MODEL RISK-SENSITIVE OFFLINE REINFORCEMENT LEARNING

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ABSTRACT

Offline reinforcement learning (RL) is becoming critical in risk-sensitive areas such as finance and autonomous driving, where incorrect decisions can lead to substantial financial loss or compromised safety. However, traditional risk-sensitive offline RL methods often struggle with accurately assessing risk, with minor errors in the estimated return potentially causing significant inaccuracies of risk estimation. These challenges are intensified by distribution shifts inherent in offline RL. To mitigate these issues, we propose a model risk-sensitive offline RL framework designed to minimize the worst-case of risks across a set of plausible alternative scenarios rather than solely focusing on minimizing estimated risk. We present a critic-ensemble criterion method that identifies the plausible alternative scenarios without introducing additional hyperparameters. We also incorporate the learned Fourier feature framework and the IQN framework to address spectral bias in neural networks, which can otherwise lead to severe errors in calculating model risk. Our experiments in finance and self-driving scenarios demonstrate that the proposed framework significantly reduces risk, by 11.2% to 18.5%, compared to the most outperforming risk-sensitive offline RL baseline, particularly in highly uncertain environments.

1 INTRODUCTION

Many application domains of offline reinforcement learning (RL), such as finance (Zhang et al., 2020; Wang & Ku, 2022), self-driving (Bernhard et al., 2019; Seres et al., 2023), and healthcare (Lu et al., 2020), are inherently risk-sensitive due to the potential for high costs or safety risks from interactions between the agent and the environment at deployment. In these scenarios, both premature and well-trained agents can make unsafe decisions, prompting active research efforts to integrate offline RL with risk-sensitive RL (Urpí et al., 2021; Ma et al., 2021; Rigter et al., 2024). These efforts aim to minimize risk instead of maximizing expected return in offline RL settings. Here, *(aleatoric) risk*[1] is defined as a real number representing the severity of potential harm associated with the return distribution. The function that maps a return distribution to its corresponding risk is called *risk measure*. For example, the risk calculated by conditional value at risk (*CV@R*) risk measure with a confidence level $\alpha \in [0, 1]$ is the negation of the average from the most pessimistic to the α-th quantile (Yoo et al., 2024).

As these risk measures are designed to emphasize rare events, while underestimating frequent ones, the estimated risk becomes highly sensitive to changes in the underlying return distribution (Cont et al., 2010; Kou et al., 2013; Embrechts et al., 2015; Pesenti et al., 2016). Such errors can lead to incorrect risk management, especially when there is a distribution shift between the training and deployment environments. This situation is common in offline RL, where the agent cannot receive real-time feedback from the environment.

In finance, the issue is addressed by minimizing the worst-case of risks across a set of plausible alternative scenarios rather than directly minimizing the risk estimated by the model (Bernard et al., 2023). This worst-case is referred to as *model risk*, which quantifies the worst-case consequence of using incorrect models (Breuer & Csiszár, 2016; Bernard et al., 2023). Unlike traditional approaches that rely solely on the estimated risk and are vulnerable to model errors, the model risk acknowledges

[1] Whenever the context is clear, we will refer to aleatoric risk as risk.

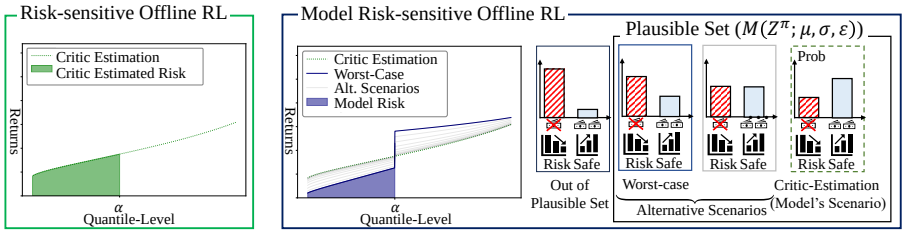


Figure 1: Difference between risk and model risk measured by $CV@R_\alpha$. The negation of the shaded area represents the risk.

the potential for inaccuracies in the model, enabling more robust and reliable risk-sensitive decision-making. Inspired by this, our work in risk-sensitive offline RL shifts the focus from conventional risk minimization to model risk minimization. In doing so, we address the common issue of estimating return distribution statistics (mean and the standard deviation) in RL contexts, where the statistics are often unreliable or unknown. This contrasts with finance, where the mean and deviation are assumed to be known.

Figure 1 illustrates the difference between traditional risk-sensitive offline RL approaches (left-side in the figure) and our model risk-sensitive offline RL (right-side in the figure). In the left-side, existing risk-sensitive offline RL approaches minimize the risk estimated by critic (green area). In the right-side, our model risk-sensitive RL minimizes risk (blue area) of the worst-case scenario (blue line) among the plausible alternative scenarios set (gray lines). By minimizing the model risk, the agent accounts for the possibility of being incorrect, aiming to achieve the best possible results despite potential errors.

We present a model risk-sensitive offline RL framework that accounts for model errors, enhancing robustness in highly uncertain environments. To the best of our knowledge, this is the first approach that incorporates model risk into offline RL algorithms. Specifically, we develop a critic-ensemble criterion method that identifies plausible alternative scenarios, enabling the calculation of model risk without introducing additional hyperparameters. Achieving this requires accurate return distribution statistics, particularly the mean and deviation. To enhance their precision, which is essential for identifying plausible alternative scenarios, we adopt a Fourier feature network for quantile regression. Through case studies in finance and self-driving scenarios, we demonstrate that our model risk-sensitive offline RL framework is more robust than other approaches, achieving an 11.2% to 18.5% reduction in risk. Notably, our approach exhibits strong advantages in highly uncertain environments with a large possibility of model errors.

Our contributions are as follows.

- We present the model risk-sensitive offline framework that minimizes the model risk, enabling more robust decision-making compared to conventional risk-sensitive offline RL approaches.
- We introduce the critic-ensemble criterion, which adjusts a set of alternative scenarios by expanding or contracting it based on the level of uncertainty, thereby enabling the computation of model risk.
- We devise the Fourier feature quantile regression network to precisely estimate the statistics of distributions, which are required to calculate the model risk.
- Through several case studies, we demonstrate the robustness of our approach in highly uncertain environments.

2 Related Work

Risk-sensitive RL. Traditional research on risk-sensitive RL is based on risk measures that can be computed without estimating whole return distributions (Howard & Matheson, 1972; Sato et al., 2001; Mihatsch & Neuneier, 2002; Tamar et al., 2015; Chow et al., 2017). With the rise of distributional RL, Dabney et al. (2018a) developed IQN framework, which enables the direct computation of risk via quantile regression, thereby unifying the risk-sensitive RL framework. Other

approaches include the direct calculation of *CV@R* policy gradient without calculating the return distribution (Tamar et al., 2015; Chow et al., 2017), entropy-based methods (Mihatsch & Neuneier, 2002), dynamic *CV@R*(Du et al., 2022; Lim & Malik, 2022), and entropic value at risk (*EV@R*, Ni & Lai (2022)). Dynamic *CV@R* recursively accounts for risk at each timestep and is related to 1R2R (Rigter et al., 2024). *EV@R* is a risk measure that is out of our scope, but it is closely related to CODAC (Ma et al., 2021). Another interesting approach is integrating risk-sensitive RL with safe RL. For example, Kim & Oh (2022) integrate the risk-sensitive approach to the safe RL, constraining the *CV@R* of costs instead of expected costs. Meanwhile, Ying et al. (2021); Greenberg et al. (2022) constrain the risk using the safe RL method, while maximizing return.

Risk-sensitive Offline RL. Research in risk-sensitive offline RL relies on integrating IQN and offline RL frameworks. Urpí et al. (2021) first proposed a risk-sensitive offline RL framework ORAAC by combining IQN and BCQ (Fujimoto et al., 2019). ORAAC is hindered by excessive constraints on the behavior policy, limiting its capability to generate better trajectories. To address the limitation, CODAC (Ma et al., 2021), which is based on CQL, was proposed. Yet, CODAC inherits the issues from CQL, such as hyperparameter sensitivity (Tarasov et al., 2024) [2]. Another approach, 1R2R (Rigter et al., 2024), is a model-based RL method that estimates risk with worst-case perturbations of the transition. However, its main limitation is that it is only guaranteed when transition probabilities follow a normal distribution.

Model Risk. Our work is built on the concept of model risk, which has been extensively studied in the field of mathematical finance (Bernard et al., 2023; Breuer & Csiszár, 2016; Blanchet & Murthy, 2019; Glasserman & Xu, 2014). Focusing on the worst-case expectation among scenarios aligns with Breuer's approach to aleatoric risk using a coherent risk measure (Breuer & Csiszár, 2016; Glasserman & Xu, 2014). However, we focus on the worst-case of *aleatoric risk* among alternative scenarios, following Bernard et al. (2023), which leads to a different formulation. This approach allows comparison of value distributions with different support, which is crucial in offline RL, since the worst-case distribution may not match the model's support (Bernard et al., 2023).

3 Background and Problem Formulation

3.1 Background

Risk-sensitive RL. We consider a Markov decision process $(\mathcal{S}, \mathcal{A}, R, \mathcal{P}, \gamma)$ where $\mathcal{S}$ is a set of states, $\mathcal{A}$ is a set of actions, $R : \mathcal{S} \times \mathcal{A} \to \triangle(\mathbb{R})$ is an immediate reward which may be random, and $\mathcal{P} : \mathcal{S} \times \mathcal{A} \to \triangle(\mathcal{S})$ is a transition probability. Here $\triangle(\mathcal{X})$ denotes a set of random variables whose supports are subsets of $\mathcal{X}$. Further, we assume that all random attributes' means and variances are finite. An optimal risk-sensitive policy is formulated as

$$\pi^* = \arg\min_{\pi} H_\phi \left(Z^\pi(s,a)\right) \text{ where } Z^\pi(s,a) = \sum_{t=0}^{T} R(s_t, a_t)\gamma^t | s_0 = s, a_0 = a. \tag{1}$$

Here $H_\phi(Z^\pi(s,a))$ denotes the risk of policy π with respect to its return $Z^\pi(s,a)$ and H_ϕ denotes a risk measure given by the user or environment. As we are interested in spectral risk measures (Adam et al. (2008), in Def. 2), we use the quantile function of a random variable.

Definition 1 *(Koenker, 2005) Let Z be a random variable. A quantile function of the random variable Z, $F_Z^{-1} : [0,1] \to \mathbb{R}$, is defined as the left inverse of cumulative distribution of Z. Specifically,*

$$F_Z^{-1}(p) := \inf\{x \in \mathbb{R} \mid p \leq F_Z(x)\}, \tag{2}$$

where $F_Z : \mathbb{R} \to [0,1]$ is the cumulative distribution function of Z. For readability, we denote $F_Z^{-1}(\cdot; s, a)$ as the quantile function of dependent random variable, $Z|s,a$, rather than $F_{Z|s,a}^{-1}(\cdot)$.

The spectral risk is the weighted integral of the quantile function.

[2] This is also a problem commonly seen in GAN-based approaches. See Appendix for an explanation of the relationship between CQL and GAN.

Definition 2 *(Adam et al., 2008) A function $H_\phi : L^1 \to (\mathbb{R} \cup \{\infty\})$ is called spectral risk measure which has the following representation*

$$H_\phi(Z) = \underbrace{-\int_0^1 F_Z^{-1}(p)\phi(p)dp = -\int_0^1 F_Z^{-1}(F_\phi^{-1}(u))du}_{u:=F_\phi(p) \Rightarrow du=dF_\phi(p)=\phi(p)dp \ \ with \ \ p=F_\phi^{-1}(u)} = -\mathbb{E}_{u\sim U[0,1]}[F_Z^{-1}(F_\phi^{-1}(u))], \quad (3)$$

where $\phi : [0,1] \to [0,\infty]$ is a weight function; right-continuous, non-increasing with $\int_0^1 \phi(p)dp = 1$; i.e, ϕ is a density function for some random variable. Further, the value of this function given Z, $H_\phi(Z)$, is called the risk of Z. Here L^1 is a set of random variables whose means are finite. The right most term is followed by and inverse transform sampling (Miller et al., 2010; Dabney et al., 2018a) and Monte-Carlo integral.

Since ϕ is non-increasing, the risk measure necessarily prioritizes worse outcomes, as intended. The negation is required in the definition, since we aim to *minimize* the risk. For example, when we want to optimize for the worst-case risk measure (i.e., ϕ is a Dirac-delta function), we minimize the negation of the worst-case outcome, the risk.

To calculate the risk $H_\phi(Z^\pi(s,a))$, we use the quantile function of $Z^\pi(s,a)$, $F_{Z^\pi}^{-1}(\cdot, s, a)$. To learn the quantile function, we use the following distributional Bellman equation (Bellemare et al., 2017; Dabney et al., 2018b;a).

$$Z^\pi(s_t, a_t) \overset{\text{distr.}}{=} R(s_t, a_t) + \gamma Z^\pi(s_{t+1}, a), \quad (4)$$

where $a = \arg\min_{a'} H_\phi(Z(s_{t+1}, a'))$. Specifically, we use the quantile regression method to estimate the quantile of return $Z^\pi(s,a)$. The quantile regression loss (Koenker, 2005; Dabney et al., 2018b) of the critic is

$$\mathcal{L}_{\text{crit.}}(\theta) = \frac{1}{NN'} \sum_{j=1}^{N'} \sum_{i=1}^{N} \underbrace{|p_i - \mathbf{1}(\delta_{ij}^\theta < 0)|}_{\text{Asymmetric weighted}} \cdot \underbrace{|\delta_{ij}^\theta|}_{L^1\text{loss}}, \quad (5)$$

where δ_{ij}^θ is the distributional Bellman residual and θ is a critic's parameter. The distributional Bellman residual is defined as a pairwise difference between the target quantile and the critic's quantile as

$$\delta_{ij}^\theta = R(s_t, a_t) + \gamma F_{Z^\pi}^{-1}(p_j; s_{t+1}, a, \theta_{\text{tar}}) - F_{Z^\pi}^{-1}(p_i; s_t, a_t, \theta), \quad (6)$$

where p_i, p_j denote N, N' numbers of independent random variables which follow $U[0,1]$ and θ_{tar} denotes the parameter of target-critic.

Risk-sensitive Offline RL. In offline settings, an agent is not allowed to interact directly with the environment and can only access a transition dataset $\mathcal{D} := \{(s_i, a_i, r_i, s_i')\}_{i=1}^{|\mathcal{D}|}$. Due to this constraint, naive RL approaches suffer from overestimation of $Z^\pi(s,a)$ from distribution shifts (Fujimoto et al., 2019). Thus, an auxiliary regulation is commonly applied. The objective of offline RL is defined as

$$\pi^* = \arg\max_\pi \mathbb{E}[Z^\pi(s,a)] \text{ constraint to } D(\pi \| \pi_\mathcal{D}) < \epsilon, \quad (7)$$

where $D(\cdot\|\cdot)$ is some divergence (or metric), $\pi_\mathcal{D}$ is the behavior policy for the dataset $\mathcal{D}$, and $\epsilon > 0$ is a tolerance. Incorporating this regulation, the objective of risk-sensitive offline RL is extended from Eq.(1) as

$$\pi^* = \arg\min_\pi H_\phi\left(Z^\pi(s,a)\right) \text{ constraint to } D(\pi \| \pi_\mathcal{D}) < \epsilon. \quad (8)$$

3.2 Problem Formulation

The primary concern of risk-sensitive offline RL is that the error of computing $H_\phi(Z^\pi(s,a))$ can be significant, as the agent cannot receive direct feedback from the environment in offline settings. To address this, we instead compute the worst-case of risks among alternative scenarios (i.e., model risk) and focus on minimizing the model risk. The model risk is defined as the highest possible risk within a $\sqrt{\varepsilon}$-ball with mean μ and standard deviation σ. The random variables in the ball specify the plausible alternative scenarios.

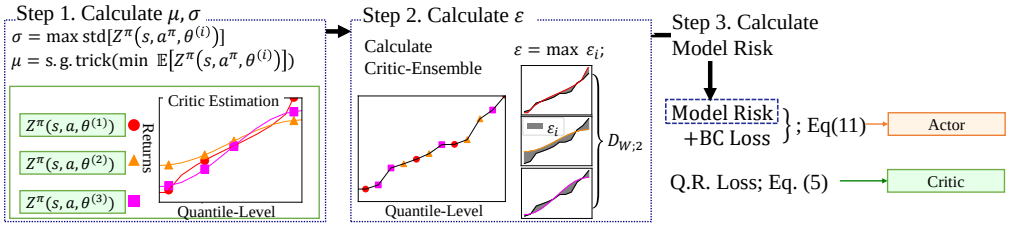


Figure 2: Overall framework of MR-IQN. Q.R. loss is abbreviation of quantile regression loss and s.g. trick is stop-gradient trick. Note that the stop-gradient trick does not change the value.

Definition 3 *(Bernard et al. (2023)) The model risk $MR_\phi(Z^\pi(s,a); \mu, \sigma, \varepsilon)$ is defined as*

$$MR_\phi(Z^\pi(s,a); \mu, \sigma, \varepsilon) := \sup_{X \in M(Z^\pi(s,a); \mu, \sigma, \varepsilon)} H_\phi(X), \tag{9}$$

where $M(Z^\pi(s,a); \mu, \sigma, \varepsilon) = \{X \in L^2 \mid \mathbb{E}[X] = \mu, \mathbb{V}ar[X] = \sigma^2, D_{W;2}(X, Z^\pi(s,a)) < \sqrt{\varepsilon}\}$ and L^2 is a set of random variables whose mean and variance are finite. Here, $D_{W;2}$ is the Wasserstein 2-distance defined as $D_{W;2}(X,Y) = \sqrt{\int_0^1 |F_X^{-1}(p) - F_Y^{-1}(p)|^2 dp}$. The distribution of random variable $Z^\pi(s,a)$ is called reference distribution.

Our goal is to find a policy π^* such that

$$\pi^* = \arg\min_\pi \mathrm{MR}_\phi(Z^\pi(s,a); \mu, \sigma, \varepsilon) \text{ constraint to } D(\pi \| \pi_\mathcal{D}) < \epsilon. \tag{10}$$

4 Model Risk-Sensitive Offline RL Algorithm

To achieve a model risk-sensitive offline RL agent satisfying the objective in Eq.(10), we introduce Model Risk IQN (MR-IQN), which integrates the TD3+BC framework (Fujimoto & Gu, 2021) with the model risk objective from Eq.(10). The MR-IQN consists of distributional critics and a deterministic actor. The critic is trained to minimize the loss function in Eq.(5), similar to other approaches (Urpí et al., 2021; Dabney et al., 2018a; Kuznetsov et al., 2020). Specifically, the critics are implemented using IQN (Dabney et al., 2018a) and TQC (Kuznetsov et al., 2020), following Yoo et al. (2024). Meanwhile the actor is trained to minimize the loss function,

$$\mathcal{L}(\pi) = \underbrace{\lambda_{\text{q.learn}} \mathrm{MR}_\phi(Z^\pi(s,a); \mu, \sigma, \varepsilon)}_{\text{Model risk}} + \underbrace{(a - a_\mathcal{D})^2}_{\text{BC loss}}. \tag{11}$$

Here $\lambda_{\text{q.learn}} > 0$ is a scale parameter, $a_\mathcal{D}$ is a batch action, and $a = \pi(s)$.

Figure 2 describes the MR-IQN framework. We extract μ and σ from the critic estimation. To precisely estimate the statistics, μ and σ, we leverage the Fourier feature architecture (Tancik et al., 2020) for the critics, as in Li & Pathak (2021). Next, we calculate ε as the distance between the ensemble of the critics obtained by quantile mixture of individual critics. Finally, we calculate $\mathrm{MR}_\phi(Z^\pi(s, \pi(s)))$ and minimize the loss function in Eq.(11). To do so, we first introduce the theorem and corollary below for calculating model risk.

Theorem 1 *(Bernard et al., 2023) Let Z be a random variable. For $\varepsilon > (\mu_Z - \mu)^2 + (\sigma_Z - \sigma)^2$, there exists a unique quantile function $h_\lambda : [0,1] \to \mathbb{R}$ which satisfies*

$$MR_\phi(Z; \mu, \sigma, \varepsilon) = -\int_0^1 h_\lambda(p)\phi(p)dp, \tag{12}$$

where μ_Z is the mean of Z, σ_Z is the standard deviation of Z. We have

$$h_\lambda(p) = \mu - \sigma\left(\frac{\phi(p) + \lambda F_Z^{-1}(p) - a_\lambda}{b_\lambda}\right), \tag{13}$$

where $\lambda \geq 0$ is some constant, and $a_\lambda = \mathbb{E}[\phi(U) + \lambda F_Z^{-1}(U)], b_\lambda = std[\phi(U) + \lambda F_Z^{-1}(U)]$. Here U denotes the uniform random variable $U[0,1]$ and std denotes the standard deviation operator. If $\varepsilon \leq (\mu_Z - \mu)^2 + (\sigma_Z - \sigma)^2$ holds, the problem is ill-posed or it has only a trivial solution.

Let $c_0 = corr(F_Z^{-1}(U), \phi(U))$. If $\varepsilon < (\mu_Z - \mu)^2 + (\sigma_Z - \sigma)^2 + 2\sigma\sigma_Z(1 - c_0)$ holds, we achieve

$$\lambda = \frac{K}{\sigma_Z^2}\sqrt{\frac{V\sigma_Z^2 - C_{\phi,Z}^2}{\sigma^2\sigma_Z^2 - K^2}} - \frac{C_{\phi,Z}^2}{\sigma_Z}, \tag{14}$$

where $V = \mathbb{V}ar[\phi(U)]$, $C_{\phi,Z} = \mathbb{C}ov[F_Z^{-1}(U), \phi(U)]$, and $K = \frac{1}{2}\left((\mu_Z - \mu)^2 + \sigma^2 + \sigma_Z^2 - \varepsilon\right)$. Otherwise, $\lambda = 0$. Here, corr is the Pearson correlation.

Corollary 1 *(Bernard et al., 2023) The solution of Eq.(9) exists and unique whenever h_λ exists as*

$$MR_\phi(Z^\pi(s,a); \mu, \sigma, \varepsilon) = -(\underbrace{\mu - \sigma std[\phi(U)] corr[\phi(U), \phi(U) + \lambda F_{Z^\pi}^{-1}(U; s, a)]}_{\int_0^1 h_\lambda(p)\phi(p)dp \text{ in } Eq.(12)}), \tag{15}$$

where $\lambda \geq 0$ is the constant determined by Eq.(14) in Theorem 1.

4.1 Critic-Ensemble Criterion

To calculate the model risk for policy gradient using Eq.(15), it is required to determine μ, σ and tolerance ε. For ε, we want to set ε large when there is a large inconsistency between an individual critic while setting ε small when the inconsistency is small. For conservatism, we choose μ as the minimum value among the critics' estimation and σ as the maximum to reduce overestimation bias as discussed by Fujimoto et al. (2018). From now on, we use K to denote the number of critics.

Step 1. Calculating $\boldsymbol{\mu, \sigma}$. To ensure conservatism, we select μ_0, σ as the smallest expectation and the largest deviation of Z^π, as follows.

$$\mu_0 := \min_{i=1,\ldots,K} \mathbb{E}[Z^\pi(s,a,\theta^{(i)})], \sigma := \max_{i=1,\ldots,K} \text{std}[Z^\pi(s,a,\theta^{(i)})] \tag{16}$$

When $\lambda = 0$, the gradient of the policy in Eq.(15) solely depends on the mean-deviation risk measure, which often leads to local optima. To mitigate this, we apply the stop gradient trick to μ as

$$\mu := -H_\phi(Z^\pi(s,a)) + \text{stop-grad}(H_\phi(Z^\pi(s,a)) + \mu_0), \tag{17}$$

where $H_\phi(Z^\pi(s,a)) = \max_{i=1,\ldots,K} H_\phi(Z^\pi(s,a,\theta^{(i)}))$, i.e., the most conservative critic.

Step 2. Calculating ε through quantile mixture. We construct an ensemble of critics as a single quantile function by mixing their quantiles and calculate the distance between this ensemble and individual critics.

We first form the critic ensemble by sorting the estimates from all critics and interpolating them over an equally divided space (i.e., quantile mixture). Next, we calculate the Wasserstein-2 distance between the ensemble of critics and individual critics. The largest of these distances is selected as ε.

The first step involves calculating the sorted critic values y_k defined as

$$\{y_1, \ldots, y_{NK}\} = \cup_{i=1}^K \cup_{j=1}^N F_{Z^\pi}^{-1}(p_j; s, a, \theta^{(i)}),\ y_1 \leq y_2 \leq \cdots \leq y_{NK}, \tag{18}$$

where N represents the number of random variables $p_j \sim U[0,1]$. Let X be an equally divided set of $[0,1]$ defined as $X = \{x_1 = 0, x_2 = 1/(NK-1), \ldots, x_{NK} = 1\}$. By interpolating pairs $\{(x_1, y_1), \ldots, (x_{NK}, y_{NK})\}$, we construct the ensemble of critics, $F_{\text{ens.}}^{-1}$. Figure 3 depicts the procedure of calculating $F_{\text{ens.}}^{-1}$. Finally, ε is computed by

$$\varepsilon = \max_{i=1,\ldots,K} \int_0^1 |F_{\text{ens.}}^{-1}(x; s, a) - F_{Z^\pi}^{-1}(x; s, a, \theta^{(i)})|^2 dx = D_{W;2}^2(F_{Z^\pi}^{-1}(\cdot; s, a, \theta^i), F_{\text{ens.}}^{-1}(\cdot)). \tag{19}$$

We use the trapezoid method to numerically calculate the integral in Eq.(19).

Step 3. Calculate the model risk. We take the most conservative critic and take the reference distribution of the critic estimates in the place of F_Z^{-1} in Eq.(14) and Eq.(15).

$$F_{\text{ref}}^{-1}(\cdot) = F_{Z^\pi}^{-1}(\cdot; s, a, \theta^{(*)}),\ \text{where } \theta^{(*)} = \arg\max_{\theta^{(i)}} H_\phi(Z^\pi(s,a,\theta^{(i)})). \tag{20}$$

Algorithm 1 Critic-Ensemble Model Risk

Inputs: Critic Parameters $\theta^{(i)}$, Risk Measure's Density ϕ, N the number of samples.
Output: Model Risk $\text{MR}_\phi(Z^\pi(s,a), \mu, \sigma, \varepsilon)$
Step1 Calculate μ, σ:
Sample p_j from $U[0,1]$ with $j = 1, \ldots, N$. ▷ To estimate risk and mean by inv. trans. sampling.
$\mu_0 \leftarrow \min_{i=1\ldots K} \mathbb{E}_{p_j}[F^{-1}_{Z^\pi}(p_j; s, a, \theta^{(i)})]$ ▷ Get minimal mean over critics; mean over p_j.
$p'_j \leftarrow F^{-1}_\phi(p_j)$, $H_\phi(Z^\pi(s,a)) \leftarrow \max_{i=1,\ldots,K} -\mathbb{E}_{p'_j}[F^{-1}_{Z^\pi}(p'_j; s, a, \theta^{(i)})])$ ▷ Eq.(3)
$\mu \leftarrow -H_\phi(Z^\pi(s,a)) + \text{stop-grad}(H_\phi(Z^\pi(s,a)) + \mu_0)$ ▷ Stop-Gradient Trick in Eq.(17)
$\sigma \leftarrow \max_{i=1,\ldots,K} \text{std}_{p_j}[F^{-1}_{Z^\pi}(p_j, s, a, \theta^{(i)})]$ ▷ Get maximal standard deviation over critics.
Step 2 Calculate ε:
$\{y_k\}_{k=1}^{NK} \leftarrow \text{sort}(\cup_{i=1}^{K} \cup_{j=1}^{N} \{F^{-1}_{Z^\pi}(p_j; s, a, \theta^{(i)})\})$ ▷ Sort
$\{x_1 \leftarrow 0, x_2 \leftarrow \frac{1}{NK-1}, \ldots, x_{NK-1} \leftarrow \frac{NK-2}{NK-1}, x_{NK} \leftarrow 1\}$ ▷ $x \leftarrow \text{linspace}(0, 1, NK)$
$F^{-1}_{\text{ens.}} \leftarrow \text{interpolation}(\{(x_k, y_k)\}_{k=1}^{NK})$ ▷ Rearrange and Interpolate
$\varepsilon = \max_{i=1,\ldots,K} \int_0^1 |F^{-1}_{\text{ens}}(x) - F^{-1}_{Z^\pi}(x; s, a, \theta^{(i)})|^2 dx$ ▷ Calculate via numerical integral.
Step 3 Calculate Model Risk:
$\theta^{(*)} \leftarrow \arg\max_{\theta^{(i)}} H_\phi(Z^\pi(s,a,\theta^{(i)}))$, $F^{-1}_{\text{ref}} \leftarrow F^{-1}_{Z^\pi}(\cdot; s, a, \theta^{(*)})$ ▷ Take the most conservative.
$\lambda \leftarrow$ calculate lambda$(F^{-1}_{\text{ref}}, \mu, \sigma, \varepsilon, \phi)$ ▷ using Eq.(14).
$\text{MR}_\phi(Z^\pi(s,a)) \leftarrow$ calculate model risk$(F^{-1}_{\text{ref}}, \mu, \sigma, \varepsilon, \phi)$ ▷ using Eq.(15)
Return: $\text{MR}_\phi(Z^\pi(s,a))$

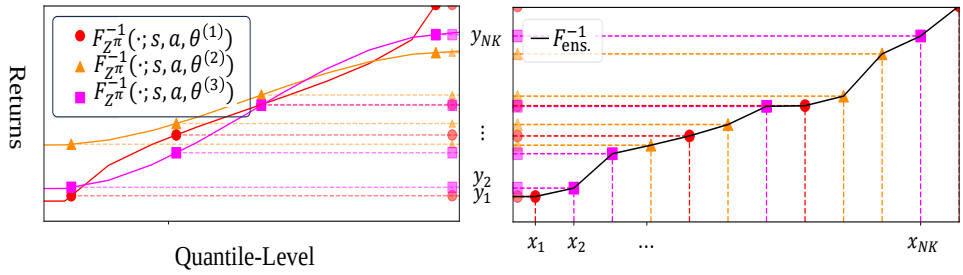


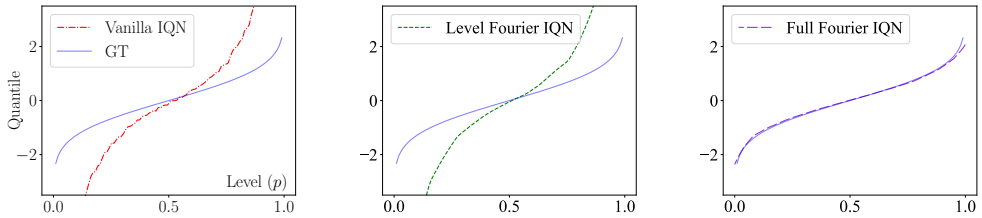
Figure 3: Procedure of calculating $F^{-1}_{\text{ens.}}$ with three critics. $F^{-1}_{\text{ens.}}$ is a quantile mixture of critics, therefore, it shows a mid-level pessimism similar to $F^{-1}_{Z^\pi}(\cdot, s, a, \theta^{(3)})$.

If a solution does not exist, (i.e., $\varepsilon < (\mu_Z - \mu)^2 + (\sigma_Z - \sigma)^2$), we substitute $H_\phi(Z^\pi(s,a))$ for $\text{MR}_\phi(Z^\pi(s,a); \mu, \sigma, \epsilon)$ to preserve the gradient signal.

Algorithm 1 outlines the procedure of computing model risk throughout the critic-ensemble criterion. First, the most conservative μ, σ among critics, are chosen. Next, we compute ε by measuring the $D_{W;2}$ distance between each individual critic and the ensemble, which is obtained by quantile mixture via sort and rearrange. Finally, we compute the model risk using the parameters obtained from the previous steps.

4.2 Addressing Spectral Bias of Quantile Regression

Although our framework accounts for model errors, it is essential to estimate the mean (μ) and deviation (σ) as accurately as possible, since these statistics are assumed to approximate the ground truth. While quantile regression yields an unbiased estimator (Koenker, 2005), due to spectral bias (Rahaman et al., 2019), conventional neural networks often fail to capture these statistics effectively. Spectral bias refers to the phenomenon where neural networks struggle to learn high-frequency components of patterns. For risk-sensitive RL to be effective, Z^π must converge in the distributional sense to ensure proper signals can be back-propagated. According to the Lévy convergence theorem (Williams, 1991), this is equivalent to the pointwise convergence of the Fourier transform of Z^π, emphasizing the critical role of the high-frequency domain (and thus spectral bias) in convergence. The Fourier feature network (Tancik et al., 2020; Li & Pathak, 2021) is a known solution to the spectral bias problem.



(a) Vanilla IQN
$D_{W;1}$: 2.23 ± 0.05
Crossing(%): 6.07 ± 2.08

(b) Fourier feature in level domain
$D_{W;1}$: 2.33 ± 0.14
Crossing(%): 0.01 ± 0.00

(c) Full Fourier feature
$D_{W;1}$: 0.06 ± 0.02
Crossing(%): 0.01 ± 0.00

Figure 4: Results of quantile regression. $D_{W;1}$ is the Wasserstein 1-distance between ground truth quantile, and Crossing is the ratio of the numbers of crossing quantiles (%).

Figure 4 demonstrates the importance of the Fourier feature network in quantile regression. Here the networks are trained to estimate $F^{-1}_{\mathcal{N}(\mu,\sigma^2)}$ given $\mu \sim \mathcal{N}(0,1)$, $\log\sigma \sim \mathcal{N}(0,1)$. (a) is the result of a network trained by Vanilla IQN. (b) is the case when we handle the spectral bias in the level of quantile. (c) shows the case when we handle the spectral bias of all input domains. As shown, the spectral bias is the cause of inaccurate distribution estimation. Note that applying Fourier features in the level domain[3] helps reduce the crossing quantile, but it does not improve overall accuracy. This is because the cosine embedding used in IQN already functions as a type of Fourier feature network. Figure 5 depicts the implementation of the Fourier feature network in (c). For parameters ξ initialized from $\mathcal{N}(0, \sigma_{ff})$ and input X_0, the Fourier feature is

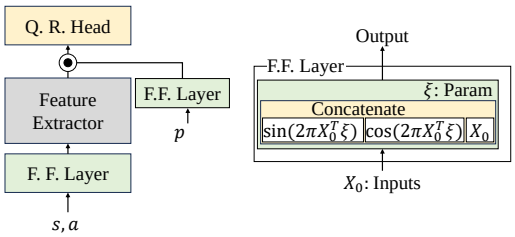


Figure 5: Fourier feature quantile regression network. $\odot$ is pairwise multiplication operator.

$$X_1 = \sin(2\pi X_0^T \xi) \# \cos(2\pi X_0^T \xi) \# X_0, \tag{21}$$

where $\#$ denotes the concatenation operator, and X_1 denotes the feature passed to the next layer. We also leverage using the Fourier feature IQN instead of cosine embedding IQN to reduce crossing quantile error. The remaining process is the same as conventional MLP.

5 Experiments

5.1 Baselines and Criterion

Risk Measures. For the experiments, we are interested in following risk measures.

(i) *CV@R*. considers average over below α's quantile. The risk measure's density $\phi_\alpha^{\text{CV@R}}$ parameterized by $\alpha \in [0,1]$ of is

$$\phi_\alpha^{CV@R}(p) = \frac{1}{\alpha}\mathbf{1}_{[0,\alpha]}(p). \tag{22}$$

(ii) *Wang*. is a risk measure motivated by areas where one's cost becomes another's benefit, like an option trading (Wang, 2002). The density of *Wang* risk measure, $\phi_\alpha^{\text{wang}}$, parameterized by $\alpha < 0$ is

$$\phi_\alpha^{wang}(p) = \frac{\varphi(F^{-1}_{\mathcal{N}(0,1)}(1-p)+\alpha)}{\varphi(F^{-1}_{\mathcal{N}(0,1)}(1-p))}, \tag{23}$$

where $F^{-1}_{\mathcal{N}(0,1)}$ and φ are the quantile and density of standard normal distribution, respectively.

[3]The input domain of quantile function; i.e., the space of p in $F_Z^{-1}(p)$.

Baseline Algorithms. ORAAC (Urpí et al., 2021) is a risk-sensitive variant of BCQ (Fujimoto et al., 2019). It achieves the risk-sensitive offline RL objective by only having slight modifications in the dataset's actions. **CODAC** (Ma et al., 2021) is a risk-sensitive variant of CQL (Kumar et al., 2020). It achieves a dataset constraint by minimizing KL-divergence between the RL agent's actions and the dataset's actions using the Donsker-Varadhan representation. **IQN-TD3+BC** is a risk-sensitive variant of the TD3+BC (Fujimoto & Gu, 2021) algorithm, where the IQN-Critic is used. The optimal actor $\pi_{\text{TD3+BC}}$ in IQN-TD3+BC is defined as

$$\pi_{\text{TD3+BC}} = \arg\min_{\pi} \lambda_{\text{q.learn}} H_\phi(Z^\pi(s,a)) + (a - a_{\mathcal{D}})^2. \tag{24}$$

Here $\lambda_{\text{q.learn}} > 0$ is a scale hyperparameter, $a_{\mathcal{D}}$ is a batch action, same as in Eq.(11).

5.2 Main Results

We evaluate MR-IQN on finance and self-driving scenarios comparing with the baselines above. The label Mean is the average of the mean score over seeds and $-H_\phi(Z^\pi)$ is the mean negative risk over seeds. All reported scores are averaged across 5 seeds. Numerical descriptions provided without explicit criteria are the results of comparing the negative risk. We also present the D4RL results, comparing the baselines and 1R2R, in Appendix A.1. Dataset details are provided in Appendix B.

Table 1: Performance comparison in finance scenarios.

Target Risk Measure		*CV@R* (50%)		*Wang* (−0.5)	
Env.	Algorithm	Mean	$-H_\phi(Z^\pi)$	Mean	$-H_\phi(Z^\pi)$
Forex Env. (High)	CODAC	1577.47 ± 41.52	249.53 ± 28.33	1591.76 ± 8.12	892.34 ± 54.06
	ORAAC	302.37 ± 19.27	-99.90 ± 0.01	304.89 ± 38.05	70.47 ± 17.22
	IQN-TD3+BC	2103.16 ± 103.43	454.16 ± 11.76	2181.16 ± 123.41	1307.52 ± 90.05
	MR-IQN (ours)	$\mathbf{2537.52 \pm 289.57}$	$\mathbf{556.94 \pm 172.92}$	$\mathbf{2681.40 \pm 114.00}$	$\mathbf{1639.79 \pm 57.88}$
Stock Env. (Low)	CODAC	37.67 ± 7.54	13.98 ± 2.34	33.43 ± 7.01	21.89 ± 4.65
	ORAAC	$\mathbf{78.48 \pm 3.27}$	26.07 ± 2.25	78.57 ± 6.09	50.97 ± 3.98
	IQN-TD3+BC	$\mathbf{80.49 \pm 1.87}$	$\mathbf{30.92 \pm 1.15}$	83.10 ± 3.05	55.55 ± 1.76
	MR-IQN (ours)	74.78 ± 2.10	$\mathbf{31.16 \pm 1.02}$	$\mathbf{87.03 \pm 2.62}$	$\mathbf{58.56 \pm 1.61}$

Finance. To investigate the real-world scenarios, we conduct experiments using the trading log data ranging from 2023-Feb-3rd to 2023-December-1st. The trading environment is based on the MetaTrader Simulator (Amin, 2021). We evaluate 1000 episodes for each seed to calculate *CV@R* and *Wang* negative risk. For the details, see Appendix. Table 1 depicts the results.

In the experiment, MR-IQN shows the highest performance in both Forex and Stock environments. In Forex environment, where uncertainty is high due to high leverage (High), the model risk approach effectively covers tail risk, maintaining high average returns and low risk, compared to the baseline approaches without model risk. IQN-TD3+BC, which is comparable to MR-IQN shows degraded performance (18.5%-11.2%) in Forex, indicating that MR-IQN is particularly useful in handling higher uncertainty and errors. Meanwhile, MR-IQN shows a slightly degraded average compared to ORAAC and IQN-TD3+BC in Stock environment. This indicates that there is a trade-off between conservatism and average performance.

Self-Driving. To simulate the self-driving scenario, we utilize the Airsim environment (Shah et al., 2017). In this scenario, RL agents are tasked with driving a quadcopter drone to a designated goal while avoiding obstacles, with wind disturbances affecting the drone's movement. The agents receive rewards based on the cosine of the angle between their movement direction and the goal. We evaluate performance in 100 episodes for each seed to measure negative risk. Table 2 depicts the results.

MR-IQN outperforms other baselines, showing the highest success rate (87.4%-88.8%) and the lowest collision rate (10.6%-11.6%). CODAC showed a relatively low average of rewards and a higher success rate compared to ORAAC. This implies that CODAC fails to generalize efficient actions and hovers over. The high collision rate of ORAAC implies that it fails to avoid obstacles because of the strict constraint to the batch action.

Table 2: Performance comparison in self-driving scenarios

Risk Measure	Algorithm	Mean	$-H_\phi(Z^\pi)$	Success(%)	Collision(%)
CV@R (50%)	CODAC	43.56 ± 89.45	-425.72 ± 97.73	65.0 ± 13.1	21.4 ± 5.5
	ORAAC	92.37 ± 40.51	-88.37 ± 4.20	23.0 ± 15.7	68.0 ± 10.2
	IQN-TD3+BC	$\mathbf{411.78 \pm 33.10}$	$\mathbf{265.51 \pm 45.25}$	81.8 ± 5.6	$\mathbf{14.8 \pm 5.6}$
	MR-IQN(ours)	$\mathbf{431.44 \pm 30.41}$	$\mathbf{290.96 \pm 46.95}$	$\mathbf{88.0 \pm 4.6}$	$\mathbf{11.6 \pm 4.9}$
Wang (-0.5)	CODAC	6.67 ± 1.02	-2.15 ± 0.75	52.6 ± 19.3	31.2 ± 27.6
	ORAAC	156.22 ± 32.09	51.73 ± 27.53	36.2 ± 6.4	58.4 ± 8.1
	IQN-TD3+BC	$\mathbf{412.98 \pm 26.68}$	311.20 ± 31.03	$\mathbf{83.8 \pm 5.1}$	14.4 ± 3.9
	MR-IQN(ours)	$\mathbf{436.17 \pm 19.16}$	$\mathbf{341.26 \pm 18.36}$	$\mathbf{87.4 \pm 1.8}$	$\mathbf{10.6 \pm 2.6}$

5.3 Ablation Study

Table 3: Ablation results. ✓ represents presence and ✗ represents absence. M.R. and F.F. are the abbreviations for model risk and Fourier feature, respectively.

Components			*CV@R* (50%)		*Wang* (-0.5)	
M.R.	F.F.	TQC	Mean	$-H_\phi(Z^\pi)$	Mean	$-H_\phi(Z^\pi)$
✓	✓	✓	$\mathbf{2537.52 \pm 289.57}$	$\mathbf{556.94 \pm 172.92}$	$\mathbf{2681.40 \pm 114.00}$	$\mathbf{1639.79 \pm 57.88}$
✓	✓	✗	2003.37 ± 214.76	464.84 ± 57.01	1975.85 ± 104.49	1160.65 ± 87.82
✓	✗	✓	$\mathbf{2410.83 \pm 86.94}$	$\mathbf{513.02 \pm 77.54}$	2324.31 ± 146.95	1376.27 ± 122.96
✓	✗	✗	1763.21 ± 278.16	278.16 ± 94.26	1607.13 ± 76.54	872.54 ± 56.23
✗	✓	✓	2103.16 ± 103.43	454.16 ± 11.76	2181.16 ± 123.41	1307.52 ± 90.05
✗	✓	✗	1712.79 ± 75.40	295.10 ± 49.83	1650.28 ± 51.48	934.79 ± 30.00
✗	✗	✓	2170.68 ± 186.55	476.23 ± 75.94	2188.57 ± 134.40	1297.15 ± 88.15
✗	✗	✗	1454.30 ± 76.87	153.84 ± 38.57	1414.44 ± 40.00	750.22 ± 37.45

We also conduct an ablation study in the Forex environment, identifying model risk, Fourier features, and TQC as key contributors to the model's improvement. Table 3 presents the results. MR-IQN achieves an 8.6%-19.8% performance gain from the Fourier feature and a 19.8%-41.3% performance gain from TQC. The performance gain from model risk is reported in Section 5.2. Notably, model risk enhances performance when combined with other factors, due to the design of the critic ensemble criterion and the requirement for accurate estimation of μ and σ, as discussed in Section 4.2.

Our framework achieves approximately 118.6%-262.0% performance gains compared to the naive implementation, which omits the key components. The reason why TQC is introduced to our framework is to construct a target quantile which is similar to $F^{-1}_{\text{ens.}}$. As a result, the calculated model risk tolerance, ε, becomes more aligned with our intention, i.e., large ε for high inconsistency and vice versa. However, the performance improvement of TQC without other factors (72.9%-209.6%) implies that TQC itself is not a negligible factor.

6 Conclusion

We proposed a model risk-sensitive offline RL framework, devising the critic-ensemble criterion to capture model risk effectively. To ensure the precision of model risk calculation, we employed Fourier feature networks, which accurately estimate both the mean and standard deviation—critical components for calculating the model risk. The framework ensures more reliable decision-making in risk-sensitive applications by accounting for potential model errors and striving to make the best decisions despite them. While our framework is limited to spectral risk measures and cannot accommodate those outside this class, such as *CMV* (Vadori et al., 2020) or *EV@R* (Ni & Lai, 2022), the broad applicability of spectral risk measures covers many practical real-world problems. Our future work involves extending the framework to partially observable Markov decision processes, including embodied control and decision-making in mission-critical business applications.

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