Contextual Dynamic Pricing with Unknown Noise: Explore-then-UCB Strategy and Improved Regrets

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Abstract

Dynamic pricing is a fast-moving research area in machine learning and operations management. A lot of work has been done for this problem with known noise. In this paper, we consider a contextual dynamic pricing problem under a linear customer valuation model with an unknown market noise distribution F. This problem is very challenging due to the difficulty in balancing three tangled tasks of revenue-maximization, estimating the linear valuation parameter θ_0, and learning the nonparametric F. To address this issue, we develop a novel *Explore-then-UCB* (ExUCB) strategy that includes an exploration for θ_0-learning and a followed UCB procedure of joint revenue-maximization and F-learning. Under Lipschitz and concavity assumptions on F, ExUCB is the first approach to achieve the $\tilde{O}(T^{2/3})$ regret rate. Under the Lipschitz assumption only, ExUCB can match the best existing regret of $\tilde{O}(T^{3/4})$. Furthermore, for regret lower bounds under the nonparametric F, not much work has been done beyond only assuming Lipschitz. To fill this gap, we provide the first $\tilde{\Omega}(T^{3/5})$ lower bound under Lipschitz and concavity assumptions.

1 Introduction

Dynamic pricing is a process of continuously adjusting the prices by learning from the customers' feedback. The feedback usually depends on the pricing action. To maximize the overall revenues in a sales horizon, a pricing policy should well balance between learning the customers' demands (exploration) and setting revenue-maximizing prices based on the current knowledge (exploitation). There is a rich literature on this information-regret tradeoff under different settings [9, 30, 18, 14, 22].

In this paper, we consider an important setting in dynamic pricing where some contextual information is available in each selling time period. It is interesting and challenging to improve the revenues by well exploiting the sales-relevant contextual information such as product features and market environments.

To model the influence of the contextual information, we adopt a binary feedback model which incorporates a comparison between the customer's valuation and the seller's price [4, 28, 47, 23, 35]. Specifically, in the selling time period t with the associated context $x_t \in \mathbb{R}^{d_0}$, the customer's valuation v_t of the product is assumed to be linear with repect to x_t, together with some noise z_t, i.e., $v_t = v(x_t) = x_t^\top \theta_0 + z_t$. Here the noises $\{z_t\}_{t\in[T]}$ across the time horizon $[T] = \{1, \ldots, T\}$ are independent and identically distributed (i.i.d.) from a Cumulative Distribution Function (CDF) F. Then under the seller's price p_t, a binary feedback $y_t = 1_{\{v_t \geq p_t\}}$ representing the customer's purchasing decision is observed by the seller. Namely, the purchase at time period t happens if and only if the customer's valuation v_t is greater than or equal to the seller's price p_t. The seller then collects the current feedback y_t which can help the pricing decision at the next time period. Note

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Table 1: Regret bounds under different smoothness assumptions on noise.

Smoothness Assumptions on Unknown Noise CDF F	*Upper Bound*	*Lower Bound*
$m(\geq 2)$ times continuously differentiable	$\tilde{O}(T^{\frac{2m+1}{4m-1}})$[23]	
Arbitrary	$\tilde{O}(T^{\frac{3}{4}})$ [48]	$\tilde{\Omega}(T^{\frac{2}{3}})$[48]
Lipschitz	$\tilde{O}(T^{\frac{3}{4}})$ **(This work)**	
Lipschitz and concavity	$\tilde{O}(T^{\frac{2}{3}\vee(1-\alpha)})$[35]	$\tilde{\Omega}(T^{\frac{3}{5}})$ **(This work)**
	$\tilde{O}(T^{\frac{2}{3}})$ **(This work)**	

that the binary feedback structure $y_t = 1_{\{v_t \geq p_t\}}$ is critical to this online pricing problem's bandit nature. We cannot observe the full information v_t at each time period. Instead, we can only observe the partial information $y_t = 1_{\{v_t \geq p_t\}}$ which varies with different pricing action p_t.

At each time period t, the expected reward of any price p is $\mathbb{E}\big(p1_{\{v_t \geq p\}}\big)$. Thus, the optimal pricing depends on the distribution of the valuation $v_t = v(x_t) = x_t^\top \theta_0 + z_t$. We assume both unknown parameter θ_0 and unknown noise CDF F at the beginning of the horizon. Therefore, the seller needs to gradually learn both θ_0 and F for better pricing. In this paper, we investigate two different smoothness assumptions for the noise CDF F. The first one assumes Lipschitz continuity and concavity, while the second only assumes Lipschitz.

To tackle these two sub-problems under different noise smoothness assumptions, we propose two sub-policies unified under a single pricing policy framework, namely *Explore then Upper Confidence Bound* (ExUCB). In particular, our proposed ExUCB first estimates θ_0 through some random pricing exploration phase, and then uses this estimate to drive a UCB procedure that well balances revenue-maximizing and F-learning.

The same pricing problem has been investigated in the literature [35, 23, 48] under a variety of smoothness assumptions on noise. The existing regret bounds are presented in Table 1 along with our new regret results in this paper. In summary, our theoretical contributions are threefold.

1. Under the **Lipschitz and concavity** smoothness assumptions, our proposed ExUCB policy is the first procedure to achieve $\tilde{O}(T^{2/3})$ regret rate. We prove valid θ_0 estimation accuracy and quantify its influence on the regret of the UCB procedure driven by the θ_0-estimate. This helps us determine an optimal balance between random pricing exploration and the followed UCB phase. Our $\tilde{O}(T^{2/3})$ regret improves the existing regret bound of $\tilde{O}(T^{2/3\vee(1-\alpha)})$ in [35] which has an indeterministic α. It also improves the $\tilde{O}(T^{5/7})$ regret in [23] which assumes a stronger smoothness assumption of twice continuously differentiable F.
2. Under the **Lipschitz and concavity** smoothness assumptions, we obtain the lower bound of $\tilde{\Omega}(T^{3/5})$ by constructing the instances such that any policy cannot perform well on all of them. To our limited knowledge, this is the first lower bound result under such smoothness assumptions. To construct instances under the Lipschitz and concavity assumptions, the core step is to build "bump towers" by piling up an infinite series of "quadratically shrinking" basic bump functions. Note that the $\tilde{\Omega}(T^{2/3})$ lower bound in [48] only applies to the Lipschitz assumption since their constructed instances do not satisfy the concavity assumption.
3. Under the **Lipschitz** smoothness assumption, ExUCB can match the best existing upper bound of $\tilde{O}(T^{3/4})$ in [48], which shows the adaptivity of ExUCB to different levels of smoothness assumptions on the noise.

Our improved regret over existing results in [35, 23] demonstrates the methodological novelty of the proposed *Explore-then-UCB* strategy. In [35], the authors implemented the UCB idea but did not apply random pricing explorations. Thus they can only use adaptive data to estimate θ_0, which results in an indeterministic α in their regret of $\tilde{O}(T^{2/3\vee(1-\alpha)})$. In contrast, by using a random exploration phase, ExUCB achieves the exact regret of $\tilde{O}(T^{2/3})$. In [23], the authors proposed an Explore-then-Commit type of policy that first estimates θ_0 and F in an exploration phase and then commits to these estimates for pricing in the exploitation phase. In comparison, ExUCB imposes

a UCB procedure to adaptively balance between revenue-maximizing and F-learning. The regret advantage of ExUCB indicates the importance of our UCB procedure after the exploration.

2 Related Works

In this section, we discuss how our work relates to the literature of dynamic pricing and bandit.

Non-Contextual Dynamic Pricing. Extensive investigations have been conducted on dynamic pricing problems without contextual information [9, 11, 46, 10, 16]. Due to the bandit nature of online pricing problems, the Upper Confidence Bound (UCB) idea [5, 1] has been successfully applied in several different non-contextual settings [32, 37, 44]. However, these methods are not able to utilize the potential contextual information for better pricing.

Contextual Dynamic Pricing. There are significant interests among researchers in contextual dynamic pricing [41, 27, 36, 38, 8, 43, 7, 21, 45, 29, 15, 17]. With fully nonparametric expected revenue functions, [15] designed the pricing policy with $\tilde{O}(T^{(d_0+2)/(d_0+4)})$ regret based on the adaptive binning idea from nonparametric contextual bandit [40]. In [43], the authors adopted a log-linear valuation model with a very special property that the linear structure and noise distribution have separable influences on optimal prices. They proposed a pricing algorithm with $\tilde{O}(T^{1/2})$ regret that took advantage of this property. Some recent literature [28, 25, 26, 47, 23, 35, 48] adopted the same linear valuation model $v_t = v(x_t) = x_t^\top \theta_0 + z_t$ as in this work. By assuming a known noise distribution, [28, 47] designed algorithms with $O(\log T)$ regret. For noise in a known parametric family, [28] proposed a pricing policy with $O(T^{1/2})$ regret. In [26], the authors assumed a known ambiguity set that contains the noise distribution and proposed an algorithm with a $\tilde{O}(T^{2/3})$ regret with respect to a robust benchmark corresponding to the ambiguity set. With unknown noise distribution, the ambiguity set would be extremely large and the robust benchmark can be far from the benchmark corresponding to the true optimal policy. In [25], the authors considered unknown noise with a "full information" feedback structure that the seller can observe all valuations (bids) and proposed an algorithm with $\tilde{O}(T^{1/2})$ regret.

The most related works to ours are [23, 35, 48], which considered binary cencored feedback and unknown noise distributions under different smoothness assumptions. In [23], the authors proposed an Explore-then-Commit pricing policy with an $\tilde{O}(T^{\frac{2m+1}{4m-1}})$ regret under $m(\geq 2)$ times continuously differentiable F. Their twice continuously differentiable assumption ($m = 2$) would imply the Lipschitz and concavity assumptions. Thus our proposed ExUCB algorithm can achieve a lower regret of $\tilde{O}(T^{2/3})$ than the $\tilde{O}(T^{5/7})$ regret in [23] even under weaker smoothness assumptions. In [35], the authors proved an $\tilde{O}(T^{2/3\vee(1-\alpha)})$ regret. The value of α depends on the convergence rate of their proposed linear classification θ_0-estimation method and is indeterministic because the estimation is based on adaptive data. In comparison, our proposed policy achieves an exact regret of $\tilde{O}(T^{2/3})$. In [48], the authors developed an adaptive pricing policy that achieved an $\tilde{O}(T^{3/4})$ regret for adversarial contexts and arbitrary bounded noise distributions. In comparison, our ExUCB policy matches the $\tilde{O}(T^{3/4})$ regret under a Lipschitz F and can improve the regret to $\tilde{O}(T^{2/3})$ with an additional concavity assumption.

Bandit Algorithms. Bandit-type feedback structure is very natural in dynamic pricing problems [32, 15, 48]. The bandit literature provides a significant variety of methods to resolve the exploration-exploitation tradeoff that arises with the bandit feedback [12, 33]. A key tool we used in this paper is the perturbed linear bandit [35]. It is related to the misspecified linear bandits [34, 39, 24] and non-stationary linear bandits [19, 42, 49]. In addition, because of the continuous nature of the pricing actions, dynamic pricing is closely related to continuum-armed bandit [2, 31, 6, 13]. The lower bound we proved borrows the "needle in haystack" idea that is widely applied in the continuum-armed bandit literature [31, 48].

3 Preliminaries

Problem Setting The sales time horizon is $[T] = \{1, \ldots, T\}$ with the initial time period $t = 1$. We present the online pricing procedure as follows.

1. At time period t, the seller observes a context $x_t \in \mathbb{R}^{d_0}$.
2. The customer valuates the product at $v_t = x_t^\top \theta_0 + z_t$, where $z_t \overset{\text{i.i.d.}}{\sim} F$.
3. The seller sets a price p_t based on x_t and the past sales data $\{(x_s, p_s, y_s)\}_{s \le t-1}$.
4. The seller observes the binary feedback $y_t = 1_{\{v_t \ge p_t\}}$ and collects the revenue $p_t y_t$.
5. Let $t = t + 1$ and go back to Step 1.

Regret Definition Given the context x_t, the probability of a purchase is $1 - F(p_t - x_t^\top \theta_0)$ and thus the expected reward of setting the price p_t is $p_t\big(1 - F(p_t - x_t^\top \theta_0)\big)$. Define the optimal price given the context x as $p^*(x) = \arg\max_{p \ge 0} p\big(1 - F(p - x^\top \theta_0)\big)$. Then the regret r_t at time period t is defined as the expected revenue loss with respect to the optimal price $p_t^* = p^*(x_t)$, i.e., $r_t = p_t^*\big(1 - F(p_t^* - x_t^\top \theta_0)\big) - p_t\big(1 - F(p_t - x_t^\top \theta_0)\big)$.

Definition 1 *The cumulative regret across the horizon is defined as*

$$R_T = \sum_{t=1}^{T} r_t = \sum_{t=1}^{T} \Big[p_t^*\big(1 - F(p_t^* - x_t^\top \theta_0)\big) - p_t\big(1 - F(p_t - x_t^\top \theta_0)\big)\Big].$$

The expected cumulative regret $\mathbb{E}(R_T)$ is obtained from taking expectation over the potential randomness of the data and the pricing policy. Our goal is to minimize $\mathbb{E}(R_T)$ by dynamically setting the price p_t under unknown θ_0 and F.

Perturbed Linear Bandit Definition We use the Perturbed Linear Bandit (PLB) [35] formulation to tackle the balance between F-learning and revenue-maximizing. We introduce its definition here.

Definition 2 *The rewards Z_t, parameters ξ_t and action sets $\mathcal{A}_t$ form a perturbed linear bandit with a perturbation constant C_p, if*

$$Z_t = \langle \xi_t, A_t \rangle + \eta_t$$

for any selected action $A_t \in \mathcal{A}_t$, and any two parameters are close to each other, i.e., $||\xi_s - \xi_t||_\infty \le C_p, \forall s, t \in \mathbb{N}^+$. Here η_t is σ-sub-Gaussian conditional on the filtration $\mathcal{F}_{t-1} = \sigma(\xi_1, A_1, Z_1, \ldots, \xi_t, A_t)$.

The "all-close-to-each-other" condition on the linear parameters ξ_t's implies the existence of a "central" parameter ξ^* such that $||\xi_t - \xi^*||_\infty \le \frac{C_p}{2}$ for any t. Thus the reward structure at time t regulated by ξ_t can be viewed as a perturbation from that regulated by ξ^*. The linear bandit [1, 20, 3] is a zero-perturbation PLB with $\xi_t = \xi^*$ for any t.

3.1 Technical Assumptions

We summarize our main assumptions as follows.

Assumption 1 *(Bounded contexts and parameter) The covariates x_t are bounded as $||x_t||_\infty \le 1$. The ℓ_1 norm $||\theta_0||_1$ of θ_0 is bounded by a constant W.*

Assumption 2 *(i.i.d. contexts) The covariates $x_t \overset{\text{i.i.d.}}{\sim} \mathbb{P}_x$ with the support $\mathcal{X}$ and the matrix $\Sigma = \mathbb{E}\big((1, x_t^\top)^\top (1, x_t^\top)\big)$ satisfies that $\Sigma - c_0 \mathbb{I}$ is positive-definite for some positive constant c_0.*

Assumption 3 *(Bounded noise mean) The absolute value of the noise expectation $|\mathbb{E}(z)|$ is bounded by a constant U, where $z \sim F$.*

Assumption 4 *(Bounded valuations) All realized valuations $v_t \in [0, B]$ for a known constant B.*

Assumption 5 *(Bounded optimal prices) Any optimal price $p^*(x)$ for $x \in \mathcal{X}$ satisfies $p^*(x) \in (0, p_{\max})$ for a known constant $p_{\max}$.*

Assumption 6 *(Lipschitz Continuity) The noise CDF F is Lipschitz continuous with a constant L, i.e., $|F(x) - F(y)| \le L|x - y|, \forall x, y \in \mathbb{R}$.*

Assumption 7 *(Concavity) Define the general expected revenue functions associated with the noise distribution F as $f_q(p) = p(1 - F(p - q))$. There exists a positive constant C such that for any $x \in \mathcal{X}$ and $q = x^\top \theta_0$, we have $f_q(p^*(x)) - f_q(p) \leq C(p^*(x) - p)^2, \forall p \in [0, p_{\max}]$.*

Assumptions 1 – 2 are standard assumptions in dynamic pricing literature [28, 15, 26, 23, 35, 48]. Assumption 3 is mild and weaker than the bounded noise assumption in [26, 23, 48]. Assumption 4 assumes bounded valuations [26, 23] while Assumption 5 assumes bounded optimal prices [15, 35]. As mentioned in [15, 26, 23, 35], for real-life products, it is reasonable to know the upper bounds of customer valuations and optimal prices. The Lipschitz condition in 6 has also been considered in [35, 48] and is a basic smoothness assumption. Assumption 7 assumes local concavity of the expected revenue functions around the optimal prices and has also been imposed in [15, 35]. It is satisfied with bounded second derivatives of F [35] but fits for a much broader class of F that is not everywhere differentiable. Both Assumptions 6 – 7 are satisfied by $m(\geq 2)$ times continuously differentiablility of F as considered in [23].

In this paper, we investigate two smoothness levels on the unknown noise distribution F.

Case (A) Lipschitz and concavity, i.e., both Assumptions 6 – 7 hold.

Case (B) Lipschitz-only, i.e., only Assumption 6 holds.

For these two scenarios, we design respective algorithms that are unified under a single ExUCB policy framework.

4 Algorithm

We first propose the general ExUCB policy in Algorithm 1. Without the knowledge of the horizon length T, we utilize the doubling trick [33] to cut the horizon into episodes. Each episode consists of an exploration phase and a followed UCB phase. Denote the first episode length as α_1 and the number of episodes as $n(T, \alpha_1)$. The schematic of ExUCB is displayed in Figure 1.

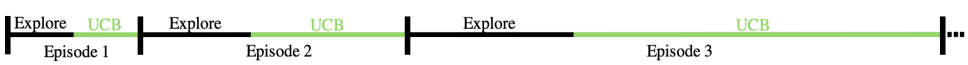


Figure 1: Schematic Representation of *Explore-then-UCB* Policy.

Algorithm 1 *Explore-then-UCB* (ExUCB)

Input: (at time 0) $p_{\max}, B, \alpha_1, C_1, C_2, \beta, \gamma, \lambda$
Input: (arrives over time) covariates $\{x_t\}_{t \in [T]}$
For episodes $k = 1, 2, \ldots, n(= n(T, \alpha_1))$, **do**
  Set the (projected) length of the k-th episode as $\ell_k = 2^{k-1}\alpha_1$.
  (Exploration Phase)
  For time $t \in \mathcal{E}_k := \{\sum_{i=1}^{k-1} \ell_i + 1, \ldots, \sum_{i=1}^{k-1} \ell_i + \lceil C_1 \ell_k^\beta \rceil\}$, **do**
    Set a price p_t uniformly randomly from $(0, B)$.
    Receive a binary response y_t.
  Calculate the θ_0-estimate $\hat{\theta}_k$ by

$$(\hat{\mu}_k, \hat{\theta}_k) = \arg\min_{\mu, \theta} \frac{1}{|\mathcal{E}_k|} \sum_{t \in \mathcal{E}_k} \left(B y_t - (1, x_t^\top)(\mu, \theta^\top)^\top \right)^2.$$

  (UCB Phase)
  For time $t \in \mathcal{U}_k := \{\sum_{i=1}^{k-1} \ell_i + \lceil C_1 \ell_k^\beta \rceil + 1, \ldots, \sum_{i=1}^{k-1} \ell_i + \ell_k\}$, **do**
    Apply the **Inner UCB Algorithm** on the coming sequential covariates $\{x_t\}_{t \in \mathcal{U}_k}$ with the θ_0-estimate $\hat{\theta}_k$, the discretization number $d_k = \lceil C_2 (\ell_k - \lceil C_1 \ell_k^\beta \rceil)^\gamma \rceil$, the optimal price bound $p_{\max}$, the projected length $(\ell_k - \lceil C_1 \ell_k^\beta \rceil)$, and the regularization parameter λ.

In the exploration phase, we conduct random pricing and finally obtain a θ_0 estimate. In the followed UCB phase, we implement a UCB procedure that uses the θ_0 estimate to drive the balance between F-learning and revenue-maximizing. We discuss the two components in the following.

Estimation of θ_0. At the beginning of each episode, we impose a random pricing exploration phase to generate data for θ_0 estimation. The exploration phase length $\lceil C_1 \ell_k^{\beta} \rceil$ is set as some β order of the episode length ℓ_k. By uniformly random pricing in the bounded valuation range, there arises a linear regression structure [26, 23] involving the signal of θ_0. Specifically, we have

$$\mathbb{E}(By_t) = B\mathbb{E}\big(\mathbb{E}(1_{\{p_t \le x_t^\top \theta_0 + z_t\}} | x_t, z_t)\big) = B\mathbb{E}(\frac{x_t^\top \theta_0 + z_t}{B}) = (1, x_t^\top)(\mu, \theta_0^\top)^\top.$$

Thus we are able to provide guarantees for $\hat{\theta}_k$ using classical analysis techniques on linear regression. In contrast, [35] applied a linear classification method on the adaptive data of the previous episode to obtain a θ_0-estimate, which lacks theoretical guarantees for the estimation accuracy.

$\hat{\theta}_k$-driven UCB Procedure. In the UCB phase of episode k, we use the obtained linear regression estimate $\hat{\theta}_k$ to guide a balance between F-learning and revenue-maximizing. Different from [35], our UCB procedure does not start at the beginning of each episode. In addition, we offer a different source of θ_0 estimate that is independent of previous episodes and allow more general discretizations tuned by a parameter γ.

Algorithm 2 Inner UCB Algorithm

1: **Input: (arrives over time)** covariates $\{x_t\}_{t \in [T_0]}$; θ_0-estimate $\hat{\theta}$; discretization number d; optimal price bound $p_{\max}$; projected length T_0; regularization parameter λ in the UCB formulation 1
2: Cut the F-learning interval $G(\hat{\theta}) = [-||\hat{\theta}||_1, p_{\max} + ||\hat{\theta}||_1]$ into d same-length sub-intervals with their midpoints denoted as $m_1, \ldots, m_d$.
3: **For** time $t = 1, \ldots, T_0$, **do**
4: Construct the candidate price set $\mathcal{S}_t = \{m_j + x_t^\top \hat{\theta} | j \in [d], m_j + x_t^\top \hat{\theta} \in (0, p_{\max})\}$.
5: Determine the available arm set $\mathcal{B}_t = \{j \in [d] : m_j + x_t^\top \hat{\theta} \in (0, p_{\max})\}$.
6: Calculate $\text{UCB}_t(1 - F(m_j))$ for $j \in \mathcal{B}_t$ as in (1).
7: Calculate $j_t \in \arg\max_{j \in \mathcal{B}_t} (m_j + x_t^\top \hat{\theta}) \text{UCB}_t(1 - F(m_j))$.
8: Set the price $p_t = m_{j_t} + x_t^\top \hat{\theta}$ and receive a binary response y_t.

The Inner UCB Algorithm is explicitly presented as Algorithm 2 above. In summary, the knowledge of F is continuously updated and the prices are set accordingly. Since any potential optimal price $p \in (0, p_{\max})$, we only need to explore those F-values on the potential range of $p - x_t^\top \theta_0$, i.e., $[-||\theta_0||_1, p_{\max} + ||\theta_0||_1]$. Thus we first restrict our F-learning attention to the interval $G(\hat{\theta}) = [-||\hat{\theta}||_1, p_{\max} + ||\hat{\theta}||_1]$. Then we implement a key discretization procedure of $G(\hat{\theta})$ and further focus the learning of F on the midpoints $\{m_j\}_{j \in [d]}$ of the discretized sub-intervals.

Now at each time period t, we only consider those prices p_t that would require some value of $\{1 - F(m_j)\}_{j \in [d]}$ to calculate its expected revenue. We construct the candidate price set $\mathcal{S}_t = \{m_j + x_t^\top \hat{\theta} | j \in [d], m_j + x_t^\top \hat{\theta} \in (0, p_{\max})\}$. Then for any candidate price $p_t = m_j + x_t^\top \theta_0 \in \mathcal{S}_t$, its associated purchasing probability $(1 - F(p_t - x_t^\top \theta_0))$ would be close to $(1 - F(m_j))$, on which we have accumulated much knowledge from the past data. Denote $\mathcal{D}_{t-1,j} := \{s : 1 \le s \le t-1, j_s = j\}$ as all past time periods s such that $p_s - x_t^\top \hat{\theta} = m_j$. Then we construct the UCB of $1 - F(m_j)$ as

$$\text{UCB}_t(1 - F(m_j)) = \begin{cases} \frac{\sum_{s \in \mathcal{D}_{t-1,j}} p_s^2 y_s}{\lambda + \sum_{s \in \mathcal{D}_{t-1,j}} p_s^2} + \sqrt{\frac{\beta_t}{\lambda + \sum_{s \in \mathcal{D}_{t-1,j}} p_s^2}}, & \text{for } \mathcal{D}_{t-1,j} \neq \emptyset; \\ +\infty, & \text{for } \mathcal{D}_{t-1,j} = \emptyset. \end{cases} \tag{1}$$

Note that for $s \in \mathcal{D}_{t-1,j}$, $y_s \sim \text{Ber}(1 - F(p_s - x_s^\top \theta_0)) \approx \text{Ber}(1 - F(m_j))$. Therefore, the first term in the right-hand side of (1) for $\mathcal{D}_{t-1,j} \neq \emptyset$ is a regularized weighted average of those y_s and thus an estimate of $1 - F(m_j)$. The second term is an associated confidence interval length. Based on these

upper confidence bounds for $1 - F(m_j)$, we select the price $p_t = m_j + x_t^\top \hat{\theta}$ from the candidate set $\mathcal{S}_t$ that maximizes the optimism expected revenue $(m_j + x_t^\top \hat{\theta})\text{UCB}_t(1 - F(m_j))$.

It remains to specify the choice of β_t in Equation (1). We will discuss its choice by the following perturbed linear bandit formulation of the pricing problem in Inner UCB Algorithm.

Perturbed Linear Bandit Formulation With the restriction on the candidate sets $\mathcal{S}_t$, we are able to formulate the pricing problem guided by $\hat{\theta}$ as a perturbed linear bandit. By specifying the linear parameter $\xi_t = (1 - F(m_1 + x_t^\top \hat{\theta} - x_t^\top \theta_0), \ldots, 1 - F(m_d + x_t^\top \hat{\theta} - x_t^\top \theta_0))^\top \in \mathbb{R}^d$ and mapping each candidate price $p_t = m_j + x_t^\top \hat{\theta}$ to an action vector A_t with a single non-zero j-th element p_t, the expected revenue of setting price p_t can be rewritten as

$$p_t(1 - F(p_t - x_t^\top \theta_0)) = \langle \xi_t, A_t \rangle.$$

In addition, all linear parameters ξ_t are around the central parameter $\xi^* = (1 - F(m_1), \ldots, 1 - F(m_d))^\top$ and thus close to each other with a perturbation constant $2L||\hat{\theta} - \theta_0||_1$. This implies a lower ℓ_1 estimation error of $\hat{\theta}$ would lead to less perturbations and probably incur less regret.

Under the PLB formulation, the Inner UCB Algorithm is indeed equivalent to a modified version of the LinUCB algorithm [1, 33, 35]. Thus we specify β_t in (1) as $\beta_t = \beta_t^* = p_{\max}^2(1 \vee (\frac{1}{p_{\max}}\sqrt{\lambda d} + \sqrt{2\log(T_0) + d\log(\frac{d\lambda + (t-1)p_{\max}^2}{d\lambda})})^2)$.

5 Regret Analysis

In this section, we analyze the regret of our proposed *Explore-then-UCB* algorithm. For both Case **(A)** and Case **(B)**, we specify the parameters β and γ in Algorithm 1 appropriately and prove the respective upper bounds. Furthermore, we prove the first regret lower bound for Case **(A)**.

5.1 Upper Bounds

For upper bounds, we first analyze a single episode in Algorithm 1 and then extend it to the entire horizon. Firstly, we provide the ℓ_1 estimation error of our θ_0 estimation procedure.

Lemma 1 *Under Assumptions 1 – 4, there exists positive constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ such that for any episode k with the exploration phase length $n_k \geq \tilde{c}_3(d_0 + 1)^3$, we have with probability at least $1 - \frac{2}{n_k} - \tilde{c}_1 e^{-\frac{\tilde{c}_2}{(d_0+1)^2} n_k}$ that*

$$||\hat{\theta}_k - \theta_0||_1 \leq \frac{8(B + U + W)(d_0 + 1)}{c_0}\sqrt{\frac{\log n_k}{n_k}}.$$

Therefore, with an exploration phase length n_k that scales as ℓ_k^β, we obtain an estimate $\hat{\theta}_k$ with a high probability ℓ_1-error of $\tilde{O}(\ell_k^{-\beta/2})$. As this $\hat{\theta}_k$ will guide the UCB procedure, we need to investigate how the estimation error propagates into the UCB phase regret. A lower error rate is expected to cause less regret. On the other hand, a lower error rate requires a larger β and costs a longer exploration phase with more regret. Therefore, besides the "inner" balance between revenue-maximizing and F-learning in the UCB phase, there is an "outer" balance between the exploration phase regret and the UCB phase regret, both associated with θ_0-learning. This outer balance is regulated by the value of β, which should be set differently under Case **(A)** and Case **(B)** to achieve the optimal balance.

Secondly, we analyze the regret for the Inner UCB Algorithm. By defining the discrete best prices $\tilde{p}_t^* := \arg\max_{p \in \mathcal{S}_t} p(1 - F(p - x_t^\top \theta_0))$ in $\mathcal{S}_t$, the time t regret r_t can be decomposed as

$$\underbrace{\tilde{p}_t^*(1 - F(\tilde{p}_t^* - x_t^\top \theta_0)) - p_t(1 - F(p_t - x_t^\top \theta_0))}_{r_{t,1}} + \underbrace{p_t^*(1 - F(p_t^* - x_t^\top \theta_0)) - \tilde{p}_t^*(1 - F(\tilde{p}_t^* - x_t^\top \theta_0))}_{r_{t,2}}.$$

We refer to $R_{T_0,1} = \sum_{t=1}^{T_0} r_{t,1}$ and $R_{T_0,2} = \sum_{t=1}^{T_0} r_{t,2}$ as the discrete-part and continuous-part regrets. By the PLB formulation of the pricing problem, there is a correspondence between candidate prices and PLB actions. Thus the discrete-part regret is equivalent to the PLB regret. To bound the

Table 2: Regret components' rates for Case **(A)** and Case **(B)**.

	Exploration Phase	UCB Phase		
		Discrete-part		Continuous-part
		θ_0 Estimation Error	F-Learning	
		$O(\|\|\hat{\theta}-\theta_0\|\|_1 T_0)$	$\tilde{O}(d\sqrt{T_0})$	
Case **(A)**	$O(T_0^{\beta})$	$O(T_0^{1-\frac{\beta}{2}})$	$\tilde{O}(T_0^{\gamma+\frac{1}{2}})$	$O(\frac{T_0}{d^2}) = O(T_0^{1-2\gamma})$
Case **(B)**	$O(T_0^{\beta})$	$O(T_0^{1-\frac{\beta}{2}})$	$\tilde{O}(T_0^{\gamma+\frac{1}{2}})$	$O(\frac{T_0}{d}) = O(T_0^{1-\gamma})$

discrete-part regret of the Inner UCB Algorithm, we prove the following Proposition 1 based on Theorem 1 in [35].

Proposition 1 *Under Assumptions 1 and 6, there exists positive constants $C_1^{'}, C_2^{'}$ such that with probability at least $1-\frac{1}{T_0}$, the Inner UCB Algorithm yields a discrete-part regret*

$$R_{T_0,1} \leq C_1^{'} d\sqrt{T_0}\log(C_2^{'}T_0) + C_3^{'} L||\hat{\theta}-\theta_0||_1 T_0.$$

In Proposition 1, the first $\tilde{O}(d\sqrt{T_0})$ component is typical in linear bandit and attributes to the lack of knowledge for F and the central parameter ξ^* in the PLB formulation. The second $L||\hat{\theta}-\theta_0||_1 T_0$ component is proportional to the PLB perturbation constant $2L||\hat{\theta}-\theta_0||$. It demonstrates how estimation errors influence the regret upper bounds, which matches with the intuition that a better estimation would incur a lower regret in the UCB phase.

The discretization number d plays a critical role in balancing the discrete-part and continuous-part regret. A larger d leads to a higher discrete-part regret as indicated by Proposition 1, which is intuitive since more discretizations yield more candidate prices and hence a more challenging search process. On the other hand, a larger d and a denser discretization would make the discrete best price "closer" to the overall best price and decrease the continuous-part regret. Specifically, under Case **(A)** and Case **(B)**, we can bound the continuous-part regret with the order $O(\frac{T_0}{d^2})$ and $O(\frac{T_0}{d})$ respectively. Namely, we could achieve a lower rate with the extra concavity Assumption 7. Since we choose the discretization number d_k in episode k to scale as $(u_k)^{\gamma}$ where u_k denotes the UCB phase length, γ regulates the balance between the discrete-part and continuous-part regret in the UCB phase and should be set differently for optimal balances in Case **(A)** and Case **(B)**.

Now we are ready to present the two regret upper bounds for Case **(A)** and Case **(B)**.

Theorem 1 *Under Assumptions 1 – 7, by choosing $\beta = \frac{2}{3}$ and $\gamma = \frac{1}{6}$ in Algorithm 1, the expected regret satisfies $\mathbb{E}(R_T) = \tilde{O}(T^{2/3})$.*

Theorem 2 *Under Assumptions 1 – 6, by choosing $\beta = \frac{3}{4}$ and $\gamma = \frac{1}{4}$ in Algorithm 1, the expected regret satisfies $\mathbb{E}(R_T) = \tilde{O}(T^{3/4})$.*

We illustrate the regret components' orders in Table 2 for a single episode with a generic length T_0. Table 2 explains the different choices of β and γ in the two cases to minimize the overall regret rates. It clearly demonstrates that β balances between the exploration phase regret and the discrete-part regret due to θ_0 estimation error; while γ balances between the continuous-part regret and the discrete-part regret due to F-learning. As shown in Theorem 1, through optimal balance, ExUCB improves the existing regret of $\tilde{O}(T^{2/3\vee(1-\alpha)})$ in [35] and $\tilde{O}(T^{5/7})$ in [23] to $\tilde{O}(T^{2/3})$ under Lipschitz and concavity assumptions. In addition, an optimal choice of β and γ in Theorem 2 helps ExUCB match the best existing regret of $\tilde{O}(T^{3/4})$ [48] under the Lipschitz-only assumption.

5.2 Lower Bound

We next prove a regret lower bound of $\tilde{\Omega}(T^{3/5})$ for Case **(A)**. To our limited knowledge, this is the first lower bound result under the Lipschitz and concavity assumptions. Note that there is a gap between our proved upper and lower bounds for Case **(A)**. Indeed, such a gap also happens in other works [23, 48] on the unknown noise pricing problems. This may be due to the inherent difficulties of these problems, e.g., in learning both θ_0 and F.

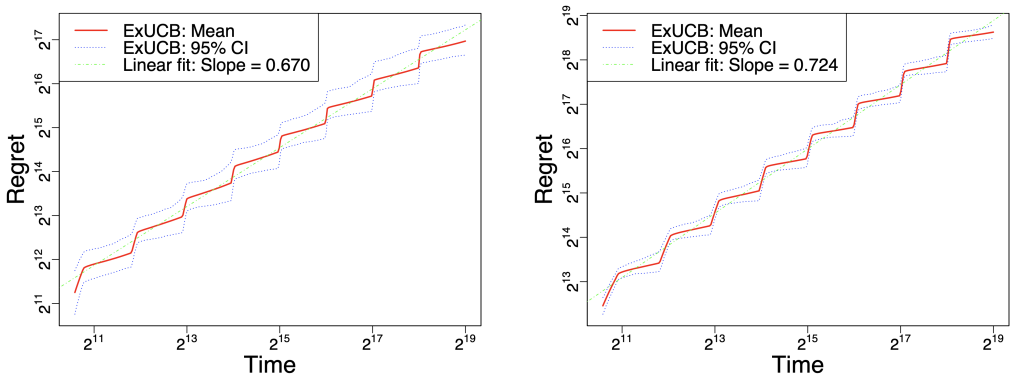
Theorem 3 *For any $\delta > 0$, no policy can achieve an $O(T^{3/5-\delta})$ regret for the dynamic pricing problem under Assumptions 1 – 7.*

Remark 1 *In [48], the authors proved an $\tilde{\Omega}(T^{2/3})$ lower bound with constructed instances that can fit into Case **(B)** but does not apply to Case **(A)**. Our proved lower bound rate is lower since our instances need to satisfy more assumptions and lie in a more benign class. Another similar lower bound of $\Omega(T^{3/5})$ is proved in [15] for 1-dimensional nonparametric contextual expected revenue functions. However, their instances are constructed in a local-bin fashion with respect to the covariate and thus are outside our revenue function class. Therefore, their lower bound does not apply to our setting.*

Proof Sketch of Theorem 3. We follow similar ideas in [31, 48] to construct the instances. Firstly we set $\theta_0 = 0$ and ignore the difficulty from contexts. Namely, the instances we construct are indeed non-contextual. Secondly, we construct a series of "bump towers" and then transform them to valid expected revenue functions in the form of $p(1 - F(p))$, while still preserving the intended properties. To construct the series of "bump towers", we construct an infinite number of infinitely-nested interval series $[0, 1] = [a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_k, b_k] \supset \ldots$. Specifically, we divide $[a_k, b_k]$ with length $w_k = 3^{-k!}$ into three same-length sub-intervals and further divide the middle sub-interval into $\frac{w_k}{w_{k+1}}$ candidate intervals. Then each of these candidate intervals forms one version of $[a_{k+1}, b_{k+1}]$. For each of the infinitely-nested interval series, we add up bump functions on all these nested intervals. Different from [48], to construct the expected revenue functions that satisfy the concavity assumption, we adopt a different basic bump function and develop a "quadratically-shrinking" adding pattern of the bump functions. With these constructions, we prove that any policy will miss the peaks of some revenue function instances often enough, thus accumulating an inevitable amount of the regret.

6 Numerical Experiments

We conduct numerical experiments to support our theoretical regret bounds of ExUCB under both Case **(A)** and Case **(B)**. We consider a total horizon length $T = \sum_{i=1}^{10} 2^{8+i}$ that is divided into 10 episodes with the first episode length $\alpha_1 = 2^9$. For both cases, we specify the linear parameter $\theta_0 = 30$ and sample the i.i.d. covariates as $x_t \sim \text{Unif}(1/2, 1)$. For Case **(A)**, the noise distribution is set as a Uniform mixture $\frac{3}{4}\text{Unif}(-15, 0) + \frac{1}{4}\text{Unif}(0, 15)$; while for Case **(B)**, we adopt another Uniform mixture $\frac{1}{4}\text{Unif}(-15, 0) + \frac{3}{4}\text{Unif}(0, 15)$. It can be verified that the first distribution satisfies the Lipschitz and concavity assumptions while the second one only satisfies the Lipschitz assumption. We apply ExUCB with different choices of β, γ specified in Theorems 1 – 2 to these two instances. For both cases, we set the constants in Algorithm 1 as $p_{\max} = 50, B = 50, C_1 = 1, C_2 = 20, \lambda = 0.1$. With 100 replications, we plot the log-log scale of average accumulative regrets versus the time periods in Figure 2 along with the 95% confidence intervals. The linear fits extract a slope of 0.670 for Case **(A)** and a slope of 0.724 for Case **(B)**, which indicates that our proved regrets of $\tilde{O}(T^{2/3})$ for Case **(A)** and $\tilde{O}(T^{3/4})$ for Case **(B)** are sharp.



(a) Case **(A)**: ExUCB with $\beta = 2/3, \gamma = 1/6$. (b) Case **(B)**: ExUCB with $\beta = 3/4, \gamma = 1/4$.

Figure 2: Regret rates of ExUCB for Case **(A)** and Case **(B)**.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] We summarized our main contributions at the end of Section 1. To support these results, we rigorously specified the problem settings, presented the algorithms and theoretical analysis, and conducted numerical experiments.
   (b) Did you describe the limitations of your work? [Yes] In Section 5.2, we discussed the gap between our proved upper and lower bounds. It is still an open problem to further close the gaps.
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] In Appendix C, we discussed the issue of pricing discriminations. However, this issue naturally exists and our algorithms would not worsen the cases. In fact, our proposed algorithms may help identify the right prices for both buyers and sellers, and thus improve the market efficiency.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes] Our numerical experiments only used simulated data. The simulated data has no relations with natural sciences or human subjects. The numerical simulation is only for validating our theoretical results.
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] We stated a full set of seven assumptions in Section 3 "Technical Assumptions". In each of our lemma, proposition and theorems, we stated the exact assumptions it requires among these seven assumptions.
   (b) Did you include complete proofs of all theoretical results? [Yes] In Appendix A, we completely proved all theoretical results we claim.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] We included all codes in the supplementary material. The codes are accompanied with instant explanations. We also provided a "Readme" document with instructions on running the codes to reproduce our results. There is no input data for our codes as we simulated the data in our codes.

(b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] As we considered online learning problems, we did not pre-train the algorithms. However, we did specify the hyperparameters and the way they were chosen in Section 6.

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] We report all error bars with 0.95 coverage by using the 0.025 and 0.975 quantiles of the 100 replication results.

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] We ran all numerical experiments on a laptop. We reported the time consumed for each replication in our codes.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [N/A] We did not use existing assets.

(b) Did you mention the license of the assets? [N/A]

(c) Did you include any new assets either in the supplemental material or as a URL? [N/A]

(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A] We did not use crowdsourcing or conduct research with human subjects.

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]