Generalization Analysis on Learning with a Concurrent Verifier

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Abstract

Machine learning technologies have been used in a wide range of practical systems. In practical situations, it would be natural to expect input-output pairs of a machine learning model to satisfy some requirements. However, it is hard to obtain a model satisfying the requirements by just learning from examples. A simple solution is to add a module that checks whether the input-output pairs meet the requirements and then modifies the model's outputs. Such a module, we call *concurrent verifier* (CV), can give a certification, but how the generalizability of the machine learning model changes by using a CV is unclear. This paper gives a generalization analysis of learning with a CV. We analyze how the learnability of a machine learning model changes when we use a CV, and show the condition where we can obtain a guaranteed hypothesis using a verifier only in the inference time. We also show that typical error bounds based on the Rademacher complexity will be no larger than that of the original model when using a CV in multi-class classification and structured prediction settings. Therefore, using a verifier in a learning phase will not hurt the generalizability of the model.

1 Introduction

As machine learning technology matures, many systems exploiting machine learning models have been developed. When developing a system that uses a machine learning model, a model with merely a small prediction error would not be satisfactory due to real-field requirements. For example, an object recognition model sensitive to slight noise would cause security issues [4, 26], or a model with unexpected output would increase the system's cost for dealing with them. Thus, we want input-output pairs of a machine learning model to satisfy some *requirements*. However, it is hard to obtain a model satisfying the requirements by just learning from examples. Moreover, since the learned models tend to be complex and the input domain tends to be quite large, it would not be realistic to certify that every input-output pair satisfies requirements. In addition, even if we find an input-output pair that does not satisfy the requirements, it would not be easy to modify a model since we have to re-estimate it from training examples.

This paper considers a way to obtain a machine learning model that makes its input-output pairs satisfy required properties. Here, we consider the following assumptions on the situation where a machine learning model is used. First, we can certainly judge whether an input-output pair $(x, h(x))$ satisfies the requirements or not, where $h : \mathcal{X} \to \mathcal{Y}$ is a machine learning model or a hypothesis. As we will show later, important use cases fit this setting. Second, there already exists a machine learning model whose prediction error is small enough, but its input-output pairs are not guaranteed to satisfy the requirements. This second assumption is also reasonable since modern machine learning models show high enough prediction accuracy in various tasks. Under these assumptions, a practical choice for addressing this problem is not to change the machine learning model but adding a module that

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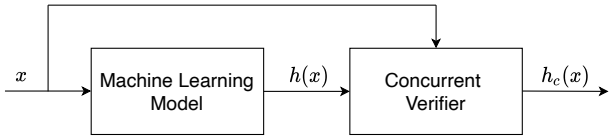

Figure 1: An overview of a machine learning model with a concurrent verifier. The verifier checks whether input-output pairs of a model satisfy requirements.

checks input-output pairs of machine learning model h. We call the module *a concurrent verifier* (CV). Figure 1 shows the system configuration of a machine learning model with a CV. The verifier checks whether the input-output pair $(x, h(x))$ satisfies the required properties. If it satisfies requirements, the verifier outputs $h(x)$. If not, then it rejects $h(x)$ and modifies or asks the learning model to modify its output. A pair of a machine learning model and a verifier can be seen as another machine learning model whose input-output pairs are guaranteed to satisfy the required conditions.

Although a model with a verifier can guarantee that its input-output pairs satisfy requirements, its effect on prediction error is unclear. This paper gives theoretical analyses of generalization errors of a machine learning model with a CV. We focus on how the learnability of the original model, denoted as hypothesis class $\mathcal{H}$, can change by using the verifier. First, we consider the situation where we use a CV only in the inference phase. This setting corresponds to the case where required properties are unknown when we are in the training phase. We show that if the hypothesis class is PAC learnable, we can obtain a guaranteed hypothesis by using a verifier only in the inference time.

Second, we consider the situation where we know the requirements when learning the model. This situation corresponds to viewing the learnability of hypothesis set $\mathcal{H}_c$, which is obtained by modifying every hypothesis $h \in \mathcal{H}$ to satisfy the requirements. Hence we compare the generalization error upper bounds of $\mathcal{H}_c$ with those of $\mathcal{H}$. On the multi-class classification setting, we show that existing error bounds [13, 16] based on the Rademacher complexity of $\mathcal{H}$ are also bounds of the modified hypothesis $\mathcal{H}_c$ for any input-output requirements. Moreover, we give similar analyses for the structured prediction task. A structured prediction is a kind of multi-class classification where the set of classes $\mathcal{Y}$ can be decomposed into substructures, and it is worth analyzing the task since there are many works considering constraints in structured prediction. Some works give error bounds for structured prediction tasks, which are tighter than simply applying the bound for multi-class classification task [14, 6, 17]. Similar to the case of multi-class classification, we show that existing Rademacher complexity-based bounds for structured prediction of $\mathcal{H}$ are also the bounds for $\mathcal{H}_c$.

Our main contributions are as follows: a) We introduce a concurrent verifier, a model-agnostic way to guarantee machine learning models satisfy required properties. Although a similar mechanism was used in some existing models, our model gives a generalization analysis that does not depend on a specific model. b) We show that if hypothesis class $\mathcal{H}$ is PAC-learnable, then using a verifier at inference time can give a hypothesis with a guarantee in its generalization error. Interestingly, if H is not PAC learnable, we might fail to obtain a guaranteed hypothesis even if requirements are consistent with distribution $\mathcal{D}$. c) We show that if we use a CV in a learning phase of multi-class classification tasks, then the theoretical error bounds of $\mathcal{H}$ based on the Rademacher complexity will not increase with any input-output requirements. We also give similar results for structured prediction tasks.

1.1 Use Cases of a Concurrent Verifier

We show some typical use cases of CVs.

Error-sensitive applications A typical situation where we want to use a verifier is that some prediction errors might cause severe effects, and we want to avoid them. For example, a recommender system might limit the set of candidate items depending on user attributes. Although such a rule might degrade the prediction accuracy, practically a safer model is preferable.

Controlling outputs of structured prediction Constraints are frequently used in structured prediction tasks for improving the performance or improving the controllability of outputs. For example, [21, 5] exploits constraints on sequence labeling tasks for reflecting background knowledge to improve the prediction results. More recently, there are works [9, 2] exploiting constraints in language generation

tasks, including image captioning and machine translation, where they restrict a model to output a sentence that includes given keywords. Since the constraints used in this previous work can be written as a logical formula, our CV model can represent them as requirements.

Robustness against input perturbations If a machine learning model changes its output if we modify its input x to x' that is very close to x, then the model is called sensitive against a small change [25]. It might be a security risk if a model is sensitive since its behavior is unpredictable. Therefore, some methods evaluate and verify the robustness of neural networks against small perturbations [26, 4]. Existing verification method checks a machine learning model's robustness around input x by searching whether there exists x' that is close to x and the model f gives different outputs, i.e., $h(x) \neq h(x')$, for verification samples $x_1, \ldots, x_n$. However, although these verification methods can test a model, they do not directly show a way to obtain a robust model.

A CV can be used to fix a model to achieve robustness around samples $x_1, \ldots, x_n$ by setting a rule of the form: "$h(x')$ must be equal to $h(x_i)$ if x' is close to x_i". Although this solution might not guarantee robustness where samples are scarce, adding enough non-labeled verification samples can be a reasonable choice.

2 Related work

Machine learning models that can exploit constraints have been investigated in many research fields, including statistical symbolic learning and structured prediction. For example, Markov logic networks [20], Problogs [8], and probabilistic circuit models [11] integrate statistical models with symbolic logic formulations. Since these models can incorporate hard constraints represented by symbolic logic, they can guarantee input-output pairs. However, previous research focused on its practical performance and gave little theoretical analysis of its learnability when hard constraints are used. Moreover, previous works integrated the ability to exploit constraints into specific models. In contrast, our CV is model-agnostic and can be used in combination with a wide range of machine learning models.

Recently, verification of machine learning models has gathered attention. There are attempts to check whether a machine learning model has the desired property[4, 26, 10, 24]. There are exact verification methods using integer programming (MIP) [26], constraint satisfaction (SAT) [18], and satisfiable module theory (SMT) solver [10] to assess the robustness of a neural network model against input noise. These approaches aim at obtaining models that fulfill required properties. However, verification methods cannot help modify models if the model does not satisfy the requirements. If we want ML models to meet requirements, post-processing would be needed as our concurrent verification model.

There are methods that can give upper bounds on generalization error, including VC-dimension [27] and its extensions [7, 19], Rademacher complexity [3, 12], stability [23], and PAC-Bayes [15, 1]. We use Rademacher complexity in the following analysis since they are among the most popular tools for giving theoretical upper bounds on the generalization error. Moreover, Rademacher complexity has some extensions, including local Rademacher complexity [13] and factor graph Rademacher complexity [6]. We can provide theoretical guarantees on these extended measures.

3 Preliminaries

Our notation follows [22]. We first introduce the notations used in the following sections. Let $\mathcal{X}$ denote the domain of inputs, $\mathcal{Y}$ be the domain of label, and Z be the domain of examples defined as $Z \coloneqq \mathcal{X} \times \mathcal{Y}$. Let $\mathcal{H}$ be a hypothesis class, and $\ell : \mathcal{H} \times Z \to \mathbb{R}_+$ be loss function. A sample $S = (z_1, \ldots, z_m) \in Z^m$ is a finite sequence of size m drawn i.i.d. from a fixed but unknown probability distribution $\mathcal{D}$ on Z. A learning algorithm A maps a sample to a hypothesis h. We use the notation $A(S)$ to denote the hypothesis that the learning algorithm A returns upon receiving S. We represent set $\{1, \ldots, K\}$ as $[K]$.

Given distribution $\mathcal{D}$ on Z, we denote by $L_{\mathcal{D}}(h)$ *the generalization error* and by $L_S(h)$ *the empirical error* of h over S defined by:

$$L_{\mathcal{D}}(h) \coloneqq \mathop{\mathbb{E}}_{z \sim \mathcal{D}}[\ell(h, z)], \qquad L_S(h) \coloneqq \frac{1}{m} \sum_{i=1}^{m} \ell(h, z_i). \tag{1}$$

PAC learnability We introduce PAC learnability and agnostic PAC learnability as follows.

Definition 3.1. (Agnostic PAC learnability) Hypothesis class $\mathcal{H}$ is *agnostic PAC learnable* if there exists a function $m_{\mathcal{H}} : (0,1)^2 \to \mathbb{N}$ and a learning algorithm A with the following property: For every $\epsilon, \delta \in (0,1)$ and distribution $\mathcal{D}$ over Z, if S is a sample consists of $m \geq m_{\mathcal{H}}(\epsilon, \delta)$ i.i.d. examples generated by $\mathcal{D}$, then, with at least probability $1-\delta$, the following holds:

$$L_{\mathcal{D}}(A(S)) \leq \min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') + \epsilon \,. \tag{2}$$

A distribution $\mathcal{D}$ is *realizable* by hypothesis set $\mathcal{H}$ if there exists $h^* \in \mathcal{H}$ such that $L_{\mathcal{D}}(h^*) = 0$. If $\mathcal{D}$ is realizable by agnostic PAC learnable hypothesis $\mathcal{H}$, then we say $\mathcal{H}$ is *PAC learnable*. If $\mathcal{H}$ is PAC learnable, then (2) becomes: $L_{\mathcal{D}}(A(S)) \leq \epsilon$ since $\min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') = 0$.

Rademacher complexity In the following sections, we use Rademacher complexity for deriving generalization bounds. Given loss function $\ell(h, z)$ and hypothesis class $\mathcal{H}$, let us denote $\mathcal{G}$ as

$$\mathcal{G} \coloneqq \ell \circ \mathcal{H} \coloneqq \{z \mapsto \ell(h, z) : h \in \mathcal{H}\}$$

Definition 3.2. (Empirical Rademacher complexity) Let $\mathcal{G}$ be a family of functions mapping from Z to $\mathbb{R}$ and $S = (z_1, \ldots, z_m) \in Z^m$ be a sample of size m. Then, *the empirical Rademacher complexity* of $\mathcal{G}$ with respect to sample S is defined as:

$$R_S(\mathcal{G}) \coloneqq \mathop{\mathbb{E}}_{\boldsymbol{\sigma}} \left[\sup_{g \in \mathcal{G}} \sum_{i=1}^{m} \sigma_i g(z_i) \right] ,$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in \{\pm 1\}^m$ are random variables distributed i.i.d. according to $\mathbb{P}[\sigma_i = 1] = \mathbb{P}[\sigma_i = -1] = 1/2$. *The Rademacher complexity* of $\mathcal{G}$ is defined as the expected value of the empirical Rademacher complexity, defined as:

$$R_m(\mathcal{G}) \coloneqq \mathop{\mathbb{E}}_{S \sim \mathcal{D}^m} [R_S(\mathcal{G})] \,.$$

4 Concurrent Verifier

We give a formal definition of a CV. A CV works with a machine learning model, which is a function $h : \mathcal{X} \to \mathcal{Y}$. If x is given to the model and the model outputs $h(x)$, then the verifier checks whether $(x, h(x))$ satisfies the required property. We assume that the required property can be represented as *requirement function* $c : (\mathcal{X} \times \mathcal{Y}) \to \{0, 1\}$. If $c(x, h(x)) = 1$ then the pair satisfies the property and if $c(x, h(x)) = 0$ then does not satisfy. Requirement function c can be represented by a set of deterministic rules. For example, if $\mathcal{X} = \mathbb{R}$ and $\mathcal{Y} = \{0, 1\}$, then requirements can be in the form: "if $x > 0$, then $y \neq 0$". We assume that for all possible input $x \in \mathcal{X}$, there exists $y \in \mathcal{Y}$ such that $c(x, y) = 1$ for avoiding the situation that the requirements are unsatisfiable for any output y. It is not difficult to relax this assumption if we all a machine learning model to reject unsatisfiable input x.

After checking the input-output pair, a verifier modifies output $h(x)$ depending on the value of $c(x, h(x))$. If $c(x, h(x)) = 1$, the verifier outputs $h(x)$ since it satisfies requirements. If $c(x, h(x)) = 0$, then the verifier will modify $h(x)$ to some $y \in \mathcal{Y}$ satisfying $c(x, y) = 1$. Therefore, if we use a verifier with a machine learning model corresponding to h, then the combination of the model and the verifier can be seen as a function $h_c : \mathcal{X} \to \mathcal{Y}$ defined as:

$$h_c(x) \coloneqq \begin{cases} h(x) & \text{if } c(x, h(x)) = 1 \\ y_c & \text{if } c(x, h(x)) = 0 \end{cases} , \tag{3}$$

where $y_c \in \mathcal{Y}$ satisfies $c(x, y_c) = 1$ and is selected deterministically. When $\mathcal{Y} = [K]$, an example for selecting minimum $i \in [K]$ satisfying $c(x, i) = 1$ as y_c would be a reasonable choice. When $\mathcal{Y} = [K]$ and $h(x)$ is made by scoring functions $h(x, y) : (\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}$, it would be also reasonable to select y^* such that $y^* = \operatorname{argmax}_{y \in \mathcal{Y}, c(x,y)=1} h(x, y)$. Learning a model corresponds to selecting a hypothesis h from hypothesis class $\mathcal{H}$. Therefore, learning a model with CV corresponds to choosing a hypothesis from the modified hypothesis class: $\mathcal{H}_c = \{h_c : h \in \mathcal{H}\}$. By definition, every hypothesis in $\mathcal{H}_c$ satisfies the requirements, and thus we can guarantee the model satisfies the condition if we select a hypothesis from $\mathcal{H}_c$. In the following sections, we will analyze the learnability of $\mathcal{H}_c$ by comparing it with that of $\mathcal{H}$.

5 Inference Time Verification

We first analyze the change of generalization errors when we use a verifier only in an inference phase. In other words, requirements are unknown in the learning phase and we estimate hypothesis $\hat{h} = A(S)$ from hypothesis class $\mathcal{H}$ by using training sample S and algorithm A. In the inference phase, we use a CV to modify $\hat{h}$ to $\hat{h}_c$. We call this setting *inference time verification* (ITV). This class of situations contains many exciting settings. Typical settings are 1) pre-trained machine learning models used in a wide range of applications, and 2) models that are hard to replace, which might encounter different requirements from ones at the learning time in the long run.

In this section, we will give analyses on a multi-class classification setting. We set $\mathcal{Y} = [K]$ and hypothesis class $\mathcal{H}$ is a set of mappings $h : \mathcal{X} \to [K]$. We also assume that the loss function ℓ is 0-1 loss defined as $\ell_{\text{0-1}}(h, (x, y)) \coloneqq \mathbf{1}_{h(x) \neq y}$, where $\mathbf{1}$ is an indicator function.

The following theorem shows the situation where ITV works well; In other words, a situation where the generalization error of $\hat{h}_c$ is not larger than that of other hypotheses in $\mathcal{H}_c$ with high probability.

Theorem 5.1. *If $\mathcal{Y} = [K]$ and hypothesis class $\mathcal{H}$ is PAC learnable with 0-1 loss $\ell_{\text{0-1}}$, training example S, and algorithm A. Suppose $\hat{h} = A(S)$ be the hypothesis estimated form S satisfying $L_{\mathcal{D}}(\hat{h}) \leq \epsilon$ for some parameter $\epsilon \in (0, 1)$. Then, for any requirement c, the hypothesis $\hat{h}_c$ obtained by modifying $\hat{h}$ with a CV satisfies*

$$L_{\mathcal{D}}(\hat{h}_c) \leq \min_{h_c \in \mathcal{H}_c} L_{\mathcal{D}}(h_c) + \epsilon \,.$$

We give a proof in Appendix A. The theorem suggests that if $\mathcal{H}$ is PAC-learnable, inference time verification is sufficient to obtain a hypothesis with a small generalization error in $\mathcal{H}_c$. We should note that the generalization error might increase by using a verifier, and the amount of increase is always larger than $L_{\mathcal{D}}(f_c)$. Therefore, $L_{\mathcal{D}}(f_c)$ can be seen as representing the discrepancy between the data distribution $\mathcal{D}$ and requirement c. We say requirement c is *consistent* with f if $c(x, f(x)) = 1$ for all x. If c is consistent with f, then $L_{\mathcal{D}}(f_c) = 0$ and we can certify that $L_{\mathcal{D}}(\hat{h}_c) \leq \epsilon$.

The above theorem shows ITV works when $\mathcal{H}$ is PAC-learnable. However, this will not hold if $\mathcal{D}$ is not realizable with $\mathcal{H}$.

Theorem 5.2. *Suppose $\mathcal{D}$ is not realizable with hypothesis class $\mathcal{H}$. Then, there exists $\mathcal{D}$, training example S, algorithm A, requirements c and $\epsilon \in (0, 1)$ such that $\hat{h} = A(S)$ satisfies $L_{\mathcal{D}}(\hat{h}) \leq \min_{h \in \mathcal{H}} L_{\mathcal{D}}(h) + \varepsilon$ but $L_{\mathcal{D}}(\hat{h}_c) > \min_{h_c \in \mathcal{H}_c} L_{\mathcal{D}}(h_c) + \epsilon$.*

We give a proof in Appendix B. The above theorems show that the realizability of $\mathcal{H}$ is the key factor that distinguishes the cases where ITV works well. Moreover, different from the realizable case, Theorem 5.2 holds even if requirement c is consistent with distribution $\mathcal{D}$. Let $f_{\mathcal{D}} : \mathcal{X} \to \mathcal{Y}$ be the Bayes optimal predictor defined as:

$$f_{\mathcal{D}}(x) \coloneqq \operatorname*{argmax}_{y \in \mathcal{Y}} \mathbb{P}[y \mid x] \,.$$

The Bayes optimal predictor is optimal, in the sense that for every other classifier $g : \mathcal{X} \to \mathcal{Y}$, $L_{\mathcal{D}}(f_{\mathcal{D}}) \leq L_{\mathcal{D}}(g)$. Theorem 5.2 holds if c is consistent with $f_{\mathcal{D}}$. These results show that existing methods [9, 2] using constraints only in the inference time might fail to select the best hypothesis.

Running time analysis Using a CV will increase the time needed for inference. Suppose that a verifier is an oracle that can answer the query of asking the value of $c(x, y)$. To achieve the modification procedure shown in (3), we need at most K queries.

6 Learning Time Verification

In Section 5, we show that if $\mathcal{H}$ is PAC learnable with 0-1 loss, then modifying a hypothesis at an inference time is enough to obtain a hypothesis with the smallest generalization error while satisfying requirements. If $\mathcal{H}$ is not PAC learnable, then the ITV scheme might fail to obtain a hypothesis with a small generalization error. In this section, we show the generalization error can be bounded when we use a CV in the learning phase. We call this setting *learning time verification* (LTV).

We can see that the LTV scheme corresponds to the learning task where the hypothesis class is $\mathcal{H}_c$. Therefore, we analyze the learnability of $\mathcal{H}_c$ using the standard tools for generalization analyses. This paper provides analyses based on the Rademacher complexity since they are widely used tools and can give tight bounds for both data-dependent and data-independent cases. Moreover, some previous work gives bounds of structured prediction tasks using the Rademacher complexity. In the literature, constraints are actively used in structured prediction tasks, including language generation and sequence labeling. Therefore, analyzing the generalization error when using a CV on structured prediction tasks is important.

In the following, we first show that the upper bounds of generation error based on the Rademacher complexity of $\mathcal{H}_c$ in a multi-class classification task (§6.1, 6.2) and a structured prediction setting (§6.3). Our main finding is that the upper bounds based on the Rademacher complexity of $\mathcal{H}_c$ is always less than or equal to those of $\mathcal{H}$. Therefore, adding a CV to a machine learning model will not degrade its learnability.

6.1 Multi-class Classification

We first give the Rademacher complexity-based error bounds on a multi-class classification task, i.e., $\mathcal{Y} = [K]$. In this section, we show that a standard upper bound [16] based on Rademacher complexity of $\mathcal{H}$ can be used as an upper bound of $\mathcal{H}_c$ for any requirement c. In the next section, we will show a state-of-the-art error bound, which is based on the local Rademacher complexity $\mathcal{H}$, which can also be used as an upper bound of $\mathcal{H}_c$.

Following previous works, let $h : (\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}$ be a scoring function, and hypothesis class $\mathcal{H}$ is defined as a set of scoring functions. A scoring function defines a mapping from $\mathcal{X}$ to $\mathcal{Y}$:

$$x \mapsto \operatorname*{argmax}_{y \in \mathcal{Y}} h(x, y) \,.$$

Let $\rho_h(x, y)$ be *the margin of function* of h defined as:

$$\rho_h(x, y) \coloneqq h(x, y) - \max_{y' \neq y} h(x, y') \,.$$

A hypothesis h misclassifies the labeled example (x, y) if $\rho_h(x, y) \leq 0$. Thus, by using a margin function, the 0-1 loss can be represented as $\ell_{0\text{-}1}(h, z) = \mathbf{1}_{\rho_h(x,y) \leq 0}$. Since 0-1 loss is hard to handle in learning, we use margin loss $\ell_\rho(h, (x, y)) = \Phi_\rho(\rho_h(x, y))$, where $\Phi_\rho(t)$ is defined as:

$$\Phi_\rho(t) = \min(1, \max(0, 1 - t/\rho)) \,.$$

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *μ-Lipschitz* if $|f(t) - f(t')| \leq \mu|t - t'|$ for any $t, t' \in \mathbb{R}$. Φ_ρ is an $1/\rho$-Lipschitz function. The *empirical margin loss* of hypothesis h is defined as:

$$L_{S,\rho}(h) \coloneqq \frac{1}{m} \sum_{i=1}^{m} \Phi_\rho(\rho_h(x_i, y_i)) \,.$$

As same in the case of ITV, introducing a CV to a machine learning model corresponds to modify its corresponding hypothesis class $\mathcal{H}$ to hypothesis class $\mathcal{H}_c$ that is consistent with requirement c. If h is a score function, then we define the consistent function h_c as:

$$h_c(x, y) = \begin{cases} h(x, y) & \text{if } c(x, y) = 1 \\ -M & \text{if } c(x, y) = 0 \end{cases} , \tag{4}$$

where M is a positive constant satisfying $M > |\max_{(x,y) \in Z} h(x, y)|$. As stated in Section 4, we assume that there exists $y \in \mathcal{Y}$ satisfying $c(x, y) = 1$ for all $x \in \mathcal{X}$. Therefore, we can guarantee that $\rho_{h_c}(x, y) < 0$ if $c(x, y) = 0$.

Followings are the main results of the general multi-class learning problem, which is based on the margin bound shown in Theorem 9.2 of [16]. Our main finding is that the generalization error of any hypothesis, h_c is bounded by the Rademacher complexity of the hypothesis set $\mathcal{H}$, which suggests that if we have a tight bound for hypothesis class $\mathcal{H}$, then we can expect to find a good hypothesis from $\mathcal{H}_c$ under any requirements c.

Theorem 6.1. *Let $\mathcal{H} \subseteq \mathbb{R}^{\mathcal{X}\times\mathcal{Y}}$ be a hypothesis class with $\mathcal{Y} = [K]$, and c be a requirement. Fix $\rho > 0$. Then, for any $\delta > 0$, with probability at least $1-\delta$, the following bound holds for all $h_c \in \mathcal{H}_c$:*

$$L_{\mathcal{D}}(h_c) \leq L_{S,\rho}(h_c) + \frac{4K}{\rho} R_m(\Pi_1(\mathcal{H})) + \sqrt{\frac{\log \frac{1}{\delta}}{2m}} ,$$

where $\Pi_1(\mathcal{H})$ is defined as:

$$\Pi_1(\mathcal{H}) \coloneqq \{x \mapsto h(x,y) : y \in \mathcal{Y}, h \in \mathcal{H}\} .$$

We give a proof in Appendix C. The parameter ρ sets the margin value. Following the technique shown in [16], we can obtain a generalized bound that holds uniformly for all $\rho > 0$. The above theorem suggests that using a CV at a learning phase would not worsen the error bound for any requirement c. Intuitively, the theorem seems reasonable since requirements c impose a restriction on $\mathcal{H}$ and thus the complexity of $\mathcal{H}_c$ is not larger than $\mathcal{H}$. However, we should note that it is not so trivial since $\mathcal{H}_c \subseteq \mathcal{H}$ is not always true.

Running time analysis We analyze the number of evaluations $c(x,y)$ required for learning with a CV. Let S_1 be a sub-sequence of training example S such that $c(x_i,y_i) = 1$, and S_0 be a sub-sequence such that $c(x_i,y_i) = 0$. If we use a 0-1 loss function, then the empirical loss of hypothesis h_c is:

$$L_S(h_c) = \frac{1}{m} \sum_{(x_i,y_i)\in S_1} \mathbf{1}_{h_c(x_i)\neq y_i} + \frac{|S_0|}{m} ,$$

since every h_c misclassifies examples in S_0. Therefore, we need at most $K|S_1| + |S_0|$ queries for the learning process. It is also true when we use a margin loss function. On the other hand, the problem of estimating the best hypothesis can be harder than the original problem depending on the requirement c.

6.2 A Tighter Bound Based on Local Rademacher Complexity

The bound for $\mathcal{H}_c$ we showed in the previous section is relatively simple, and tighter bounds of $\mathcal{H}$ based on the Rademacher complexity have been developed in the literature. In this section, we show the state-of-the-art error bound for $\mathcal{H}$ based on *the local Rademacher complexity* can be used as a bound for $\mathcal{H}_c$ for any requirements c.

Definition 6.2. Let $\mathcal{G}$ be a family of functions from Z to $\mathbb{R}$, and S be a sample of size m. Then, for any $r > 0$, *the empirical local Rademacher complexity* of $\mathcal{G}$ is defined as:

$$R_S(\mathcal{G}; r) = R_S\left(\{g \in \mathcal{G} : \mathbb{E}[f^2] \leq r\}\right) .$$

[13] shows a tighter generalization bound for a multi-class classification problem using the local Rademacher complexity when a hypothesis class is the ℓ_p norm hypothesis space with kernel κ defined as:

$$\mathcal{H}_{p,\kappa} \coloneqq \{h = (\langle \mathbf{w}_1, \phi(x)\rangle, \ldots, \langle \mathbf{w}_K, \phi(x)\rangle) : \|\mathbf{w}\|_{2,p} \leq 1, 1 \leq p \leq 2\} ,$$

where h is represented as a vector valued function $(h_1, \ldots, h_K)$ with $h_j(x) = h(x,j), \forall j = 1, \ldots, K$, and $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a Mercer kernel with the associated feature map ϕ, i.e., $\kappa(x,x') = \langle \phi(x), \phi(x')\rangle$. $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_K)$ and $\|\mathbf{w}\| = \left[\sum_{i=1}^K \|\mathbf{w}\|_2^p\right]^{\frac{1}{p}}$ is the $\ell_{2,p}$-norm. For any $p \geq 1$, let q be the dual exponent of p satisfying $1/p + 1/q = 1$. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a loss function satisfying: 1) $\mathbf{1}_{t\leq 0}(t) \leq \Phi(t)$ for all t; 2) $\Phi(t)$ is decreasing and it has a zero point c_Φ; 3) Φ is ζ-smooth, that is, $|\Phi'(t) - \Phi'(t')| \leq \zeta|t - t'|$. We let $\mathcal{L}_c \coloneqq \{(x,y) \mapsto \Phi(\rho_{h_c}(x,y)) : h_c \in \mathcal{H}_{p,\kappa,c}\}$.

Theorem 6.3. *Let $\mathcal{H}_{p,\kappa,c}$ be the set of hypothesis obtained by modifying hypothesis $h \in \mathcal{H}_{p,\kappa}$ with requirement c. For any $\delta > 0$, with probability at least $1-\delta$, the following bound holds:*

$$R_m(\mathcal{L}_c; r) \leq \frac{C_{d,\vartheta}\xi(K)\sqrt{\zeta r}\log^{\frac{3}{2}}(m)}{\sqrt{m}} + \frac{4\log\frac{1}{\delta}}{m} ,$$

where $\vartheta = \sup_{x\in\mathcal{X}} \kappa(x,x) < \infty$, $d = \sup_{t\in\mathbb{R}} \Phi(t) < \infty$, and $C_{d,\vartheta}$ is a constant. $\xi(K)$ is:

$$\xi(K) = \begin{cases} \sqrt{e}(4\log K)^{1+\frac{1}{2\log K}} & \text{if } q \geq 2\log K, \\ (2q)^{1+\frac{1}{q}} K^{\frac{1}{q}} & \text{otherwise} . \end{cases}$$

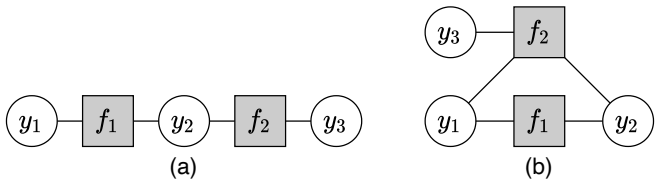

Figure 2: Example of factor graphs. (a) represents the decomposition: $h(x, y) = h_{f_1}(x, y_1, y_2) + h_{f_2}(x, y_2, y_3)$ (b) represents the decomposition: $h(x, y) = h_{f_1}(x, y_1, y_3) + h_{f_2}(x, y_1, y_2, y_3)$.

We give a proof in Appendix D. The bound equals that of $R_m(\mathcal{L}; r)$ (Theorem 1 of [13]) for any requirement c and any hypothesis $\mathcal{H}_{\rho,\kappa}$. Therefore, the generalization error bounds in [13] based on Theorem 1 of the paper holds for any requirement c.

6.3 Analyses of Structured Prediction

Structured prediction is a kind of multi-class classification task, where the label set $\mathcal{Y}$ might be a set of sequences, images, graphs, trees, or other objects admitting some possibly overlapping structure. As mentioned in Section 1, previous works try to impose some constraints on the output of structured prediction tasks. Thus it is also useful to derive error bounds for structured prediction tasks when we use a CV. In the following, we show that the Rademacher complexity-based generalization error bounds derived in the seminal work [6] also hold if we use a CV. Although tighter bounds are given in more recent work [17, 14], we give bounds based on [6] due to their simplicity.

We give some definitions for the structured prediction task. Following previous work, we assume that $\mathcal{Y}$ is decomposable along with substructures: $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_l$. Here, $\mathcal{Y}_k$ is the set of possible labels that can be assigned to the k-th substructure. We denote by $\mathsf{L} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$ a loss function measuring the dissimilarity of two elements of the output space $\mathcal{Y}$. L is *definite*, that is, $\mathsf{L}(y, y') = 0$ iff $y = y'$. A typical definite loss function for a structured prediction task is the Hamming loss defined by $\mathsf{L}(y, y') = \frac{1}{l}\sum_{k=1}^{l} \mathbf{1}_{y_k \neq y'_k}$ for all $y = (y_1, \ldots, y_l)$ and $y' = (y'_1, \ldots, y'_l)$, with $y_k, y'_k \in \mathcal{Y}_k$. Other typical examples of loss functions can be seen in [6]. Using loss function L, the generalization error and the empirical error of h is defined as:

$$L_{\mathcal{D}}(h) = \mathop{\mathbb{E}}_{(x,y)\sim\mathcal{D}}[\mathsf{L}(\mathsf{h}(x), y)], \qquad L_S(h) = \frac{1}{m}\sum_{i=1}^{m} \mathsf{L}(\mathsf{h}(x), y)\,.$$

Same with the multi-class classification task, hypothesis class $\mathcal{H}$ can be represented as a set of scoring function $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$. We use $\mathsf{h}(x)$ to represent the predictor defined by $h \in \mathcal{H}$: $\mathsf{h}(x) \coloneqq \operatorname{argmax}_{y\in\mathcal{Y}} h(x, y)$ for all $x \in X$. Following the previous work, we assume that each scoring function can be decomposed as a sum, and such decomposition follows a *factor graph*. A factor graph G is a tuple $G = (V, F, E)$, where V is a set of variable nodes, F is a set of factor nodes, and E is a set of undirected edges between a variable node and a factor node. Every node in V corresponds to a substructure index, that is, $V = \{1, \ldots, l\}$.

For any factor node f, we denote by $\mathcal{N}(f) \subseteq V$ the set of variable nodes connected to f and define $\mathcal{Y}_f$ as the substructure set cross-product $\mathcal{Y}_f = \prod_{k\in\mathcal{N}(f)} \mathcal{Y}_k$. Then, h admits the following decomposition as a sum of functions h_f, each taking as argument a pair of $(x, y_f) \in \mathcal{X} \times \mathcal{Y}_f$:

$$h(x, y) = \sum_{f\in F} h_f(x, y_f)\,. \tag{5}$$

Figure 2 shows examples of decompositions based on factor graphs. We conventionally assume that the structure of factor graphs may change depending on a particular example (x_i, y_i): $G(x_i, y_i) = G_i = ([l_i], F_i, E_i)$. A special case of this setting is when the size l_i of each example is allowed to vary. In such a case, the number of possible labels $\mathcal{Y}$ is potentially infinite.

Following the multi-class classification, our CV maps hypothesis h to h_c to satisfy the requirements. The definition of h_c follows (4). We should note that the definition does not require h_c to have a factored representation.

For analyzing the complexity, [6] introduces *the empirical factor graph Rademacher complexity* $R_S^G(\mathcal{H})$ of a hypothesis class $\mathcal{H}$ for a sample $S = (x_1, \ldots, x_m)$ and factor graph G as follows:

$$R_S^G(\mathcal{H}) = \frac{1}{m} \mathop{\mathbb{E}}_{\boldsymbol{\epsilon}} \left[\sup_{h \in \mathcal{H}} \sum_{i=1}^{m} \sum_{f \in F_i} \sum_{y \in \mathcal{Y}_f} \sqrt{|F_i|} \epsilon_{i,f,y} h_f(x_i, y) \right],$$

where $\boldsymbol{\epsilon} = (\epsilon_{i,f,y})_{i \in [m], f \in F_i, y \in \mathcal{Y}_f}$ and every $\epsilon_{i,f,y}$ is i.i.d. Rademacher random variable. The factor graph Rademacher complexity of $\mathcal{H}$ for a factor graph G is defined as the expectation:

$$R_m^G(\mathcal{H}) \coloneqq \mathop{\mathbb{E}}_{S \sim \mathcal{D}^m} [R_S^G(\mathcal{H})].$$

By using the factor graph Rademacher complexity, [6] gives bounds for structured prediction task with the following additive and multiplicative empirical losses:

$$L_{S,\rho}^{\mathrm{add}}(h) \coloneqq \frac{1}{m} \sum_{i=1}^{m} \left[\Phi^* \left(\max_{y' \neq y_i} \mathsf{L}(y', y_i) - \frac{1}{\rho} \left(h(x_i, y_i) - h(x_i, y') \right) \right) \right]$$

$$L_{S,\rho}^{\mathrm{mult}}(h) \coloneqq \frac{1}{m} \sum_{i=1}^{m} \left[\Phi^* \left(\max_{y' \neq y_i} \mathsf{L}(y', y_i) \left(1 - \frac{1}{\rho} \left(h(x_i, y_i) - h(x_i, y') \right) \right) \right) \right],$$

where $\Phi^*(t) = \min(B, \max(0, t))$ for all t, with $B = \max_{y,y'} \mathsf{L}(y, y')$. As shown in [6], these loss functions cover typical surrogate loss function used in structured prediction tasks. We show the following bound for structured prediction.

Theorem 6.4. *Fix $\rho > 0$. For any $\delta > 0$ and requirement c, with probability at least $1 - \delta$ over the draw of a sample S of size m from distribution $\mathcal{D}$, the following holds for all $h_c \in \mathcal{H}_c$:*

$$L_{\mathcal{D}}(h_c) \leq L_{\mathcal{D},\rho}^{\mathrm{add}}(h_c) \leq L_{S,\rho}^{\mathrm{add}}(h_c) + \frac{4\sqrt{2}}{\rho} R_m^G(\mathcal{H}) + B \sqrt{\frac{\log \frac{1}{\delta}}{2m}}$$

$$L_{\mathcal{D}}(h_c) \leq L_{\mathcal{D},\rho}^{\mathrm{mult}}(h_c) \leq L_{S,\rho}^{\mathrm{mult}}(h_c) + \frac{4\sqrt{2}B}{\rho} R_m^G(\mathcal{H}) + B \sqrt{\frac{\log \frac{1}{\delta}}{2m}},$$

We give a proof in Appendix E. ρ is a parameter that determines the margin. Similar to the case of the multi-class classification, we can derive a bound that holds for any $\rho > 0$ following the derivation of [6]. The above result indicates that the bound will not change if we use a CV for any requirement c. This is interesting since the above result holds even if we do not have a factored representation of $h_c(x, y)$ similar to (5), while the derived bound depends on the factor graph Rademacher complexity, which depends on the factored representation of $h(x, y)$.

We analyze the overhead in running time for evaluating loss function $L_{S,\rho}^{\mathrm{add}}(h_c)$ and $L_{S,\rho}^{\mathrm{mult}}(h_c)$ for hypothesis h_c. Evaluating them needs to solve an optimization problem of finding $y \in \mathcal{Y}$ that maximizes $h_c(x, y)$. For the structured prediction task, it is known that originally tractable optimization problems can be intractable if we put additional constraints [21]. Similarly, the overhead in running time for evaluating loss function when we use a CV depends on both the model and the type of requirements.

7 Conclusion

This paper gives a generalization analysis when there are requirements that input-output pairs of a machine learning model must satisfy. We introduce a concurrent verifier, a simple module that enables us to guarantee that the input-output pairs of a machine learning model satisfy requirements. We show the situation where we can obtain a hypothesis with a small error when we use a verifier only in an inference phase. Interestingly, if $\mathcal{H}$ is not PAC learnable, we might fail to obtain a guaranteed hypothesis even if requirements are consistent with distribution $\mathcal{D}$. We also give generalization bounds based on the Rademacher complexity when we use a verifier in a learning phase. We find that the obtained bounds are less than or equal to existing bounds, independent of the machine learning model and type of requirements.

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Checklist

The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. For example:

- Did you include the license to the code and datasets? [Yes]
- Did you include the license to the code and datasets? [No] The code and the data are proprietary.
- Did you include the license to the code and datasets? [N/A]

Please do not modify the questions and only use the provided macros for your answers. Note that the Checklist section does not count towards the page limit. In your paper, please delete this instructions block and only keep the Checklist section heading above along with the questions/answers below.

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] We discuss conditions a verifier does not work well in Section 5.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] We give proof in Appendix
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [N/A]

(b) Did you mention the license of the assets? [N/A]

(c) Did you include any new assets either in the supplemental material or as a URL? [N/A]

(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]