla et al., 2020; Chen et al., 2020a), which we discuss in Appendix A.3. For ease of discussion, we refer to the distances between pairs of embeddings from the same class as *positives*, and to the distances between pairs of embeddings from different classes as *negatives*.

Importantly, the concepts of support set and query set of Sec. 3.1 and 3.2 do not apply. More simply, the images and labels constituting the batch $B = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_b, y_b)\}$ are sampled i.i.d. from the training dataset, as normally happens in standard supervised learning. Considering that in PN there is no parameter adaptation happening at the level of each episodic batch $B_E$, $S$ and $Q$ *do not* have a functional role in the algorithm, and their defining values $\{w, m, n\}$ can be interpreted as hyper-parameters controlling the data-loader during training. More specifically, PN differs from NCA in three key aspects:

1. PN relies on the creation of prototypes, while NCA does not.
2. Due to the nature of episodic learning, PN only considers pairwise distances between the query and the support set; the NCA instead uses *all* the distances within a batch and treats each example in the same way.
3. Because of how $L$, $S$ and $Q$ are sampled in episodic learning, some images will be inevitably sampled more frequently than others, and some will likely never be seen during training (this corresponds to sampling "with replacement"). NCA instead visits every image of the dataset once and only once within each epoch (sampling "without replacement").

Fig. 1 illustrates the difference in batch exploitation. For PN (*left*) and a training episode of $w$=3 ways, $n$=3 shots and $m$=1 queries, we would only have $wm = 3$ positives and $w(w-1)m = 6$ negatives. When considering the number of distances contributing to the loss, points within the support set count independently and thus we get to $wmn = 9$ positives and $w(w-1)mn = 18$ negatives. Instead, if we consider the *same* training batch in a non-episodic way, when computing the NCA loss (Fig. 1 *right*) we have $\binom{m+n}{2}w = 18$ positives and $\binom{w}{2}(m+n)^2 = 48$ negatives. This is a big difference, which increases for larger values of $n$, $w$ and $m$. To investigate the effect of these three key differences between PN and NCA, in Sec. 4 we conduct a wide range of experiments.

### 3.4 Few-shot classification during evaluation

Once $f_\theta$ has been trained, there are many possible ways to perform few-shot classification during evaluation. In this paper we consider three simple approaches that are particularly aligned for embeddings learned via metric-based losses such as Eq. 2 or Eq. 3.

**$k$-NN.** To classify an image $\mathbf{q}_i \in Q$, we first compute the Euclidean distance to each support point $\mathbf{s}_j \in S$: $d_{ij} = \|f_\theta(\mathbf{q}_i) - f_\theta(\mathbf{s}_j))\|^2$. Then, we simply assign $y(\mathbf{q}_i)$ to be majority label of the $k$ nearest neighbours. A downside here is that $k$ is a hyper-parameter that has to be chosen, although a reasonable choice in the FSL setup is to set it equal to the number of "shots" $n$.

**1-NN with Class Centroids.** Similar to $k$-NN, we can perform classification by inheriting the label of the closest class centroid, i.e. $y(\mathbf{q}_i) = \arg\min_{j \in \{1,\dots,k\}} \|f_\theta(\mathbf{x}_i) - \mathbf{c}_j\|$. This is the approach used at test-time by Snell et al. (2017) and Wang et al. (2019).

**Soft Assignments.** This is what the original NCA paper (Goldberger et al., 2005) used for evaluation. To classify an image $\mathbf{q}_i \in Q$, we compute the values $p_{ij} = \exp(-\|f_\theta(\mathbf{q}_i) -$

$f_\theta(\mathbf{s}_j))\|^2)/\sum_{\mathbf{s}_k \in S} \exp(-\|f_\theta(\mathbf{q}_i) - f_\theta(\mathbf{s}_k)\|^2)$ for all $\mathbf{s}_j \in S$, which is the probability that image $i$ is sampled from image $j$. We then compute the likelihood for each class $k$: $\sum_{s_j \in S_k} p_{ij}$, and choose the class with the highest likelihood $y(\mathbf{q}_i) = \arg\max_k \sum_{s_j \in S_k} p_{ij}$. This approach is the only one closely aligned with the training procedure, and has a direct probabilistic interpretation.

We also experimented with using the NCA loss on the support set to perform adaptation at test time, which we discuss in Appendix A.1.

# 4 EXPERIMENTS

In the following, Sec. 4.1 describes our experimental setup. Sec. 4.2 shows the effect of the hyper-parameters controlling the creation of episodes in PN. In Sec. 4.3 we perform a set of ablation studies to better illustrate the relationship between PN and the NCA. Finally, in Sec. 4.4 we compare our version of the NCA to several recent methods on three FSL benchmarks.

## 4.1 EXPERIMENTAL SETUP

We conduct our experiments on *mini*ImageNet (Vinyals et al., 2016), CIFAR-FS (Bertinetto et al., 2019) and *tiered*ImageNet (Ren et al., 2018), using the ResNet12 architecture introduced by Lee et al. (2019) as embedding function $f_\theta$. A detailed description of benchmarks, architecture and implementation details is deferred to Appendix A.4, while below we discuss the most important choices of the experimental setup.

Like Wang et al. (2019), for all our experiments (including those with Prototypical Networks) we centre and normalise the feature embeddings before performing classification, as it is considerably beneficial for performance. After training, we compute the mean feature vectors of all the images in the training set: $\bar{\mathbf{x}} = \frac{1}{|\mathcal{D}^{\text{train}}|} \sum_{\mathbf{x} \in \mathcal{D}^{train}} \mathbf{x}$. Then, all feature vectors in the test set are updated as $\mathbf{x}_i \leftarrow \mathbf{x}_i - \bar{\mathbf{x}}$, and normalised by $\mathbf{x}_i \leftarrow \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|}$.

We compared the inference methods discussed in Sec. 3.4 on *mini*ImageNet and CIFAR-FS. Results can be found in Table 4 in Appendix A.4. We chose to use 1-NN with class centroids in all our experiments, as it performs significantly better than $k$-NN or Soft Assignment. This might sound surprising, as the Soft Assignment approach closely reflects the training protocol and outputs class probabilities. We speculate its inferior performance could be caused by poor model calibration (Guo et al., 2017), which for NCA would affect the contribution of the images of the support set. However, it is not immediately clear how one could use the tools provided by the calibration literature given that the set of classes between dataset splits are disjoint.

As standard, performance is assessed on episodes of 5-way, 15-query and either 1- or 5-shot. Each model is evaluated on 10,000 episodes sampled from the test set (or the validation set, in some experiments). To further reduce the variance, we trained each model five times with five different random seeds, for a total of 50,000 episodes per configuration, from which error bars are computed.

## 4.2 EPISODES HYPER-PARAMETERS

Despite Prototypical Networks being one of the simplest FSL methods, the creation of episodes requires the use of several hyper-parameters ($\{w, m, n\}$, Sec. 3.1) which can significantly affect performance. Snell et al. (2017) state that the number of shots $n$ between training and testing should match and that one should use a higher number of ways $w$ during training time. In their experiments, they train 1-shot models with $w = 30$, $n = 1$, $m = 15$ and 5-shot models with $w = 20$, $n = 5$, $m = 15$. This makes the corresponding batch sizes of these episodes 480 and 400, respectively. Since, as seen in Sec. 3.3, the number of positives and negatives grows rapidly for both PN and NCA (although at a different rate), this makes a fair comparison between models trained on different types of episodes difficult.

We investigate the effect of changing these hyper-parameters in a systematic manner. To compare configurations fairly across episode/batch sizes, we define each configuration by its number of shots $n$, the batch size $b$ and the total number of images per class $a$ (which accounts for the sum between support and query set, $a = n + m$). For example, if we train a 5-shot model with $a = 8$ and

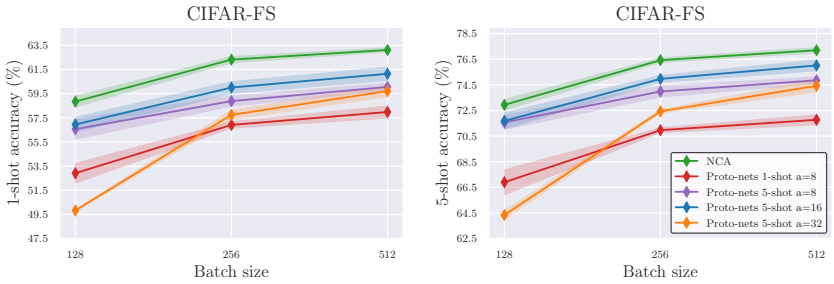


Figure 2: 1-shot (left) and 5-shot accuracies (right) on the val. set of CIFAR-FS. Models trained using NCA, or Proto-nets with different configurations: 1-shot with $a$=8 and 5-shot with $a$=8, 16 or 32. Values correspond to the mean accuracy of five models trained with different random seeds. Please see Sec. 4.2 for details.

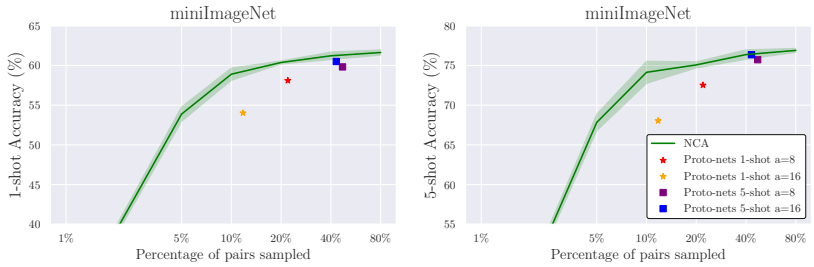


Figure 3: 1-shot and 5-shot accuracies on the *mini*ImageNet test set for models trained with a batch size of 256 while only sampling a % of the total number of available pairs. Reported values correspond to the mean accuracy of five models trained with different random seeds. Individual points contain models trained using PNs which are plotted on the x-axis based on the relative percentage of distance pairs that are used in their computation compared to NCA on the same batch size. Please see Sec. 4.2 for details.

$b = 256$, it means that its corresponding training episodes will have $n = 5$, $q = 8 - 5 = 3$, and $w = 256/8 = 32$. Using this notation, we train configurations of PN covering several combinations of these hyper-parameters so that the resulting batch size corresponds to an episode is 128, 256 or 512. Then, we train three configurations of NCA, where the only hyper-parameter is the batch size.

Results for CIFAR-FS can be found in Fig. 2, where we report results for NCA and PN with $a = 8$, 16 or 32. Results for *mini*ImageNet observe the same trend and are deferred to Appendix A.5. Note that the results of 1-shot with $a = 16$ and $a = 32$ are not reported, as they fare significantly worse. Several things can be noticed. First, NCA performs better than *all* PN configurations, no matter the batch size. Second, PN is very sensitive to different hyper-parameter configurations. For instance, with batches of 128 PN trained with episodes of 5-shot and $a = 32$ performs significantly worse than a PN trained with episodes of 5-shot and $a = 16$. Finally, we can also notice that, contrary to what has been previously reported (Snell et al., 2017; Cao et al., 2020), the 5-shot model with the best configuration is always strictly better than any 1-shot configuration. We speculate that this is probably due to the fact that we compare configurations keeping the batch size constant.

**Considerations on computational efficiency.** Despite the NCA completely subsuming all PN configurations (Fig. 2), one might posit that by using distances within a batch, NCA is more computationally demanding and thus the episodic strategy of PN can be advantageous in some practical scenarios (e.g. real-time applications with large batches). To investigate this, we perform an experiment where we train NCA models by randomly sampling a fraction of the total number of distances used in the loss. Then, for comparison, we include different PN models after having computed to which percentage of discarded pairs (in a normal batch) their episodic batch corresponds to.

Results can be found in Fig. 3. As expected, we can see how sub-sampling a fraction of the total available number of pairs within a batch negatively affects performance. To answer to possible concerns on computational efficiency, we can notice that the PN points lie very close to the under-sampling version of the NCA, which signals that the episodic strategy of PN is roughly equivalent to train with the NCA and only exploiting a fraction of the distances available in a batch.

Figure 4: Ablation experiments on NCA and Prototypical Networks, both on batches or episodes of size 256 on the validation set of *mini*ImageNet and CIFAR-FS. Please refer to Sec. 4.3 for details.

### 4.3 Ablation experiments

To better analyse why NCA performs better than PN, in this section we consider the three key differences discussed earlier with a series of ablations on models trained on batches of size 256. Results are summarised in Fig. 4. We refer the reader to Appendix A.1 to obtain detailed steps describing how these ablations affect Eq. 2 and Eq. 3.

First, we compare two variants of NCA: one in which the sampling of the training batches happens sequentially and without replacement, as it is standard in supervised learning, and one where the batches are sampled with replacement. Interestingly, this modification (row 1 and 2 of Fig. 4) has a negligible effect across the two datasets and four splits considered, meaning that the replacement sampling introduced by episodic learning will not interfere with the other ablations.

We then perform a series of ablations on episodic batches, i.e. sampled with the method described in Sec. 3.1. For each ablation, we perform experiments on PN trained with 1-shot and 5-shot, both with $a = 8$. This means that both the 1-shot and 5-shot models have 32 classes and 8 images per class, allowing a fair comparison. The batch size is 256 for the NCA too. We first train standard PN models. Next, we train a PN model where "prototypes" are not computed (point 1 of Sec. 3.3), meaning that distances are considered between individual points, but a separation between query and support set remains. Then, we perform an ablation where we ignore the separation between support and query set (point 2 of Sec. 3.3), but still compute prototypes for the points that would belong to the support set. Last, we perform an ablation where we consider all the previous points together: we sample with replacement, we ignore the separation between support and query set and we do not compute prototypes. This amounts to the NCA loss, except that it is computed on batches with a fixed number of classes and a fixed number of images per class. Notice that in Fig. 4 there is only one row dedicated to 1-shot models. This is because we cannot generate prototypes from 1-shot models, so we cannot have a "no proto" ablation. Furthermore, for 1-shot models the "no S/Q" ablation is equivalent to the NCA with a fixed batch composition. From Fig. 4, we can see that disabling prototypes (row 6) negatively affects the performance of 5-shot (row 5), albeit slightly. On the other hand, enabling the computation between all pairs increases the performance (last row), Importantly, enabling all the ablations (row 3) *completely* recovers the performance lost by PN.

These experiments clearly highlight that the separation of roles between the images belonging to support and query set, which is typical of episodic learning (Vinyals et al., 2016), is detrimental for the performance of Prototypical Networks. Instead, using the NCA loss on standard mini-batches allows to fully exploit the training data and significantly improve performance. Moreover, the NCA has the advantage of simplifying the overall training procedure, as the hyper-parameters for the creation of episodes $\{w, n, m\}$ no longer need to be considered.

### 4.4 Comparison with the state-of-the-art

We now benchmark our models on three FSL datasets. When considering which methods to compare against, we chose those *a)* which have been recently published, *b)* that are using a ResNet12 and *c)* with a setup that is not significantly more complicated than ours. For example, we only report the results of the main approach proposed by Tian et al. (2020) and not their *sequential self-distillation* (Furlanello et al., 2018) variant, which is computationally expensive and can be applied to most methods. For NCA, we found that concatenating the output of intermediate layers modestly

| | *mini*ImageNet | | CIFAR-FS | |
|---|---|---|---|---|
| **method** | **1-shot** | **5-shot** | **1-shot** | **5-shot** |
| **Episodic Methods** | | | | |
| adaResNet (Munkhdalai et al., 2018) | $56.88 \pm 0.62$ | $71.94 \pm 0.57$ | - | - |
| TADAM(Oreshkin et al., 2018) | $58.50 \pm 0.30$ | $76.70 \pm 0.30$ | - | - |
| Shot-Free (Ravichandran et al., 2019) | $60.71\pm$ n/a | $77.64\pm$ n/a | $69.2\pm$ n/a | $84.7\pm$ n/a |
| TEAM (Qiao et al., 2019) | $60.07\pm$ n/a | $75.90\pm$ n/a | - | - |
| MTL (Sun et al., 2019) | $61.20 \pm 1.80$ | $75.50 \pm 0.80$ | - | - |
| TapNet (Yoon et al., 2019) | $61.65 \pm 0.15$ | $76.36 \pm 0.10$ | - | - |
| MetaOptNet-SVM(Lee et al., 2019) | $62.64 \pm 0.61$ | $78.63 \pm 0.46$ | $72.0 \pm 0.7$ | $84.2 \pm 0.5$ |
| Variatonal FSL (Zhang et al., 2019) | $61.23 \pm 0.26$ | $77.69 \pm 0.17$ | - | - |
| **Simple Baselines** | | | | |
| Transductive finetuning (Dhillon et al., 2020) | $62.35 \pm 0.66$ | $74.53 \pm 0.54$ | $70.76 \pm 0.74$ | $81.56 \pm 0.53$ |
| RFIC-simple (Tian et al., 2020) | $62.02 \pm 0.63$ | $79.64 \pm 0.44$ | $71.5 \pm 0.8$ | $86.0 \pm 0.5$ |
| Meta-Baseline (Chen et al., 2020b) | $63.17 \pm 0.23$ | $79.26 \pm 0.17$ | - | - |
| *Our implementations:* | | | | |
| Proto-nets (Snell et al. (2017) setup) | $59.93 \pm 0.23$ | $75.89 \pm 0.16$ | $70.20 \pm 0.22$ | $83.96 \pm 0.16$ |
| Proto-nets (our setup) | $61.32 \pm 0.23$ | $77.77 \pm 0.15$ | $70.41 \pm 0.31$ | $84.46 \pm 0.29$ |
| SimpleShot (Wang et al., 2019) | $62.16 \pm 0.23$ | $78.33 \pm 0.17$ | $70.01 \pm 0.21$ | $84.50 \pm 0.11$ |
| **NCA (ours)** | $62.52 \pm 0.24$ | $78.3 \pm 0.14$ | $72.48 \pm 0.40$ | $85.13 \pm 0.29$ |
| **NCA (ours) multi-layer** | $63.21 \pm 0.08$ | $79.27 \pm 0.08$ | $72.44 \pm 0.36$ | $85.42 \pm 0.29$ |

Table 1: Comparison of methods that use ResNet12 on *mini*ImageNet and CIFAR-FS (test set).

| | *tiered*ImageNet | |
|---|---|---|
| **method** | **1-shot** | **5-shot** |
| Shot-Free (Ravichandran et al., 2019) | $63.52\pm$ n/a | $82.59\pm$ n/a |
| RFIC-simple (Tian et al., 2020) | $69.74 \pm 0.72$ | $84.41 \pm 0.55$ |
| Meta-Baseline (Chen et al., 2020b) | $68.62 \pm 0.27$ | $83.29 \pm 0.18$ |
| MetaOptNet-SVM(Lee et al., 2019) | $65.99 \pm 0.72$ | $81.56 \pm 0.53$ |
| *Our implementations:* | | |
| Proto-nets (Snell et al. (2017) setup) | $65.45 \pm 0.23$ | $81.14 \pm 0.17$ |
| SimpleShot (Wang et al., 2019) | $66.17 \pm 0.15$ | $80.64 \pm 0.20$ |
| **NCA (ours)** | $68.36 \pm 0.11$ | $83.20 \pm 0.18$ |
| **NCA (ours) multi-layer** | $68.76 \pm 0.07$ | $83.38 \pm 0.11$ |

Table 2: Comparison of methods that use ResNet12 on *tiered*ImageNet (test set).

improves performance at (almost) no additional cost. We use the output of the average pool layers from all ResNet blocks except from the first and we refer to this variant as NCA multi-layer.

Results can be found in Table 1 for *mini*ImageNet and CIFAR-FS and Table 2 for *tiered*ImageNet. In Table 1 we report Prototypical Networks results for both the episodic setup from Snell et al. (2017) and the best one (batch size 512, 5-shot, $a$=16) found from the experiment of Fig. 2, which brings a considerable improvement over the original. We did not compute a new setup for Prototypical Networks for *tiered*ImageNet, since the larger dataset and the higher number of classes would have made the hyper-parameter search too demanding. Notice how our variants of the NCA are comparable or superior to the current state-of-the-art, despite being extremely simple. It fairs surprisingly well both against methods that use meta-learning and episodic learning, and also against the recently proposed high-performing simple baselines based on pre-training with the cross-entropy loss.

## 5 Conclusion

Towards the aim of understanding the reasons behind the poor competitiveness of meta-learning methods with respect to simple baselines, in this paper we start by investigating the role of episodes in the popular Prototypical Networks. We found that its performance is highly sensitive to the set of hyper-parameters used to sample the episodes. By replacing the Prototypical Networks loss with the classic Neighbourhood Component Analysis, we are able to ignore these hyperparameters while significantly improving the few-shot classification accuracy. With a series of experiments, we found out that the performance discrepancy mostly arises from the separation between support and query set within each episode, and that Prototypical Networks' episodic strategy is almost empirically equivalent to randomly discarding a large fraction of distances within a standard mini-batch. Finally, we show that our variant of the NCA achieves an accuracy on multiple popular FSL benchmarks that is comparable or superior with state-of-the-art methods of similar complexity.

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# A APPENDIX

## A.1 ADAPTING TO THE SUPPORT SET

Prototypical Networks does not perform any kind of parameter adaptation at test time. On the one hand this is convenient, as it allows fast inference; on the other hand, useful information from the support set $S$ might remain unexploited.

In the 5-shot case it is possible to minimise the NCA loss since it can directly be computed on the support set: $\mathcal{L}_{\mathrm{NCA}}(S)$. We tried training a positive semi-definite matrix $A$ on the outputs of the trained neural network, which corresponds to learning a Mahalanobis distance metric as in Goldberger et al. (2005). However, we found that there was no meaningful increase in performance. Differently, we did find that fine-tuning the whole neural network $f_\theta$ by $\arg\min_\theta \mathcal{L}_{\mathrm{NCA}}(S)$ brought a slight improvement. However, given the computational cost, we opted for non performing adaptation to the support sets in our experiments of Sec. 4. Results on *mini*ImageNet an CIFAR-FS are shown in Table 3.

## A.2 DETAILS ABOUT THE ABLATION STUDIES OF SECTION 4.3

Referring to the three key differences between the Prototypical Networks and the NCA losses listed in Sec. 3.3, in this section we detail how to obtain the ablations we used to perform the experiments of Sec. 4.3.

We can "disable" the creation of prototypes (point 1), which will change the prototypical loss of Eq. 2 to

$$\mathcal{L}(S,Q) = -\frac{1}{|Q|+|S|} \sum_{(\mathbf{q}_i, y) \in Q} \log \left( \frac{\sum\limits_{(\mathbf{s}_j, y') \in S_y} \exp -\|\mathbf{q}_i - \mathbf{s}_j\|^2}{\sum\limits_{(\mathbf{s}_k, y'') \in S} \exp -\|\mathbf{q}_i - \mathbf{s}_k\|^2} \right). \tag{4}$$

This is similar to $\mathcal{L}_{\mathrm{NCA}}$ (Eq. 3), where the positives are represented by the distances from $Q$ to $S_k$, and the negatives by the distances from $Q$ to $S \setminus S_k$. The only difference now is the separation of the batch into a query and support set.

Independently, we can "disable" point 2, which gives us

$$\mathcal{L}(S,Q) = -\frac{1}{|Q|+|S|} \sum_{(\mathbf{z}_i, y_i) \in Q \cup C} \log \left( \frac{\sum\limits_{\substack{(\mathbf{z}_j, y_j) \in Q \cup C \\ y_j = y_i \\ i \neq j}} \exp -\|\mathbf{z}_i - \mathbf{z}_j\|^2}{\sum\limits_{\substack{(\mathbf{z}_k, y_k) \in Q \cup C \\ k \neq i}} \exp -\|\mathbf{z}_i - \mathbf{z}_k\|^2} \right), \tag{5}$$

| | *mini*ImageNet | | CIFAR-FS | |
|---|---|---|---|---|
| **method** | **1-shot** | **5-shot** | **1-shot** | **5-shot** |
| **NCA** | $62.52 \pm 0.24$ | $78.3 \pm 0.14$ | $72.48 \pm 0.40$ | $85.13 \pm 0.29$ |
| **NCA multi-layer** | $63.21 \pm 0.08$ | $79.27 \pm 0.08$ | $72.44 \pm 0.36$ | $85.42 \pm 0.29$ |
| **NCA (ours) multi-layer + ss** | - | $\mathbf{79.79 \pm 0.08}$ | - | $\mathbf{85.66 \pm 0.32}$ |

Table 3: Comparison NCA, NCA using multiple evaluation layers and support set (ss) optimisation on *mini*ImageNet and CIFAR-FS datasets. NCA can only be optimised in 5-shot case since there are no positives in 1-shot support set. Support set is optimised for 5 epochs using an Adam optimiser with learning rate 0.0001 and weight decay 0.0005. For details, see Sec. A.1

which essentially combines the prototypes with the query set, and computes the NCA loss on that total set of embeddings.

Finally, we can "disable" both point 1 and 2, which gives us

$$\mathcal{L}(S, Q) = -\frac{1}{|Q| + |S|} \sum_{(\mathbf{z}_i, y_i) \in Q \cup S} \log \left( \frac{\sum\limits_{\substack{(\mathbf{z}_j, y_j) \in Q \cup S \\ y_j = y_i \\ i \neq j}} \exp - \|\mathbf{z}_i - \mathbf{z}_j\|^2}{\sum\limits_{\substack{(\mathbf{z}_k, y_k) \in Q \cup S \\ k \neq i}} \exp - \|\mathbf{z}_i - \mathbf{z}_k\|^2} \right). \tag{6}$$

This almost gives us exactly the NCA loss, where the only difference is the construction of batches with a fixed number of classes and a fixed number of images per class.

### A.3 Differences between the NCA and contrastive losses

Eq. 3 is similar to the contrastive loss functions (Khosla et al., 2020; Chen et al., 2020a) that are used in self-supervised learning and representation learning. The main differences are that 1.) The denominator only contains negative pairs and 2.) the inner sum in the numerator is moved outside of the logarithm in the supervised contrastive loss function from Khosla et al. (2020). We opt to work with the NCA loss because we found it performs better than the supervised constrastive loss. Using the supervised contrastive loss we only managed to obtain 51.05% 1-shot and 63.36% 5-shot performance on the *mini*Imagenet test set.

### A.4 Implementation details

**Benchmarks.** In our experiments, we use three popular FSL benchmarks. ***mini*ImageNet** (Vinyals et al., 2016) is a subset of ImageNet generated by randomly sampling 100 classes, each with 600 randomly sampled images. We adopt the commonly used splits of Ravi & Larochelle (2017) who use 64 classes for meta-training, 16 for meta-validation and 20 for meta-testing. **CIFAR-FS** was proposed by Bertinetto et al. (2019) as an anagolous version of *mini*Imagenet for CIFAR-100. It uses the same sized splits and same number of images per split as *mini*ImageNet. ***tiered*ImageNet** (Ren et al., 2018) is also constructed from ImageNet, but contains 608 classes, with 351 training classes, 97 validation classes and 160 test classes. The class split have been generated using WordNet (Miller, 1995) to ensure that the training classes are semantically "distant" to the validation and test classes. For all datasets, we use images of size $84 \times 84$.

**Architecture.** In all our experiments, $f_\theta$ is represented by a ResNet12 with widths $[64, 160, 320, 640]$. We chose this architecture, initially introduced by Lee et al. (2019), as it is the one which is most frequently adopted by recent FSL methods. Unlike most methods, we do not use a DropBlock regulariser (Ghiasi et al., 2018), as we did not notice it to meaningfully contribute to performance.

**Optimisation.** To train all the models used for our experiments, unless differently specified, we used a SGD optimiser with Nesterov momentum, weight decay of 0.0005 and initial learning rate of $0.1$. For *mini*ImageNet and CIFAR-FS we decrease the learning rate by a factor of 10 after 70% of

| | miniImageNet | |
|---|---|---|
| **method** | **5-shot val** | **5-shot test** |
| Soft Assignment | $79.11 \pm 0.27$ | $77.16 \pm 0.10$ |
| $k$-NN | $75.82 \pm 0.21$ | $73.52 \pm 0.12$ |
| 1-NN centroid | $\mathbf{80.61 \pm 0.20}$ | $\mathbf{78.30 \pm 0.14}$ |
| | CIFAR-FS | |
| Soft Assignment | $76.12 \pm 0.32$ | $83.31 \pm 0.37$ |
| $k$-NN | $73.46 \pm 0.39$ | $80.94 \pm 0.38$ |
| 1-NN centroid | $\mathbf{77.80 \pm 0.35}$ | $\mathbf{85.13 \pm 0.32}$ |

Table 4: Comparison of different evaluation methods. Average results over 5 trained models.

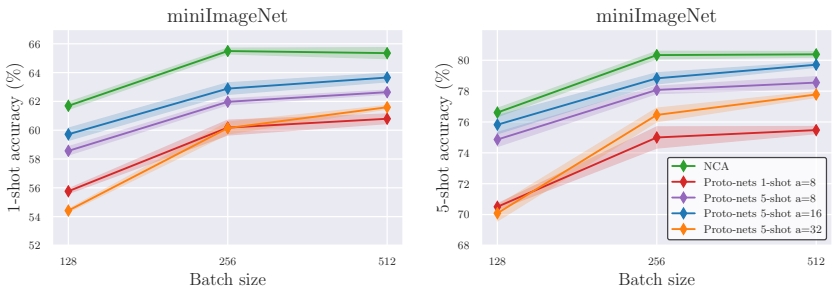


Figure 5: 1-shot (left) and 5-shot accuracies (right) on the validation set of *mini*ImageNet for different batch sizes. Models are trained using NCA or Proto-nets with different configurations: 1-shot with $a = 8$ and 5-shot with $a = 8$, 16 or 32. Reported values correspond to the mean accuracy of five models trained with different random seeds. Please see Sec. 4.2 for details.

epochs have been trained, and train for a total of 120 epochs. As data augmentations, we use random horizontal flipping and centre cropping.

Only for the experiments of 4.4, we slightly change our training setup. On CIFAR-FS, we increase the number of training epochs from 120 to 240, which improved accuracy of about 0.5%. For *tiered*ImageNet, we train for 120 epochs and decrease the learning rate by a factor of 10 after 50% and 75% of the training progress. These changes affect all our methods and baselines: NCA, Prototypical Networks (with both old and new batch setup), and SimpleShot (Wang et al., 2019).

**Projection network.** Similarly to (Khosla et al., 2020; Chen et al., 2020a), we also experimented with a *projection network* (but *only* for the comparison of Sec. 4.4). The projection network is a single linear layer $A \in \mathbb{R}^{M \times P}$ that is placed on top of $f_\theta$ at training time, where $M$ is the output dimension of the neural network $f_\theta$ and $P$ is output dimension of $A$, which can be considered as a hyper-parameter. The output of $A$ is used for minimising Eq 1. At test time, we do not use the output of $A$ and directly use the output of $f_\theta$. For CIFAR-FS and *tiered*ImageNet, we found this did not help performance. For *mini*ImageNet however we found that this improved performance, and we set $P = 128$.

Note, that this is not an unfair advantage over other methods. Compared to SimpleShot (Wang et al., 2019) and other simple baselines, we actually use fewer parameters without the projection network (effectively making our ResNet12 a ResNet11) since they use an extra fully connected layer to minimise cross entropy during pre-training.

### A.5 Additional results for Section 4.2

Fig. 5 complements the results of Fig. 2 from Sec. 4.2