Benefits of Permutation-Equivariance in Auction Mechanisms

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Abstract

Designing an incentive-compatible auction mechanism that maximizes the auctioneer's revenue while minimizes the bidders' ex-post regret is an important yet intricate problem in economics. Remarkable progresses have been achieved through learning the optimal auction mechanism by neural networks. In this paper, we consider the popular *additive valuation* and *symmetric valuation* setting; *i.e.*, the valuation for a set of items is defined as the sum of all items' valuations in the set, and the valuation distribution is invariant when the bidders and/or the items are permutated. We prove that permutation-equivariant neural networks have significant advantages: the permutation-equivariance decreases the expected ex-post regret, improves the model generalizability, while maintains the expected revenue invariant. This implies that the permutation-equivariance helps approach the theoretically optimal *dominant strategy incentive compatible* condition, and reduces the required sample complexity for desired generalization. Extensive experiments fully support our theory. To our best knowledge, this is the first work towards understanding the benefits of permutation-equivariance in auction mechanisms. Code will be released publicly.

1 Introduction

Optimal auction design [27] has wide applications in economics, including online advertising [17], resource allocation [15], and supply chain [4]. In an auction, every bidder has a private valuation profile over all items, and accordingly, submits her bid profile. An auctioneer collects the bids from all bidders, and determines a feasible item allocation to the bidders as well as the prices the bidders need to pay. Consequently, every bidder receives her utility. From the auctioneer's perspective, the optimal auction mechanism is required to maximize her *revenue*, defined as the sum of all bidders' payments. From the aspect of the bidders, the optimal auction mechanism needs to incentivize every bidder to bid according to their truthful valuation profiles (truthful bidding). This is summarized as the *dominant strategy incentive compatible* (DSIC) condition; *i.e.*, the truthful bidding is always the dominant strategy for every bidder [23].

The optimal auction mechanism can be approximated via neural networks [12, 26, 10]. The "approximation error", or the "distance" to the DSIC condition, is usually measured by the *ex-post regret*, defined as the gap between the bidder's utility of truthful bidding and the utility when her bid profile is only the best to herself (selfish bidding), while the bid profiles of all other bidders are fixed in both cases [12]. When a bidder's ex-post regret is always equal to 0, the truthful bidding is her dominant strategy. Therefore, the optimal auction design can be modeled as a linear programming problem, where the object is to maximize the expected revenue subject to the expected ex-post regret being

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0 [12]. Another major consideration in learning the optimal mechanism is the generalizability to unseen data, usually measured by the *generalization bound*, *i.e.*, the upper bound of the gap between the expected revenue/ex-post regret and their empirical counterparts on the training data [12].

In this paper, we consider the popular setting of *additive valuation* and *symmetric valuation* [12, 26, 10]. The additive valuation condition defines the valuation for a set of items as the sum of the valuations for all items in this set. The symmetric valuation condition assumes the joint distribution of all bidders' valuation profiles to be invariant when bidders and/or items are permutated. This setting covers many applications in practice. For example, when items are independent, the additive valuation condition holds. Moreover, if the auction is anonymous or the order of the items is not prior-known, the symmetric valuation condition holds.

We demonstrate that permutation-equivariant models have significant advantages in learning the optimal auction mechanism as follows. **(1)** We prove that the permutation-equivariance in auction mechanisms decreases the expected ex-post regret while maintaining the expected revenue invariant. Conversely, and equivalently, the permutation-equivariance promises a larger expected revenue, when the expected ex-post regret is fixed. **(2)** We show that the permutation-equivariance of auction mechanisms reduces the required sample complexity for a desirable generalizability. We prove that the $l_{\infty,1}$-distance between any two mechanisms in the mechanism space decreases when they are projected to the permutation-equivariant mechanism (sub-)space. This smaller distance implies a smaller covering number of the permutation-equivariant mechanism space, which further leads to a small generalization bound [12].

We further provide an explanation for the learning process of non-permutation-equivariant neural networks (NPE-NNs). In learning the optimal auction mechanism by an NPE-NN, we show that an extra positive term exists in the quadratic penalty of the ex-post regret based on the result **(1)**. This term serves as a regularizer to penalize the "non-permutation-equivariance". Moreover, this regularizer also interferes the revenue maximization, and thus affects the learning performance of NPE-NNs. This further explains the advantages of permutation-equivariance in auction design.

Experiments in extensive auction settings are conducted to verify our theory. We design permutation-equivariant versions of RegretNet (RegretNet-test and RegretNet-PE) by projecting the RegretNet [12] to the permutation-equivariant mechanism space in the training and test stage. The empirical results show that permutation-equivariance helps: (1) significantly improve the revenue while maintain the same ex-post regret; (2) record the same revenue with a significantly lower ex-post regret; and (3) narrow the generalization gaps between the training ex-post regret and its test counterpart. These results fully support our theory.

Related works. Myerson completely solves the optimal auction design problem in one-item auctions [21]. However, solutions are not clear when the number of bidders/items exceeds one [12]. Initial attempts have been presented on the characterization of optimal auction mechanisms [19, 24, 9] and algorithmic solutions [1, 2, 25]. Remarkable advances have been made on the required sample complexity for learning the optimal auction mechanism in various settings, including single-item auctions [5, 20, 15], multi-item single-bidder auctions [11], combinatorial auctions [3, 30], and allocation mechanisms [22]. Machine learning-based auction design (automated auction design) have obtained considerable progresses [6, 7, 28]. The optimal auction design is modelled as a linear programming problem [6, 7]. However, early works suffer from the scalablility issues that the number of the constraints grows exponentically when the bidder number and the item numbers increase. To address this issue, recent works propose to learn the optimal auction mechanism by deep learning. RegretNet is designed for the multi-bidder and multi-item settings [12]. Then, RegretNet is developed to meet more restrictive constraints, such as the budget condition [14] and the cerifying strategyproof condition [8]. Rahme *et al.* [26] propose the first equivariant neural network-based auction mechanism design method with significant empirical advantages. Ivanov *et al.* [16] propose a RegretFormer which (1) introduces attention layers to RegretNet to learn permutation-equivariant auction mechanisms, and (2) adopts a new interpretable loss function to control the revenue-regret trade-off. Duan *et al.* [10] extend the applicable domain to contextual settings. All these works make remarkable contributions in designing new algorithms from the empirical aspect only. However,

the theoretical foundations are still elusive. To our best knowledge, our paper is the first work on theoretically studying the benefits of permutation equivariance in auction design via deep learning.

2 Notations and Preliminaries

Auction. Suppose n bidders are bidding m items in an auction. Every bidder i has her bidder-context (feature) $x_i \in \mathcal{X}$, while every item j is associated with its item-context (feature) $y_j \in \mathcal{Y}$. The bidder i has a private valuation $v_{ij} \in \mathcal{V} \subset \mathbb{R}_{\geq 0}$ for the item j, which is sampled from a conditioned distribution $\mathbb{P}(\cdot|x_i, y_j)$. The value profile $v_i = (v_{i1}, \ldots, v_{im})$ is unknown to the auctioneer. For the simplicity, we define $x = (x_1^T, \ldots, x_n^T)^T$, $y = (y_1, \ldots, y_m)$, $v = (v_1^T, \ldots, v_n^T)^T$, $v_{-i} = (v_1^T, \ldots, v_{i-1}^T, v_{i+1}^T, \ldots, v_n^T)^T$, and $(v_i', v_{-i}) = (v_1^T, \ldots, (v_i')^T \ldots, v_n^T)^T$.

Every bidder submits a bid profile b_i to the auctioneer according to her valuation profile. Then, the auctioneer determines a feasible item allocation $g(b, x, y)$ and corresponding payments $p(b, x, y)$ as per an auction mechanism (g, p). Consequently, every bidder receives her utility $u_i(v_i, b, x, y) = \sum_{j=1}^m g_{ij}(b, x, y) \cdot v_{ij} - p_i(b, x, y)$. The auction mechanism (g, p) consists of an allocation rule $g : \mathbb{R}^{n\times m} \times \mathcal{X}^n \times \mathcal{Y}^m \to \mathbb{R}^{n\times m}$ and a payment rule $p : \mathbb{R}^{n\times m} \times \mathcal{X}^n \times \mathcal{Y}^m \to \mathbb{R}^{n\times m}$, where g_{ij} is the probability of allocating item j to the bidder i, and $p_i = \sum_{j=1}^m p_{ij}$ is the price that the bidder i should pay. To avoid allocating an item over once, the allocation rule is constrained such that $\sum_{i=1}^n g_{ij}(b, x, y) \leq 1$ for all $j \in [m]$. Every v_i in our notations can be replaced by b_i. Thus, we can define the similar notations: $b_{-i} = (b_1^T, \ldots, b_{i-1}^T, b_{i+1}^T, \ldots, b_n^T)^T$, $(b_i, v_{-i}) = (v_1^T, \ldots, (b_i)^T \ldots, v_n^T)^T$ and $(v_i, b_{-i}) = (b_1^T, \ldots, (v_i)^T \ldots, b_n^T)^T$.

Optimal auction mechanism. An auction mechanism (g, p) is defined to be *dominant strategy incentive compatible* (DSIC), if truthful bidding is always a dominant strategy of every bidder; *i.e.*, $u_i(v_i, (v_i, b_{-i}), x, y) \geq u_i(v_i, b, x, y)$ for all $i \in [n]$, $v, b \in \mathcal{V}^{n\times m}$, $x \in \mathcal{X}^n$ and $y \in \mathcal{Y}^m$. In addition, an auction mechanism (g, p) is called *individually rational* (IR), if for any bidder-contexts $x \in \mathcal{X}^n$, any item-contexts $y \in \mathcal{Y}^m$, and any bidder $i \in [n]$ $x \in \mathcal{X}^n$, valuation profile and bid profile $v, b \in \mathcal{V}^{n\times m}$, truthful bidding always leads to a non-negative utility; *i.e.*, $u_i(v_i, (v_i, b_{-i}), x, y) \geq 0$. If an auction mechanism is DSIC and IR, a rational bidder with an obvious dominant strategy will play it (bidding truthfully). Moreover, an optimal auction mechanism is required to maximize the auctioneer's expected revenue $rev = \mathbb{E}_{(v,x,y)}\big[\sum_{i=1}^n p_i(v, x, y)\big]$.

Auction design. The ex-post regret $reg_i(v, x, y)$ for the bidder i is defined as $\max_{b_i' \in \mathcal{V}^m} u_i(v_i, (b_i', v_{-i}), x, y) - u_i(v_i, v, x, y)$. An auction mechanism (g, p) is DSIC, if and only if $\sum_{i=1}^n reg_i(v, x, y) = 0$ for any value profile $v \in \mathcal{V}^{n\times m}$, bidder-context $x \in \mathcal{X}^n$, and item-context $y \in \mathcal{Y}^m$. Suppose the payment rule p satisfy $p_i(b, x, y) \leq \sum_{j=1}^m g_{ij}(b, x, y) b_{ij}$. Then, the auction design can be modeled as a linear programming problem that maximizes the expected revenue $\mathbb{E}_{(v,x,y)}[\sum_{i=1}^n p_i(v, x, y)]$ subject to the expected ex-post regret $\mathbb{E}_{(v,x,y)}[\sum_{i=1}^n reg_i(v, x, y)] = 0$. Without loss of generality, the ex-post regret may refer to the sum of all bidders' ex-post regrets.

Definition 2.1. Suppose the network's parameter is ω, and the bidder i's empirical payment and ex-post regret are defined as $\frac{1}{L}\sum_{l=1}^L p_i^\omega(v^{(l)}, x^{(l)}, y^{(l)})$, and $\widehat{reg}_i(\omega) = \frac{1}{L}\sum_{l=1}^L reg_i^\omega(v^{(l)}, x^{(l)}, y^{(l)})$, where the sample set $\{(v^{(l)}, x^{(l)}, y^{(l)})\}_{l=1}^L$ is *i.i.d.* sampled from the following prior distribution, $\mathbb{P}(v, x, y) = \prod_{i,j=1}^{n,m} \mathbb{P}(v_{ij}|x_i, y_j)\mathbb{P}_{X_i}(x_i)\mathbb{P}_{Y_j}(y_j)$.

Equivariant mapping. We define a mapping f as *G-equivariant* if $\psi_g \circ f = f \circ \rho_g$ for two chosen group linear representations ρ and ψ and any g in group G.

Definition 2.2. (Permutaion-Equivariant Mapping) A permutation-equivariant mapping is defined to be $f : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ that for any instance $x \in \mathbb{R}^{n\times m}$, and permutation matrices $\sigma_n \in \mathbb{R}^{n\times n}$ and $\sigma_m \in \mathbb{R}^{m\times m}$, we have $f(\sigma_n x \sigma_m) = \sigma_n f(x) \sigma_m$.

In this paper, we consider the bidder-permutation $\sigma_n \in \mathbb{R}^{n\times n}$ and item-permutation $\sigma_m \in \mathbb{R}^{m\times m}$. Specifically, we define a mapping f is bidder-symmetric or item-symmetric, if $f(\sigma_n x) = \sigma_n f(x)$ or $f(x\sigma_m) = f(x)\sigma_m$, respectively. Moreover, we define an auction mechanism (g, p) as bidder-

symmetric or item-symmetric, if the allocation rule g and the payment rule p are both bidder-symmetric or item-symmetric.

Orbit averaging. For any feature mapping $f : \mathcal{F} \to \mathcal{G}$, the orbit averaging $\mathcal{Q}$ on f is defined as $\mathcal{Q}f = \frac{1}{|G|}\sum_{g\in G} \psi_g^{-1} \circ f \circ \rho_g$, where ρ and ψ are two chosen group representations acting on the feature spaces $\mathcal{F}$ and $\mathcal{G}$, respectively. Orbit averaging can project any mapping to be equivariant:

Proposition 2.3. *Orbit averaging $\mathcal{Q}$ is a projection to the equivariant mapping space $\{f : \psi \circ f = f \circ \rho\}$, i.e., $\psi \circ \mathcal{Q}f = \mathcal{Q}f \circ \rho$ and $\mathcal{Q}^2 = \mathcal{Q}$. In particular, if f is already equivariant, then $\mathcal{Q}f = f$.*

Moreover, $\mathcal{Q}u$ and $\mathcal{Q}reg$ refer to the utility and the ex-post regret induced by $\mathcal{Q}g$ and $\mathcal{Q}p$. For the simplicity, we denote the orbit averagings that modify the auction mechanism to be bidder-symmetric, item-symmetric, and bidder/item-symmetric by bidder averaging $\mathcal{Q}_1$, item averaging $\mathcal{Q}_2$, and bidder-item aggregated averaging $\mathcal{Q}_3$. The proof of the feasibility can be found in Appendix A.1.

Hypothesis complexity. The generalizatbility to unseen data is usually measured by the generalization bound, which depends on the hypothesis set's complexity. To characterize the complexity of the hypothesis set, we introduce the following definitions of *covering number* $\mathcal{N}_{\infty,1}$ and its corresponding distance $l_{\infty,1}$. Based on the covering number, we can obtain a generalization bound in Theorem 3.6.

Definition 2.4. ($l_{\infty,1}$-distance) Let $\mathcal{X}$ be a feature space and $\mathcal{F}$ a space of some functions from $\mathcal{X}$ to $\mathbb{R}^n$. The $l_{\infty,1}$-distance on the space $\mathcal{F}$ is defined as $l_{\infty,1}(f, g) = \max_{x\in\mathcal{X}}(\sum_{i=1}^n |f_i(x) - g_i(x)|)$.

Definition 2.5. (Covering number) Covering number $\mathcal{N}_{\infty,1}(\mathcal{F}, r)$ is the minimum number of balls with radius r that can cover $\mathcal{F}$ under $l_{\infty,1}$-distance.

3 Theoretical Results

This section presents the theoretical results. For simplicity, we view $p = (p_1, \ldots, p_n)^T$ as a $n \times 1$ matrix to present the prices the bidders should pay. We first prove that the permutation-equivariance induces a same expected revenue and a smaller expected ex-post regret in Section 3.1. Next in Section 3.2, we prove that the permutation-equivariant mechanism space has a smaller covering number, which promises a smaller required sample complexity and a better generalization. Detailed proofs are omitted from the main text and given in supplementary materials due to space limitation.

3.1 Benefits for Revenue and Ex-Post Regret

In this section, we discuss the benefits for the revenue and the ex-post regret in the conditions of bidder-symmetry and item-symmetry separately, and then discuss the benefits when both of them hold. Based on these results, we also study the learning process of non-permutation-equivariant neural networks for auction design.

3.1.1 Benefits in the Bidder/Item-Symmetry Condition

When the bidders come from the same distribution, the joint valuation distribution f is invariant under bidder-permutation, *i.e.* $f(\sigma_n v, \sigma_n x, y) = f(v, x, y)$ for any $\sigma_n \in S_n$. Meanwhile, when the items are indistinguishable, the joint distribution f is invariant under item-permutation, *i.e.*, $f(v\sigma_m, x, y\sigma_m) = f(v, x, y)$ for any $\sigma_m \in S_m$. Both conditions do not always hold simultaneously. In this section, we study them separately.

To measure the "non-permutation-equivariance" of the mechanism, we introduce the conception of *regret gap* between the projected mechanism and the original mechanism for as below,

$$\Delta_{\cdot}(g, p; v, x, y) = \max_{v'\in\mathcal{V}^{n\times m}} \sum_{i=1}^n u_i(v_i, (v'_i, v_{-i}), x, y) - \max_{v'\in\mathcal{V}^{n\times m}} \sum_{i=1}^n [\mathcal{Q}_{\cdot}u]_i(v_i, (v'_i, v_{-i}), x, y),$$

where v is the valuation profiles, v_i is valuation profile of bidder i, x is the bidder-context, y is the item-context, and the orbit averaging $\mathcal{Q}_{\cdot}$ can be the bidder averaging $\mathcal{Q}_1$ or the item averaging $\mathcal{Q}_2$.

The bidder averaging $\mathcal{Q}_1$ and the item averaging $\mathcal{Q}_2$ acting on the allocation rule g and the payment rule p, respectively, are as below,

$$\mathcal{Q}_1 g(v,x,y) = \frac{1}{n!}\sum_{\sigma_n \in S_n} \sigma_n^{-1} g(\sigma_n v, \sigma_n x, y), \quad \mathcal{Q}_1 p(v,x,y) = \frac{1}{n!}\sum_{\sigma_n \in S_n} \sigma_n^{-1} p(\sigma_n v, \sigma_n x, y),$$

$$\mathcal{Q}_2 g(v,x,y) = \frac{1}{m!}\sum_{\sigma_m \in S_m} g(v\sigma_m, x, y\sigma_m)\sigma_m^{-1}, \text{ and } \mathcal{Q}_2 p(v,x,y) = \frac{1}{m!}\sum_{\sigma_m \in S_m} p(v\sigma_m, x, y\sigma_m).$$

We thus can prove the following theorem that characterizes the benefits of permutation-equivariance for revenue and ex-post regret in the condition of bidder/item-symmetry.

Theorem 3.1 (Benefits for revenue and ex-post regret in the condition of bidder/item-symmetry)**.** *When the valuation distribution is invariant under permutations of bidders/items, the projected mechanism has the same expected revenue and a smaller expected ex-post regret, that is,*

$$\mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n}[\mathcal{Q}_{\cdot}p]_i(v,x,y)\right] = \mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n} p_i(v,x,y)\right], \text{ and} \tag{1}$$

$$\mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n} reg_i(v,x,y)\right] - \mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n}[\mathcal{Q}_{\cdot}reg]_i(v,x,y)\right] = \mathbb{E}_{(v,x,y)}\left[\Delta_{\cdot}(g,p;v,x,y)\right] \geq 0, \tag{2}$$

where p is the payment rule, reg is the ex-post regret, and $\mathcal{Q}_{\cdot}$ is the bidder/item averaging.

A smaller expected ex-post regret implies this mechanism is closer to the *dominant strategy incentive compatible* condition. Conversely, and equivalently, when the expected ex-post regrets are fixed, the projected auction mechanism has a larger expected revenue. For any auction mechanism, in the bidder/item-symmetry condition, we can project it through the bidder/item averaging.

Remark 3.2. The mechanism space can be decompose into the direct sum of the permutation-equivariant mechanism space $\{\mathcal{M} : \mathcal{Q}\mathcal{M} = \mathcal{M}\}$ and the complementary space $\{\mathcal{N} : \mathcal{Q}\mathcal{N} = 0\}$ [13]. Thus, an mechanism $\mathcal{M}$ has an unique decomposition: $\mathcal{M} = \mathcal{Q}\mathcal{M} + \mathcal{N}$. The pure permutation-equivariant part $\mathcal{Q}\mathcal{M}$ contains all and only the "permutation-equivariance" of the mechanism $\mathcal{M}$. The pure non-permutation-equivariant part $\mathcal{N}$ is independent from the permutation-equivairance. In this way, we may study the influence of permutation-equivariance by comparing the mechanism $\mathcal{M}$ and its permutation equivariant part $\mathcal{Q}\mathcal{M}$.

3.1.2 Interplay between Bidder-Symmetry and Item-Symmetry.

If the valuation distribution is invariant under both bidder-permutation and item-permutation, we can project the mechanism to be permutation-equivariant with respect to both bidder and item in two steps (by mapping $\mathcal{Q}_1 \circ \mathcal{Q}_2$ or mapping $\mathcal{Q}_2 \circ \mathcal{Q}_1$). Consequently, the projected mechanism has the same expected revenue and a smaller expected ex-post regret. Equivalently, we can also project an auction mechanism to be bidder-symmetric and item-symmetric immediately by the bidder-item aggregated averaging $\mathcal{Q}_3$ as below,

$$\mathcal{Q}_3 g(v,x,y) = \frac{1}{n!m!}\sum_{\sigma_n \in S_n}\sum_{\sigma_m \in S_m} \sigma_n^{-1} g(\sigma_n v\sigma_m, \sigma_n x, y\sigma_m)\sigma_m^{-1}, \text{ and}$$

$$\mathcal{Q}_3 p(v,x,y) = \frac{1}{n!m!}\sum_{\sigma_n \in S_n}\sum_{\sigma_m \in S_m} \sigma_n^{-1} p(\sigma_n v\sigma_m, \sigma_n x, y\sigma_m).$$

We can prove that the bidder-item aggregated averaging $\mathcal{Q}_3$ is the composition of the orbit averaging operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$, as shown in the following lemma. This lemma shows that the order of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ would not influence their composition.

Lemma 3.3. *The bidder-item aggregated averaging is the composition of bidder averaging and item averaging: $\mathcal{Q}_3 = \mathcal{Q}_1 \circ \mathcal{Q}_2 = \mathcal{Q}_2 \circ \mathcal{Q}_1$.*

Based on this lemma, we can prove the following theorem on the benefits of permutation-equivariance for revenue and ex-post regret in the condition of both bidder-symmetry and item-symmetry.

Theorem 3.4 (Benefits for revenue and ex-post regret in the condition of both bidder-symmetry and item-symmetry)**.** *When the valuation distribution is invariant under both item-permutation and item-permutation, then the projected mechanism has a same expected revenue and a smaller expected ex-post regret, that is,*

$$\mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n}\mathcal{Q}_3 p_i(v,x,y)\right] = \mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n} p_i(v,x,y)\right] \text{ and}$$

$$\mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n} reg_i(v,x,y)\right] - \mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n}[\mathcal{Q}_3 reg]_i(v,x,y)\right] = \mathbb{E}_{(v,x,y)}\Big[\Delta_3(g,p;v,x,y)\Big] \geq 0,$$

where p is the payment rule, reg is the ex-post regret, and $\mathcal{Q}_3$ is the bidder-item aggregated averaging.

The difference between bidder-symmetry and item-symmetry is significant in practice. For example, for a symmetric valuation distribution, when the mechanism is already bidder-symmetric but not item-symmetric, we can project it to be item-symmetric to gain an extra benefit from item-symmetry. That means, the two regret gaps induced by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are "additive" as below,

$$\Delta_3(g,p;v,x,y) = \Delta_1(g,p;v,x,y) + \Delta_2(\mathcal{Q}_1 g, \mathcal{Q}_1 p; v,x,y).$$

In general, $\mathbb{E}[\Delta_2(g,p;v,x,y)] \neq \mathbb{E}[\Delta_2(\mathcal{Q}_1 g, \mathcal{Q}_1 p; v,x,y)]$ and thus $\mathbb{E}[\Delta_3(g,p;v,x,y)] \neq \mathbb{E}[\Delta_1(g,p;v,x,y)] + \mathbb{E}[\Delta_2(g,p;v,x,y)]$. That means, the benefits from bidder-symmetry and item-symmetry are "additive" but not "independent".

3.1.3 Insights on Training Non-Permutation-Equivariant Mechanism

Because the expected revenue is always same for the original mechanism and the projected permutation-equivariant mechanism, we only consider the gredient caused by the expected ex-post regret. We can decompose the original expected ex-post regret into the sum of the expected ex-post regret of the projected mechanism and the expectation of the regret gap as below,

$$\mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n} reg_i(v,x,y)\right] = \mathbb{E}_{(v,x,y)}\left[\sum_{i=1}^{n}[\mathcal{Q}_3 reg]_i(v,x,y)\right] + \mathbb{E}_{(v,x,y)}\Big[\Delta_3(g,p;v,x,y)\Big].$$

The regret gap $\Delta_3(g,p;\cdot)$ follows from the "non-permutation-equivariance" of the mechanism $\mathcal{M}$. When the distance $l(\mathcal{M}, \mathcal{Q}\mathcal{M})$ tends to 0, the regret gap converges to 0. When the auction mechanism has a negligible ex-post regret, the expectation of the regret gap is also close to 0. That means, the mechanism is closing to being permutation-equivariant. However, even using symmetric dataset or adopting data augmentation in training, the learned mechanism will not be permutation-equivariant in general [18]. As a result, to achieve negligible ex-post regret, the non-permutation-equivariant models need to learn more samples to approach permutation-equivariance. That is because the non-permutation-equivariant part (expected regret gap) would mislead the gradient of the expected regret but have no benefit to the expected revenue and the expected ex-post regret.

On the other hand, the regret gap can be viewed as a regularizer in the ex-post regret to penalize the "non-permutation-equivariance" of the mechanism. When the optimizer tries to minimize the ex-post regret, the auction mechanism approaches to be permutation-equivariant. Therefore, if the mechanism achieves a negligible ex-post regret, it is almost to be permutationn-equivariant. This result can explain why RegretNet struggles to find permutation-equivariant auction mechanisms [26]. However, in complex settings, it will be harder for non-permutation-equivariant models to approach the negligible ex-post regret. It can explain why the permutation-equivariant models show a significant improvement in complex setting, compared with that they have similar performances in simple settings [10], which shows the great importance of adopting permutation-equivariant models in complex settings.

3.2 Benefits for Generalization

In this section, we studies permutation-equivariance from the aspect of generalizability [29], which characterizes the performance gap of a learned mechanism on collected training data and unseen data.

We first study the covering number of the permutation-equivariant mechanism space. Let $\mathcal{U} = \{u^\omega : \omega \in \Omega\}$ and $\mathcal{P} = \{p^\omega : \omega \in \Omega\}$ be the sets of all possible utilities and payment rules, and $\mathcal{Q}.\mathcal{U} = \{\mathcal{Q}.u : u \in \mathcal{U}\}$ and $\mathcal{Q}.\mathcal{P} = \{\mathcal{Q}.p : p \in \mathcal{P}\}$ the spaces of all projected utilities and payment rules. In addition, let $\mathcal{N}_{\infty,1}(\mathcal{U}, r)$ and $\mathcal{N}_{\infty,1}(\mathcal{P}, r)$ be the minimum numbers of balls with radius r that can cover $\mathcal{U}$ and $\mathcal{P}$ under $l_{\infty,1}$-distance, respectively. We obtain the following result, which indicates the projected permutation-equivariant mechanism space has smaller covering numbers.

Theorem 3.5 (Covering number of the permutation-equivariant mechanism space)**.** *The space of all projected bidder-symmetric mechanisms has smaller covering numbers, that is,*

$$\mathcal{N}_{\infty,1}(\mathcal{Q}_1\mathcal{U}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{U}, r) \quad \textit{and} \quad \mathcal{N}_{\infty,1}(\mathcal{Q}_1\mathcal{P}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{P}, r).$$

The space of all projected item-symmetric mechanisms has smaller covering numbers, that is,

$$\mathcal{N}_{\infty,1}(\mathcal{Q}_2\mathcal{U}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{U}, r) \ \textit{and} \ \mathcal{N}_{\infty,1}(\mathcal{Q}_2\mathcal{P}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{P}, r).$$

Intuitively, the orbit averaging $\mathcal{Q}$ narrows the distance between two mechanisms: $l(\mathcal{Q}\mathcal{M}, \mathcal{Q}\mathcal{M}') \leq l(\mathcal{M}, \mathcal{M}')$, for any two mechanisms. Then, any r-cover $r > 0$ $\mathcal{A}$ for space $\mathcal{U}$ or space $\mathcal{P}$ induces an r-cover $\mathcal{Q}\mathcal{A}$ for space $\mathcal{Q}\mathcal{U}$ or space $\mathcal{Q}\mathcal{P}$.

Combining with Lemma 3.3, we have the following results,

$$\mathcal{N}_{\infty,1}(\mathcal{Q}_3\mathcal{U}, r) = \mathcal{N}_{\infty,1}(\mathcal{Q}_1\mathcal{Q}_2\mathcal{U}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{Q}_2\mathcal{U}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{U}, r) \ \text{and}$$
$$\mathcal{N}_{\infty,1}(\mathcal{Q}_3\mathcal{P}, r) = \mathcal{N}_{\infty,1}(\mathcal{Q}_1\mathcal{Q}_2\mathcal{P}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{Q}_2\mathcal{P}, r) \leq \mathcal{N}_{\infty,1}(\mathcal{P}, r).$$

We then prove that two generalization bounds of permutation-equivariant mechanisms which characterize the gap between the expected revenue/ex-post regret and their empirical counterparts.

Theorem 3.6 (Generalization bounds of permutation-equivariant mechanisms)**.** *If for any bidder, her valuation satisfies that $v_i(S) \leq 1$ for any $S \subset [m]$, then with probability at least $1 - \delta$, we have the following inequalities with $\epsilon \geq \sqrt{\frac{9n^2}{2L}(\log\frac{4}{\delta} + \max\{\log\mathcal{N}_{\infty,1}(\mathcal{P}, \frac{\epsilon}{3}), log\mathcal{N}_{\infty,1}(\mathcal{U}, \frac{\epsilon}{6})\})}$,*

$$\left|\mathbb{E}\left[\sum_{i=1}^{n} p_i^\omega(v, x, y)\right] - \frac{1}{L}\sum_{l=1}^{L}\sum_{i=1}^{n} p_i^\omega(v^{(l)}, x^{(l)}, y^{(l)})\right| \leq \epsilon, \ \textit{and} \tag{3}$$

$$\left|\mathbb{E}\left[\sum_{i=1}^{n} reg_i^\omega(v, x, y)\right] - \sum_{i=1}^{n} \widehat{reg}_i(\omega)\right| \leq \epsilon, \tag{4}$$

where L is the number of samples, $\mathcal{U}$ and $\mathcal{P}$ are the spaces of all possible utilities and payment rules.

Equivalently, we can rewrite this result in the form of the sample complexity,

Corollary 3.7. *For any $\epsilon > 0$, $\delta \in (0, 1)$, and mechanism parameter ω, when the sample complexity $L \geq \frac{9n^2}{2\epsilon^2}\left(\log\frac{4}{\delta} + \max\left\{\log\mathcal{N}_{\infty,1}(\mathcal{P}, \frac{\epsilon}{3}), log\mathcal{N}_{\infty,1}(\mathcal{U}, \frac{\epsilon}{6})\right\}\right)$, with probability at least $1 - \delta$, the generalization bounds, eqs. (3) and (4), hold.*

Remark 3.8. Combining Theorem 3.5, we have proved that the permutation-equivariance can improve the generalizability.

4 Experiments

This section presents our experimental results. More details and results are presented in the supplementary materails. Code will be released publicly.

Model architecture. We project RegretNet [12] to the permutation-equivariant mechanism space via orbit averaging: (1) using bidder averaging for the bidder-symmetry condition, (2) applying item

Table 1: Experimental results. "$n \times m\ Uniform$" refers that there are n bidders and m items, and the valuations are i.i.d. drawn from the uniform distribution $U[0, 1]$. To simplify, we multiply all results by a factor of 10^5 for the ex-post regret and generalization error (GE).

Method	$2 \times 1\ Uniform$			$3 \times 1\ Uniform$			$5 \times 1\ Uniform$		
	Revenue	Regret	GE	Revenue	Regret	GE	Revenue	Regret	GE
Optimal	0.417	0	-	0.531	0	-	0.672	0	-
RegretNet	0.415	17.4	6.00	0.535	18.3	11.4	0.658	15.9	6.40
RegretNet-Test	0.415	16.3	-	0.535	13.3	-	0.658	6.50	-
RegretNet-PE	0.420	14.6	3.90	0.541	16.4	10.2	0.677	13.2	5.10
Method	$2 \times 2\ Uniform$			$2 \times 5\ Uniform$			$5 \times 3\ Uniform$		
	Revenue	Regret	GE	Revenue	Regret	GE	Revenue	Regret	GE
RegretNet	0.870	70.2	31.0	2.24	104	86.4	1.56	28.4	19.4
RegretNet-Test	0.870	53.8	-	2.24	74.0	-	1.56	8.60	-
RegretNet-PE	0.913	67.3	15.1	2.38	89.9	24.9	1.85	20.1	11.8

Table 2: Experimental results. "$n \times m\ Uniform$" refers that there are n bidders and m items, and the valuations are i.i.d. sampled from the uniform distribution $U[0, 1]$.

Method	$1 \times 2\ Uniform$		$2 \times 2\ Uniform$	
	Revenue	Regret	Revenue	Regret
RegretNet	0.549	0.00069	0.870	0.00070
EquivariantNet	0.551	0.00013	0.873	0.00100
RegretNet-Test	0.549	0.00062	0.870	0.00054
RegretNet-PE	0.643	0.00022	0.913	0.00067

averaging for the item-symmetry condition, and (3) employing bidder-item aggregated averaging for the bidder-symmetry and item-symmetry condition. The projected model is called RegretNet-PE. Specifically, RegretNet is an auction mechanism defined as (g^ω, p^ω), in which both of the allocation rule g^ω and the payment rule p^ω are neural networks that consist of three fully-connected layers, and ω is the overall model parameter of the auction mechanism. The detailed architecture is given in the supplementary materials.

Comparison with EquivariantNet. RegretNet uses two feed-forward full-connected networks to learn the allocation rule and payment rule, respectively. We denote the weight matrix in the layer ℓ as $W^{(\ell)}$. Both EquivariantNet and RegretNet-PE inherit the architecture of RegretNet (with some modifications), but utilize different approaches to realize the permutation-equivariance. EquivariantNet applies parameter-sharing in every layer during training, to constrain $W^{(\ell)}$ to be equivariant. In contrast, RegretNet-PE employs orbit averaging to be permutation-equivariant. Specifically, RegretNet-PE adopts a weight matrix $I_K \otimes W^{(\ell)}(\rho_{g_1}^T \dots \rho_{g_K}^T)^T$ in the first layer, weight matrices $I_K \otimes W^{(\ell)}$ in the following layers, and multiples a matrix $(\rho_{g_1}^{-1}, \dots, \rho_{g_K}^{-1})$ to the output layer, where K is the scale of the group $G = \{g_1, \dots, g_K\}$, ρ_{g_k} represents the permutation operator on bidders and items, I_K is an identity matrix, and $\otimes$ is the Kronecker product. It is worth noting that RegretNet-PE is only designed for verifying our theory.

Auction settings. We adopt two families of auction settings in our experiments: (1) single-item auctions, including two-bidder, three-bidder and five-bidder settings; and (2) multi-item auctions, including two-bidder and two-item, two-bidder and five-item, and five-bidder and three-item settings. The optimal auction mechanism for any single-item auction is known [21]. We thus compare our method with the optimal auction mechanisms in the single-item settings. Also, we compare RegretNet-PE and EquivariantNet in the one-bidder, two-item setting, and the two-bidder, two-item setting.

Besides, we employ a multivariate uniform distribution $U[0,1]^m$ to model the bidder valuation profiles. We sample 5,000 data points for training and 100 data batches of size 100 (totally 10,000 data points) for test in every setting.

Model training. We optimize the auction mechanism model via solving the following optimization problem, following the standard settings [12, 26, 10],

$$\mathcal{L}_\rho(\omega,\lambda) = -\frac{1}{L}\sum_{l=1}^{L}\sum_{i=1}^{n} p_i^\omega(v^{(l)}, x^{(l)}, y^{(l)}) + \sum_{i=1}^{n}\lambda_i \widehat{reg}_i(\omega) + \frac{\rho}{2}\Big(\sum_{i=1}^{n}\widehat{reg}_i(\omega)\Big)^2,$$

where $\lambda \in \mathbb{R}^n$ is the Lagrange multiplier and $\rho > 0$ is the factor of the quadratic regularization term. During the training process, the objective function $\mathcal{L}_\rho(\omega,\lambda)$ is minimized via Adam with learning rate 0.001 with respect to the model parameter ω and the Largrange multiplier λ is updated via SGD once in every 100 iterations, until the ex-post regret is smaller than 0.001. The regularization factor ρ is is set to 1000 and 5 initially for single-item settings and multiple and gradually increased along the training process. In calculating the best bid profile v_i' of every bidder i, we first randomly initialize the bid profiles for 1,000 times, optimize each of them individually via Adam with the same settings, and take the best one as the approximated best bidding.

Evaluation. We leverage three metrics to evaluate the performance of the auction mechanism, which are: (1) the empirical revenue $\widehat{rev}$, (2) the empirical ex-post regret across all bidders, *i.e.*, $\sum_{i=1}^n \widehat{reg}_i$, and (3) the generalization error defined on top of regrets, *i.e.*, $GE = |\widehat{reg}_{test} - \widehat{reg}_{train}|$.

Computing resource. The experiments are conducted on 1 GPU (NVIDIA® Tesla® V100 16GB) and 10 CPU cores (Intel® Xeon® Processor E5-2650 v4 @ 2.20GHz).

Experimental results. We train a RegretNet and a RegretNet-PE on the training data. The well-trained RegretNet is then projected to be permutation-equivariant, denoted as "RegretNet-Test". The results are collected in Tables 1 and 2.

From Tables 1 and 2, we observe that (1) compared to RegretNet, RegretNet-PE has a significantly higher revenue with a lower ex-post regret, and narrows the generalization gap between the training ex-post regret and its test counterpart; (2) compared to RegretNet, RegretNet-Test receives the same revenue with a significantly lower ex-post regret; and (3) under comparable ex-post regrets, RegretNet-PE has a considerably higher revenue than EquivariantNet, while all permutation-equivariant models (EquivariantNet, RegretNet-Test, and RegretNet-PE) can outperform RegretNet. These results show significant benefits of permutation-equivariance on revenue, ex-post regret, and generalizability, which fully supports our theoretical findings in Theorems 3.1, 3.4, and 3.5.

5 Conclusion and future works

In this paper, under *additive valuation* and *symmetric valuation* setting, we study the benefits of permutation-equivariance in auction mechanisms in two aspects: a better performance and a better generalization. First, we prove a smaller expected ex-post regret and a same expected revenue when projecting an mechanism to be permutation-equivariant. Next, we propose the permutation-equivariant mechanism space has a smaller covering number, which promises the permutation-equivariant models a better generalization. Extensive experiments are conducted to verify our theoretical results. Our results help understand the optimal auction mechanisms' characterization and the learning processes' difference between non-equivariant models and equivaraint models.

Beyond the additive valuation setting, an interesting direction is to extend our results to other conditions, including the combinatorial and the unit-demand auctions. Meanwhile, the understanding of the difference in the aspect of the training process between non-equivariant models and equivariant models is still elusive.

Social impact. Our results can help understand and design optimal auction mechanisms for symmetric valuation distribution. As a result, our work could inspire more near-optimal auction mechanisms and promote economic growth. No potential negative social impact is identified.

References

[1] S. Alaei. Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. *The 16th Annual Symposium on Foundations of Computer Science*, 2011.

[2] S. Alaei, H. Fu, N. Haghpanah, and J. Hartline. The simple economics of approximately optimal auctions. *The 16th Annual Symposium on Foundations of Computer Science,*, 2013.

[3] M.-F. F. Balcan, T. Sandholm, and E. Vitercik. Sample complexity of automated mechanism design. *Neural Information Processing Systems (NeurIPS)*, 2016.

[4] L. Chen, L. Wang, and Y. Lan. Auction models with resource pooling in modern supply chain management. *Modern Supply Chain Research and Applications*, 2019.

[5] R. Cole and T. Roughgarden. The sample complexity of revenue maximization. In *ACM symposium on Theory of computing*, 2014.

[6] V. Conitzer and T. Sandholm. Complexity of mechanism design. *Proceedings of the Eighteenth Conference on Uncertainty in Artificial Intelligence (UAI)*, 2002.

[7] V. Conitzer and T. Sandholm. Self-interested automated mechanism design and implications for optimal combinatorial auctions. In *ACM Conference on Electronic Commerce*, 2004.

[8] M. Curry, P.-Y. Chiang, T. Goldstein, and J. Dickerson. Certifying strategyproof auction networks. *Neural Information Processing Systems (NeurIPS)*, 2020.

[9] C. Daskalakis, A. Deckelbaum, and C. Tzamos. Strong duality for a multiple-good monopolist. *Econometrica*, 2017.

[10] Z. Duan, J. Tang, Y. Yin, Z. Feng, X. Yan, M. Zaheer, and X. Deng. A context-integrated transformer-based neural network for auction design. *arXiv preprint arXiv:2201.12489*, 2022.

[11] S. Dughmi, L. Han, and N. Nisan. Sampling and representation complexity of revenue maximization. *Web and Internet Economic*, 2014.

[12] P. Dütting, Z. Feng, H. Narasimhan, D. Parkes, and S. S. Ravindranath. Optimal auctions through deep learning. In *International Conference on Machine Learning (ICML)*, 2019.

[13] B. Elesedy and S. Zaidi. Provably strict generalisation benefit for equivariant models. In *International Conference on Machine Learning (ICML)*, 2021.

[14] Z. Feng, H. Narasimhan, and D. C. Parkes. Deep learning for revenue-optimal auctions with budgets. In *the 17th International Conference on Autonomous Agents and Multiagent Systems*, 2018.

[15] J. Huang, Z. Han, M. Chiang, and H. V. Poor. Auction-based resource allocation for cooperative communications. *IEEE Journal on Selected Areas in Communications*, 2008.

[16] D. Ivanov, I. Safiulin, K. Balabaeva, and I. Filippov. Optimal-er auctions through attention. *arXiv preprint arXiv:2202.13110*, 2022.

[17] B. J. Jansen and T. Mullen. Sponsored search: an overview of the concept, history, and technology. *International Journal of Electronic Business*, 2008.

[18] C. Lyle, M. van der Wilk, M. Kwiatkowska, Y. Gal, and B. Bloem-Reddy. On the benefits of invariance in neural networks. *arXiv preprint arXiv:2005.00178*, 2020.

[19] A. M. Manelli and D. R. Vincent. Bundling as an optimal selling mechanism for a multiple-good monopolist. *Journal of Economic Theory*, 2006.

[20] M. Mohri and A. M. Medina. Learning theory and algorithms for revenue optimization in second-price auctions with reserve. *International Conference on Machine Learning (ICML)*, 2013.

[21] R. B. Myerson. Optimal auction design. *Mathematics of operations research*, 1981.

[22] H. Narasimhan and D. C. Parkes. A general statistical framework for designing strategy-proof assignment mechanisms. *Uncertainty in Artificial Intelligence (UAI)*, 2016.

[23] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani. Algorithmic game theory. Cambridge University Press, 2007.

[24] G. Pavlov. Optimal mechanism for selling two goods. *The BE Journal of Theoretical Economics*, 2011.

[25] G. Popescu. An algorithmic characterization of multi-dimensional mechanisms. *Computing Reviews*, 2012.

[26] J. Rahme, S. Jelassi, J. Bruna, and S. M. Weinberg. A permutation-equivariant neural network architecture for auction design. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2021.

[27] T. Roughgarden. Twenty lectures on algorithmic game theory. *Cambridge Books*, 2016.

[28] T. Sandholm and A. Likhodedov. Automated design of revenue-maximizing combinatorial auctions. *Operations Research*, 2015.

[29] A. Sannai, M. Imaizumi, and M. Kawano. Improved generalization bounds of group invariant/equivariant deep networks via quotient feature spaces. In *International Conference on Machine Learning (ICML)*, 2021.

[30] V. Syrgkanis. A sample complexity measure with applications to learning optimal auctions. *Neural Information Processing Systems (NeurIPS)*, 2017.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] Our contributions are stated in the 4th-7th sentences of the abstract, and the 4th-7th paragraphs of the introduction. The scope is described in the related work section.
   (b) Did you describe the limitations of your work? [Yes] We describes the problem setting we considered in the 3rd paragraph of the introduction. Our contributions are for this problem setting.
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] Our work is committed to learn the optimal auction mechanism. No negative societal impact is identified.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes] We have read the ethics review guidelines. We ensure that our paper conforms all of them.
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] We state all of our assumptions in the third paragraph of introduction, the first paragraph of preliminary, Theorem 3.1, and Theorem 3.2. Wherever we use an assumption, we always state it again.

(b) Did you include complete proofs of all theoretical results? [Yes] We give a proof sketch in Section 3 due to space limitation. Full proofs are given in the supplemental material.

3. If you ran experiments...

   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No] We promise to release our code publicly. We are associated in a industrial organization. The code is currently under review of our organization.

   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] We mostly follow the setting of the RegretNet. Other details are given in Section 4.

   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No]

   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] They are given in Section 4.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

   (a) If your work uses existing assets, did you cite the creators? [Yes] They are given in Section 4.

   (b) Did you mention the license of the assets? [Yes] They are given in Section 4.

   (c) Did you include any new assets either in the supplemental material or as a URL? [No] We promise to release our code publicly. We are associated in a industrial organization. The code is currently under review of our organization.

   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [Yes] The data is synthetic in our data.

   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [Yes] The data is synthetic in our data. No personally identifiable information or offensive content is contained.

5. If you used crowdsourcing or conducted research with human subjects...

   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]