Context-Based Dynamic Pricing with Partially Linear Demand Model

Jinzhi Bu [†], **David Simchi-Levi** [‡§], **Chonghuan Wang** [§]
[†] Department of Logistics and Maritime Studies, Faculty of Business,
The Hong Kong Polytechnic University
[‡] Institute for Data, Systems, and Society, Department of Civil and Environmental Engineering,
Operations Research Center, MIT
[§] Laboratory for Information and Decision Systems, MIT
jinzhi.bu@polyu.edu.hk, dslevi@mit.edu, chwang9@mit.edu

Abstract

In today's data-rich environment, context-based dynamic pricing has gained much attention. To model the demand as a function of price and context, the existing literature either adopts a parametric model or a non-parametric model. The former is easier to implement but may suffer from model mis-specification, whereas the latter is more robust but does not leverage many structural properties of the underlying problem. This paper combines these two approaches by studying the context-based dynamic pricing with online learning, where the unknown expected demand admits a semi-parametric partially linear structure. Specifically, we consider two demand models, whose expected demand at price $p \in R^+$ and context $x \in \mathbb{R}^d$ is given by $bp + g(x)$ and $f(p) + a^\top x$ respectively. We assume that $g(x)$ is β-Hölder continuous in the first model, and $f(p)$ is kth-order smooth with an additional parameter δ in the second model. For both models, we design an efficient online learning algorithm with provable regret upper bounds, and establish matching lower bounds. This enables us to characterize the statistical complexity for the two learning models, whose optimal regret rates are $\widetilde{\Theta}(\sqrt{T} \vee T^{\frac{d}{d+2\beta}})$ and $\widetilde{\Theta}(\sqrt{T} \vee (\delta T^{k+1})^{\frac{1}{2k+1}})$ respectively. The numerical results demonstrate that our learning algorithms are more effective than benchmark algorithms, and also reveal the effects of parameters d, β and δ on the algorithm's empirical regret, which are consistent with our theoretical findings.

1 Introduction

The recent success of online retailing has provided a data-rich environment for sellers to take better advantage of contextual information to dynamically improve their pricing strategies. Examples of such contextual information include products' characteristics, seasonality, festival information, economic indicators, etc. A fundamental question stemming from the presence of contexts is how to select a predictive model to capture the relationship among demand, price, and context.

In the current literature, there are two modeling approaches. The first approach is to adopt a parametric model, among which the linear model, i.e., demand = b $\times$ price + $\langle$a, context$\rangle$, is arguably the most fundamental one. The parametric assumption brings great convenience to practical implementation, and inspires many efficient learning algorithms. However, it may also lead to a serious issue of model mis-specification, as discussed in [25]. The second approach is to employ a fully non-parametric model, i.e., demand $= \mu$(price, context) for some unknown function $\mu(\cdot)$ without any parametric assumption. The non-parametric model can be quite robust to different application contexts. However,

36th Conference on Neural Information Processing Systems (NeurIPS 2022).

it does not leverage many structural properties of the underlying problem that may have been detected from huge amounts of historical data, resulting in much complexity for learning the function $\mu(\cdot)$.

Motivated by the idea of combining these two approaches, in this paper, we study context-based dynamic pricing with online learning where the unknown expected demand admits a semi-parametric partially linear structure. Such a structure preserves the advantages of simple interpretability from linear models and great flexibility from non-parametric models. Thus, it has also attracted significant attention in many other fields, especially when there exist high-dimensional contexts, such as statistics ([13]), econometrics ([36]), and biomedicines ([37]).

Formally, consider a seller selling products over a finite selling horizon of T periods. At the beginning of each period t, the seller observes an independently and identically distributed (i.i.d.) context vector $x_t \in [0,1]^d \in \mathbb{R}^d$ encoding the product's characteristics and other exogenous information in period t. The seller then chooses a price p_t from the feasible price range $[\underline{p}, \overline{p}] \subset (0, \infty)$ and customers observe the posted price p_t. The random demand D_t is then generated according to the following function: $D_t(p_t) = \mu(p_t, x_t) + \varepsilon_t$, where $\mu(p_t, x_t)$ is an unknown function and $\{\varepsilon_t\}_{t\geq 1}$ is a sequence of i.i.d. sub-Gaussian random variables (r.v.'s) with zero mean and variance proxy σ^2, i.e., $\mathbb{E}[e^{\lambda \varepsilon_t}] \leq e^{\frac{\lambda^2\sigma^2}{2}}$ for any $\lambda \in \mathbb{R}$. We consider two models for $\mu(p_t, x_t)$:

Dynamic Pricing with Linear Pricing Effect (DPLPE): $\mu(p_t, x_t) = bp_t + g(x_t)$ for unknown price elasticity $b < 0$ and β-Hölder continuous function $g : [0,1]^d \to \mathbb{R}$ with smoothness parameter $0 < \beta \leq 1$ (see Assumption 1);

Dynamic Pricing with Linear Contextual Effect (DPLCE): $\mu(p_t, x_t) = f(p_t) + a^\top x_t$ for unknown parameter $a \in \mathbb{R}^d$ and kth-order smooth function $f : [\underline{p}, \overline{p}] \to \mathbb{R}$ with degree $k \geq 1$ and parameter $\delta > 0$ that capture how tightly f can be approximated by a polynomial function (see Assumption 2).

The seller observes demand D_t and collects revenue $p_t D_t$. The expected revenue at price p and context x_t is denoted by $r(p, x_t) = p\mu(p, x_t)$ for any $p \in [\underline{p}, \overline{p}]$. An *admissible* pricing policy π is defined as a sequence of functions $\{\pi_t(\cdot) : t \geq 1\}$, where each $\pi_t(\cdot)$ maps the historical information $(x_1, p_1, D_1, \ldots, x_{t-1}, p_{t-1}, D_{t-1}, x_t)$ observed up to period t and possibly some external randomness to a feasible price in $[\underline{p}, \overline{p}]$. A clairvoyant optimal policy is assumed to know function $d(\cdot)$ and therefore always chooses the optimal price $p_t^* := \arg\max_{p\in[\underline{p},\overline{p}]} r(p, x_t)$ in each period t. The performance of a pricing policy π is measured by *regret* $R^\pi(T)$, defined as the difference between the T-period expected revenues generated by the clairvoyant optimal policy and by the pricing policy π. That is, $R^\pi(T) = \sum_{t=1}^T \mathbb{E}[r(p_t^*, x_t) - r(p_t, x_t)]$, where the expectation is taken over all the randomness associated with the environment and the algorithm.

1.1 Main results

Our main contributions lie in systematically characterizing the statistical complexity of the context-based dynamic pricing with partially linear demand models DPLPE and DPLCE. Throughout the paper, we use notations $\mathcal{O}(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ to hide constant factors, and use $\widetilde{\mathcal{O}}(\cdot)$, $\widetilde{\Omega}(\cdot)$ and $\widetilde{\Theta}(\cdot)$ to hide both constant and logarithmic factors.

Statistical complexity for DPLPE. For the first model DPLPE, we design an online learning algorithm that integrates the ideas of random shock pricing in [25] and contextual space binning and approximation. We prove that our algorithm achieves the theoretical regret upper bound $\widetilde{\mathcal{O}}(\sqrt{T} \vee T^{\frac{d}{d+2\beta}})$. We also establish the information-theoretic lower bound $\Omega(\sqrt{T} \vee T^{\frac{d}{d+2\beta}})$ for any admissible policy, which matches the regret upper bound up to logarithmic factors. The key novelty in our lower bound analysis is the construction of a family of high-dimensional β-Hölder continuous functions. To the best of our knowledge, this is also the first model that generalizes the commonly used Lipschitz continuous assumption to Hölder continuity in context-based dynamic pricing, with tight characterizations on the regret upper and lower bounds.

Statistical complexity for DPLCE. For the second model DPLCE, we devise a different learning algorithm based on the biased linear contextual bandit borrowed from [32] and our new idea of being more optimistic to chase the context-dependent optimal price. We prove that our algorithm achieves the regret upper bound $\widetilde{\mathcal{O}}(\sqrt{T} \vee (\delta T^{k+1})^{\frac{1}{2k+1}})$, and also establish a matching lower bound $\Omega(\sqrt{T} \vee (\delta T^{k+1})^{\frac{1}{2k+1}})$. Since δ is a parameter that depends on the specific problem instance $f(\cdot)$,

we refer to the regret bounds we derive as the *instance-dependent* bounds. To our knowledge, this is the first tight instance-dependent regret bound in context-based dynamic pricing with online learning. To achieve this, in the regret lower bound analysis, we need to construct a family of kth-order smooth functions that are adapted to parameters k and δ, which is also new to dynamic pricing literature.

1.2 Related work

Non-contextual dynamic pricing. There is a vast literature on dynamic pricing with online demand learning without context. These works can be classified into two categories: parametric models (see, e.g., [7], [18], [38]) and non-parametric models (see, e.g., [4], [5], [33],[32], [23]). We refer the reader to [12] for a comprehensive review on this stream of literature. Among these studies, [32] is the most relevant to our paper. [32] study dynamic pricing without context, and assume the expected demand is kth-order smooth in price for some integer k. They design a learning algorithm by applying the Optimism-in-the-Face-of-Uncertainty (OFU) principle borrowed from bandits, and prove the optimal regret is $\widetilde{\Theta}(T^{\frac{k+1}{2k+1}})$. The idea of algorithmic design in our DPLCE model is inspired by [32] but there are two major differences. First, the optimal price is stationary in [32], but can change over time in our setting. Therefore, the idea of multi-armed bandit protocol proposed in their paper for learning the stationary optimal price does not work. To overcome this difficulty, we propose a novel idea of being more optimistic over OFU in price selection to chase the context-dependent optimal price. Second, our notion of kth-order smoothness is more general than theirs by allowing k not necessarily to be an integer and δ not necessarily to be a constant independent of T. The general values of k and δ require a more careful selection of the parameters used by the algorithm, and a more sophisticated construction for the hard instances in the lower bound analysis.

Context-based dynamic pricing. There is a growing body of literature on context-based dynamic pricing (e.g., [27], [16], [22], [3], [24]). We refer to [3] for a recent review and briefly discuss the works related to this paper. [27] consider the demand model $a^\top x + bp$ in an incumbent-price setting, i.e., the expected demand for some given price is exactly known, and show that the greedy algorithm achieves the regret $\mathcal{O}(\log T)$. [25] study the same demand model as our DPLPE, and prove that the regret of a random price shock (RPS) algorithm compared with the clairvoyant who knows the best *linear* approximation to $g(x)$ is $\mathcal{O}(\sqrt{T})$. We borrow their RPS idea to design the learning algorithm for our DPLPE model, but analyze the regret of our algorithm relative to the optimal policy that knows the *true* demand model. [8] consider a non-parametric demand model in context-based dynamic pricing. For the Lipschitz continuous and locally concave revenue function, they prove that the optimal regret is $\widetilde{\Theta}(T^{\frac{d+2}{d+4}})$. This regret rate is strictly higher than $\widetilde{\Theta}(\sqrt{T} \vee T^{\frac{d}{d+2}})$ in our DPLPE model with $\beta = 1$ due to the separated effects of the price and context on demand we assume.

There is also a stream of works formulating demand using a binary choice model (see, e.g., [16], [21], [10], [30], [34], [14], [35], [20]). Every customer buys the product with probability $1 - F\left(p_t - \theta^\top x_t\right)$, where $F(\cdot)$ is the CDF of random noise. [14], [35] and [20] are the closest to our work. They also consider a semi-parametric form where $F(\cdot)$ is unknown non-parametric and θ is unknown parametric. Conceptually, one can transform our demand model into the one where each customer buys the product with probability $bp_t + g\left(x_t\right)$ (for DPLPE) or $f\left(p_t\right) + a^\top x_t$ (for DPLCE), assuming $bp_t + g\left(x_t\right)$ and $f\left(p_t\right) + a^\top x_t$ fall in $[0, 1]$. Since the modelling approaches are different, we are also facing different challenges. Different from the binary model, we need to estimate b and the non-parametric function $g\left(\cdot\right)$ at the same time for DPLPE (or a and the non-parametric function $f\left(\cdot\right)$ for DPLCE). Despite that we have an additive structure, it is still challenging to decouple bp_t and $g\left(x_t\right)$ for DPLPE (or $f(p_t)$ and $a^\top x_t$ for DPLCE) with only a buying/not-buying feedback. In our model, the distribution of random shock makes no difference to the optimal pricing strategy. Thus, whether the distribution is parametric/known or not is not important in our formulation.

Bandits with contextual information. The most studied model in contextual bandit is the linear model (see, e.g., [2], [11], [29], [9], [1]), where the expected reward is a linear combination of contexts. The algorithms developed in these works are mostly built upon the celebrated idea of the OFU principle. In our DPLCE model, we borrow the OFU idea to design a learning algorithm. There are also a substantial amount of literature considering non-parametric reward feedback under Hölder continuous assumption (see, e.g., [28], [31], [26], [15]). Among these studies, [31] assuming a continuous action space is the most relevant to this work. For the general Lipschitz reward function, [31] proves that the optimal regret is $\widetilde{\Theta}(T^{\frac{d+d_p+1}{d+d_p+2}})$, where d and d_p are the dimensions for the context

space and action space respectively. Letting $d_p = 1$ leads to the optimal regret rate $\widetilde{\Theta}(T^{\frac{d+2}{d+3}})$ for a dynamic pricing problem. This rate is strictly higher than the optimal regret $\widetilde{\Theta}(\sqrt{T} \vee T^{\frac{d}{d+2}})$ in our DPLPE model with $\beta = 1$, and higher than $\widetilde{\Theta}(T^{\frac{2}{3}})$ in our DPLCE with $k = 1$ and $\delta = \Theta(1)$.

Table 1 provides a summary for the main results of this work and the mostly related literature.

Table 1: Summary of the main results in this work and existing results in literature

Paper	Demand model	Regret upper bound	Regret lower bound	Key assumption
Dynamic pricing without context				
[18]	$a + bp$	$\widetilde{\mathcal{O}}(\sqrt{T})$	$\Omega(\sqrt{T})$	Linear demand model
[32]	$f(p)$	$\widetilde{\mathcal{O}}(T^{\frac{k+1}{2k+1}})$	$\Omega(T^{\frac{k+1}{2k+1}})$	f: kth-order smooth
Dynamic pricing with context				
[27]	$a^\top x + bp$	$\mathcal{O}(\log T)$	$\Omega(\log T)$	Known incumbent price
[25]	$bp + g(x)$	$\mathcal{O}(\sqrt{T})$	$\Omega(\sqrt{T})$	Benchmark: the best linear model
[31]	$d(p, x)$	$\mathcal{O}(T^{\frac{d+2}{d+3}})$	$\Omega(T^{\frac{d+2}{d+3}})$	d: Lipschitz continuous
[8]	$d(p, x)$	$\mathcal{O}((\log T)^2 T^{\frac{d+2}{d+4}})$	$\Omega(T^{\frac{d+2}{d+4}})$	d: Lipschitz continuous, smooth and locally concave revenue
[14]	$\mathbb{I}_{\{\theta^\top x + \epsilon \geq p\}}$	$\widetilde{\mathcal{O}}(T^{\frac{2k+1}{4k-1}})$	N.A.	ϵ's CDF: kth-order smooth ($k \geq 2$)
[35]	$\mathbb{I}_{\{\theta^\top x + \epsilon \geq p\}}$	$\widetilde{\mathcal{O}}(T^{\frac{3}{4}})$	$\Omega(T^{\frac{2}{3}})$	No distributional assumptions
This work (Sec. 2 and Sec. 3)	$bp + g(x)$	$\widetilde{\mathcal{O}}(\sqrt{T} \vee T^{\frac{d}{d+2\beta}})$	$\Omega(\sqrt{T} \vee T^{\frac{d}{d+2\beta}})$	g: β-Hölder continuous in $[0,1]^d$,
	$f(p) + a^\top x$	$\widetilde{\mathcal{O}}(\sqrt{T} \vee (\delta T^{k+1})^{\frac{1}{2k+1}})$	$\Omega(\sqrt{T} \vee (\delta T^{k+1})^{\frac{1}{2k+1}})$	f: kth-order smooth with parameter δ

2 Dynamic Pricing with Linear Pricing Effect (DPLPE)

In this section, we study the semi-parametric demand model with linear pricing effect, i.e., $\mu(p_t, x_t) = bp_t + g(x_t)$. We assume that the unknown price elasticity b belongs to some known interval $[\underline{b}, \overline{b}] \subseteq (-\infty, 0)$, and the optimal price $p_t^* = -\frac{g(x_t)}{2b} \in [\underline{p}, \overline{p}]$ for any $b \in [\underline{b}, \overline{b}]$ and $x_t \in [0,1]^d$. We now formally introduce our assumption on function $g(\cdot)$.

Assumption 1 *The function $g : \mathbb{R}^d \to \mathbb{R}$ is β-Hölder continuous for some parameter $0 < \beta \leq 1$, denoted by $g \in \mathcal{G}(\beta, d)$, meaning that there exists a constant $L > 0$, such that for any $x_1, x_2 \in [0,1]^d$, $|g(x_1) - g(x_2)| \leq L \|x_1 - x_2\|^\beta$.*

When $\beta = 1$, $\mathcal{G}(\beta, d)$ is the class of Lipschitz continuous functions, which is usually assumed in dynamic pricing literature (see, e.g., [8]). We consider a more general class of Hölder continuous functions with arbitrary $0 < \beta \leq 1$.

We first propose Algorithm 1 for our DPLPE model. Our algorithm combines two main ideas: (i) context space binning and approximation; and (ii) pricing with random shock.

Context space binning and approximation. To learn function $g(\cdot)$, the β-Hölder continuous assumption guarantees that it cannot change dramatically in a local area. This inspires a natural idea of dividing the context space $[0,1]^d$ into different small bins and using a constant to approximate $g(\cdot)$ in each bin. Specifically, we divide each coordinate of $[0,1]^d$ into M equally sized segments and get M^d small bins. We then estimate $\mathbb{E}[g(x)|x \in \mathbf{M}_j]$ (and $g(x_t)$ for any $x_t \in \mathbf{M}_j$) using the sample average approximation (SAA) from the residuals $\{d_s - p_s \hat{b}_t : 1 \leq s \leq t-1, x_s \in \mathbf{M}_j\}$, where $\hat{b}_t$ is an estimate for b to be discussed in the next paragraph. It's worth noting that there exists an important trade-off for the choice of M. If M is large, the amount of data collected for each bin will be limited, discouraging the success of learning $g(\cdot)$. If M is small, we may incur a larger approximation error of $g(\cdot)$ by using a constant conditional expectation in each bin, leading to a suboptimal pricing strategy and a greater revenue loss. The specific choice of M is given in Theorem 1.

Pricing with random shock. The idea of pricing with random shock is adopted from [25]. In each period t, the algorithm first computes a greedy price p_t^g by plugging in the estimates for $g(x_t)$ and b, and then charges a price p_t by adding a random shock Δ_t, which takes the value of δ_t or $-\delta_t$ with equal probability, to the greedy price p_t^g. Following [25], instead of regressing d_t against the true price p_t, we estimate b by regressing d_t against the price shock Δ_t. Noting that $\frac{\sum_{s=1}^t \Delta_s d_s}{\sum_{s=1}^t \Delta_s^2} = \frac{\sum_{s=1}^t \Delta_s (bp_s + g(x_s) + \varepsilon_s)}{\sum_{s=1}^t \delta_s^2} = b + \frac{\sum_{s=1}^t \Delta_s (bp_s^g + g(x_s) + \varepsilon_s)}{\sum_{s=1}^t \delta_s^2}$. Since Δ_s is independent of

$bp_s^g + g(x_s) + \varepsilon_s$ for each $1 \le s \le t$ and has zero mean, $\frac{\sum_{s=1}^t \Delta_s d_s}{\sum_{s=1}^t \Delta_s^2}$ is an unbiased estimate for b. This also explains the necessity of regressing d_t against the random shock Δ_t.

Algorithm 1: Algorithm for Dynamic Pricing with Linear Price (`ADPLP`)

Input: price range $[\underline{p}, \overline{p}]$, bounds on the price coefficient $\underline{b}$ and $\overline{b}$, number of bins M
Initialization:
Partition each coordinate of $[0,1]^d$ into M segments of equal length, denoted as $\mathbf{M}_j$ for $j = 1, 2, \cdots, M^d$;
Initialize $\mathcal{D}_{1,j} = \emptyset$ for each $j \in [M^d]$ and $\hat{b}_1 = \frac{\underline{b}+\overline{b}}{2}$;
Main Steps:
for $t = 1, 2, \cdots, T$ **do**
  Set $\delta_t \leftarrow t^{-\frac{1}{4}}$;
  Observe x_t and find $j \in [M^d]$ such that $x_t \in \mathbf{M}_j$;
  If $\mathcal{D}_{t,j} = \emptyset$, set $\hat{a}_{t,j} = -\hat{b}_t(\underline{p} + \overline{p})$; otherwise, set $\hat{a}_{t,j} \leftarrow \frac{\sum_{(x_k, p_k, d_k) \in \mathcal{D}_{t,j}} (d_k - p_k \hat{b}_t)}{|\mathcal{D}_{t,j}|}$;
  Set unconstrained greedy price: $p_t^u \leftarrow -\frac{\hat{a}_{t,j}}{2\hat{b}_t}$;
  Project greedy price: $p_t^g \leftarrow \mathbf{Proj}(p_t^u, [\underline{p} + \delta_t, \overline{p} - \delta_t])$;
  Generate an independent random variable $\Delta_t \leftarrow \delta_t$ w.p. $\frac{1}{2}$ and $\Delta_t \leftarrow -\delta_t$;
  Set price $p_t \leftarrow p_t^g + \Delta_t$;
  Observe realized demand d_t;
  Update $\mathcal{D}_{t+1,j} \leftarrow \mathcal{D}_{t,j} \cup \{(x_t, p_t, d_t)\}$ and $\mathcal{D}_{t+1,i} \leftarrow \mathcal{D}_{t,i}$ for $i \neq j$;
  Update $\hat{b}_{t+1} \leftarrow \mathbf{Proj}(\frac{\sum_{s=1}^t \Delta_s d_s}{\sum_{s=1}^t (\Delta_s)^2}, [\underline{b}, \overline{b}])$;
end for

The following theorem shows the regret upper bound of Algorithm 1, whose proof is in Appendix A.

Theorem 1 *Let Algorithm 1 run with $M = \lceil T^{\frac{1}{d+2\beta}} \rceil$. Under Assumption 1 and the known bound $[\underline{b}, \overline{b}]$ of the unknown b, the regret of Algorithm 1 is*

$$\widetilde{\mathcal{O}}\Big(\sqrt{T} \vee T^{\frac{d}{d+2\beta}}\Big).$$

The bound in Theorem 1 consists of two terms $\widetilde{\mathcal{O}}(\sqrt{T})$ and $\widetilde{\mathcal{O}}(T^{\frac{d}{d+2\beta}})$. The first term $\widetilde{\mathcal{O}}(\sqrt{T})$ arises from our introduction of random shocks reflecting the complexity of learning b. For the simplest linear demand model without context, the squared estimation error of b incurred by an asymptotically optimal policy is in the order of $t^{-1/2}$ (see [18]). Therefore, a cumulative regret $\mathcal{O}(\sqrt{T})$ is in general unavoidable. The second term $\widetilde{\mathcal{O}}(T^{\frac{d}{d+2\beta}})$ is incurred by the inaccuracy of approximating the non-parametric function $g(\cdot)$ with a constant in each small bin. It captures the complexity for learning the non-parametric function $g(\cdot)$. This bound is monotonically increasing in dimension d, which agrees with the intuition that a higher dimension of the context space leads to a more challenging task of learning the function $g(\cdot)$. In addition, this bound decreases in β. This is consistent with the fact that if $g(\cdot)$ is smoother (i.e., β is larger), it can be more accurately approximated by a constant in local bins. We also note that only when $d = 1$ and $\beta > \frac{1}{2}$ is $\widetilde{\mathcal{O}}(\sqrt{T})$ larger than $\widetilde{\mathcal{O}}(T^{\frac{d}{d+2\beta}})$. Otherwise, the regret upper bound is always $\widetilde{\mathcal{O}}(T^{\frac{d}{d+2\beta}})$.

We then establish the regret lower bound. We denote the regret of policy π as $R^{\pi}_{g,b,\mathcal{P},\mathcal{D}}(T)$ when the demand function is $bp + g(x) + \varepsilon$, and the distributions of x and ε are $\mathcal{P}$ and $\mathcal{D}$ respectively. We use $\mathcal{E}(\sigma)$ to denote the class of sub-Gaussian distributions with variance proxy σ^2.

Theorem 2 *For any $\mathcal{G}(\beta, d)$, $0 < \underline{b} \le \overline{b}$ and $\sigma \ge 0$, there exists a constant $K_1 > 0$ independent of T, such that for any admissible policy π,*

$$\sup_{\substack{g \in \mathcal{G}(\beta,d), \\ b \in [\underline{b}, \overline{b}], \mathcal{P}, \mathcal{D} \in \mathcal{E}(\sigma)}} R^{\pi}_{g,b,\mathcal{P},\mathcal{D}}(T) \ge K_1 \cdot \left(\sqrt{T} \vee T^{\frac{d}{d+2\beta}}\right).$$

Theorem 2 shows that the regret upper bound achieved by Algorithm 1 in Theorem 1 is unimprovable in terms of the dependency on the learning horizon T. The first lower bound $\Omega(\sqrt{T})$ can be directly obtained from the existing literature, e.g., [17]. To prove the second lower bound $\Omega(T^{\frac{d}{d+2\beta}})$, we construct a series of Hölder continuous functions in $[0,1]^d$ that are difficult to distinguish from each other. We then apply the Bretagnolle–Huber inequality (see [6]) and KL divergence arguments to bound the regret of any algorithm from below. See our construction and detailed proof in Appendix B.

We next compare our result with two relevant papers [25] and [8]. In [25], the same demand model $bp + g(x)$ is considered, but the performance of their learning algorithm is benchmarked with the optimal policy for the "best" linear demand function. In this case, they show that the optimal regret is $\Theta(\sqrt{T})$. By contrast, our regret notion is defined against the optimal policy for the *true* demand function, and thus, our optimal regret has an extra term $\widetilde{\Theta}(T^{\frac{d}{d+2\beta}})$ capturing the complexity of learning the function $g(\cdot)$. In [8], the authors prove the optimal regret $\widetilde{\Theta}(T^{\frac{d+2}{d+4}})$ for general Lipschitz continuous demand functions (in both context and price) assuming local concavity of the revenue function. When $\beta = 1$, the demand function we consider here is a special case of theirs. For this case, our optimal regret becomes $\widetilde{\Theta}(\sqrt{T} \vee T^{\frac{d}{d+2}})$, which is strictly lower than theirs. This reduction of the optimal regret benefits from the separated effects of the price and context assumed in our model. In terms of the algorithmic design, the algorithm in [8] divides both the price space and context space into bins, and treats the learning problem in each small bin as independent ones. In our setting, we leverage all the historical data to estimate the price sensitivity and achieve *information sharing* among different bins in the context space.

3 Dynamic Pricing with Linear Contextual Effect (DPLCE)

In this section, we study the semi-parametric demand model with linear contextual effect, i.e., $\mu(p_t, x_t) = f(p_t) + a^\top x_t$. We assume $||a|| \leq \overline{a}$ for some constant $\overline{a} > 0$. We now introduce our assumption on function $f(\cdot)$.

Assumption 2 *The function $f : [\underline{p}, \overline{p}] \to \mathbb{R}$ is kth-order smooth with degree $k \geq 1$ and parameter δ, denoted by $f \in \mathcal{F}^k([\underline{p}, \overline{p}]; \delta)$, meaning that f is $\mathfrak{b}(k)$-times differentiable on $[\underline{p}, \overline{p}]$ and*

$$\left| f^{(\mathfrak{b}(k))}(p) - f^{(\mathfrak{b}(k))}(p') \right| \leq \delta \left| p - p' \right|^{k - \mathfrak{b}(k)}, \quad \forall p, p' \in [\underline{p}, \overline{p}], \tag{1}$$

where $\mathfrak{b}(k) := \sup\{i \in \mathbb{N} : i < k\}$ is the maximal integer that is strictly less than k.

Eq. (1) requires that the $\mathfrak{b}(k)$th-order derivative of $f(\cdot)$ is $(k - \mathfrak{b}(k))$-Hölder continuous. This assumption also implies that all the ith-order derivatives are bounded for $i \leq \mathfrak{b}(k)$ (i.e., $C_0 := \sup_{0 \leq i \leq \mathfrak{b}(k)} \sup_{p \in [\underline{p}, \overline{p}]} |f^{(i)}(p)| < \infty$). When k is integer, i.e., $\mathfrak{b}(k) = k - 1$, Eq. (1) becomes the Lipschitz-continuous condition and Assumption 2 reduces to Assumption 1 in [32]. Different from [32] that sees δ as a constant, we allow the order of δ to depend on T, e.g., $\delta = \mathcal{O}(T^{-\lambda})$ for some $\lambda \geq 0$. As will be seen later, δ shows how tightly $f(p)$ can be approximated by a polynomial function, and will have impacts on both the regret upper bound and lower bound.

We propose Algorithm 2 for DPLCE. The key ideas are three-fold: (i) local polynomial approximation; (ii) biased linear contextual bandit and OFU; and (iii) optimism over OFU for price selection.

Local polynomial approximation. The idea of local approximation for smooth non-parametric functions through polynomial functions is borrowed from [32]. Specifically, we partition the price interval $[\underline{p}, \overline{p}]$ into N segments of equal size, denoted as $\mathbf{I}_1, \cdots, \mathbf{I}_N$. For each price segment $\mathbf{I}_j := [a_j, b_j)$, we use a polynomial function of degree $\mathfrak{b}(k)$: $P_{\mathbf{I}_j}(p) := \sum_{i=0}^{\mathfrak{b}(k)} \frac{f^{(i)}(a_j)}{i!}(p - a_j)^i$, to locally approximate the true function $f(p)$. To learn the the coefficients of $P_{\mathbf{I}_j}(p)$, we apply the ridge regression method. By our assumption that $f^{(\mathfrak{b}(k))}(p)$ is $(k - \mathfrak{b}(k))$-Hölder continuous, it can be verified from Taylor expansion that $|f(p) - P_{\mathbf{I}_j}(p)| \leq |\frac{(p - a_j)^{\mathfrak{b}(k)}}{\mathfrak{b}(k)!}| \cdot |f^{(\mathfrak{b}(k))}(a_j') - f^{(\mathfrak{b}(k))}(a_j)| \leq \delta \cdot \frac{(\overline{p} - \underline{p})^k}{\mathfrak{b}(k)! \cdot N^k} = \mathcal{O}(\frac{\delta}{N^k})$, where $a_j' \in [a_j, p]$. Therefore, δ reflects how tightly f can be approximated by a local polynomial function of degree $\mathfrak{b}(k)$. Moreover, the number of price segments N needs to be carefully selected to balance the accuracy of local polynomial approximation and the learning efficiency for each price segment. The specific choice of N is given in Theorem 3.

Algorithm 2: Algorithm for Dynamic Pricing with Linear Context (ADPLC)

Input: time horizon T, price range $[\underline{p}, \overline{p}]$, degree k, parameter C_0, context dimension d, upper bound $\bar{a}$, number of price segments N, error control terms Δ, noise variance proxy σ

Initialization:

Partition $[\underline{p}, \overline{p}]$ into N segments of equal length, denoted as $\mathbf{I}_i$ for $i = 1, 2, \cdots, N$;

Initialize for all $i \in [N]$: $\mathcal{D}_i = \emptyset$.

Main Steps:

for $t = 1, 2, \cdots, T$ **do**

  Observe x_t;

  for $i = 1, 2, \cdots, N$ **do**

    $(\hat{\theta}_{t,i}, \hat{a}_{t,i}, \gamma_{t,i}, V_{t,i}) = \texttt{RLC}(\mathfrak{b}(k), d, \mathbf{I}_i, \mathcal{D}_i, \Delta, 1/T^2, C_0, \sigma, \bar{a})$;

    $\hat{r}_{t,i} = \max_{p \in \mathbf{I}_i} p \times \Big(\langle \hat{\theta}_{t,i}, \varphi(p) \rangle + \langle \hat{a}_{t,i}, x_t \rangle + \gamma_{t,i} \sqrt{\phi(p, x_t)^\top V_{t,i}^{-1} \phi(p, x_t)} + \Delta \Big)$;

    $\hat{p}_{t,i} = \arg\max_{p \in \mathbf{I}_i} p \times \Big(\langle \hat{\theta}_{t,i}, \varphi(p) \rangle + \langle \hat{a}_{t,i}, x_t \rangle + \gamma_{t,i} \sqrt{\phi(p, x_t)^\top V_{t,i}^{-1} \phi(p, x_t)} + \Delta \Big)$;

  end for

  Select $i_t = \arg\max_{i \leq N-1} \hat{r}_{t,i}$ and charge $p_t = \hat{p}_{t,i_t}$;

  Observe realized demand d_t;

  Update $\mathcal{D}_{i_t} \leftarrow \mathcal{D}_{i_t} \cup \{(x_t, p_t, d_t)\}$;

end for

Algorithm 3: Regression with Linear Context (RLC)

Input: polynomial degree k, context dimension d, domain $\mathbf{I} = [l, u]$, history $\mathcal{D}$, bias Δ, probability ϵ, smoothness parameter C, noise variance proxy σ, upper bound $\bar{a}$

Compute $\lambda = \frac{(u-l)^{2k}-1}{(u-l)^2-1} + d$;

Compute $\gamma = \sigma \sqrt{(d+k) \ln \left(\frac{d+k+|\mathcal{D}|}{d+k} \right) - 2 \ln \epsilon} + \lambda^{\frac{1}{2}} (C^2 k + \bar{a}^2)^{\frac{1}{2}} + \Delta \sqrt{|\mathcal{D}|}$;

Compute $V = \lambda I_{(k+d) \times (k+d)} + \sum_{(x,p,d) \in \mathcal{D}} \phi(p, x) \phi(p, x)^\top$, where $\phi(p, x) = (\varphi(p)^\top, x^\top)^\top$ and $\varphi(p) = (1, (p-l), \cdots, (p-l)^{k-1})^\top$;

Compute Ridge estimate

$(\hat{\theta}, \hat{a}) = \arg\min \sum_{(x,p,d) \in \mathcal{D}} (d - \langle \theta, \varphi(p) \rangle - \langle a, x \rangle)^2 + \lambda (\|\theta\|_2^2 + \|a\|_2^2)$;

Output: $\hat{\theta}, \hat{a}, \gamma, V$

Biased linear contextual bandit and OFU. To suggest a good price for each segment, we borrow the biased linear contextual bandit from [32]. Specifically, for each $t \geq 1$ and $p_t \in \mathbf{I}_j$, we can rewrite $D_t(p_t)$ as follows: $D_t(p_t) = P_{\mathbf{I}_j}(p_t) + a^\top x_t + \beta_t = \theta_j^\top \varphi(p_t) + a^\top x_t + \beta_t$, where $\beta_t := f(p_t) - P_{\mathbf{I}_j}(p_t) + \varepsilon_t$, $\varphi(p_t) := (1, (p_t - a_j), \cdots, (p_t - a_j)^{\mathfrak{b}(k)})$ and $\theta_j \in \mathbb{R}^{\mathfrak{b}(k)+1}$ whose ith element is $\frac{f^{(i-1)}(a_j)}{(i-1)!}$ for $1 \leq i \leq \mathfrak{b}(k) + 1$. This model is similar to the linear contextual bandit in the literature (see, e.g., [1], [9]), where the OFU idea that selects actions in an optimistic way is commonly adopted. The key difference is that β_t contains a biased term that is not mean-zero and depends on the pricing decision. We adapt the idea of adding an additional term Δ when implementing the OFU principle to compute an optimistic price and optimistic revenue from [32].

Optimism over OFU for price selection. In the absence of context, the optimal price in [32] is *fixed*. To learn this fixed optimal price, [32] propose the idea of embedding the biased linear contextual bandit into a multi-armed bandit protocol that treats each price segment as an arm. The segment to which the fixed optimal price belongs is regarded as the "best" arm. By contrast, in our context-based setting, the optimal price depends on the random context revealed in each period and changes over time. Thus, our "best" arm in each period also changes, and the idea of multi-armed bandit protocol in [32] does not work. To overcome this challenge, we propose a new idea of optimism over OFU, which proceeds as follows. We first implement the OFU principle to each price segment $\mathbf{I}_i$ and compute an optimistic price $\hat{p}_{t,i}$ and optimistic revenue $\hat{r}_{t,i}$. Then we choose the most optimistic price from the N candidate prices $\hat{p}_{t,1}, \ldots, \hat{p}_{t,N}$ that achieves the highest optimistic revenue.

The following theorem presents the regret upper bound for Algorithm 2, whose proof is in Appendix C.

Theorem 3 *Let Algorithm 2 run with* $N = \lceil \delta^{\frac{2}{2k+1}} T^{\frac{1}{2k+1}} \rceil + 1$ *and* $\Delta = \delta(\overline{p} - \underline{p})^k / N^k$. *Under Assumption 2 and* $||a|| \leq \overline{a}$, *the regret of Algorithm 2 is*

$$\widetilde{\mathcal{O}}\left(d\big(\sqrt{T} \vee (\delta T^{k+1})^{\frac{1}{2k+1}}\big)\right).$$

First, when δ is a constant and k is an integer, the regret upper bound becomes $\widetilde{\mathcal{O}}(dT^{\frac{k+1}{2k+1}})$, recovering Theorem 1 in [32] for the setting without contexts (i.e., $d = 1$ and x is a constant). As k becomes larger while keeping δ a constant, the upper bound decreases according to $\widetilde{\mathcal{O}}(dT^{\frac{k+1}{2k+1}})$. In the extreme case of $k = \infty$, the upper bound becomes $\widetilde{\mathcal{O}}(d\sqrt{T})$, which coincides with the optimal regret rate for the linear model $d(p, x) = bp + a^\top x$ (see, e.g., [3])[1]. Second, when δ is not a constant whose order can shrink as T increases, Theorem 3 shows an *instance-dependent* bound which further differentiates our result from that in [32]. We can see that when $\delta = \Omega(T^{-\frac{1}{2}})$, the regret bound is $\widetilde{\mathcal{O}}(\delta^{\frac{1}{2k+1}} T^{\frac{k+1}{2k+1}})$, and when δ decreases to $\mathcal{O}(T^{-\frac{1}{2}})$, the regret bound is always $\widetilde{\mathcal{O}}(\sqrt{T})$. This shows that the magnitude of δ greatly affects the regret upper bound, and the smaller δ is, the lower the regret bound will be. This is because when δ becomes smaller, the approximation error using a local polynomial function $\mathcal{O}(\frac{\delta}{N^k})$ becomes smaller, leading to more accurate estimates for $f(\cdot)$. Since the polynomial function is differentiable infinitely many times, its regret is $\widetilde{\mathcal{O}}(\sqrt{T})$ from Section 3.4.1 in [32]. Therefore, as δ decreases, the regret upper bound decays to $\widetilde{\mathcal{O}}(\sqrt{T})$, which is attained when $\delta = \mathcal{O}(T^{-\frac{1}{2}})$. To prove Theorem 3, by the idea of optimism over OFU, we can upper bound the regret by the length of the confidence interval which can be controlled by the well-known concentration inequality in [1].

We then establish the regret lower bound. Similar to Theorem 1, we denote the regret of policy π by $R^{\pi}_{f,a,\mathcal{P},\mathcal{D}}(T, k, \delta)$ when the demand function is $f(p) + a^\top x + \varepsilon$, where $f \in \mathcal{F}^k([\underline{p}, \overline{p}; \delta])$, and the distributions of x and ε are $\mathcal{P}$ and $\mathcal{D}$ respectively.

Theorem 4 *For any* $\mathcal{F}^k([\underline{p}, \overline{p}; \delta)$, $\bar{a} \geq 0$, *and* $\sigma \geq 0$, *there exists a constant* $K_2 > 0$ *independent of* T *and* δ, *such that for any admissible policy* π,

$$\sup_{\substack{f \in \mathcal{F}^k([\underline{p}, \overline{p}]; \delta), \\ \|a\| \leq \bar{a}, \mathcal{P}, \mathcal{D} \in \mathcal{E}(\sigma)}} R^{\pi}_{f,a,\mathcal{P},\mathcal{D}}(T, k, \delta) \geq K_2 \cdot \left(\sqrt{T} \vee \left(\delta T^{k+1}\right)^{\frac{1}{2k+1}}\right). \tag{2}$$

Theorem 4 shows that the regret upper bound achieved by Algorithm 2 in Theorem 3 is unimprovable in terms of its dependency on the learning horizon T and parameter δ. Note that the upper bound in Theorem 3 grows linearly in d, but the lower bound in Theorem 4 is independent of d. This is because in the lower bound analysis, we simply consider $a = \mathbf{0}$. We leave the problem of analyzing the more complicated dimension-dependent lower bound as future research. The first lower bound $\Omega(\sqrt{T})$ in Eq. (2) directly follows from the existing literature (e.g., [17]). The proof of the second lower bound $\Omega((\delta T^{k+1})^{\frac{1}{2k+1}})$ relies on constructing a series of kth-order smooth functions with parameter δ and applying Pinsker's inequality to bound the total variation of two probability measures through their KL divergence. See our construction and detailed proof in Appendix D.

4 Numerical Study

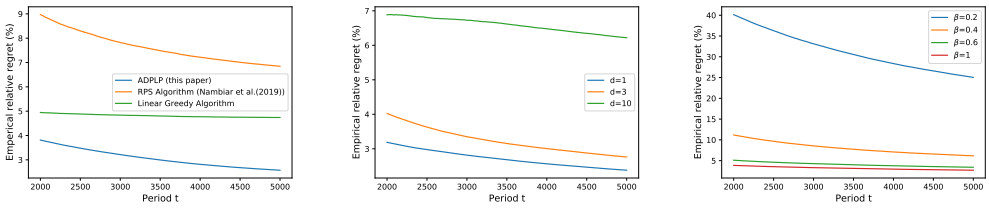
In this section, we conduct a numerical study to test the empirical performances of our algorithms. We measure the performance of a learning algorithm π by the relative regret defined as follows: $\frac{\sum_{t=1}^{T} \mathbb{E}[p_t^* d(p_t^*, x_t) - p_t d(p_t, x_t)]}{\sum_{t=1}^{T} \mathbb{E}[p_t^* d(p_t^*, x_t)]} \times 100\%$. For both models, we compare our algorithms with the linear greedy algorithm that estimates the demand function by a linear function and myopically selects the optimal price that maximizes the proxy revenue in each period t. For our DPLPE model, we also compare our algorithm with the random price shock (RPS) algorithm proposed by [25]. For each instance, we repeat our experiments for 50 independent runs, and approximate the relative regret by the empirical relative regret averaged over 50 runs.

[1]Note that the linear model is a special case of DPLCE by letting $k = \infty$.

Numerical results for DPLPE model. For our DPLPE model with demand function $d(p, x_t) = bp + g(x_t)$, we set $b = -5$, $\varepsilon \sim \mathcal{N}(0, 5)$, $[\underline{b}, \overline{b}] = [-8, -3]$, $[\underline{p}, \overline{p}] = [1, 19]$, $x_t \sim \text{Uniform}([0, 1]^d)$, and consider the following form of function $g(x)$:

$$g(x) := \begin{cases} 100 \times (D(x, \partial[0,1]^d))^\beta + 100 & \text{if } D(x, \partial[0,1]^d) \leq \frac{1}{4}, \\ 100 \times (\frac{1}{4})^\beta - 300 \times (D(x, \partial[0,1]^d) - \frac{1}{4})^\beta + 100 & \text{otherwise,} \end{cases}$$

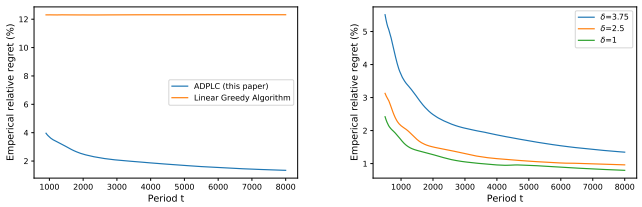
where $D(x, \partial[0, 1]^d)$ denotes the Euclidean distance between x and the boundary of $[0, 1]^d$. In Figure 1(a), we compare the empirical relative regret of our ADPLP with those of RPS algorithm and the linear greedy algorithm. The figure shows that our algorithm performs much better than both algorithms. In Figure 1(b), we investigate the effect of d on the empirical relative regret of ADPLP by varying d in the set $\{1, 3, 10\}$ while keeping $\beta = 1$. As shown in the figure, as d decreases, the regret decreases, which is consistent with our theoretical results. In Figure 1(c), we test the effect of β on the empirical relative regret of ADPLP by varying β in the set $\{0.2, 0.4, 0.6, 1\}$ while keeping $d = 1$. When β is relatively small, a slight increment of β, e.g., $\beta = 0.2$ to $\beta = 0.4$, will lead to significant speed-up of the convergence. The results of $\beta = 0.6$ and $\beta = 1$ are very close, which indicates that when β is relatively large, increasing β may only bring limited improvement.



(a) Regret under $d = 2$, $\beta = 1$ (b) Effect of d on ADPLP (c) Effect of β on ADPLP

Figure 1: Empirical relative regret for DPLPE model

Numerical results for DPLCE model. For our DPLCE model, we consider the following demand: $D_t(p_t) = -\frac{4}{15}\delta p_t^{2.5} + 30 + \frac{1}{d}\mathbf{1}_d \cdot x_t + \varepsilon_t$, where $\mathbf{1}_d$ denotes $(1, 1, \cdots, 1) \in \mathbb{R}^d$, x_t is uniformly distributed on $[0, 1]^d$ independently and $\varepsilon_t \sim \mathcal{N}(0, 0.1^2)$. It can be verified that $k = 2.5$ in this case.In Figure 2(a), we fix $d = 2$, $\delta = 3.75$ and $[\underline{p}, \overline{p}] = [2.6, 3.8]$, and compare the empirical relative regret of our ADPLC with that of the linear greedy algorithm. The linear greedy algorithm has almost a constant empirical relative regret, showing that it fails to identify the optimal solution. By contrast, the empirical relative regret of ADPLC decreases as t increases. We also test the effect of δ on the empirical relative regret of ADPLC by varying $\beta \in \{3.75, 2.5, 1\}$. As shown in Figure 2(b), as δ becomes smaller, the empirical relative regret decreases, which is consistent with our theoretic results.



(a) Regret under $\delta = 3.75$ (b) Effect of δ on ADPLC

Figure 2: Empirical relative regret for DPLCE model

5 Conclusion

In this paper, we study the context-based dynamic pricing with online learning, where the unknown expected demand admits a partially linear semi-parametric structure. For both models DPLPE and DPLCE, we develop an efficient learning algorithm and characterize the optimal regret by proving matching upper and lower bounds. We note that our algorithms need to have the preliminary knowledge about the smoothness parameters β in DPLPE model and k and δ in DPLCE model.

One important direction for future research is to explore whether parameter-free algorithms can be developed and still achieve the optimal regret rate for both models.

References

[1] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In *Advances in Neural Information Processing Systems*, pages 2312–2320, 2011.

[2] Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.

[3] Gah-Yi Ban and N Bora Keskin. Personalized dynamic pricing with machine learning: High-dimensional features and heterogeneous elasticity. *Management Science*, 67(9):5549–5568, 2021.

[4] Omar Besbes and Assaf Zeevi. Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms. *Operations Research*, 57(6):1407–1420, 2009.

[5] Omar Besbes and Assaf Zeevi. Blind network revenue management. *Operations research*, 60(6):1537–1550, 2012.

[6] J Bretagnolle and C Huber. Estimation des densités: risque minimax. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 47(2):119–137, 1979.

[7] Josef Broder and Paat Rusmevichientong. Dynamic pricing under a general parametric choice model. *Operations Research*, 60(4):965–980, 2012.

[8] Ningyuan Chen and Guillermo Gallego. Nonparametric pricing analytics with customer covariates. *Operations Research*, 69(3):974–984, 2021.

[9] Wei Chu, Lihong Li, Lev Reyzin, and Robert Schapire. Contextual bandits with linear payoff functions. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics*, pages 208–214. JMLR Workshop and Conference Proceedings, 2011.

[10] Maxime C Cohen, Ilan Lobel, and Renato Paes Leme. Feature-based dynamic pricing. *Management Science*, 66(11):4921–4943, 2020.

[11] Varsha Dani, Thomas P Hayes, and Sham M Kakade. Stochastic linear optimization under bandit feedback. In *Proceedings of the 21st Conference on Learning Theory*, 2008.

[12] Arnoud V den Boer. Dynamic pricing and learning: historical origins, current research, and new directions. *Surveys in operations research and management science*, 20(1):1–18, 2015.

[13] Robert F Engle, Clive WJ Granger, John Rice, and Andrew Weiss. Semiparametric estimates of the relation between weather and electricity sales. *Journal of the American statistical Association*, 81(394):310–320, 1986.

[14] Jianqing Fan, Yongyi Guo, and Mengxin Yu. Policy optimization using semiparametric models for dynamic pricing. *Available at SSRN 3922825*, 2021.

[15] Yichun Hu, Nathan Kallus, and Xiaojie Mao. Smooth contextual bandits: Bridging the parametric and non-differentiable regret regimes. In *Conference on Learning Theory*, pages 2007–2010. PMLR, 2020.

[16] Adel Javanmard and Hamid Nazerzadeh. Dynamic pricing in high-dimensions. *The Journal of Machine Learning Research*, 20(1):315–363, 2019.

[17] N Bora Keskin. Optimal dynamic pricing with demand model uncertainty: A squared-coefficient-of-variation rule for learning and earning. *Available at SSRN 2487364*, 2014.

[18] N.B. Keskin and A. Zeevi. Dynamic pricing with an unknown demand model: Asymptotically optimal semi-myopic policies. *Operations Research*, 62(5):1142–1167, 2014.

[19] David A Levin and Yuval Peres. *Markov chains and mixing times*, volume 107. American Mathematical Soc., 2017.

[20] Yiyun Luo, Will Wei Sun, et al. Distribution-free contextual dynamic pricing. *arXiv preprint arXiv:2109.07340*, 2021.

[21] Jieming Mao, Renato Leme, and Jon Schneider. Contextual pricing for lipschitz buyers. *Advances in Neural Information Processing Systems*, 31, 2018.

[22] Sentao Miao, Xi Chen, Xiuli Chao, Jiaxi Liu, and Yidong Zhang. Context-based dynamic pricing with online clustering. *arXiv preprint arXiv:1902.06199*, 2019.

[23] Sentao Miao and Yining Wang. Network revenue management with nonparametric demand learning:\sqrt {T}-regret and polynomial dimension dependency. *Available at SSRN 3948140*, 2021.

[24] Jonas W Mueller, Vasilis Syrgkanis, and Matt Taddy. Low-rank bandit methods for high-dimensional dynamic pricing. *Advances in Neural Information Processing Systems*, 32, 2019.

[25] Mila Nambiar, David Simchi-Levi, and He Wang. Dynamic learning and pricing with model misspecification. *Management Science*, 65(11):4980–5000, 2019.

[26] Vianney Perchet and Philippe Rigollet. The multi-armed bandit problem with covariates. *The Annals of Statistics*, 41(2):693–721, 2013.

[27] Sheng Qiang and Mohsen Bayati. Dynamic pricing with demand covariates. *Available at SSRN 2765257*, 2016.

[28] Philippe Rigollet and Assaf Zeevi. Nonparametric bandits with covariates. *arXiv preprint arXiv:1003.1630*, 2010.

[29] Paat Rusmevichientong and John N Tsitsiklis. Linearly parameterized bandits. *Mathematics of Operations Research*, 35(2):395–411, 2010.

[30] Virag Shah, Ramesh Johari, and Jose Blanchet. Semi-parametric dynamic contextual pricing. *Advances in Neural Information Processing Systems*, 32, 2019.

[31] Aleksandrs Slivkins. Contextual bandits with similarity information. In *Proceedings of the 24th annual Conference On Learning Theory*, pages 679–702. JMLR Workshop and Conference Proceedings, 2011.

[32] Yining Wang, Boxiao Chen, and David Simchi-Levi. Multimodal dynamic pricing. *Management Science*, 67(10):6136–6152, 2021.

[33] Zizhuo Wang, Shiming Deng, and Yinyu Ye. Close the gaps: A learning-while-doing algorithm for single-product revenue management problems. *Operations Research*, 62(2):318–331, 2014.

[34] Jianyu Xu and Yu-Xiang Wang. Logarithmic regret in feature-based dynamic pricing. *Advances in Neural Information Processing Systems*, 34, 2021.

[35] Jianyu Xu and Yu-Xiang Wang. Towards agnostic feature-based dynamic pricing: Linear policies vs linear valuation with unknown noise. In *International Conference on Artificial Intelligence and Statistics*, pages 9643–9662. PMLR, 2022.

[36] Adonis Yatchew and Joungyeo Angela No. Household gasoline demand in canada. *Econometrica*, 69(6):1697–1709, 2001.

[37] Scott L Zeger and Peter J Diggle. Semiparametric models for longitudinal data with application to cd4 cell numbers in hiv seroconverters. *Biometrics*, pages 689–699, 1994.

[38] Feng Zhu and Zeyu Zheng. When demands evolve larger and noisier: Learning and earning in a growing environment. In *International Conference on Machine Learning*, pages 11629–11638. PMLR, 2020.

Checklist

The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. For example:

- Did you include the license to the code and datasets? [Yes] See Section 0.
- Did you include the license to the code and datasets? [No] The code and the data are proprietary.
- Did you include the license to the code and datasets? [N/A]

Please do not modify the questions and only use the provided macros for your answers. Note that the Checklist section does not count towards the page limit. In your paper, please delete this instructions block and only keep the Checklist section heading above along with the questions/answers below.

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See the last paragraph of Section 3 and Section 5.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Sections 2 and 3.
   (b) Did you include complete proofs of all theoretical results? [Yes] See Appendix.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See Section 4.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 4.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

Appendix A. Proof of Theorem 1

Here we provide a proof for Theorem 1. For each $t \in [T]$, let j_t be the index of the bin that x_t belongs. Conditional on $j_1, j_2, \ldots, j_T$, the total regret can be decomposed as follows:

$$
\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}[r_t | j_1, j_2, \ldots, j_T] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\max_{p \in [\underline{p}, \overline{p}]} p(bp + g(x_t)) - p_t(bp_t + g(x_t)) \big| j_1, j_2, \ldots, j_t \right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[-\frac{g(x_t)}{2b} \left(b \cdot \big(-\frac{g(x_t)}{2b} \big) + g(x_t) \right) - p_t \left(bp_t + g(x_t) \right) \big| j_1, j_2, \ldots, j_t \right] \\
&= |b| \sum_{t=1}^{T} \mathbb{E}\left[\left(-\frac{g(x_t)}{2b} - p_t \right)^2 \Big| j_1, j_2, \ldots, j_t \right] \\
&\leq 2|b| \sum_{t=1}^{T} \mathbb{E}\left[\left(-\frac{g(x_t)}{2b} - p_t^0 \right)^2 + \left(p_t^0 - p_t \right)^2 \Big| j_1, j_2, \ldots, j_t \right] \\
&\leq 2|b| \sum_{t=1}^{T} \mathbb{E}\left[\left(-\frac{g(x_t)}{2b} - p_t^u \right)^2 + \left(p_t^0 - p_t \right)^2 \Big| j_1, j_2, \ldots, j_t \right] \\
&\leq 6|b| \sum_{t=1}^{T} \mathbb{E}\Bigg[\underbrace{\Big(\frac{-g(x_t) + \mathbb{E}[g(x)|x \in \mathbf{M}_{j_t}]}{2b} \Big)^2}_{\text{regret from discretization error of } g(\cdot)} + \underbrace{\Big(\frac{\mathbb{E}[g(x)|x \in \mathbf{M}_{j_t}]}{2b} - \frac{\mathbb{E}[g(x)|x \in \mathbf{M}_{j_t}]}{2\hat{b}_t} \Big)^2}_{\text{regret from estimation error of } b} \\
&\qquad + \underbrace{\Big(\frac{\mathbb{E}[g(x)|x \in \mathbf{M}_{j_t}] - \hat{a}_{t,j_t}}{2\hat{b}_t} \Big)^2}_{\text{regret from SAA}} + \underbrace{\left(p_t^0 - p_t \right)^2}_{\text{regret from random shock}} \Bigg| j_1, j_2, \ldots, j_t \Bigg], \qquad (3)
\end{aligned}
$$

where $p_t^0 := \text{Proj}(p_t^u, [\underline{p}, \overline{p}])$, the first identity holds since x_t and p_t are independent of $j_{t+1}, \ldots, j_T$, the first and the third inequalities follow from Cauchy-Schwarz inequality, and the second inequality is due to $-\frac{g(x_t)}{2b} \in [\underline{p}, \overline{p}]$.

The first term on the RHS of (3) arises from approximating $g(\cdot)$ using a constant in each local bin, and can be further upper bounded as follows due to Assumption 1:

$$
\mathbb{E}\left[\Big(\frac{-g(x_t) + \mathbb{E}[g(x)|x \in \mathbf{M}_j]}{2b} \Big)^2 \Big| j_1, \ldots, j_t \right] \leq \frac{\max_{x,y \in \mathbf{M}_{j_t}} (g(x) - g(y))^2}{4\overline{b}^2} \leq \frac{L^2 d^\beta}{4\overline{b}^2 M^{2\beta}}. \qquad (4)
$$

The second term on the RHS of (3) is due to the estimation error of price sensitivity b because

$$
\begin{aligned}
&\mathbb{E}\left[\Big(\frac{\mathbb{E}[g(x)|x \in \mathbf{M}_j]}{2b} - \frac{\mathbb{E}[g(x)|x \in \mathbf{M}_j]}{2\hat{b}_t} \Big)^2 \Big| j_1, \ldots, j_t \right] \\
&\leq \frac{\max_{x \in [0,1]^d} (g(x))^2}{4\overline{b}^4} \mathbb{E}[(\hat{b}_t - b)^2 | j_1, \ldots, j_t]. \qquad (5)
\end{aligned}
$$

The third term on the RHS of (3) represents the estimation error of $\mathbb{E}[g(x)|x \in \mathbf{M}_j]$ using the SAA method. We establish the following equation for this term whose proof is deferred to the end of this section:

$$
\begin{aligned}
\mathbb{E}\left[\Big(\frac{\mathbb{E}[g(x)|x \in \mathbf{M}_{j_t}] - \hat{a}_{t,j_t}}{2\hat{b}_t} \Big)^2 \Big| j_1, \ldots, j_t \right] &= 1_{\{|\mathcal{D}_{t,j_t}| \geq 1\}} \times \mathcal{O}\left(\mathbb{E}\left[(b - \hat{b}_t)^2 \big| j_1, \ldots, j_t \right] + \frac{1}{|\mathcal{D}_{t,j_t}|} \right) \\
&\quad + 1_{\{|\mathcal{D}_{t,j_t}| = 0\}} \times \mathcal{O}(1). \qquad (6)
\end{aligned}
$$

The last term on the RHS of (3) comes from the regret of the random shock added to the greedy policy for exploration, and can be bounded by $\mathcal{O}(t^{-1/2})$:

$$
\mathbb{E}[\left(p_t^0 - p_t \right)^2 | j_1, \ldots, j_T] \leq 2\mathbb{E}[(p_t^0 - p_t^g)^2 + (p_t^g - p_t)^2 | j_1, \ldots, j_T] \leq 4\delta_t^2 = \frac{4}{\sqrt{t}}. \qquad (7)
$$

As seen from Eq. (5) and Eq. (6), the regret bound depends on the estimation error of b. To proceed, we establish the following upper bound on the estimation error of b, whose proof can be checked at the end of this section:

$$\mathbb{E}[(b-\hat{b}_t)^2|j_1,\ldots,j_t] = \mathcal{O}\big(\frac{1}{\sqrt{t}}\big). \tag{8}$$

Putting Eq. (3) to Eq. (8) together and after some calculation, we can then establish the following upper bound on the total expected regret, whose detailed proof is also provided later in this section:

$$\sum_{t=1}^{T}\mathbb{E}[r_t] = \mathcal{O}\Big(\frac{T}{M^{2\beta}}\Big) + \mathcal{O}(\sqrt{T}) + \mathcal{O}(M^d \ln T). \tag{9}$$

By setting $M = \lceil T^{\frac{1}{d+2\beta}} \rceil$, we obtain the upper bound $\widetilde{\mathcal{O}}(\sqrt{T} + T^{\frac{d}{d+2\beta}}) = \widetilde{\mathcal{O}}(\sqrt{T} \vee T^{\frac{d}{d+2\beta}})$ in Theorem 1.

Now, it suffices to prove Eqs. (6), (8) and (9).

Proof of Eq. (6). Note that for each $t \geq 1$, when $|\mathcal{D}_{t,j_t}| \geq 1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\big(\hat{a}_{t,j_t} - \mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)^2\big|j_1,\ldots,j_t\right] \\
&= \mathbb{E}\left[\Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}}\Big(d_k - p_k\hat{b}_t\Big) - \mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\Big)^2\Big|j_1,\ldots,j_t\right] \\
&= \mathbb{E}\left[\Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}}\Big(bp_k + g(x_k) + \varepsilon_k - p_k\hat{b}_t\Big) - \mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\Big)^2\Big|j_1,\ldots,j_t\right] \\
&\leq 3\mathbb{E}\Bigg[\Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}} p_k(b-\hat{b}_t)\Big)^2 + \Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}}\big(g(x_k) - \mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)\Big)^2 \\
&\qquad + \Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}}\varepsilon_k\Big)^2\Big|j_1,\ldots,j_t\Bigg] \\
&\leq 3\mathbb{E}\left[\overline{p}^2(b-\hat{b}_t)^2 + \Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}}\big(g(x_k) - \mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)\Big)^2 + \frac{\sigma^2}{|\mathcal{D}_{t,j_t}|}\Big|j_1,\ldots,j_t\right],
\end{aligned} \tag{10}$$

where the first identity follows from the definition of $\hat{a}_{t,j_t}$ and the inequality follows from Cauchy-Schwaz inequality.

We then bound the second term on the RHS of (10). Denoting the time indices when the feature vector falls into $\mathbf{M}_{j_t}$ by $1 \le s_1 < s_2 < \ldots < s_{|\mathcal{D}_{t,j_t}|} \le t-1$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{(x_k,p_k,d_k)\in\mathcal{D}_{t,j_t}}\big(g(x_k)-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)\Big)^2\Big|j_1,\ldots,j_t\Big]\\
&=\mathbb{E}\Big[\Big(\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{k=1}^{|\mathcal{D}_{t,j_t}|}\big(g(x_{s_k})-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)\Big)^2\Big|x_{s_k}\in\mathbf{M}_{j_t},\forall 1\le k\le|\mathcal{D}_{t,j_t}|\Big]\\
&=\mathrm{Var}\Big[\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{k=1}^{|\mathcal{D}_{t,j_t}|}\big(g(x_{s_k})-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)\Big|x_{s_k}\in\mathbf{M}_{j_t},\forall 1\le k\le|\mathcal{D}_{t,j_t}|\Big]\\
&=\frac{1}{|\mathcal{D}_{t,j_t}|^2}\sum_{k=1}^{|\mathcal{D}_{t,j_t}|}\mathrm{Var}\Big[\big(g(x_k)-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)\Big|x_{s_k}\in\mathbf{M}_{j_t}\Big]\\
&=\frac{1}{|\mathcal{D}_{t,j_t}|}\mathrm{Var}\big[g(y)-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big|y\in\mathbf{M}_{j_t}\big]\\
&\le\frac{1}{4\,|\mathcal{D}_{t,j_t}|}\max_{x,y\in\mathbf{M}_{j_t}}(g(x)-g(y))^2\\
&\le\frac{1}{4\,|\mathcal{D}_{t,j_t}|}\max_{x,y\in[0,1]^d}(g(x)-g(y))^2,
\end{aligned}
\tag{11}
$$

where the second identity holds since conditional on $x_{s_k}\in\mathbf{M}_{j_t}$ for each $1\le k\le|\mathcal{D}_{t,j_t}|$, $\frac{1}{|\mathcal{D}_{t,j_t}|}\sum_{k=1}^{|\mathcal{D}_{t,j_t}|}\big(g(x_{s_k})-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]\big)$ is a mean zero random variable, the fourth identity holds since $x_{s_1},x_{s_2},\ldots,x_{s_{|\mathcal{D}_{t,j_t}|}}$ are i.i.d. random variables, and the first inequality follows from Hoeffding's lemma.

Therefore, we have the following upper bound on the third term of (3):

$$
\begin{aligned}
&\mathbb{E}\left[\Big(\frac{\hat{a}_{t,j_t}-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]}{2\hat{b}_t}\Big)^2\Big|j_1,\ldots,j_t\right]\\
&=\mathbb{E}\left[1_{\{|\mathcal{D}_{t,j_t}|\ge1\}}\times\Big(\frac{\hat{a}_{t,j_t}-\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]}{2\hat{b}_t}\Big)^2\Big|j_1,\ldots,j_t\right]+\mathcal{O}(1)\times1_{\{|\mathcal{D}_{t,j_t}|=0\}}\\
&\le1_{\{|\mathcal{D}_{t,j_t}|\ge1\}}\frac{3}{4\overline{b}^2}\mathbb{E}\left[\overline{p}^2(b-\hat{b}_t)^2+\frac{\frac{1}{4}\max_{x,y\in[0,1]^d}(g(x)-g(y))^2+\sigma^2}{|\mathcal{D}_{t,j_t}|}\Big|j_1,\ldots,j_t\right]\\
&\quad+1_{\{|\mathcal{D}_{t,j_t}|=0\}}\times\mathcal{O}(1),
\end{aligned}
\tag{12}
$$

where the first identity holds since if $|\mathcal{D}_{t,j_t}|=0$, a constant loss is incurred since $\hat{a}_{t,j_t}$, $\hat{b}_t$ and $\mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]$ are all bounded, and the inequality follows from (10) and (11) for $|\mathcal{D}_{t,j_t}|\ge1$. This completes the proof of Eq. (6).

Proof of Eq. (8). We next bound $\mathbb{E}[(b-\hat{b}_t)^2|j_1,\ldots,j_t]$ and prove Eq. (8). When $t \geq 2$, we have

$$\begin{aligned}
\mathbb{E}[(b-\hat{b}_t)^2|j_1,\ldots,j_t] &= \mathbb{E}\left[\Big(\frac{\sum_{s=1}^t \Delta_s(bp_s^g + g(x_s)+\varepsilon_s)}{\sum_{s=1}^t \delta_s^2}\Big)^2\Big|j_1,\ldots,j_t\right] \\
&= \frac{1}{(\sum_{s=1}^t s^{-\frac{1}{2}})^2}\mathbb{E}\left[\Big(\sum_{s=1}^t \Delta_s(bp_s^g + g(x_s)+\varepsilon_s)\Big)^2\Big|j_1,\ldots,j_t\right] \\
&= \frac{1}{(\sum_{s=1}^t s^{-\frac{1}{2}})^2}\mathbb{E}\left[\sum_{s=1}^t \Delta_s^2(bp_s^g + g(x_s)+\varepsilon_s)^2\Big|j_1,\ldots,j_t\right] \\
&\leq \frac{3}{(\sum_{s=1}^t s^{-\frac{1}{2}})^2}\sum_{s=1}^t \mathbb{E}\left[\sum_{s=1}^t s^{-\frac{1}{2}}\big((bp_s^g)^2 + (g(x_s))^2 + \varepsilon_s^2\big)|j_1,\ldots,j_t\right] \\
&\leq \frac{3\big(\underline{b}^2\overline{p}^2 + \max_{x\in[0,1]^d}(g(x))^2 + \sigma^2\big)}{\sum_{s=1}^t s^{-\frac{1}{2}}} \\
&\leq \frac{3\big(\underline{b}^2\overline{p}^2 + \max_{x\in[0,1]^d}(g(x))^2 + \sigma^2\big)}{\sqrt{t}},
\end{aligned} \tag{13}$$

where the first identity follows from plugging in the definition of $\hat{b}_t$, the second identity holds since $\delta_s = s^{-\frac{1}{2}}$ by its definition, the third identity holds since when $s \neq k$, Δ_s is independent of $\Delta_k(bp_s^g + g(x_s)+\varepsilon_s)(bp_k^g + g(x_k)+\varepsilon_k)$, and from $\mathbb{E}[\Delta_s] = 0$, we have $\mathbb{E}[\Delta_s\Delta_k(bp_s^g + g(x_s)+\varepsilon_s)(bp_k^g + g(x_k)+\varepsilon_k)] = 0$, the first inequality follows from $\Delta_s \in \{-s^{-\frac{1}{4}}, s^{\frac{1}{4}}\}$ and Cauchy-Schwarz inequality, and the last inequality holds since when $t \geq 2$, $\sum_{s=1}^t s^{-\frac{1}{2}} \geq \int_1^{t+1} s^{-\frac{1}{s}}ds = 2(\sqrt{t+1}-1) \geq \sqrt{t}$. This completes the proof of Eq. (8).

Proof of Eq. (9). Before we proceed to bound the total regret an prove Eq. (9), we note the following inequality on the summation for $1_{\{|\mathcal{D}_{t,j_t}|\geq 1\}}\frac{1}{|\mathcal{D}_{t,j_t}|}$:

$$\begin{aligned}
\sum_{t=1}^T 1_{\{|\mathcal{D}_{t,j_t}|\geq 1\}}\frac{1}{|\mathcal{D}_{t,j_t}|} &= \sum_{j=1}^{M^d}\sum_{t=1}^T 1_{\{j_t=j,|\mathcal{D}_{t,j}|\geq 1\}}\frac{1}{|\mathcal{D}_{t,j}|} = \sum_{j=1}^{M^d}\sum_{k=1}^{|\mathcal{D}_{T,j}|}\frac{1}{k} \\
&\leq \sum_{j=1}^{M^d}(1+\log|\mathcal{D}_{T,j}|) = M^d + \log\Big(\prod_{j=1}^{M^d}|\mathcal{D}_{T,j}|\Big) \\
&\leq M^d + \log\Big(\frac{1}{M^d}\sum_{j=1}^{M^d}|\mathcal{D}_{T,j}|\Big)^{M^d} \leq M^d\Big(1+\log T\Big),
\end{aligned} \tag{14}$$

where the second inequality holds since geometric mean is less than the arithmetic mean, and the last inequality holds since $\sum_{j=1}^{M^d}|\mathcal{D}_{T,j}| \leq T$.

Combining (12) and (14), we then obtain

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E}\left[\Big(\frac{\hat{a}_{t,j_t} - \mathbb{E}[g(x)|x\in\mathbf{M}_{j_t}]}{2\hat{b}_t}\Big)^2\Big|j_1,\ldots,j_t\right] \\
&\leq \frac{3}{4\underline{b}^2}\sum_{t=1}^T \mathbb{E}\left[\overline{p}^2(b-\hat{b}_t)^2\Big|j_1,\ldots,j_t\right] + \left(\frac{1}{4}\max_{x,y\in[0,1]^d}(g(x)-g(y))^2 + \sigma^2\right)\sum_{t=1}^T 1_{\{|\mathcal{D}_{t,j_t}|\geq 1\}}\frac{1}{|\mathcal{D}_{t,j_t}|} + \mathcal{O}(M^d) \\
&\leq \frac{3}{4\underline{b}^2}\sum_{t=1}^T \mathbb{E}\left[\overline{p}^2(b-\hat{b}_t)^2\Big|j_1,\ldots,j_t\right] + \left(\frac{1}{4}\max_{x,y\in[0,1]^d}(g(x)-g(y))^2 + \sigma^2\right)M^d(1+\ln T) + \mathcal{O}(M^d),
\end{aligned} \tag{15}$$

where the first inequality follows from (12) and $\sum_{t=1}^T 1_{\{|\mathcal{D}_{t,j_t}|=0\}} \leq M^d$, and the second inequality follows from (14).

Finally, putting Eqs. (3), (4), (7), (12), (13) and (15) together, we obtain the following upper bound on the total regret conditional on $j_1, \ldots, j_T$:

$$\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}[r_t | j_1, \ldots, j_T] \\
&\leq 6|\underline{b}| \left(\frac{c_1 T}{M^{2\beta}} + c_2 \sum_{t=1}^{T} \mathbb{E}[(b - \hat{b}_t)^2 | j_1, \ldots, j_t] + 4 \sum_{t=1}^{T} t^{-\frac{1}{2}} + c_3 (1 + \ln T) M^d \right) + \mathcal{O}(M^d) \\
&\leq 6|\underline{b}| \left(\frac{c_1 T}{M^{2\beta}} + c_4 \sum_{t=1}^{T} t^{-\frac{1}{2}} + c_3 (1 + \ln T) M^d \right) + \mathcal{O}(M^d) \\
&\leq 6|\underline{b}| \left(\frac{c_1 T}{M^{2\beta}} + c_4 (2\sqrt{T} - 1) + c_3 (1 + \ln T) M^d \right) + \mathcal{O}(M^d), \qquad (16)
\end{aligned}$$

where $c_1 = \frac{L^2 d^\beta}{4\underline{b}^2}$, $c_2 = \frac{1}{4\underline{b}^2}(\max_{x \in [0,1]^d} (g(x))^2 + 3\overline{p}^2)$, $c_3 = \frac{3}{4\underline{b}^2}(\frac{1}{4} \max_{x,y \in [0,1]^d} (g(x) - g(y))^2 + \sigma^2)$ and $c_4 = 3c_2(\underline{b}^2 \overline{p}^2 + \max_{x \in [0,1]^d} (g(x))^2 + \sigma^2) + 4$. This completes the proof of Eq. (9).

By setting $M = \lceil T^{\frac{1}{d+2\beta}} \rceil$ and taking the expectation with respect to $j_1, \ldots, j_T$ on both sides of (16), we obtain the following upper bound on the total expected regret:

$$\sum_{t=1}^{T} \mathbb{E}[r_t] \leq 6|\underline{b}| \left((c_1 + 2^d c_3 (1 + \ln T)) T^{\frac{d}{d+2\beta}} + c_4 (2\sqrt{T} - 1) \right) = \widetilde{\mathcal{O}}(\sqrt{T} \vee T^{\frac{d}{d+2\beta}}),$$

which completes the proof of Theorem 1.

Appendix B. Proof of Theorem 2

The first lower bound $\Omega(\sqrt{T})$ is directly implied from the existing results (e.g., Theorem 1 in [17]) by letting $g(x)$ be a constant function (which belongs to $\mathcal{G}(\beta, d)$). To show the second lower bound $\Omega(T^{\frac{d}{d+2\beta}})$, we first need to construct a series of Hölder continuous functions that are "similar" to each other and therefore difficult to distinguish. We partition the context space $[0,1]^d$ into M^d equally sized bins, denoted as $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_{M^d}$, by dividing each dimension of the context space into M intervals of equal length. We then construct a series of functions $g_{\mathbf{w}}(\cdot)$ indexed by a tuple $\mathbf{w} \in \{0,1\}^{M^d}$, and the j-th component of $\mathbf{w}$ determines the value of $g_{\mathbf{w}}(x)$ for $x \in \mathbf{M}_j$ as follows:

$$g_{\mathbf{w}}(x) = \begin{cases} |\underline{b}| \, (\underline{p} + \overline{p}) & w_j = 0 \\ |\underline{b}| \, (\underline{p} + \overline{p}) + (D(x, \partial \mathbf{M}_j))^\beta & w_j = 1 \text{ and } D(x, \partial \mathbf{M}_j) \leq \frac{1}{4^{1/\beta} M} \\ |\underline{b}| \, (\underline{p} + \overline{p}) + \frac{1}{4M^\beta} & w_j = 1 \text{ and } D(x, \partial \mathbf{M}_j) > \frac{1}{4^{1/\beta} M}, \end{cases} \qquad (17)$$

where $\partial \mathbf{M}_j$ denotes the boundary of the bin $\mathbf{M}_j$, and $D(x, \mathbf{M}_j) := \inf\{\|x - y\| : y \in \partial \mathbf{M}_j\}$ denotes the Euclidean distance between x and $\partial \mathbf{M}_j$. The following lemma shows that each $g_{\mathbf{w}}$ is Hölder continuous, whose proof is more sophisticated than [28] and [8], and deferred to Appendix B.1.

Lemma 1 *For each* $\mathbf{w} \in \{0,1\}^{M^d}$, $g_{\mathbf{w}}(\cdot)$ *defined in* (17) *belongs to* $\mathcal{G}(\beta, d)$ *with* $L = 2$.

Let $\mathcal{P}$ be a uniform distribution on $\bigcup_{j=1}^{M^d} \{x \in \mathbf{M}_j : D(x, \partial \mathbf{M}_j) > \frac{1}{4^{1/\beta} M}\}$. By setting $b = \underline{b}$, whenever x falls into $\mathbf{M}_j$, the optimal price $p^*(x)$ associated with any function $g_{\mathbf{w}}(\cdot)$ is

$$p^*(x) = \begin{cases} \frac{\underline{p} + \overline{p}}{2} & w_j = 0 \\ \frac{\underline{p} + \overline{p}}{2} + \frac{1}{8M^\beta |\underline{b}|} & w_j = 1. \end{cases} \qquad (18)$$

We then consider two demand functions $g_{(\mathbf{w}_{-j}, w_j)}$ for $w_j = 0, 1$, where we use $(\mathbf{w}_{-j}, w_j)$ to denote an index $\mathbf{w} \in \{0,1\}^{M^d}$ whose j-th coordinate is w_j and the other coordinates are $\mathbf{w}_{-j}$. Eq. (18) indicates that if the price charged by an algorithm in period t is greater than $\frac{\underline{p} + \overline{p}}{2} + \frac{1}{16M^\beta |\underline{b}|}$, its gap with the optimal price under demand function $g_{(\mathbf{w}_{-j}, 0)}$ is greater than $\frac{1}{16M^\beta |\underline{b}|}$; and if the price

charged by the algorithm is less than $\frac{\underline{p}+\overline{p}}{2} + \frac{1}{16M^\beta|\underline{b}|}$, its gap with the optimal price under the other function $g_{(\mathbf{w}_{-j},1)}$ is still greater than $\frac{1}{16M^\beta|\underline{b}|}$. Bretagnolle–Huber inequality (see [6]) guarantees that the minimal error of making one type of the mistakes depends on how well the algorithm can distinguish between the two demand functions. This is formalized in the following inequality:

$$\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}\left(p_t \geq \frac{\underline{p}+\overline{p}}{2} + \frac{1}{16M^\beta\,|\underline{b}|}\middle| x_t \in \mathbf{M}_j\right) + \mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}}\left(p_t < \frac{\underline{p}+\overline{p}}{2} + \frac{1}{16M^\beta\,|\underline{b}|}\middle| x_t \in \mathbf{M}_j\right)$$
$$\geq \frac{1}{2}\exp\left(-\mathrm{KL}\left(\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}, \mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}}\right)\right), \tag{19}$$

where $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},w_j)}}$ denotes the probability measure under policy π up to period t when the true demand function is $g_{(\mathbf{w}_{-j},w_j)}$. We then apply the chain rule of KL divergence to establish the following upper bound on the KL divergence between $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}$ and $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}}$:

$$\mathrm{KL}(\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}, \mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}}) \leq \frac{1}{32\sigma^2 M^{d+2\beta}} t. \tag{20}$$

With a careful decomposition of the regret, by applying Eqs. (19) and (20) with $M = \lceil T^{\frac{1}{d+2\beta}}\rceil$, we will obtain the desired lower bound $\Omega(T^{\frac{d}{d+2\beta}})$.

In the subsequent analysis, we fix the price elasticity to be $\underline{b}$, the distribution of contexts to be a uniform distribution on $\bigcup_{j=1}^{M^d}\{x \in \mathbf{M}_j : D(x, \partial\mathbf{M}_j) > \frac{1}{4^{1/\beta}M}\}$, and the distribution of random noise to be normal distribution. For the ease of presentation, we omit the dependency of the regret and the expectation on these terms. For any policy π, we establish the following lower bound on its worst-case regret by restricting to the functions $g_{\mathbf{w}}(\cdot)$ constructed in (17):

$$\begin{aligned}
\sup_{g\in\mathcal{G}(\beta,d)} R^\pi_g(T) &\geq \sup_{g\in\{g_{\mathbf{w}}:\mathbf{w}\in\{0,1\}^{M^d}\}} R^\pi_g(T) \\
&= |\underline{b}| \sup_{g\in\{g_{\mathbf{w}}:\mathbf{w}\in\{0,1\}^{M^d}\}} \sum_{t=1}^T \mathbb{E}^\pi_g\left[(p^*(x_t)-p_t)^2\right] \\
&\geq \frac{|\underline{b}|}{2^{M^d}} \sum_{\mathbf{w}\in\{0,1\}^{M^d}} \sum_{t=1}^T \mathbb{E}^\pi_{g_{\mathbf{w}}}\left[(p^*(x_t)-p_t)^2\right] \\
&\geq \frac{|\underline{b}|}{2^{M^d}} \sum_{\mathbf{w}\in\{0,1\}^{M^d}} \sum_{t=1}^T\sum_{j=1}^{M^d} \mathbb{E}^\pi_{g_{\mathbf{w}}}\left[(p^*(x_t)-p_t)^2\mathbb{I}_{\{x_t\in\mathbf{M}_j\}}\right] \\
&= \frac{|\underline{b}|}{2^{M^d}} \sum_{j=1}^{M^d} \sum_{\mathbf{w}_{-j}\in\{0,1\}^{M^d-1}} \sum_{w_j\in\{0,1\}} \sum_{t=1}^T \mathbb{E}^\pi_{g_{(\mathbf{w}_{-j},w_j)}}\left[(p^*(x_t)-p_t)^2\mathbb{I}_{\{x_t\in\mathbf{M}_j\}}\right] \\
&= \frac{|\underline{b}|}{2^{M^d}M^d} \sum_{j=1}^{M^d} \sum_{\mathbf{w}_{-j}\in\{0,1\}^{M^d-1}} \sum_{w_j\in\{0,1\}} \sum_{t=1}^T \mathbb{E}^\pi_{g_{(\mathbf{w}_{-j},w_j)}}\left[(p^*(x_t)-p_t)^2 \mid x_t\in\mathbf{M}_j\right],
\end{aligned} \tag{21}$$

where in the second and third identities, we use $(\mathbf{w}_{-j}, w_j)$ to denote an index $\mathbf{w}$ whose j-th coordinate is w_j and the other coordinates are $\mathbf{w}_{-j}$, and in the third identity, we use the fact that $\mathbb{E}^\pi_{g_{\mathbf{w}}}[(p^*(x_t)-p_t)^2\mathbb{I}_{\{x_t\in\mathbf{M}_j\}}] = \mathbb{E}^\pi_{g_{\mathbf{w}}}[(p^*(x_t)-p_t)^2 \mid x_t\in\mathbf{M}_j]\mathcal{P}_u(x_t\in\mathbf{M}_j)$ and $\mathcal{P}_u(x_t\in\mathbf{M}_j) = \frac{1}{M^d}$ by our construction.

Noting the expression of the optimal price in Eq. (18), we have the following lower bound on $\mathbb{E}^\pi_{g_{(\mathbf{w}_{-j},w_j)}}[(p^*(x_t)-p_t)^2 \mid x_t\in\mathbf{M}_j]$ in the RHS of (21): for each $w_j\in\{0,1\}$,

$$\mathbb{E}^\pi_{g_{(\mathbf{w}_{-j},0)}}\left[(p^*(x_t)-p_t)^2\middle| x_t\in\mathbf{M}_j\right] \geq \left(\frac{1}{16M^\beta\,|\underline{b}|}\right)^2 \mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}\left(\left\{p_t \geq \frac{\underline{p}+\overline{p}}{2} + \frac{1}{16M^\beta\,|\underline{b}|}\right\}\middle| x_t\in\mathbf{M}_j\right), \tag{22}$$

$$\mathbb{E}^{\pi}_{g_{(\mathbf{w}_{-j},1)}}\left[(p^*(x_t)-p_t)^2 \middle| x_t\in\mathbf{M}_j\right] \geq \left(\frac{1}{16M^{\beta}\,|\underline{b}|}\right)^2 \mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}}\left(\left\{p_t \leq \frac{\underline{p}+\overline{p}}{2}+\frac{1}{16M^{\beta}\,|\underline{b}|}\right\}\middle| x_t\in\mathbf{M}_j\right), \tag{23}$$

where $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},w_j)}}$ denotes the probability measure for $X_1,P_1,d_1,\ldots,X_t,P_t,d_t$ under policy π and demand function $g_{(\mathbf{w}_{-j},w_j)}$.

Due to the above Eqs. (22) and (23) and Eq. (19), to further bound the RHS of (21), we next focus on analyzing the KL-divergence between the two probability measures $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}(\cdot \mid x_t\in\mathbf{M}_j)$ and $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}}(\cdot \mid x_t\in\mathbf{M}_j)$, and prove Eq. (20). Noting the following identity

$$\begin{aligned}&\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},w_j)}}(X_1,P_1,d_1,\cdots,X_t,P_t \mid x_t\in\mathbf{M}_j)\\ &=\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}}(X_1,P_1,d_1,\cdots,X_{t-1},P_{t-1},d_{t-1})\times\pi(X_t,P_t\mid x_t\in\mathbf{M}_j,X_1,P_1,d_1,\cdots,X_{t-1},P_{t-1},d_{t-1}),\end{aligned}$$

and denoting $\pi(X_t,P_t\mid x_t\in\mathbf{M}_j,X_1,P_1,d_1,\cdots,X_{t-1},P_{t-1},d_{t-1})$ as π_t^j, we obtain

$$\begin{aligned}&\mathrm{KL}\big(\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}}(\cdot\mid x_t\in\mathbf{M}_j),\mathbb{P}^{\pi}_{g_{(\mathbf{w}_{-j},1)}}(\cdot\mid x_t\in\mathbf{M}_j)\big)\\ &\qquad=\mathrm{KL}(\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}}\times\pi_t^j,\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},1)}}\times\pi_t^j)\\ &\qquad=\mathrm{KL}(\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},1)}})+\mathbb{E}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}}[\mathrm{KL}(\pi_t^j,\pi_t^j)]\\ &\qquad=\mathrm{KL}(\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},1)}}),\end{aligned} \tag{24}$$

where the second identity follows from the chain rule of the KL divergence. Moreover, $\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},w_j)}}$ can be further decomposed as follows:

$$\begin{aligned}&\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},w_j)}}(X_1,P_1,d_1,\cdots,X_{t-1},P_{t-1},d_{t-1})\\ &=\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},w_j)}}(X_1,P_1,d_1,\cdots,X_{t-2},P_{t-2},d_{t-2})\times\mathcal{P}_u(X_{t-1})\\ &\quad\times\pi(P_{t-1}\mid X_1,P_1,d_1,\cdots,X_{t-2},P_{t-2},d_{t-2},X_{t-1})\times\mu_{g_{(\mathbf{w}_{-j},w_j)}}(d_{t-1}|P_{t-1},X_{t-1}).\end{aligned} \tag{25}$$

Denoting $\pi(P_{t-1}\mid X_1,P_1,d_1,\cdots,X_{t-2},P_{t-2},d_{t-2},X_{t-1})$ by $\pi_{\mathcal{F}_{t-2},x_{t-1}}$, we get

$$\begin{aligned}&\mathrm{KL}(\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},1)}})\\ &=\mathrm{KL}(\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},1)}})+\mathbb{E}^{\pi,t-2}_{g_{\mathbf{w}_{-j},0}}[\mathrm{KL}(\mathcal{P}_u\times\pi_{\mathcal{F}_{t-2},x_{t-1}}\times\mu_{g_{(\mathbf{w}_{-j},0)}},\mathcal{P}_u\times\pi_{\mathcal{F}_{t-2},x_{t-1}}\times\mu_{g_{(\mathbf{w}_{-j},1)}})]\\ &=\mathrm{KL}(\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},1)}})+\mathbb{E}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},0)}}[\mathrm{KL}(\mathcal{P}_u\times\pi_{\mathcal{F}_{t-2},x_{t-1}},\mathcal{P}_u\times\pi_{\mathcal{F}_{t-2},x_{t-1}})]\\ &\quad+\mathbb{E}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},0)}}[\mathbb{E}_{\mathcal{P}_u\times\pi_{\mathcal{F}_{t-2},x_{t-1}}}[\mathrm{KL}(\mu_{g_{(\mathbf{w}_{-j},0)}},\mu_{g_{(\mathbf{w}_{-j},1)}})]\\ &=\mathrm{KL}(\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},1)}})+\mathbb{E}^{\pi,t-2}_{g_{(\mathbf{w}_{-j},0)}}[\mathbb{E}_{\mathcal{P}_u\times\pi_{\mathcal{F}_{t-2},x_{t-1}}}[\mathrm{KL}(\mu_{g_{(\mathbf{w}_{-j},0)}},\mu_{g_{(\mathbf{w}_{-j},1)}})],\end{aligned} \tag{26}$$

where the first and second identities follow from the chain rule of KL divergence. Since we have assumed that ε follows a normal distribution with variance σ^2, the following equations hold:

$$\begin{aligned}&\mathrm{KL}(\mu_{g_{(\mathbf{w}_{-j},0)}}(\cdot\mid p_{t-1},x_{t-1}),\mu_{g_{(\mathbf{w}_{-j},1)}}(\cdot\mid p_{t-1},x_{t-1}))\\ &=\frac{1}{2\sigma^2}\left(\underline{b}p_{t-1}+g_{(\mathbf{w}_{-j},0)}(x_{t-1})-\underline{b}p_{t-1}-g_{(\mathbf{w}_{-j},1)}(x_{t-1})\right)^2\\ &=\frac{1}{2\sigma^2}\left(g_{(\mathbf{w}_{-j},0)}(x_{t-1})-g_{(\mathbf{w}_{-j},1)}(x_{t-1})\right)^2\\ &\leq\frac{1}{2\sigma^2}\left(\frac{1}{4M^{\beta}}\right)^2\mathbb{I}_{\{x_{t-1}\in\mathbf{M}_j\}},\end{aligned}$$

where the last inequality holds because $g_{(\mathbf{w}_{-j},0)}$ and $g_{(\mathbf{w}_{-j},1)}$ only differ in $\mathbf{M}_j$. Plugging the above equation into Eq. (26), we obtain

$$\begin{aligned}\mathrm{KL}(\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t}_{g_{(\mathbf{w}_{-j},1)}})&=\mathrm{KL}(\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},0)}},\mathbb{P}^{\pi,t-1}_{g_{(\mathbf{w}_{-j},1)}})+\frac{1}{2\sigma^2M^d}\left(\frac{1}{4M^{\beta}}\right)^2\\ &=\frac{1}{32\sigma^2M^{d+2\beta}}t.\end{aligned} \tag{27}$$

where the first identity is holds since $\mathbb{E}_{\mathcal{P}_u \times \pi_{\mathcal{F}_{t-2}, x_{t-1}}}[\mathbb{I}_{\{x_{t-1} \in \mathbf{M}_j\}}] = \frac{1}{M^d}$, and the second identity follows by repeatedly applying the first identity. This completes the proof of Eq. (20).

Combining Eqs. (21), (22), (19), (24) and (27), we have

$$\sup_{g \in \mathcal{G}(\beta,d)} R_g^{\pi}(T) \geq \frac{|\underline{b}|}{2^{M^d} M^d} \sum_{j=1}^{M^d} \sum_{\mathbf{w}_{-j} \in \{0,1\}^{M^d-1}} \sum_{t=1}^{T} \frac{1}{2} \left(\frac{1}{16 M^{\beta} |\underline{b}|} \right)^2 \exp\left(-\frac{1}{32\sigma^2 M^{d+2\beta}} t \right)$$
$$\geq \frac{|\underline{b}|}{4} T \left(\frac{1}{16 M^{\beta} |\underline{b}|} \right)^2 \exp\left(-\frac{1}{32\sigma^2 M^{d+2\beta}} T \right).$$

By letting $M = \lceil T^{\frac{1}{d+2\beta}} \rceil$ in the above inequality, we obtain the lower bound $\Omega(T^{\frac{d}{d+2\beta}})$.

B.1. Proof of Lemma 1

We prove that for any $x, y \in [0,1]^d$, $|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| \leq 2||x-y||^{\beta}$ by considering two cases: x and y fall into the same bin in case 1, and x and y fall into different bins in case 2.

Case 1: $x, y \in \mathbf{M}_j$ for some $j \in [M^d]$. When x and y fall into the same bin $\mathbf{M}_j$, we divide the proof into four subcases.

Subcase 1.1: $w_j = 0$, or $w_j = 1$, $D(x, \partial \mathbf{M}_j) > \frac{1}{4^{1/\beta} M}$ and $D(y, \partial \mathbf{M}_j) > \frac{1}{4^{/\beta} M}$. In this subcase, we have $g_{\mathbf{w}}(x) = g_{\mathbf{w}}(y)$, and the result is trivial.

Subcase 1.2: $w_j = 1$, $D(x, \partial \mathbf{M}_j) \leq \frac{1}{4^{1/\beta} M}$ and $D(y, \partial \mathbf{M}_j) \leq \frac{1}{4^{1/\beta} M}$. Without loss of generality, we assume that $g_{\mathbf{w}}(x) \leq g_{\mathbf{w}}(y)$. Then we have the following equation:

$$\begin{aligned} ||x-y||^{\beta} + g_{\mathbf{w}}(x) &= ||x-y||^{\beta} + (D(x, \partial \mathbf{M}_j))^{\beta} + |\underline{b}|(\underline{p} + \overline{p}) \\ &= ||x-y||^{\beta} + \min_{z \in \partial \mathbf{M}_j} ||z-x||^{\beta} + |\underline{b}|(\underline{p} + \overline{p}) \\ &= \min_{z \in \partial \mathbf{M}_j} (||x-y||^{\beta} + ||z-x||^{\beta}) + |\underline{b}|(\underline{p} + \overline{p}). \end{aligned} \tag{28}$$

If the following inequality holds: for $0 < \beta \leq 1$,

$$||a+b||^{\beta} \leq ||a||^{\beta} + ||b||^{\beta}, \ \forall a, b \in \mathbb{R}^d, \tag{29}$$

then we have from (28) that

$$||x-y||^{\beta} + g_{\mathbf{w}}(x) \geq \min_{z \in \partial \mathbf{M}_j} ||z-y||^{\beta} + |\underline{b}|(\underline{p} + \overline{p}) = (D(y, \partial \mathbf{M}_j))^{\beta} + |\underline{b}|(\underline{p} + \overline{p}) = g_{\mathbf{w}}(y), \tag{30}$$

which then implies $|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| \leq ||x-y||^{\beta}$.

We now show (29). Note that (29) is simply the triangle inequality when $\beta = 1$. When $0 < \beta < 1$, let $\beta' \in \mathbb{R}^+$ be such that $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. By applying Hölder's inequality, we have for any $a, b \in \mathbb{R}^+$,

$$(a+b)^{\beta} \leq \left((a^{\beta} + b^{\beta})^{\frac{1}{\beta}} (1^{\beta'} + 1^{\beta'})^{\frac{1}{\beta'}} \right)^{\beta} = 2^{\beta-1}(a^{\beta} + b^{\beta}) < a^{\beta} + b^{\beta}, \tag{31}$$

Applying (31) and the triangle inequality, we have for any $a, b \in \mathbb{R}^d$,

$$||a+b||^{\beta} \leq (||a|| + ||b||)^{\beta} < ||a||^{\beta} + ||b||^{\beta},$$

which finishes the proof of (29).

Subcase 1.3: $w_j = 1$, $D(x, \partial \mathbf{M}_j) \leq \frac{1}{4^{1/\beta} M}$ and $D(y, \partial \mathbf{M}_j) > \frac{1}{4^{1/\beta} M}$. Let $\hat{\mathbf{M}}_j := \{z \in \mathbf{M}_j : D(z, \partial \mathbf{M}_j) > \frac{1}{4^{1/\beta} M}\}$. Since $\text{Proj}(x, \hat{\mathbf{M}}_j) \in \partial \hat{\mathbf{M}}_j$, then we have $g_{\mathbf{w}}(y) = g_{\mathbf{w}}(\text{Proj}(x, \hat{\mathbf{M}}_j)) = |\underline{b}|(\underline{p} + \overline{p}) + \frac{1}{4M^{\beta}}$ and

$$|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| = |g_{\mathbf{w}}(x) - g_{\mathbf{w}}(\text{Proj}(x, \hat{\mathbf{M}}_j))|. \tag{32}$$

Since $\partial\hat{\mathbf{M}}_j = \{z \in \mathbf{M}_j : D(z, \partial\mathbf{M}_j) = \frac{1}{4^{1/\beta}M}\}$ and $\text{Proj}(x, \hat{\mathbf{M}}_j) \in \partial\hat{\mathbf{M}}_j$, it then follows that $D(\text{Proj}(x, \hat{\mathbf{M}}_j), \partial\mathbf{M}_j) = \frac{1}{4^{1/\beta}M}$. Since $D(x, \partial\mathbf{M}_j) \leq \frac{1}{4^{1/\beta}M}$ from the assumption, by applying the result in subcase 1.2, we obtain

$$|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(\text{Proj}(x, \hat{\mathbf{M}}_j))| \leq ||x - \text{Proj}(x, \hat{\mathbf{M}}_j)||^\beta = \min_{z \in \hat{\mathbf{M}}_j} ||x - z||^\beta \leq ||x - y||^\beta, \tag{33}$$

where the last inequality holds due to $y \in \hat{\mathbf{M}}_j$. Combining (32) with (33), we obtain $|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| \leq ||x - y||^\beta$.

Subcase 1.4: $w_j = 1$, $D(x, \partial\mathbf{M}_j) > \frac{1}{4^{1/\beta}M}$ and $D(y, \partial\mathbf{M}_j) \leq \frac{1}{4^{1/\beta}M}$. The proof of this subcase is similar to subcase 1.3, and is omitted for brevity.

Case 2: $x \in \mathbf{M}_i$ and $y \in \mathbf{M}_j$ for $i \neq j$. When x and y fall into different bins, we divide the proof into five subcases.

Subcase 2.1: $w_i = w_j = 0$, or if $w_i = w_j = 1$, $D(x, \partial\mathbf{M}_i) > \frac{1}{4^{1/\beta}M}$ and $D(y, \partial\mathbf{M}_j) > \frac{1}{4^{1/\beta}M}$. In this subcase, we have $g_{\mathbf{w}}(x) = g_{\mathbf{w}}(y)$ and the result is trivial.

Subcase 2.2: $w_i = 0$ and $w_j = 1$. In this subcase, we have

$$\begin{aligned}
|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| &\leq (D(y, \partial\mathbf{M}_j))^\beta && (34)\\
&\leq ||\text{Proj}(x, \partial\mathbf{M}_j) - y||^\beta && (35)\\
&\leq ||\text{Proj}(x, \partial\mathbf{M}_j) - x||^\beta + ||x - y||^\beta && (36)\\
&= ||\text{Proj}(x, \mathbf{M}_j) - x||^\beta + ||x - y||^\beta && (37)\\
&\leq 2||x - y||^\beta. && (38)
\end{aligned}$$

In the above equations, Eq. (34) holds since under the assumption $w_i = 0$ and $w_j = 1$, if $D(y, \partial\mathbf{M}_j) \leq \frac{1}{4^{1/\beta}M}$, $|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| = (D(y, \partial\mathbf{M}_j))^\beta$, and if $D(y, \partial\mathbf{M}_j) > \frac{1}{4^{1/\beta}M}$, $|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| = \frac{1}{4M^\beta} \leq (D(y, \partial\mathbf{M}_j))^\beta$. Eq. (35) follows from the definition of $D(y, \partial\mathbf{M}_j)$ and $\text{Proj}(x, \partial\mathbf{M}_j) \in \partial\mathbf{M}_j$. Eq. (36) follows from (29). Eq. (37) holds since if $\text{Proj}(x, \mathbf{M}_j)$ is an interior point of $\mathbf{M}_j$, since $\mathbf{M}_j$ is a cubic, one can always construct a ball inside $\mathbf{M}_j$ with the center $\text{Proj}(x, \mathbf{M}_j)$, and the intersected point between the ball and the line connecting x and $\text{Proj}(x, \mathbf{M}_j)$ has a strictly shorter distance to x than $\text{Proj}(x, \mathbf{M}_j)$, leading to contradiction with the fact that $\text{Proj}(x, \mathbf{M}_j)$ is the closest point in the bin $\mathbf{M}_j$ to x. Thus, $\text{Proj}(x, \mathbf{M}_j)$ must be at the boundary $\partial\mathbf{M}_j$ and $\text{Proj}(x, \mathbf{M}_j) = \text{Proj}(x, \partial\mathbf{M}_j)$. Eq. (38) follows from $y \in \mathbf{M}_j$ and the definition of $D(x, \partial\mathbf{M}_j)$.

Subcase 2.3: $w_i = 1$ and $w_j = 0$. The proof of this subcase is similar to subcase 2.2 and is omitted.

Subcase 2.4: $w_i = w_j = 1$, $D(x, \partial\mathbf{M}_i) \leq \frac{1}{4^{1/\beta}M}$ and $D(y, \partial\mathbf{M}_j) \leq \frac{1}{4^{1/\beta}M}$. Without loss of generality, we assume $g_{\mathbf{w}}(y) \geq g_{\mathbf{w}}(x)$. Then we have

$$||x - y||^\beta + g_{\mathbf{w}}(x) = \min_{z \in \partial\mathbf{M}_i} (||x - y||^\beta + ||z - x||^\beta) + |\underline{b}|(\underline{p} + \overline{p}) \geq \min_{z \in \partial\mathbf{M}_i} ||z - y||^\beta + |\underline{b}|(\underline{p} + \overline{p}), \tag{39}$$

where the inequality follows from (29).

On the other hand, when K is sufficiently large, we have

$$\begin{aligned}
g_{\mathbf{w}}(y) &= \min_{z \in \partial\mathbf{M}_j} ||z - y||^\beta + |\underline{b}|(\underline{p} + \overline{p})\\
&= \min_{z \in \partial([-K,K]^d \setminus \text{int}(\mathbf{M}_j))} ||z - y||^\beta + |\underline{b}|(\underline{p} + \overline{p}) && (40)\\
&= \min_{z \in [-K,K]^d \setminus \text{int}(\mathbf{M}_j)} ||z - y||^\beta + |\underline{b}|(\underline{p} + \overline{p}) && (41)\\
&\leq \min_{z \in \partial\mathbf{M}_i} ||z - y||^\beta + |\underline{b}|(\underline{p} + \overline{p}). && (42)
\end{aligned}$$

In the above equations, Eq. (40) holds since $\partial([-K, K]^d \setminus \text{int}(\mathbf{M}_j)) = \partial([-K, K]^d) \cup \partial\mathbf{M}_j$, and when K is sufficiently large, $D(y, \partial\mathbf{M}_j) < D(y, \partial([-K, K]^d))$ and $\text{Proj}(y, \partial(\mathbf{M}_j)) =$

$\text{Proj}(y, \partial([-K,K]^d) \cup \partial \mathbf{M}_j)$. Eq. (41) holds due to the same reason as (37). Eq. (42) holds since $\partial(\mathbf{M}_i) \subset [-K,K]^d$ and $\partial(\mathbf{M}_i) \cap \text{int}(\mathbf{M}_j) = \emptyset$ imply that $\partial(\mathbf{M}_i) \subset [-K,K]^d \setminus \text{int}(\mathbf{M}_j)$.

Combining Eqs. (39) and (42), we obtain $|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(x)| \leq ||x-y||^\beta$.

Subcase 2.5: $w_i = w_j = 1$, $D(x, \partial \mathbf{M}_j) \leq \frac{1}{4^{1/\beta} M}$ and $D(y, \partial \mathbf{M}_j) > \frac{1}{4^{1/\beta} M}$. In this subcase, we have

$$|g_{\mathbf{w}}(x) - g_{\mathbf{w}}(y)| = \left| (D(x, \partial \mathbf{M}_j))^\beta - \frac{1}{4M^\beta} \right| = \frac{1}{4M^\beta} - (D(x, \partial \mathbf{M}_j))^\beta < (D(y, \partial \mathbf{M}_j))^\beta - (D(x, \partial \mathbf{M}_j))^\beta.$$

The remaining analysis is similar to subcase 2.4, and is omitted for brevity.

Appendix C. Proof of Theorem 3

We first establish the following lemma showing that the true demand function $f(p) + a^\top x$ can be well-approximated by the linear function $\hat{\theta}_{t,j}^\top \varphi(p) + \hat{a}_{t,j}^\top x$ within price segment $\mathbf{I}_j$ after running Algorithm 3. This result is quite standard and can be obtained easily by modifying the analysis in [1]. For completeness, we provide the details in Appendix C.1.

Lemma 2 *For each $j \in [N]$, with probability at least $1 - \epsilon$, the following event holds: for any $t \in [T]$, $p \in \mathbf{I}$ and $x \in [0,1]^d$,*

$$\left| f(p) + a^\top x - \left(\hat{\theta}_{t,j}^\top \varphi(p) + \hat{a}_{t,j}^\top x \right) \right| \leq \gamma_{t,j} \sqrt{\phi(p,x)^\top V_{t,j}^{-1} \phi(p,x)} + \Delta. \tag{43}$$

Now we highlight our key idea for analyzing the regret in each period $t \in [T]$. Let $p_t^* := \arg\max_{p \in [\underline{p}, \overline{p}]} p(f(p) + a^\top x_t)$ be the optimal price for period t and $i_t^* \in [N]$ denote the index for the price segment p_t^* belongs to. Conditioning on the events guaranteed by Lemma 2 for each price segment $i \in [N]$, we have

$$\begin{aligned}
r_t &= p_t^* \left(f(p_t^*) + a^\top x_t \right) - p_t \left(f(p_t) + a^\top x_t \right) \\
&\leq \max_{p \in \mathbf{I}_{i_t^*}} p \left(\langle \hat{\theta}_{t,i_t^*}, \varphi(p) \rangle + \langle \hat{a}_{t,i_t^*}, x_t \rangle + \gamma_{t,i_t^*} \sqrt{\phi(p,x_t)^\top V_{t,i_t^*}^{-1} \phi(p,x_t)} + \Delta \right) - p_t \left(f(p_t) + a^\top x_t \right) \\
&\leq \max_{i \in [N]} \max_{p \in \mathbf{I}_i} p \left(\langle \hat{\theta}_{t,i}, \varphi(p) \rangle + \langle \hat{a}_{t,i}, x_t \rangle + \gamma_{t,i} \sqrt{\phi(p,x_t)^\top V_{t,i}^{-1} \phi(p,x_t)} + \Delta \right) - p_t \left(f(p_t) + a^\top x_t \right) \\
&= p_t \left(\langle \hat{\theta}_{t,i_t}, \varphi(p_t) \rangle + \langle \hat{a}_{t,i_t}, x_t \rangle + \gamma_{t,i_t} \sqrt{\phi(p_t,x_t)^\top V_{t,i_t}^{-1} \phi(p_t,x_t)} + \Delta \right) - p_t \left(f(p_t) + a^\top x_t \right) \\
&\leq p_t \left| \langle \hat{\theta}_{t,i_t}, \varphi(p_t) \rangle + \langle \hat{a}_{t,i_t}, x_t \rangle - \left(f(p_t) + a^\top x_t \right) \right| + p_t \left(\gamma_{t,i_t} \sqrt{\phi(p_t,x_t)^\top V_{t,i_t}^{-1} \phi(p_t,x_t)} + \Delta \right) \\
&\leq 2\overline{p} \left(\gamma_{t,i_t} \sqrt{\phi(p_t,x_t)^\top V_{t,i_t}^{-1} \phi(p_t,x_t)} + \Delta \right),
\end{aligned} \tag{44}$$

where the first inequality follows from Eq. (43) in Lemma 2 and $p_t^* \in \mathbf{I}_t^*$ by definition, the second equality is based on the design of our Algorithm 2 (line 13), and the last inequality again follows from Eq. (43) in Lemma 2. Therefore, by optimizing the revenue within each price segment in an optimistic way and choosing the price from all optimistic prices with the highest optimistic revenue, we reduce estimating the regret in period t to bounding $\gamma_{t,i_t} \sqrt{\phi(p_t,x_t)^\top V_{t,i_t}^{-1} \phi(p_t,x_t)} + \Delta$. The subsequent analysis requires applying the elliptical potential lemma (see, e.g., [1]) to bound $\sqrt{\phi(p_t,x_t)^\top V_{t,i_t}^{-1} \phi(p_t,x_t)}$ and carefully choosing the number of price segments N to balance the bias of polynomial approximation for $f(\cdot)$ and learning efficiency within each price segment. The details are given in Appendix B.1.

Let $n_{T,j} := \sum_{t=1}^T \mathbb{I}_{p_t \in \mathbf{I}_j}$ be the number of times for which prices $p_1, p_2, \ldots, p_T$ selected by our algorithm fall into $\mathbf{I}_j$, and $\mathcal{A}$ denote the event that Eq. (44) in Lemma 2 holds for any $t \in [T]$ and

$i \in [N]$. When $\mathcal{A}$ holds, from Eq. (44), the total regret can be bounded as follows:

$$\begin{aligned}
\sum_{t=1}^{T} r_t &\le 2\overline{p} \sum_{t=1}^{T} \gamma_{t,i_t} \|\phi(p_t, x_t)\|_{V_{t,i_t}^{-1}} + 2\overline{p}\Delta T \\
&= 2\overline{p} \sum_{j=1}^{N} \sum_{t=1}^{T} \gamma_{t,j} \|\phi(p_t, x_t)\|_{V_{t,j}^{-1}} \mathbb{I}_{p_t \in \mathbf{I}_j} + 2\overline{p}\Delta T \\
&\le 2\overline{p} \sum_{j=1}^{N} \gamma_{T,j} \sum_{t=1}^{T} \|\phi(p_t, x_t)\|_{V_{t,j}^{-1}} \mathbb{I}_{p_t \in \mathbf{I}_j} + 2\overline{p}\Delta T \\
&\le 2\overline{p} \sum_{j=1}^{N} \gamma_{T,j} \sqrt{n_{T,j}} \sqrt{\sum_{t=1}^{T} \|\phi(p_t, x_t)\|_{V_{t,j}^{-1}}^2 \mathbb{I}_{p_t \in \mathbf{I}_j}} + 2\overline{p}\Delta T \\
&\le 2\overline{p} \sum_{j=1}^{N} \gamma_{T,j} \sqrt{n_{T,j}} \sqrt{2(\mathfrak{b}(k) + d + 1) \ln\left(1 + \frac{n_{T,j}}{\mathfrak{b}(k) + d + 1}\right)} + 2\overline{p}\Delta T, \qquad (45)
\end{aligned}$$

where the second inequality holds since $\{\gamma_{t,j} : 1 \le t \le T\}$ is an increasing sequence for each $j \in [N]$, the third inequality follows from Cauchy-Schwarz inequality, and the last inequality holds due to $\sum_{t=1}^{T} \|\phi(p_t, x_t)\|_{V_{t,j}^{-1}}^2 \mathbb{I}_{p_t \in \mathbf{I}_j} \le 2 \ln \det(V_{T,j}) / \ln(\lambda I)$ from the elliptical potential lemma (see, e.g., Lemma 11 in [1]) and $\ln \det(V_{T,j}) \le (\mathfrak{b}(k) + d + 1) \ln(\lambda(1 + \frac{n_{T,j}}{\mathfrak{b}(k)+d+1}))$ from (51).

Since $\epsilon = T^{-2}$, we then have

$$\begin{aligned}
\gamma_{T,j} &= \sigma \sqrt{(\mathfrak{b}(k) + d + 1) \ln\left(\frac{\mathfrak{b}(k) + d + 1 + n_{T,j}}{\mathfrak{b}(k) + d + 1}\right) - 2\ln\epsilon} + \lambda^{\frac{1}{2}} (C_0^2(\mathfrak{b}(k) + 1) + \overline{a}^2)^{\frac{1}{2}} + \Delta\sqrt{n_{T,j}} \\
&\le \sigma \sqrt{2(\mathfrak{b}(k) + d + 1) \ln(T + 1)} + \lambda^{\frac{1}{2}} (C_0^2(\mathfrak{b}(k) + 1) + \overline{a}^2)^{\frac{1}{2}} + \Delta\sqrt{n_{T,j}},
\end{aligned}$$

and the first term in the RHS of (45) is bounded by

$$\begin{aligned}
&\sum_{j=1}^{N} \gamma_{T,j} \sqrt{n_{T,j}} \sqrt{2(\mathfrak{b}(k) + d + 1) \ln\left(1 + \frac{n_{T,j}}{\mathfrak{b}(k) + d + 1}\right)} \\
&\le 2\sqrt{(\mathfrak{b}(k) + d + 1) \ln(T + 1)} \left(\max\left\{\sigma\sqrt{(\mathfrak{b}(k) + d + 1) \ln(T + 1)}, \lambda^{\frac{1}{2}} (C_0^2 k + \overline{a}^2)^{\frac{1}{2}}\right\} \cdot \sum_{j=1}^{N} \sqrt{n_{T,j}} + \Delta T \right) \\
&\le 2\sqrt{(\mathfrak{b}(k) + d + 1) \ln(T + 1)} \\
&\qquad \times \left(\max\left\{\sigma\sqrt{(\mathfrak{b}(k) + d + 1) \ln(T + 1)}, \lambda^{\frac{1}{2}} (C_0^2(\mathfrak{b}(k) + 1) + \overline{a}^2)^{\frac{1}{2}}\right\} \cdot \sqrt{\sum_{j=1}^{N} n_{T,j}} \cdot \sqrt{\sum_{j=1}^{N} 1^2} + \Delta T \right) \\
&= 2\sqrt{(\mathfrak{b}(k) + d + 1) \ln(T + 1)} \Big(\max\left\{\sigma\sqrt{(\mathfrak{b}(k) + d + 1) \ln(T + 1)}, \lambda^{\frac{1}{2}} (C_0^2(\mathfrak{b}(k) + 1) + \overline{a}^2)^{\frac{1}{2}}\right\} \cdot \sqrt{TN} \\
&\quad + \Delta T \Big).
\end{aligned}$$

This, together with $\Delta = \Theta(\frac{\delta}{N^k})$ and (45), implies

$$\sum_{t=1}^{T} r_t = \mathcal{O}\left(\sqrt{(\mathfrak{b}(k) + d + 1) \ln T}\right) \cdot \mathcal{O}\left(\sqrt{(\mathfrak{b}(k) + d + 1) NT \ln T} + \frac{\delta T}{N^k}\right).$$

To balance the two terms $\sqrt{(\mathfrak{b}(k) + d + 1) NT \ln T}$ and $\frac{\delta T}{N^k}$, we let $N = \lceil (T\delta^2)^{\frac{1}{2k+1}} \rceil + 1$. When $\delta = \mathcal{O}(T^{-\frac{1}{2}})$, $N = \Theta(1)$ and thus we obtain $\sqrt{(\mathfrak{b}(k) + d + 1) NT} = \Theta(\sqrt{(\mathfrak{b}(k) + d + 1) T})$ and $\frac{\delta T}{N^k} = \mathcal{O}(\sqrt{T})$. In this case, we get $\sum_{t=1}^{T} r_t = \mathcal{O}((\mathfrak{b}(k) + d + 1)\sqrt{T} \ln T)$. When $\delta = \Omega(T^{-\frac{1}{2}})$, $N = \Theta((T\delta^2)^{\frac{1}{2k+1}})$ and we obtain $\sum_{t=1}^{T} r_t = \mathcal{O}((\mathfrak{b}(k) + d + 1)\delta^{\frac{1}{2k+1}} T^{\frac{k+1}{2k+1}} \ln T)$. Combining these two cases, we have $\sum_{t=1}^{T} r_t = \mathcal{O}((\mathfrak{b}(k) + d + 1)((\delta T^{k+1})^{\frac{1}{2k+1}} \vee \sqrt{T}) \ln T)$.

Therefore, the total expected regret is upper bounded by

$$\sum_{t=1}^{T}\mathbb{E}[r_t] = \sum_{t=1}^{T}\mathbb{E}[r_t|\mathcal{A}]\cdot\mathbb{P}(\mathcal{A}) + \sum_{t=1}^{T}\mathbb{E}[r_t|\mathcal{A}^c]\cdot\mathbb{P}(\mathcal{A}^c)$$
$$= \widetilde{\mathcal{O}}\left((\mathfrak{b}(k)+d+1)\left((\delta T^{k+1})^{\frac{1}{2k+1}} \vee \sqrt{T}\right)\right) + \mathcal{O}\left(\frac{N}{T}\right), \tag{46}$$

where the second identity holds since from the union bound, $\mathbb{P}(\mathcal{A}^c) \leq \frac{N}{\epsilon} = \frac{N}{T^2}$. Since $\mathcal{O}(\frac{N}{T}) = \mathcal{O}(T^{-\frac{2k}{2k+1}}\delta^{\frac{2}{2k+1}})$ and $\delta = \mathcal{O}(1)$, the first term in the RHS of (46) dominates.

C.1. Proof of Lemma 2

In this proof, we consider an arbitrarily fixed $\mathbf{I} \in \bigcup_{j\in[N]}\mathbf{I}_j$. Let $P_{\mathbf{I}}(p)$ be the first $\mathfrak{b}(k)+1$ terms of the Taylor expansion for function $f(p)$ at $p = l$:

$$P_{\mathbf{I}}(p) = \sum_{i=0}^{\mathfrak{b}(k)} \frac{f^{(i)}(l)}{i!}(p-l)^i, \quad \forall p \in \mathbf{I}.$$

It's easy to verify that $|f(p) - P_{\mathbf{I}}(p)| \leq \delta\frac{(u-l)^k}{\mathfrak{b}(k)!}$. For convenience, we label (x, p, d) in $\mathcal{D}$ as $\{(x_i, p_i, d_i)\}_{i=1}^t$ in chronological order. Let $\beta_i := f(p_i) - P_{\mathbf{I}}(p_i)$ for $i \in [t]$, then $|\beta_i| \leq \Delta$ and $d_i = f(p_i) + a^\top x_i + \epsilon_i = P_I(p_i) + a^\top x_i + \epsilon_i + \beta_i$.

Denote $\boldsymbol{d} = (d_i)_{i\leq t}$, $\boldsymbol{\epsilon} = (\epsilon_i)_{i\leq t}$, $\boldsymbol{\beta} = (\beta_i)_{i\leq t}$ as column vectors in $\mathbb{R}^t$. Let $\boldsymbol{X} = ((\phi(p_i, x_i))_{i\leq t}) \in \mathbb{R}^{t\times(\mathfrak{b}(k)+d+1)}$, $\theta^* = ((\frac{f^{(i)}(l)}{i!})_{0\leq i\leq \mathfrak{b}(k)}, a) \in \mathbb{R}^{\mathfrak{b}(k)+d+1}$. Thus, $\boldsymbol{d} = \boldsymbol{X}\theta^* + \boldsymbol{\epsilon} + \boldsymbol{\beta}$. The Ridge estimate $\bar{\theta} := (\hat{\theta}^\top, \hat{a}^\top)^\top$ can be written as $\bar{\theta} = V^{-1}\boldsymbol{X}^\top\boldsymbol{d}$. Note that we omit the dependency of $\bar{\theta}$, V, $\boldsymbol{X}$ and $\boldsymbol{d}$ on the cardinality of $\mathcal{D}$, i.e., t, for simplicity. By simple calculation, we have

$$\bar{\theta} - \theta^* = -\lambda V^{-1}\theta^* + V^{-1}\boldsymbol{X}^\top(\boldsymbol{\epsilon} + \boldsymbol{\beta}).$$

Multiplying $(\bar{\theta} - \theta^*)^\top V$ on both sides of the above equation, we have

$$\begin{aligned}
&(\bar{\theta} - \theta^*)^\top V(\bar{\theta} - \theta^*)\\
&= -\lambda(\bar{\theta} - \theta^*)^\top\theta^* + (\bar{\theta} - \theta^*)^\top\boldsymbol{X}^\top(\boldsymbol{\epsilon} + \boldsymbol{\beta})\\
&\leq \lambda\left\|(\bar{\theta} - \theta^*)\right\|_V \cdot \|\theta^*\|_{V^{-1}} + \left\|(\bar{\theta} - \theta^*)\right\|_V \cdot \left\|\boldsymbol{X}^\top\boldsymbol{\epsilon}\right\|_{V^{-1}} + (\bar{\theta} - \theta^*)^\top\boldsymbol{X}^\top\boldsymbol{\beta}\\
&\leq \lambda\left\|(\bar{\theta} - \theta^*)\right\|_V \cdot \|\theta^*\|_{V^{-1}} + \left\|(\bar{\theta} - \theta^*)\right\|_V \cdot \left\|\boldsymbol{X}^\top\boldsymbol{\epsilon}\right\|_{V^{-1}} + \Delta\sqrt{t}\left\|(\bar{\theta} - \theta^*)\right\|_V,
\end{aligned} \tag{47}$$

where the first inequality follows from Cauchy-Schwarz inequality, and the second inequality holds due to

$$\begin{aligned}
\left|(\bar{\theta} - \theta^*)^\top\boldsymbol{X}^\top\boldsymbol{\beta}\right| &\leq \sqrt{\sum_{1\leq\tau\leq t}\beta_\tau^2}\sqrt{\sum_{1\leq\tau\leq t}\left(\langle\phi(p_\tau, x_\tau), \bar{\theta} - \theta^*\rangle\right)^2}\\
&= \sqrt{\sum_{1\leq\tau\leq t}\beta_\tau^2}\sqrt{(\bar{\theta} - \theta^*)^\top\left(\sum_{1\leq\tau\leq t}\phi(p_\tau, x_\tau)\phi^\top(p_\tau, x_\tau)\right)(\bar{\theta} - \theta^*)}\\
&= \sqrt{\sum_{1\leq\tau\leq t}\beta_\tau^2}\sqrt{(\bar{\theta} - \theta^*)^\top(V - \lambda I)(\bar{\theta} - \theta^*)}\\
&\leq \Delta\sqrt{t}\left\|(\bar{\theta} - \theta^*)\right\|_V.
\end{aligned}$$

Dividing both sides of Eq. (47) by $\left\|\bar{\theta} - \theta^*\right\|_V$, we have

$$\left\|(\bar{\theta} - \theta^*)\right\|_V \leq \lambda\|\theta^*\|_{V^{-1}} + \left\|\boldsymbol{X}^\top\boldsymbol{\epsilon}\right\|_{V^{-1}} + \Delta\sqrt{t}. \tag{48}$$

By applying the self-normalized bound for vector-valued martingales in Theorem 1 of [1], we have with probability at least $1-\epsilon$,

$$\forall t \in [T]: \left\|\boldsymbol{X}^\top\boldsymbol{\epsilon}\right\|_{V^{-1}} \leq \sqrt{2\sigma^2\log\left(\frac{\det(V)^{\frac{1}{2}}\det(\lambda I)^{-1/2}}{\epsilon}\right)}. \tag{49}$$

Since $\|\theta^*\|_{V^{-1}}^2 \leq \frac{1}{\lambda_{\min}(V)} \|\theta^*\|_2^2 \leq \frac{1}{\lambda} \|\theta^*\|_2^2 \leq \frac{C_0^2(\mathfrak{b}(k)+1)+\overline{a}^2}{\lambda}$, it follows from Eqs. (48) and (49) that

$$\left\|(\bar{\theta}-\theta^*)\right\|_V \leq \sqrt{2\sigma^2 \log\left(\frac{\det(V)^{\frac{1}{2}} \det(\lambda I)^{-1/2}}{\delta}\right)} + \lambda^{\frac{1}{2}}(C_0^2(\mathfrak{b}(k)+1)+\overline{a}^2)^{\frac{1}{2}} + \Delta\sqrt{t}. \quad (50)$$

Moreover, let $\lambda_1, \lambda_2, \ldots, \lambda_{\mathfrak{b}(k)+d+1}$ denote the eigenvalues of matrix V. Then we have

$$\begin{aligned}
\det V &= \Pi_{i=1}^{\mathfrak{b}(k)+d+1} \lambda_i \\
&\leq \left(\frac{\sum_{i=1}^{\mathfrak{b}(k)+d+1} \lambda_i}{\mathfrak{b}(k)+d+1}\right)^{\mathfrak{b}(k)+d+1} \\
&= \left(\lambda + \frac{\sum_{i=1}^{t} \|\phi(p_i, x_i)\|_2^2}{\mathfrak{b}(k)+d+1}\right)^{\mathfrak{b}(k)+d+1} \\
&\leq \left(\lambda + \frac{|\mathcal{D}|\lambda}{\mathfrak{b}(k)+d+1}\right)^{\mathfrak{b}(k)+d+1}, \qquad (51)
\end{aligned}$$

where the first equality follows from the fact that the determinant of a matrix equals the product of its all eigenvalues, the second inequality follows from the inequality of arithmetic and geometric means, the second equality follows from the fact that the sum of eigenvalues of a matrix equals its trace, and the last inequality holds since $||\phi(p_i, x_i)||^2 = \sum_{j=0}^{\mathfrak{b}(k)} (p_i - l)^j + ||x_i||^2 \leq \frac{1-(u-l)^{2\mathfrak{b}(k)}}{1-(u-l)^2} + d \leq \frac{1}{1-(\overline{p}-\underline{p})^2/N^2} + d = \lambda$ and $t = |\mathcal{D}|$.

Therefore, with probability at least $1-\epsilon$, for any $t \in [T]$, $p \in \mathbf{I}$ and $x \in [0,1]^d$,

$$\begin{aligned}
\left|f(p) + a^\top x - (\hat{\theta}^\top \varphi(p) + \hat{a}^\top x)\right| &\leq \left|P_I(p) + a^\top x - (\hat{\theta}^\top \varphi(p) + \hat{a}^\top x)\right| + |f(p) - P_I(p)| \\
&\leq \left\|(\bar{\theta}-\theta^*)\right\|_V \|\phi(p,x)\|_{V^{-1}} + \Delta \\
&\leq \gamma \|\phi(p,x)\|_{V^{-1}} + \Delta,
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from Cauchy-Schwarz inequality, and the last inequality follows from Eqs. (50) and (51) and the definition of γ.

Appendix D. Proof of Theorem 4

The worst-case bound $\Omega(\sqrt{T})$ is directly implied from Theorem 1 in [18] by letting $a = \mathbf{0}$ and $f(p) = \alpha + \beta p$ (which belongs to $\mathcal{F}^k([\underline{p}, \overline{p}]; \delta)$ when α and β are appropriately defined to adapt to δ). To show the instance-dependent bound $\Omega((\delta T^{k+1})^{\frac{1}{2k+1}})$, we first construct a series of demand functions and use the Kullback-Leibler (KL) divergence arguments to bound the regret. Note that the smoothstep function adopted by [32] can not be used here because in our problem k is not necessarily a constant. Besides, our analysis also differs from [32] in that the constructed demand functions need to be instance-dependent such that the established lower bound achieves a tight dependency on δ.

Specifically, following from a similar idea of [15], we start with introducing an infinitely differentiable function $u(x)$ defined as

$$u(x) := \begin{cases} \exp\{-\frac{1}{x(1-x)}\} & \text{if } x \in [0,1]; \\ 0 & \text{otherwise .} \end{cases}$$

Consider $S : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$S(x) = \left(\int_0^2 u(t)dt\right)^{-1} \int_{-\infty}^{x} u(t)dt. \quad (52)$$

Note that $S(x)$ is non-decreasing infinitely differentiable function satisfying $S(x) = 0$ on $(-\infty, 0]$ and $S(x) = 1$ on $[1, \infty)$. For any integer $l \geq 1$, the l-th derivative of $S(x)$ at $x \in [0,1]$ is in the form

of $\frac{poly(x)}{(x(1-x))^{2(l-1)}} \exp\left(-\frac{1}{x(1-x)}\right)$, which is bounded in the domain. Moreover, $S^{(l)}(1) = S^{(l)}(0) = 0$. Besides, there exist a constant c_k, such that for any $x, x' \in \mathbb{R}$, $|S^{(\mathfrak{b}(k))}(x) - S^{(\mathfrak{b}(k))}(x')| \leq c_k|x - x'|^{k-\mathfrak{b}(k)}$. Based on $S(x)$, we can define $g(\cdot) : [0, 2] \to [0, \frac{1}{Z_k}]$ as follows:

$$g_k(x) = \frac{1}{Z_k}(S(x)\mathbf{1}\{x \leq 1\} + S(2-x)\mathbf{1}\{x > 1\}), \tag{53}$$

where $Z_k > 0$ is a scaling parameter which makes the l-th derivatives of $g_k(x)$ uniformly bounded by constant 1 on $[0, 2]$ for each $0 \leq l \leq \mathfrak{b}(k)$, and $|g_k^{(\mathfrak{b}(k))}(x) - g_k^{(\mathfrak{b}(k))}(x')| \leq |x - x'|^{k-\mathfrak{b}(k)}$ for any $x, x' \in \mathbb{R}$, e.g., $Z_k := \max_{0\leq k'\leq k} \max_{x\in[0,1]} |S^{(k)}(x)| \vee c_k$. Since $S(1) = 1$, $Z_k \geq 1$ naturally holds.

We partition the price range $[\underline{p}, \overline{p}]$ into J segments of equal length, denoted by $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_J$. We construct a series of revenue functions $r_0, r_1, r_2, \ldots, r_J$ as following,

$$r_j(p) := \begin{cases} \frac{1}{2}\hat{\delta} & \text{if } p \notin \mathbf{I}_j; \\ \frac{1}{2}\hat{\delta} + \eta \cdot g_k\big(2J(p - a_j)\big) & \text{if } p \in \mathbf{I}_j, \end{cases} \tag{54}$$

where $\hat{\delta} = \delta/((\sum_{i=0}^{\mathfrak{b}(k)} \frac{\mathfrak{b}(k)!}{i!}) \vee (\mathfrak{b}(k)+1)2^{\mathfrak{b}(k)-1})$. By choosing $J := \lceil 4(\mathfrak{b}(k)+1)2^{\mathfrak{b}(k)}\hat{\delta}^{\frac{2}{2k+1}}T^{\frac{1}{2k+1}}\rceil$ and $\eta = ((2\sigma) \wedge \frac{1}{2^{3k+1}})\frac{1}{((\mathfrak{b}(k)+1)2^{\mathfrak{b}(k)})^k}\hat{\delta}^{\frac{1}{2k+1}}T^{-\frac{k}{2k+1}}$, we can have the following lemma on the demand function $f_j(p) = \frac{f_j(p)}{p}$, whose proof is provided in Appendix D.1.

Lemma 3 *For each* $0 \leq j \leq J$, $f_j(p) \in \mathcal{F}^k([\underline{p}, \overline{p}]; \delta)$.

Note that for each $j \in [J]$, the induced optimal price of $f_j(p)$ belongs to $\mathbf{I}_j$ and $r_j(p)$ differs from $r_0(p)$ only in $\mathbf{I}_j$, with the maximum difference characterized by parameter η. It's important to note that η is a crucial quantity that balances the tradeoff between making it a more challenging task for the algorithm to distinguish between different demand functions (which requires η to be small) and imposing a higher regret loss if the algorithm fails to identify the true demand environment (which requires η to be large). For any policy π, consider the random variable T_j denoting the number of times the prices selected by π fall into segment $\mathbf{I}_j$. Similar to [32], we establish the following inequalities: for each $j \in [J]$,

$$\left|\mathbb{E}_j^\pi[T_j] - \mathbb{E}_0^\pi[T_j]\right| \leq \frac{1}{2}T\sqrt{\mathrm{KL}(\mathbb{P}_0^\pi(T_j)||\mathbb{P}_j^\pi(T_j))} \leq \frac{1}{4\sigma}\sqrt{\mathbb{E}_0^\pi[T_j]}T\eta. \tag{55}$$

In Eq. (55), the first inequality is obtained by bounding $\left|\mathbb{E}_j^\pi[T_j] - \mathbb{E}_0^\pi[T_j]\right|$ via the total variation of $\mathbb{P}_0^\pi$ and $\mathbb{P}_j^\pi$ and applying Pinsker's inequality that relates the total variation of two probability measures with the KL divergence, and the second inequality is due to our construction of r_j. Letting $j^* := \arg\min_{1\leq j\leq J} \mathbb{E}_0^\pi[T_j]$, we must have $\mathbb{E}_0^\pi[T_j] \leq \frac{T}{J}$.Since we set $\eta = \Theta((\delta T^{-k})^{\frac{1}{2k+1}})$ and $J = \Theta((\delta^2 T)^{\frac{1}{2k+1}})$, Eq. (55) guarantees that $\mathbb{E}_{j^*}^\pi[T_{j^*}] \leq \frac{T}{2}$. This indicates that when the true revenue function is r_{j^*}, there are at least $\frac{T}{2}$ times when the selected prices do not fall into the "best" segment $\mathbf{I}_{j^*}$, leading to the regret loss $\Omega(T\eta) = \Omega((\delta T^{k+1})^{\frac{1}{2k+1}})$.

As discussed before, it suffices to prove the lower bound $\Omega((\delta T^{k+1})^{\frac{1}{2k+1}})$. Note that we also only need to consider the case $\delta \geq T^{-\frac{1}{2}}$, since otherwise, $(\delta T^{k+1})^{\frac{1}{2k+1}} < \sqrt{T}$, and the desired lower bound in (2) becomes $\Omega(\sqrt{T})$, which is again obtained. For simplicity, we assume $[\underline{p}, \overline{p}] = [1, 2]$ and $\mathcal{D}$ is a standard normal distribution.

Based on what we have constructed, for any given policy π, let T_j be the number of times when the prices selected by π fall into segment $\mathbf{I}_j$. We then claim the following inequality for any $1 \leq j \leq J$:

$$\left|\mathbb{E}_0^\pi[T_j] - \mathbb{E}_j^\pi[T_j]\right| \leq \frac{1}{4\sigma}\sqrt{\mathbb{E}_0^\pi[T_j]}T\eta, \tag{56}$$

where $\mathbb{E}_0^\pi[\cdot]$ and $\mathbb{E}_j^\pi[\cdot]$ denote the expectation associated with the probability measure induced by policy π under demand model r_0 and r_j respectively. The proof of (56) is deferred to the last part,

and we now proceed to prove the regret lower bound based on (56). Consider the index $j^* \in [J]$ that minimizes $\mathbb{E}_0^\pi[T_{j^*}]$. By Pigeonhole principle, we have $\mathbb{E}_0^\pi[T_{j^*}] \le T/J$. From (56), we further have

$$\mathbb{E}_{j^*}^\pi[T_{j^*}] \le \frac{1}{4\sigma}\sqrt{\mathbb{E}_0^\pi[T_{j^*}]}T\eta + \mathbb{E}_0^\pi[T_{j^*}] \le \frac{1}{4\sigma}\sqrt{\frac{T}{J}}T\eta + \frac{T}{J} \le \frac{T}{2}, \tag{57}$$

where the first inequality follows from (56), the second inequality follows from the choice of j, and the last inequality holds since $\eta^2 \le (2\sigma\hat{\delta}^{\frac{1}{2k+1}}T^{-\frac{k}{2k+1}})^2 = 4\sigma^2\hat{\delta}^{\frac{2}{2k+1}}T^{-\frac{2k}{2k+1}} \le \sigma^2\frac{J}{T}$ implies $\frac{1}{4\sigma}\sqrt{\frac{T}{J}}T\eta \le \frac{1}{4}T$ and $\hat{\delta} \ge T^{-\frac{1}{2}}$ implies $\frac{T}{J} = \frac{1}{4}\hat{\delta}^{-\frac{2}{2k+1}}T^{\frac{2k}{2k+1}} \le \frac{1}{4}T$. Note that when the true demand function is $f_{j^*}(\cdot)$, in any period when policy π charges a price out of $\mathbf{I}_j$, a revenue loss η will be incurred by the definition of $r_j(\cdot)$. Hence, we have

$$\sup_{f\in\{f_1,f_2,\ldots,f_J\}} R_{f_j}^\pi(T,k,\delta) \ge R_{f_{j^*}}^\pi(T,k,\delta) \ge (T - \mathbb{E}_{j^*}^\pi[T_{j^*}])\eta \ge \frac{1}{2}T\eta = \Omega\left(\left(\delta T^{k+1}\right)^{\frac{1}{2k+1}}\right).$$

Finally, we complete the proof of Theorem 4 by proving (56). For the sake of rigor, we define a probability space as follows. Let $\Omega = ([1,2]\times\mathbb{R})^T \times \{0,1,2,\ldots,T\}$ and $\mathcal{B}(\Omega)$ be the Borel algebra on Ω. For any $t \in [T]$, let P_t and D_t be measurable functions on $(\Omega, \mathcal{B}(\Omega))$ that map each $\omega = (p_1, d_1, p_2, d_2, \ldots, p_T, d_T) \in \Omega$ to p_t and d_t respectively. For any $j \in \{0\}\cup[J]$, let T_j be a measurable function on $(\Omega, \mathcal{B}(\Omega))$ that maps $\omega = (p_1, d_1, p_2, d_2, \ldots, p_T, d_T) \in \Omega$ to the cardinality of the set $\{1 \le t \le T : p_t \in \mathbf{I}_j\}$. We also define two functions $\mu_i^\pi : ([1,2]\times\mathbb{R})^T \to \mathbb{R}^+$ and $\nu_i^\pi : ([1,2]\times\mathbb{R})^T \times \{0,1,2,\ldots,T\} \to \mathbb{R}^+$ as follows:

$$\nu_j^\pi(p_1,d_1,p_2,d_2,\ldots,p_T,d_T) = \prod_{t=1}^T\left(\mu^\pi(p_t|p_1,d_1,\ldots,p_{t-1},d_{t-1})\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(d_t-f_j(p_t))^2}{2\sigma^2}}\right),$$
$$\mu_j^\pi(p_1,d_1,p_2,d_2,\ldots,p_T,d_T,t_j) = \nu_j^\pi(p_1,d_1,p_2,d_2,\ldots,p_T,d_T)\cdot\mathbf{1}_{\{t_j=|1\le t\le T:p_t\in\mathbf{I}_j|\}},$$

where $\mu^\pi(p_t|p_1,d_1,\ldots,p_{t-1},d_{t-1})$ is the p.d.f. for p_t given $(p_1,d_1,\ldots,p_{t-1},d_{t-1})$. Let $\mathbb{P}_j^\pi(\cdot)$ be the following probability measure on $(\Omega,\mathcal{B}(\Omega))$: for any $B \in \mathcal{B}(\Omega)$, $\mathbb{P}_j^\pi(B) = \int_B \mu_j^\pi(w)dw$. Thus, $(\Omega,\mathcal{B}(\Omega),\mathbb{P}_j^\pi)$ constitute a probability space, and from the chain rule, $\nu_j^\pi(\cdot)$ and $\mu_j^\pi(\cdot)$ are the p.d.f. for $(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)$ and $(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)$ respectively. With a slight abuse of notation, we denote the distributions of T_j and $(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)$ by $\mathbb{P}_i^\pi(T_j)$ and $\mathbb{P}_i^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)$ respectively, and the conditional probability distribution of T_j given $(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)$ by $\mathbb{P}_i^\pi(T_j|P_1,D_1,P_2,D_2,\ldots,P_T,D_T)$. For any given $0 \le i \le J$ and $0 \le j \le J$, $\mathbb{E}_i^\pi[T_j]$ is then the expectation of T_j under $\mathbb{P}_i^\pi$.

Then we note that

$$\begin{aligned}\left|\mathbb{E}_0^\pi[T_j] - \mathbb{E}_j^\pi[T_j]\right| &\le \sum_{t=0}^T t\times\left|\mathbb{P}_0^\pi(t) - \mathbb{P}_j^\pi(t)\right| \le T\times\sum_{t=0}^T\left|\mathbb{P}_0^\pi(t) - \mathbb{P}_j^\pi(t)\right| \\ &= \frac{1}{2}T||\mathbb{P}_0^\pi(T_j) - \mathbb{P}_j^\pi(T_j)||_{\text{TV}} \le \frac{1}{2}T\sqrt{\frac{1}{2}\text{KL}(\mathbb{P}_0^\pi(T_j)||\mathbb{P}_j^\pi(T_j))},\end{aligned} \tag{58}$$

where the first identity follows from the property of the total variation distance for discrete random variables, see, e.g., Proposition 4.2 in [19], and the last inequality follows from Pinsker's inequality.

To further bound $\mathrm{KL}(\mathbb{P}_0^\pi(T_j)||\mathbb{P}_j^\pi(T_j))$, we note that

$$\begin{aligned}
\mathrm{KL}\Big(\mathbb{P}_0^\pi(T_j)\,||\,\mathbb{P}_j^\pi(T_j)\Big) &= \mathrm{KL}\Big(\mathbb{P}_0^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)\,||\,\mathbb{P}_j^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)\Big) \\
&\quad - \mathrm{KL}\Big(\mathbb{P}_0^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T|T_j)\,||\,\mathbb{P}_j^\pi(P_1,D_1,P_2,D_2,\ldots,p_T,D_T|T_j)\Big) \\
&\le \mathrm{KL}\Big(\mathbb{P}_0^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)\,||\,\mathbb{P}_j^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)\Big) \\
&= \mathbb{E}_0^\pi\left[\mathbb{E}_0^\pi\left[\log\frac{\mu_0^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)}{\mu_j^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T,T_j)}\Big|(P_1,D_1,\ldots,P_T,D_T)\right]\right] \\
&= \mathbb{E}_0^\pi\left[\log\frac{\nu_0^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)}{\nu_j^\pi(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)}\right] \\
&= \frac{1}{2\sigma^2}\sum_{t=1}^T \mathbb{E}_0^\pi\left[(D_t-f_j(P_t))^2-(D_t-f_0(P_t))^2\right] \\
&= \frac{1}{2\sigma^2}\sum_{t=1}^T \mathbb{E}_0^\pi\left[(f_0(P_t)-f_j(P_t))^2\right] \\
&= \frac{1}{2\sigma^2}\sum_{t=1}^T \mathbb{E}_0^\pi\left[\mathbf{1}_{\{P_t\in\mathbf{I}_j\}}(f_0(P_t)-f_j(P_t))^2\right] \\
&\le \frac{1}{2\sigma^2}\mathbb{E}_0^\pi[T_j]\cdot\max_{p\in\mathbf{I}_j}(f_0(p)-f_j(p))^2 \\
&= \frac{1}{2\sigma^2}\mathbb{E}_0^\pi[T_j]\eta^2, \qquad (59)
\end{aligned}$$

where the first identity follows from the chain rule for KL divergence, the first inequality follows from the fact that the KL divergence between any two probability distributions is non-negative, the second identity holds due to the definition of KL divergence and the law of total expectation, the third identity holds since given $(P_1,D_1,P_2,D_2,\ldots,P_T,D_T)$, T_j takes the value $|\{1\le t\le T: P_t\in\mathbf{I}_j\}|$ with probability one, and when $T_i=|\{1\le t\le T: P_t\in\mathbf{I}_j\}|$, $\mu_i^\pi(P_1,D_2,P_2,D_2,\ldots,P_T,D_T,T_j)=\nu_i^\pi(P_1,D_2,P_2,D_2,\ldots,P_T,D_T)$, the fifth identity holds since $D_t=f_0(P_t)+\varepsilon_t$, $\mathbb{E}_0^\pi[\varepsilon_t]=0$ and P_t is independent of ε_t, and the sixth identity holds since f_0 and f_j are only different in $\mathbf{I}_j$. Then (56) is obtained by combining (58) with (59).

D.1. Proof of Lemma 3

The result for $j=0$ is trivial. When $1\le j\le J$, from the properties (1) and (2) of $S(x)$, $g_k(x)$ is infinitely differentiable. Now we check the property of $f_j^{(\mathfrak{b}(k))}(p)$.

$$\begin{aligned}
\left|f_j^{\mathfrak{b}(k)}(p_1)-f_j^{\mathfrak{b}(k)}(p_2)\right| &= \left|\left(\frac{r_j(p_1)}{p_1}\right)^{(\mathfrak{b}(k))}-\left(\frac{r_j(p_2)}{p_2}\right)^{(\mathfrak{b}(k))}\right| \\
&= \left|\sum_{i=0}^{\mathfrak{b}(k)}\binom{\mathfrak{b}(k)}{i}\left(\frac{1}{p_1}\right)^{(\mathfrak{b}(k)-i)}r_j^{(i)}(p_1)-\sum_{i=0}^{\mathfrak{b}(k)}\binom{\mathfrak{b}(k)}{i}\left(\frac{1}{p_2}\right)^{(\mathfrak{b}(k)-i)}r_j^{(i)}(p_2)\right| \\
&= \left|\sum_{i=0}^{\mathfrak{b}(k)}\binom{\mathfrak{b}(k)}{i}(-1)^{(\mathfrak{b}(k)-i)}(\mathfrak{b}(k)-i)!\left(\frac{r_j^{(i)}(p_1)}{p_1^{\mathfrak{b}(k)-i+1}}-\frac{r_j^{(i)}(p_2)}{p_2^{\mathfrak{b}(k)-i+1}}\right)\right| \\
&\le \left(\sum_{i=0}^{\mathfrak{b}(k)}\frac{\mathfrak{b}(k)!}{i!}\right)\max_{0\le i\le\mathfrak{b}(k)}\left|\frac{r_j^{(i)}(p_1)}{p_1^{\mathfrak{b}(k)-i+1}}-\frac{r_j^{(i)}(p_2)}{p_2^{\mathfrak{b}(k)-i+1}}\right|, \qquad (60)
\end{aligned}$$

where the second identity follows the general Leibniz rule. Now we turn to the RHS of Eq. (60). In the following, we discuss in three cases: (1) $p_1,p_2\in\mathbf{I}_j$, (2) $p_1\in\mathbf{I}_j$ and $p_2\notin\mathbf{I}_j$, and (3) $p_1,p_2\notin\mathbf{I}_j$.

Case 1: $p_1 \in \mathbf{I}_j$ and $p_2 \in \mathbf{I}_j$.

If $0 \le i \le \mathfrak{b}(k) - 1$, we have

$$
\begin{aligned}
&\left| \frac{r_j^{(i)}(p_1)}{p_1^{\mathfrak{b}(k)-i+1}} - \frac{r_j^{(i)}(p_2)}{p_2^{\mathfrak{b}(k)-i+1}} \right| \\
&\le \left| \frac{r_j^{(i)}(p_1)}{p_1^{\mathfrak{b}(k)-i+1}} - \frac{r_j^{(i)}(p_2)}{p_1^{\mathfrak{b}(k)-i+1}} \right| + \left| \frac{r_j^{(i)}(p_2)}{p_1^{\mathfrak{b}(k)-i+1}} - \frac{r_j^{(i)}(p_2)}{p_2^{\mathfrak{b}(k)-i+1}} \right| \\
&\le \left| r_j^{(i)}(p_1) - r_j^{(i)}(p_2) \right| + \max_{p\in[1,2]} \left| r_j^{(i)}(p) \right| \left| \frac{p_1^{\mathfrak{b}(k)-i+1} - p_2^{\mathfrak{b}(k)-i+1}}{p_1^{\mathfrak{b}(k)-i+1} p_2^{\mathfrak{b}(k)-i+1}} \right| \\
&\le \max_{p\in[1,2]} \left| r_j^{(i+1)}(p) \right| |p_1 - p_2| + \max_{p\in[1,2]} \left| r_j^{(i)}(p) \right| |p_1 - p_2| \left| \sum_{q=0}^{\mathfrak{b}(k)-i} p_1^q p_2^{\mathfrak{b}(k)-i-q} \right| \\
&\le \eta(2J)^{i+1} \max_{x\in[0,2]} \left| g_k^{(i+1)}(x) \right| |p_1 - p_2| + \eta(2J)^i \max_{x\in[0,2]} \left| g_k^{(i)}(x) \right| |p_1 - p_2| \, (\mathfrak{b}(k) - i + 1) 2^{\mathfrak{b}(k)-i} \\
&\le \left(\eta(2J)^{i+1} + (\mathfrak{b}(k) - i + 1)\eta(2J)^i 2^{\mathfrak{b}(k)-i} \right) |p_1 - p_2| \\
&\le 2\eta(2J)^{i+1} |p_1 - p_2|, \qquad (61)
\end{aligned}
$$

where the third inequality follows from mean value theorem, the existence of the $i+1$-th derivatives, $p_1 \ge 1$, $p_2 \ge 1$ and the fact that $p_1^{\mathfrak{b}(k)-i} - p_2^{\mathfrak{b}(k)-i+1} = (p_1 - p_2)(\sum_{q=0}^{\mathfrak{b}(k)-i+1} p_1^q p_2^{\mathfrak{b}(k)-i-q})$, the fourth inequality holds due to $p_1 \le 2$ and $p_2 \le 2$, the fifth inequality holds by our construction that $\max_{x\in[0,2]} \left| g_k^{(i)}(x) \right| \le 1$, the last inequality follows from $(\mathfrak{b}(k) + 1)2^{\mathfrak{b}(k)} \le 2J$. Then, for $i = \mathfrak{b}(k)$,

$$
\begin{aligned}
&\left| \frac{r_j^{(\mathfrak{b}(k))}(p_1)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_2} \right| \\
&\le \left| \frac{r_j^{(\mathfrak{b}(k))}(p_1)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_1} \right| + \left| \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_2} \right| \\
&\le \eta(2J)^{\mathfrak{b}(k)} \left| g^{(\mathfrak{b}(k))}\left(2J(p_1 - a_j)\right) - g^{(\mathfrak{b}(k))}(2J(p_2 - a_j)) \right| + |p_1 - p_2| \max_{p\in[1,2]} \left| r_j^{\mathfrak{b}(k)}(p) \right| \\
&\le \eta(2J)^{\mathfrak{b}(k)} \left| 2J(p_1 - p_2)) \right|^{k-\mathfrak{b}(k)} + \eta(2J)^{\mathfrak{b}(k)} |p_1 - p_2| \\
&\le \left(\eta(2J)^k + \eta(2J)^{\mathfrak{b}(k)} \right) |p_1 - p_2|^{k-\mathfrak{b}(k)} \\
&\le 2\eta(2J)^k |p_1 - p_2|^{k-\mathfrak{b}(k)}, \qquad (62)
\end{aligned}
$$

where the second inequality holds because $\left| \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_2} \right| \le \left| \frac{1}{p_1} - \frac{1}{p_2} \right| \max_{p\in[1,2]} \left| r_j^{(\mathfrak{b}(k))}(p) \right| \le |p_1 - p_2| \max_{p\in[1,2]} \left| r_j^{(\mathfrak{b}(k))}(p) \right|$, the third inequality follows that $g^{(\mathfrak{b}(k))}(\cdot)$ is $(k - \mathfrak{b}(k))$-Hölder continuous, the fourth inequality holds because of

$|p_1 - p_2| \leq 1$ and $k - \mathfrak{b}(k) \leq 1$. Then, Eq. (60) can be simplified

$$\begin{aligned}
&\left| f_j^{\mathfrak{b}(k)}(p_1) - f_j^{\mathfrak{b}(k)}(p_2) \right| \\
&\leq \left(\sum_{i=0}^{\mathfrak{b}(k)} \frac{\mathfrak{b}(k)!}{i!} \right) \max \left\{ \max_{0 \leq i \leq \mathfrak{b}(k)-1} 2\eta(2J)^{i+1} |p_1 - p_2|, 2\eta(2J)^k |p_1 - p_2|^{k-\mathfrak{b}(k)} \right\} \\
&\leq \left(\sum_{i=0}^{\mathfrak{b}(k)} \frac{\mathfrak{b}(k)!}{i!} \right) \max \left\{ 2\eta(2J)^{\mathfrak{b}(k)} |p_1 - p_2|, 2\eta(2J)^k |p_1 - p_2|^{k-\mathfrak{b}(k)} \right\} \\
&= \left(\sum_{i=0}^{\mathfrak{b}(k)} \frac{\mathfrak{b}(k)!}{i!} \right) \left(2\eta(2J)^k |p_1 - p_2|^{k-\mathfrak{b}(k)} \right) \\
&\leq \delta |p_1 - p_2|^{k-\mathfrak{b}(k)}, \qquad (63)
\end{aligned}$$

where the last inequality holds due to $2\eta(2J)^k \leq 2\frac{1}{2^{3k+1}} \frac{1}{((\mathfrak{b}(k)+1)2^{\mathfrak{b}(k)})^k} \hat{\delta}^{\frac{1}{2k+1}} T^{-\frac{k}{2k+1}} (8(\mathfrak{b}(k) + 1)2^{\mathfrak{b}(k)} \hat{\delta}^{\frac{2}{2k+1}} T^{\frac{1}{2k+1}})^k = \hat{\delta}$.

<u>Case 2: $p_1 \in \mathbf{I}_j$ and $p_2 \notin \mathbf{I}_j$.</u>

Note that Eq. (61) still hold in this case. What we need to derive is the bound for $|\frac{r_j^{(\mathfrak{b}(k))}(p_1)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_2}|$ (i.e., Eq. (62) can not be directly applied here). Define $p_2' := \text{Proj}(p_2, \mathbf{I}_j)$. Note that p_2' is either a_j or b_j, $r_j(a_j) = r_j(b_j) = \frac{1}{2}\hat{\delta}$ and $r_j^{(i)}(a_j) = r_j^{(i)}(b_j) = 0$ for all $1 \leq i \leq \mathfrak{b}(k)$. Thus, we can have

$$\begin{aligned}
\left| \frac{r_j^{(\mathfrak{b}(k))}(p_1)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2)}{p_2} \right| &= \left| \frac{r_j^{(\mathfrak{b}(k))}(p_1)}{p_1} - \frac{r_j^{(\mathfrak{b}(k))}(p_2')}{p_2'} \right| \\
&\leq 2\eta(2J)^k |p_1 - p_2'|^{k-\mathfrak{b}(k)} \leq 2\eta(2J)^k |p_1 - p_2|^{k-\mathfrak{b}(k)}, \qquad (64)
\end{aligned}$$

where the first equality holds due to $r_j^{(\mathfrak{b}(k))}(p_2) = r_j^{(\mathfrak{b}(k))}(p_2') = 0$, the first inequality follows from Eq. (62) because p_1 and p_2' are both in $\mathbf{I}_j$, and the second inequality holds due to the projection process. Together with Eqs. (60), (61) and (64), by the same calculation of Eq. (63), we can know $|f_j^{\mathfrak{b}(k)}(p_1) - f_j^{\mathfrak{b}(k)}(p_2)| \leq \delta |p_1 - p_2|^{k-\mathfrak{b}(k)}$.

<u>Case 3: $p_1 \notin \mathbf{I}_j$ and $p_2 \notin \mathbf{I}_j$.</u>

When p_1 and p_2 are not in $\mathbf{I}_j$, $r_j^{(i)}(p_1) = r_j^{(i)}(p_2) = 0$, for all $1 \leq i \leq \mathfrak{b}(k)$. From the first two lines of Eq. (60), we can have

$$\begin{aligned}
\left| f_j^{\mathfrak{b}(k)}(p_1) - f_j^{\mathfrak{b}(k)}(p_2) \right| &= \left| \frac{r_j(p_1)}{p_1^{\mathfrak{b}(k)+1}} - \frac{r_j(p_2)}{p_2^{\mathfrak{b}(k)+1}} \right| \\
&\leq \frac{1}{2}\hat{\delta} \left| p_1^{\mathfrak{b}(k)+1} - p_2^{\mathfrak{b}(k)+1} \right| \\
&\leq \frac{1}{2}\hat{\delta} |p_1 - p_2| \left| \sum_{q=0}^{\mathfrak{b}(k)} p_1^q p_2^{\mathfrak{b}(k)-q} \right| \\
&\leq \frac{1}{2}\hat{\delta} (\mathfrak{b}(k) + 1) 2^{\mathfrak{b}(k)} |p_1 - p_2| \\
&\leq \delta |p_1 - p_2|^{k-\mathfrak{b}(k)},
\end{aligned}$$

where the third inequality holds due to $p_1 \leq 2$ and $p_2 \leq 2$, and the last inequality follows from $\hat{\delta} \leq \delta/((\mathfrak{b}(k) + 1)2^{\mathfrak{b}(k)} - 1)$.

Together with the above three cases, we draw the conclusion that for any $p_1, p_2 \in [1, 2]$, we have $|f_j^{\mathfrak{b}(k)}(p_1) - f_j^{\mathfrak{b}(k)}(p_2)| \leq \delta |p_1 - p_2|^{k-\mathfrak{b}(k)}$. We finish the proof.