Differentially Private Learning Needs Hidden State (Or Much Faster Convergence)

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Abstract

Prior work on differential privacy analysis of randomized SGD algorithms relies on composition theorems, where the implicit (unrealistic) assumption is that the internal state of the iterative algorithm is revealed to the adversary. As a result, the Rényi DP bounds derived by such composition-based analyses linearly grow with the number of training epochs. When the internal state of the algorithm is hidden, we prove a converging privacy bound for noisy stochastic gradient descent (on strongly convex smooth loss functions). We show how to take advantage of privacy amplification by sub-sampling and randomized post-processing, and prove the dynamics of privacy bound for "shuffle and partition" and "sample without replacement" stochastic mini-batch gradient descent schemes. We prove that, in these settings, our privacy bound converges exponentially fast and is substantially smaller than the composition bounds, notably after a few number of training epochs. Thus, unless the DP algorithm converges fast, our privacy analysis shows that hidden state analysis can significantly amplify differential privacy.

1 Introduction

Machine learning models leak sensitive information about their training data [39, 11]. To protect user privacy, the widely-used differentially private training algorithm, DP-SGD [1], adds carefully calibrated noise in each step of updating model parameters. This randomness guarantees that the models trained on any two neighboring datasets are indistinguishable in probability distributions. To quantify this indistinguishability, the DP analysis *bounds* the (moment of) likelihood ratio between a pair of models trained on any two neighboring datasets. Improving this privacy analysis is crucial for obtaining a higher (train and test) accuracy of the output model, under a constrained privacy budget.

The main-stream analysis of privacy loss in DP-SGD is based on composition theorems, which quantify the total privacy loss of the training process across all its iterations. Given that privacy-preserving learning via DP-SGD usually suffers from a slow convergence in empirical prediction accuracy [10], the final bound could be significantly large (and loose). This analysis worsens the privacy-accuracy trade-off by overestimating the magnitude of required noise for DP training.

To alleviate this problem, one popular direction is to design new variants of the DP-SGD algorithm, that converge faster, i.e. require a smaller number of iterations to reach a stable training accuracy. This includes works that derive privacy preserving variants of fast optimization algorithms, such as performing DP-SGD with momentum [25], adaptive gradient clipping [3, 30], adaptive selection of step-size and noise scale [4], and using pre-trained or hand-crafted features [34]. All these approaches aim for a *faster training convergence* for the DP learning algorithm, such that the composition analysis is applied to a smaller number of iterations and the total bound remains small.

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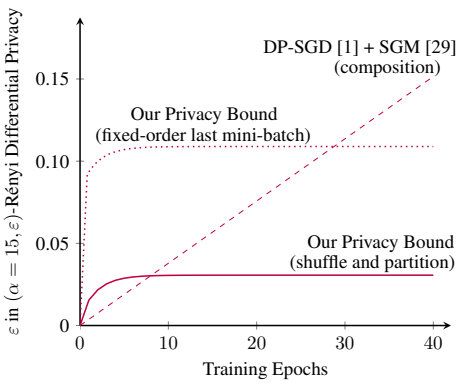
Another line of work focuses on directly improving the privacy analysis of the DP-SGD algorithm. To this end, prior work [18, 5, 15] suggest that hiding the internal state of noisy (S)GD, therefore analyzing the privacy bound for releasing only the last iterate results in a more accurate estimation of privacy loss. By applying the last-iterate privacy analysis [18], Feldman et al. [19] design new differentially private algorithms that achieve theoretically optimal excess risk with linear runtime for convex optimization, which is better than the prior privacy utility trade-off for the noisy SGD algorithm [7]. However, the analysis applies only to one *single* epoch of training, thus it is unclear how such training performs in practical learning settings (which require multiple epochs of training). Chourasia et al. [15] derive a strong privacy guarantee for hidden-state noisy GD over many training epochs, under strongly convex loss function. However, the analysis is limited to computationally expensive GD [23], and extension of it to stochastic mini-batch training would fail to model the privacy amplification by post-processing [18, 5] and mini-batch sub-sampling [12, 27, 7, 1, 5].

Thus, under the hidden-state assumption over many epochs of training, the challenge is to compute a small privacy bound for differentially private stochastic gradient descent algorithms by taking advantage of privacy amplifications due to sub-sampling and randomized updates over mini-batches.

Contributions. In this paper, we tackle this challenge, and bridge the gaps in the prior work on hidden-state (last-iterate) privacy analysis. We model the privacy dynamics of noisy SGD, and show the privacy amplification for private (hidden state) machine learning due to stochastic mini-batch selection. **(i)** As we also show in this figure, for multi-epoch noisy *stochastic mini-batch* gradient descent under *"shuffle and partition"* and *iterative "sampling without replacement"*, we prove new converging privacy bounds (for strongly convex smooth loss function) that significantly improves over the prior bounds [5, 29, 15]. Our proof relies on our new bounds for the privacy amplification by post-processing (Lemma 3.2) and sub-sampling (Theorems 4.2 and 4.3). **(ii)** For the special case of full gradient descent, our new approach (via proving better bounds for the privacy amplification by randomized post-processing) results in a strictly tighter bound than the prior work [15] with a different proof (Appendix D.7). The key insight of our new proof is that we can break down one noisy GD update into two consecutive steps: a noisy GD update with smaller noise scale, followed by pure additive Gaussian noise (randomized post-processing).



Our results show that, to obtain a tighter privacy bound, it is crucial to apply our hidden-state privacy dynamics analysis for learning tasks with slow to moderate convergence. Alternatively, it is not so costly to use composition-based privacy analysis when the training process converges very quickly.

2 An overview of the problem and our approach

We analyze differential privacy loss of the noisy (stochastic) mini-batch gradient descent algorithm, when only the last iterate parameters θ_K^0 are visible (see Algorithm 1).[1] We consider two mini-batch generation variants: **(1)** "shuffle and partition" (widely implemented in privacy libraries [40, 24]); **(2)** "sample without replacement" (analyzed extensively in prior results [12, 27, 5, 37] despite its computational cost). Our ultimate goal is to prove a *worst-case* upper bound of the Rényi divergence between distributions of *last-iterate* parameters θ_K^0 and ${\theta'}_K^0$ trained on *any* two neighboring datasets.

The prior work Chourasia et al. [15] has shown a tight converging hidden-state privacy dynamics analysis for full-batch noisy gradient descent. However, the sub-sampling steps that are unique to the stochastic mini-batch gradient descent, and their privacy benefits, are not modeled nor quantified in such bounds and the follow-up work. Would a naive extension of the prior GD analysis to the stochastic mini-batch setting result in a tight bound? To this end, we can view the updates that *involve* the (sensitive) differing data record (between neighboring datasets) as gradient descent on a *smaller*, iteratively *changing* dataset of size b (i.e., the size of a mini-batch). This extension results in the following naive privacy baseline.

[1] See Appendix B for the preliminaries about differential privacy and necessary tools for our analysis.

Algorithm 1 $\mathcal{A}_{\text{Noisy-mBGD}}$: Noisy (Stochastic) mini-batch Gradient Descent

Input: Dataset $D = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$. Parameter space $\theta \in \mathbb{R}^d$. Loss function $\ell(\theta; \mathbf{x})$. Stepsize η. Noise standard deviation σ. mini-batch size b. Initial parameters θ_0^0 sampled from an arbitrary distribution $p_0(\theta)$.
Batch Generation: obtain mini-batches B_k^j of size b, for $j = 0, \cdots, n/b-1$; for epochs $k = 0, \cdots, K-1$.
▷ **If "Shuffle and partition"**: randomly partition n data indices into $B^0, \cdots, B^{n/b-1}$, let $B_k^j = B^j$. [3]
▷ **If Sample without replacement**: resample b different indices from $\{1, \cdots, n\}$ to obtain every B_k^j.
for epoch $k = 0, 1, \cdots, K-1$ **do**
  for iteration $j = 0, 1, \cdots, n/b-1$ **do**
    $\theta_k^{j+1} = \theta_k^j - \eta \cdot g\left(\theta_k^j; B_k^j\right) + \sqrt{2\eta\sigma^2} \cdot \mathcal{N}\left(0, \mathbb{I}_d\right)$ where $g\left(\theta_k^j; B_k^j\right) = \frac{1}{b}\sum_{i \in B_k^j} \nabla \ell(\theta_k^j; \mathbf{x}_i)$
  $\theta_{k+1}^0 = \theta_k^{n/b}$
Output: θ_K^0

Theorem 2.1 (Naive Extension of Chourasia et al. [15] bound to SGD)**.** *If the loss function $\ell(\theta; \mathbf{x})$ is λ-strongly convex and β-smooth, and its gradient has ℓ_2-sensitivity S_g, then Algorithm 1 under "shuffle and partition" and step-size $\eta < \frac{2}{\lambda+\beta}$ satisfies (α, ε)-Rényi DP with $\varepsilon \leq \frac{\alpha S_g^2}{\lambda \sigma^2 b^2}(1 - e^{-\lambda\eta K/2})$.*

However, as we also show in Figure 1, this naive privacy dynamics baseline *slowly* converges to a *huge* constant, which is significantly worse than the bounds derived by composition [29, 1]. [2] This is because this naive baseline *fails to* capture the privacy amplification due to the stochasticity of mini-batch sub-sampling, and the iterative noisy updates which amplify the privacy of preceding mini-batches (referred to as amplification by post-processing). In this paper, we show how to compute a much tighter differential privacy bound for noisy SGD, under the hidden-state assumption, while taking advantage of these privacy amplifications. Our methodology is novel and quantifies the hidden-state privacy amplification due to iterative data (re)sampling throughout the training process (over multiple epochs of training). This solves the limitation of the prior work on privacy amplification (by iteration) which focuses on a single epoch, and uses composition theorems across epochs (thus, not modeling the privacy of hidden-state iterative resampling). Finally, our methodology is generic and applies to the special case of full-batch noisy gradient descent, which enables a strictly tighter privacy bound than the prior work [15] under the same assumptions (Appendix D.7).

Our approach is as follows. We decompose the distribution of last iterate parameters θ_K^0 in Algorithm 1 as a mixture of conditional distributions $p(\theta_K^0) = \sum_B p(B) \cdot p(\theta_K^0 | B)$, given any possible mini-batch sequence B. We first analyze the Rényi privacy loss for a *fixed mini-batch sequence* while modeling privacy amplification by randomized post-processing (i.e., how much the randomized gradient update improve the privacy due to updates on preceding mini-batches). We then quantify the privacy amplification by subsampling under *stochastic mini-batches*.

Privacy amplification by randomized post-processing. To start, we consider Algorithm 1 without the effect of stochasticity in the mini-batch sampling process. That is, we first assume that the mini-batch sequence used in the algorithm is fixed (by an arbitrary order). When a mini-batch contains only the indices of shared data records between two neighboring datasets, then the deterministic mini-batch gradient descent mapping does not cause any additional privacy loss. Addition of the Gaussian noise that follows this deterministic update, however, serves as a randomized post-processing which *decreases* the Rényi divergence between the two processes (associated to neighboring datasets). To compute this privacy amplification, we precisely model the change of parameter distributions with the Fokker-Planck equation (for diffusion process with zero drift).

Hidden-state privacy amplification by sub-sampling. Although privacy amplification by a *single* sub-sampling operation is well-studied [12, 27, 7, 1, 5, 37, 29, 20, 21], to the best of our knowledge, no prior bounds are applicable to multiple epochs of *hidden-state* iterative resampling. This is because

[2] A recent concurrent work [32] also follows this approach to extend the GD analysis [15] to SGLD setting, and proves a similar bound to our naive privacy dynamics baseline Theorem 2.1. However, there is a slight difference between [32, Corollary 3.3] and our Theorem 2.1, due to the flawed assumption in [32, Lemma 3.4] that the LSI constant proved in Chourasia et al. [15] (that *only* holds for GD process) also holds for SGLD process (that takes the form of a more complicated mixture distribution). See more details in Appendix C.

[3] For simplicity of presentation, we assume b divides n. If n/b is not an integer, and if the algorithm ignores the last $n - \lfloor n/b \rfloor \cdot b$ data points, then our privacy dynamics bound holds by replacing n/b with $\lfloor n/b \rfloor$.

under the hidden-state assumption, the number of mixing components in the last-iterate parameter distribution grows exponentially with the number of epochs K, i.e., [number of possible values for one mini-batch]$^{K \cdot n/b}$, which makes the mixture distribution very difficult to analyze. In this paper, to study the exponentially many mixture components, we derive recursions for the divergence between mixture distributions after one epoch. We use the joint convexity of exponentiated Rényi divergence to bound how much smaller the Rényi divergence between mixture distributions (for model parameters at one epoch) is compare to the worst case Rényi divergence across any pair of their mixture components (representing the preceding epoch). This recursion quantifies the hidden-state privacy amplification by sub-sampling, and enables a significantly smaller Rényi DP bound for noisy stochastic mini-batch gradient descent, than the composition of sub-sampled mechanisms over multiple epochs (Figure 2).

3 Privacy dynamics for fixed-ordering noisy mini-batch gradient descent

In this section, we quantify the privacy amplification by randomized post-processing in Algorithm 1, during iterations that do not access the sensitive differing data between neighboring datasets. We then combine it with RDP composition for the remaining iterations and prove a converging privacy bound.

3.1 Privacy amplification by randomized post-processing (additive Gaussian noise)

We first explain the key lemma that proves exponentially decaying Rényi privacy loss under additive Gaussian noise post-processing, when the parameter distributions satisfy the log-Sobolev inequality. This is a new bound for the well-known "privacy amplification by iteration" phenomenon [18, 5].

Lemma 3.1. *Let μ, ν be two distributions on $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping on $\mathbb{R}^d$. We denote $\mathcal{N}(0, 2t\sigma^2 \cdot \mathbb{I}_d)$ to be the standard Gaussian distribution on $\mathbb{R}^d$ with covariance matrix $2t\sigma^2 \cdot \mathbb{I}_d$. We denote $p_t(\theta)$ and $p'_t(\theta)$ to be the probability density functions for the distributions $f_\#(\mu) * \mathcal{N}(0, 2t\sigma^2\mathbb{I}_d)$ and $f_\#(\nu) * \mathcal{N}(0, 2t\sigma^2\mathbb{I}_d)$ respectively, where $f_\#(\mu), f_\#(\nu)$ denote the push forward distributions of μ, ν under mapping f. Then if μ and ν satisfy log-Sobolev inequality with constant c, and if the mapping f is L-Lipschitz, then for any order $\alpha > 1$,*

$$\frac{\partial}{\partial t} R_\alpha \left(p_t(\theta) \| p'_t(\theta)\right) \leq -c_t \cdot 2\sigma^2 \cdot \left(\frac{R_\alpha(p_t(\theta)\|p'_t(\theta))}{\alpha} + (\alpha - 1) \cdot \frac{\partial}{\partial \alpha} R_\alpha(p_t(\theta)\|p'_t(\theta)) \right), \quad (1)$$

where $c_t = \left(\frac{L^2}{c} + 2t\sigma^2\right)^{-1}$ is the log-Sobolev inequality constant for distributions $p_t(\theta)$ and $p'_t(\theta)$.

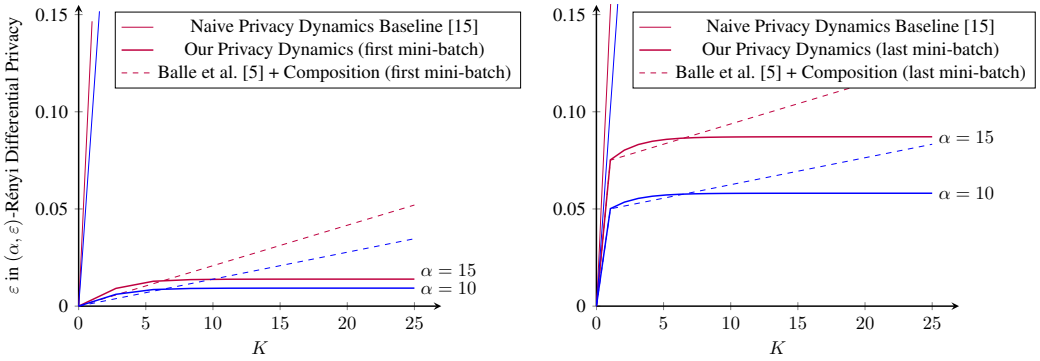
Proof Sketch. The proof starts by modelling Gaussian noise as a diffusion process with *zero* drift. We then bound the rate of Rényi divergence with the LSI constant for the process (following prior works [35, 15]). However, instead of assuming a fixed LSI constant c for all intermediate distributions, we prove a more precise LSI constant c_t that changes with t. Complete proof is in Appendix D.1. □

The partial differential inequality in Lemma 3.1 quantifies the amplification of Rényi privacy loss under additive Gaussian noise. By solving Equation (1) on $t \in [0, \eta]$, we prove the following lemma that models recursive privacy dynamics during one step of noisy mini-batch gradient descent.

Lemma 3.2. *Let D, D' be an arbitrary pair of neighboring datasets that differ in the i_0-th data point (i.e. $x_{i_0} \neq x'_{i_0}$). Let B_k^j be a fixed mini-batch used (in iteration j of epoch k) in Algorithm 1, which contains b indices sampled from $\{1, \cdots, n\}$. We denote θ_k^j and ${\theta'}_k^j$ as the intermediate parameters in Algorithm 1 on input datasets D and D', respectively. If the distributions of θ_k^j and ${\theta'}_k^j$ satisfy log-Sobolev inequality with a constant c, and if the mini-batch GD mapping $f(\theta) = \theta - \eta \cdot \frac{1}{b} \cdot \sum_{i \in B_k^j} \ell(\theta; \mathbf{x}_i)$ is L-Lipschitz for parameters θ, then the following recursive bound for Rényi divergence holds.*

$$\frac{R_\alpha(\theta_k^{j+1} \| {\theta'}_k^{j+1})}{\alpha} \leq \begin{cases} \frac{R_{\alpha'}(\theta_k^j \| {\theta'}_k^j)}{\alpha'} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} & \text{if } i_0 \notin B_k^j \\ \frac{R_\alpha(\theta_k^j \| {\theta'}_k^j)}{\alpha} + \frac{\eta S_g^2}{4\sigma^2 b^2} & \text{if } i_0 \in B_k^j \end{cases} \text{ with } \alpha' = \frac{\alpha - 1}{1 + \frac{c \cdot 2\eta\sigma^2}{L^2}} + 1. \quad (2)$$

Proof Sketch. When $i_0 \notin B_k^j$, we apply Lemma 3.1 on the deterministic mini-batch gradient descent update f, and solve Equation (1) to obtain recursive Rényi privacy bound. This proof is similar to



(a) for data points in the first mini-batch

(b) for data points in the last mini-batch

Figure 1: Rényi privacy loss of fixed-ordering noisy mini-batch gradient descent over K epochs, which repeatedly goes over a fixed sequence of mini-batch partition in each epoch. We show the privacy loss for data points in the first mini-batch B^0 and the last mini-batch $B^{n/b-1}$ of each pass. We show the ε in the (q, ε)-RDP guarantee derived by our privacy dynamics analysis (bold lines), the naive privacy dynamics baseline (thin lines), and the privacy amplification by mixing and diffusion analysis [5] combined with composition (dashed line). We evaluate under the following setting: RDP order $\alpha \in \{10, 15\}$; λ-strongly convex loss function with $\lambda = 1$; β-smooth loss function with $\beta = 4$; gradient sensitivity $S_g = 4$; size of the data set $n = 50$; step-size $\eta = 0.02$; noise variance $\sigma^2 = 4$, mini-batch size $b = 2$. We use Theorem 3.3 for our privacy dynamics; Theorem 2.1 for naive privacy dynamics baseline; and [5, Theorem 5] for Balle et al. [5] + Composition (details are in Appendix D.6).

that of [15, Theorem 2], however, dealing with the new LSI constant c_t that changes with t introduces an additional technical difficulty. When $i_0 \in B_k^j$, we use composition theorem to prove the additive recursion in the second row of Equation (2). The complete proof is in Appendix D.2. □

Comparison with prior amplification by post-processing bound. Prior to this work, for convex and smooth loss functions, Feldman et al. [18] derive a tight bound for the amplification by post-processing in noisy stochastic gradient descent. In Appendix D.4, we show that our new recursive amplification bound could recover this known *tight* bound for convex smooth loss functions, while using a different proof. For (a more restrictive setting of) *one* epoch of noisy SGD on *strongly convex* loss function, Balle et al. [5] further improve the bound in Feldman et al. [18] via a careful coupling-based approach. However, we do not see an easy way to extend the bound in Balle et al. [5] to *multiple* epochs except by using Rényi DP composition (which would give a linearly worsening Rényi DP bound with the number of epochs). By contrast, our recursive amplification bound easily applies to multiple epochs (under hidden state assumption, and without requiring composition over the epochs), and enables converging privacy dynamics for Algorithm 1 on strongly convex smooth loss functions (as discussed in Section 3.2).

3.2 Improved privacy dynamics for fixed-ordering noisy mini-batch gradient descent

By using the recursive privacy bound Lemma 3.2 and the non-overlapping property of mini-batch partitions of the dataset, we prove position-dependent Rényi DP bounds for Algorithm 1 as follows.

Theorem 3.3 (Privacy dynamics under strongly convex smooth loss)**.** *Conditioned on a fixed sequence of partitioned mini-batches $B^0, \cdots, B^{n/b-1}$ in Line 3, if the loss function is λ-strongly convex, β-smooth and its gradient has ℓ_2-sensitivity S_g, then running Algorithm 1 for $K \geq 1$ epochs with step-size $\eta < \frac{2}{\lambda+\beta}$, satisfies (α, ε)-Rényi DP for data points in the batch B^{j_0}, with*

$$\varepsilon \leq \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}} + \varepsilon_0^{n/b-j_0}(\alpha) \tag{3}$$

where $\varepsilon_0^j(\alpha) = \frac{\alpha\eta S_g^2}{4\sigma^2 b^2} \cdot (1-\eta\lambda)^{2\cdot(j-1)} \cdot \frac{1}{\sum_{s=0}^{j-1}(1-\eta\lambda)^{2s}}$ for any $j = 1, \cdots, \frac{n}{b}$ (we assume $\frac{n}{b} \geq 2$).

Proof Sketch. We first prove that the distribution of parameters θ_k^j satisfies LSI with a constant c_k^j that depends on k, j. Then we plug the LSI constants into Lemma 3.2 and prove a recursive privacy bound

for data points in each batch B^{j_0}. Finally, by carefully choosing which recursion to use and solve, we obtain the privacy bound in the theorem statement. The complete proof is in Appendix D.5. □

The above new privacy bound Theorem 3.3 quantifies the privacy amplification during iterations that only access the shared data points between neighboring datasets, and it is significantly smaller than the naive privacy dynamics baseline derived from Chourasia et al. [15] (that does not capture this additional privacy amplification). In Figure 1, we observe that for data points in the first batch B^0, our new privacy dynamics bound is smaller by a multiplicative factor of approximately n/b (where b/n is the batch sampling ratio), and the improvement for data points in the last-batch is also significant.

Bound improvement compared with Balle et al. [5]. In the first epoch (when $K = 1$), our privacy dynamics bound (based on recursive scheme) in Theorem 3.3 is of same order as the privacy amplification bound Balle et al. [5, Theorem 5], as shown in Figure 1. However, for training with multiple epochs, Balle et al. [5] use a coupling-based approach, and we do not see an easy way to extend their analysis except by using Rényi DP composition [28] over the epochs, which gives a linearly accumulating Rényi DP bound for multiple epochs (as K increases). On the contrary, as shown in Figure 1, our improved privacy dynamics bound converges to a constant, thus significantly improves over the Rényi DP composition of Balle et al. [5, Theorem 5].

4 Privacy dynamics for noisy stochastic mini-batch gradient descent

In this section, we further improve over the position-dependent privacy dynamics analysis in Section 3, by incorporating the effect of amplification of privacy loss due to the stochastic mini-batch sampling. We first prove the following lemma by the convexity of f-divergence with $f(x) = x^\alpha$.

Lemma 4.1 (Joint convexity of scaled exponentiation of Rényi divergence)**.** *Let $\mu_1, \cdots, \mu_m$ and $\nu_1, \cdots, \nu_m$ be distributions over R^d. Then for any RDP order $\alpha \geq 1$, and any coefficients $p_1, \cdots, p_m \geq 0$ that satisfy $p_1 + \cdots + p_m = 1$, the following inequality holds.*

$$e^{(\alpha-1)\cdot R_\alpha(\sum_{j=1}^m p_j\mu_j \| \sum_{j=1}^m p_j\nu_j)} \leq \sum_{j=1}^m p_j \cdot e^{(\alpha-1)\cdot R_\alpha(\mu_j\|\nu_j)} \tag{4}$$

We provide a detailed proof for Lemma 4.1 in Appendix E. This Lemma is our main tool for quantifying the additional privacy amplification under stochastic mini-batches in the rest of the section.

4.1 Privacy dynamics under shuffle and partition

The "shuffle and partition" batch generation scheme is commonly used for practical DP-SGD implementations in privacy libraries [24, 40]. By using Theorem 3.3 and new bounds for the privacy amplification by shuffling in Algorithm 1, we prove the following Rényi DP guarantee.

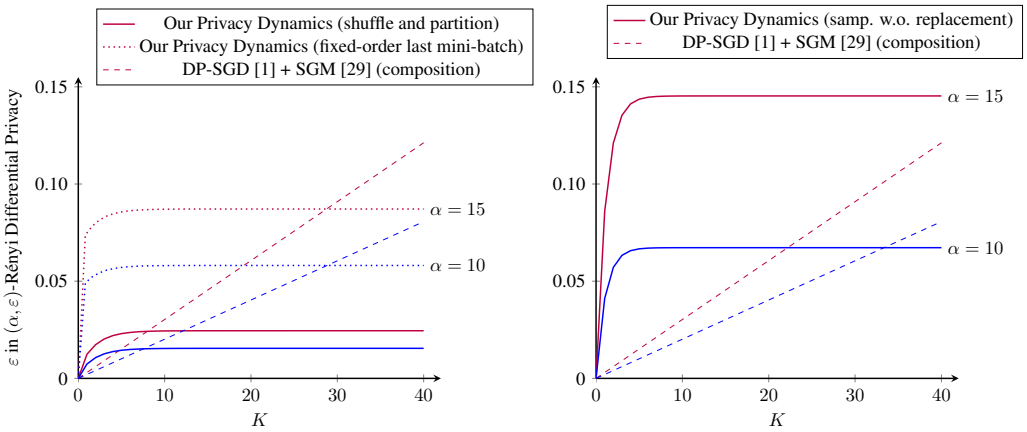
Theorem 4.2 (Privacy dynamics under "shuffle and partition")**.** *If the loss function $\ell(\theta; x)$ is λ-strongly convex, β-smooth, and if its gradient has finite ℓ_2-sensitivity S_g, then for $\frac{n}{b} \geq 2$, running Algorithm 1 for $K \geq 1$ epochs with step-size $\eta < \frac{2}{\lambda+\beta}$, under "shuffle and partition" batch generation scheme, satisfies (α, ε)-Rényi DP with*

$$\varepsilon \leq \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}} + \frac{1}{\alpha-1} \cdot \log\left(\underset{0\leq j_0<n/b}{Avg} e^{(\alpha-1)\varepsilon_0^{n/b-j_0}(\alpha)} \right) \tag{5}$$

where $\varepsilon_0^j(\alpha) = \frac{\alpha\eta S_g^2}{4\sigma^2 b^2} \cdot (1-\eta\lambda)^{2\cdot(j-1)} \cdot \frac{1}{\sum_{s=0}^{j-1}(1-\eta\lambda)^{2s}}$ for any $j = 1, \cdots, \frac{n}{b}$ (we assume $\frac{n}{b} \geq 2$).

The above Rényi DP bound Theorem 4.2 is always smaller than the Rényi DP bound in Theorem 3.3 for the worst-case fixed mini-batch sequence (i.e., when the differing data point is in the last mini-batch of each pass). Therefore, Theorem 4.2 quantifies the privacy amplification due to shuffling, when compared to the worst-case Rényi DP bound among all possible mini-batch sequences. In Figure 2 (a), we illustrate this amplification in more details.

[4] When $\alpha - 1$ and $\varepsilon_0^{n/b-j_0}(\alpha)$ are large, the second term in Equation (5) might overflow. In the experiments, we compute this log-sum-exp term with the shifted approximation $\varepsilon_0^1 + \log \underset{0\leq j_0<n/b}{Avg} e^{(\alpha-1)\left(\varepsilon_0^{n/b-j_0}(\alpha)-\varepsilon_0^1(\alpha)\right)}$.



(a) shuffle and partition

(b) (re)sample without replacement

Figure 2: Rényi privacy loss of noisy (stochastic) mini-batch gradient descent over K epochs. We show ε in the (q, ε)-RDP guarantee derived by our privacy dynamics bound under "shuffle and partition" (bold lines, left plot), our privacy dynamics bound under sampling without replacement (bold lines, right plot), our privacy dynamics bound for data points in the last batch $B^{n/b-1}$ (thin dashed lines), and the baseline composition-based bound for DP-SGD [1, 29] (thin lines). We evaluate under the following setting: RDP order $\alpha \in \{10, 15\}$; λ-strongly convex loss function with $\lambda = 1$; β-smooth loss function with $\beta = 4$; finite total gradient sensitivity $S_g = 4$; size of the data set $n = 50$; step-size $\eta = 0.02$; noise variance $\sigma^2 = 4$, batch size $b = 2$. The expressions for computing the privacy bounds are: Privacy Dynamics (shuffle and partition): Theorem 4.2 [4]; Privacy Dynamics (samp. w.o. replacement): Theorem 4.3; Privacy Dynamics (last batch): Theorem 3.3 under $j_0 = n/b - 1$; Composition: derived from Section 3.3. of Mironov et al. [29] that approximately equals $\frac{b}{n} \cdot \frac{\alpha\eta S_g^2}{4\sigma^2 b^2} \cdot K$.

Comparison with prior amplification by sub-sampling bounds [29, 20, 21]. We now investigate how our Rényi DP bound Theorem 4.2 compares with prior bounds for privacy amplification by sub-sampled mechanisms, in terms of its amplification rate (i.e. the ratio between RDP bound for noisy mini-batch GD on sub-sampled mini-batch with size b, and RDP bound for full-batch noisy GD on a small dataset with size b). Observe that only the first term of the Rényi DP bound Equation (5) increases with the number of epochs K, and its growth rate is strictly smaller than $\varepsilon_0^{\lfloor \frac{n}{b} \rfloor} < \frac{b}{n} \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}$. Here, the term $\frac{\alpha\eta S_g^2}{4\sigma^2 b^2}$ is exactly the Rényi DP bound for a *full-batch* noisy GD update on dataset with small size b. Therefore, Theorem 4.2 achieves an amplification rate of $O(\frac{b}{n})$ for *one epoch* of noisy mini-batch gradient descent under "shuffle and partition". This $O(\frac{b}{n})$ amplification rate for one epoch matches the $O(\frac{b^2}{n^2})$ amplification rate in prior bounds for one epoch of sub-sampled mechanisms [29, 20, 21], because one epoch consists of $\frac{n}{b}$ iterations.

However, prior bounds for privacy amplification by sub-sampling (such as for sub-sampled Gaussian mechanism [29] and for noisy SGD under "shuffle and partition" [20, 21]) only study a single update or a single epoch, and then rely on composition theorem for computing amplified privacy bounds for multiple epochs. Consequently, the Rényi DP bounds derived by such analyses linearly grow with the number of epochs K. On the contrary, our Rényi DP bound in Theorem 4.2 applies to multiple epochs with hidden intermediate state, and thus enables a converging Rényi DP bound that never exceeds a maximum value. Hence, our Rényi DP bound in Theorem 4.2 is strictly smaller than composition-based privacy bound for DP-SGD [1, 29] after $\frac{1}{\lambda\eta} + \frac{n}{b}$ epochs. This is because the first term in Equation (5) is strictly smaller than the composition-based privacy bound after $\frac{1}{\lambda\eta}$ epochs, and the second term in Equation (5) is strictly smaller than the composition-based privacy bound after n/b epochs (as we explain with more details in Appendix E.1).

4.2 Privacy dynamics under (re)sampling mini-batch of fixed size without replacement

We now similarly analyze privacy dynamics for another variant of noisy mini-batch gradient descent (Algorithm 1), where in each iteration of each epoch, we freshly (re)sample b indices without replacement from $\{1, \cdots, n\}$ to obtain a fixed-size mini-batch. That is, the mini-batches used in

different iterations (of one epoch) may overlap. This "sampling a mini-batch of fixed size without replacement" scheme is widely studied in the amplification by sub-sampling literature [5, 37, 26], as an attractive alternative that ensures fixed mini-batch size (when compared to Poisson sub-sampling).

Theorem 4.3 (Recursive amplification by sampling without replacement)**.** *If the loss function $\ell(\theta; x)$ is λ-strongly convex, β-smooth, and if its gradient has finite ℓ_2-sensitivity S_g, then Algorithm 1 under sampling without replacement and stepsize $\eta < \frac{2}{\lambda+\beta}$ satisfies (α, ε)-Rényi DP guarantee with*

$$\varepsilon \leq \frac{1}{\alpha - 1} \log \left(S_K^0(\alpha) \right) \tag{6}$$

where the terms $S_k^j(\alpha)$ for $k = 0, \cdots, K-1$ and $j = 0, \cdots, n/b - 1$ are recursively computed by $S_0^0(\alpha) = 1$; $S_k^{j+1}(\alpha) = \frac{b}{n} \cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}} \cdot S_k^j(\alpha) + (1 - \frac{b}{n}) \cdot S_k^j(\alpha)^{(1-\eta\lambda)^2}$; and $S_{k+1}^0(\alpha) = S_k^{n/b}(\alpha)$.

Observe that in the above recursion, the term $S_k^{j+1}(\alpha)$ is strictly smaller than $e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}} \cdot S_k^j(\alpha)$ (i.e., the Rényi DP bound for one step of noisy gradient descent on dataset with size b), thus quantifying the privacy amplification by sampling a mini-batch of fixed size without replacement. We illustrate this privacy amplification in Figure 2 (b). However, our privacy bound under "sample without replacement" in Figure 2 (b) is larger than our privacy bound under "shuffle and partition" in Figure 2 (a), which suggests room for future improvement of the amplification rate bound. Indeed, for the special case of a single update, prior privacy amplification by subsampling bounds [37, 26] achieve better amplification rates of $O(\frac{b^2}{n^2})$.

However, prior bounds for privacy amplification by sampling generally *only* apply to *a single sub-sampled update*. Therefore, for analyzing multiple iterations of updates, prior works still rely on composition theorems, which results in linear growth of Rényi DP bound with regard to number of epochs. On the contrary, our privacy amplification bound Theorem 4.3 applies to multiple iteration of hidden-state (re)sampling without replacement steps, which enables a converging Rényi DP bound even for training an infinite number of epochs. In Figure 2 (b), we compare our Rényi DP bound Theorem 4.3 with the baseline composition-based privacy bound for DP-SGD [1, 28]. We observe that for a range of RDP orders $\alpha = 10, 15, 20$, our privacy dynamics bound significantly improves over the baseline composition-based bound (after 50 epochs).

5 Example: privacy dynamics for DP-SGD on regularized logistic regression

In this section, we explain a practical setting that our improved privacy dynamics analysis is applicable: training regularized logistic regression with the DP-SGD algorithm [1] (under "shuffle and partition" mini-batch sampling scheme). Note that DP-SGD algorithm under "shuffle and partition" is equivalent to our analyzed Algorithm 1 after change of notations, as discussed in Appendix F.1. For completeness, in Algorithm 2, we also provide the pseudocode for an equivalent of DP-SGD algorithm under notations in our paper.

5.1 How to ensure strong convexity, smoothness and finite sensitivity

Suppose that we want to train models for image classification tasks. The training dataset that we take as input is $D = (\mathbf{z}_1, \cdots, \mathbf{z}_n)$, where each data record $\mathbf{z}_i = (\mathbf{x}_i, \mathbf{y}_i)$ consists of the d-dimentional data input feature vector $\mathbf{x}_i \in \mathbb{R}^d$, and the label vector $\mathbf{y}_i \in \{0, 1\}^c$ (in one-hot encoding). To satisfy the necessary conditions for our privacy dynamics bound, we use regularized logistic regression to ensure strong convexity, use feature clipping to ensure smoothness, and use gradient clipping to enforce finite gradient sensitivity, as follows.

Regularzied Logistic regression (for strong convexity). The loss function for regularized logistic regression in the multi-class setting (with per-class bias) is as follows.

$$\ell_\lambda(\theta; \mathbf{x}, \mathbf{y}) = \ell_0(\theta; \mathbf{x}, \mathbf{y}) + \frac{\lambda}{2} \|\theta\|_2^2 \tag{7}$$

where $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ is the following logistic regression loss function.

$$\ell_0(\theta; \mathbf{x}, \mathbf{y}) = -\mathbf{y}^1 \log \left(\frac{e^{\bar{x}^T \cdot \theta_1}}{e^{\bar{x}^T \cdot \theta_1} + \cdots + e^{\bar{x}^T \cdot \theta_c}} \right) - \cdots - \mathbf{y}^c \log \left(\frac{e^{\bar{x}^T \cdot \theta_c}}{e^{\bar{x}^T \cdot \theta_1} + \cdots + e^{\bar{x}^T \cdot \theta_c}} \right) \tag{8}$$

where $\bar{\mathbf{x}} = (\mathbf{x}, 1) \in \mathbb{R}^{d+1}$ denotes the concatenation of the data feature vector $\mathbf{x}$ and 1, and $\mathbf{y} = (\mathbf{y}^1, \cdots, \mathbf{y}^c)$ is the label vector. The parameter vector is $\theta = (\theta_1, \cdots, \theta_c) \in \mathbb{R}^{(d+1)\cdot c}$ that represents the weight and the per-class bias of the linear model. The logistic regression loss function is convex, and therefore the regularized logistic regression loss function is λ-strongly convex.

Feature Clipping (for bounding the smoothness constant). To ensure that the condition of loss function smoothness with regard to parameters θ is satisfied, we follow Feldman et al. [18] and normalize the data feature vector in ℓ_2 norm, such that $\|\mathbf{x}\|_2 \leq L$. Under this data feature clipping, we prove that the logistic regression loss function (8) is $(\frac{L^2+1}{2})$-smooth in the following Proposition 5.1.

Proposition 5.1. *If the data feature vector $\mathbf{x}$ has bounded ℓ_2 norm, such that $\|\mathbf{x}\|_2 \leq L$, then the unregularized logistic regression loss function $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ Equation* (8) *is convex , L-Lipschitz and β-smooth with regard to parameters θ, for*

$$L = \sqrt{2(L^2+1)} \tag{9}$$

$$\beta = \frac{L^2+1}{2} \tag{10}$$

By Proposition 5.1, the regularized logistic regression loss function Equation (7) is $(\beta + \lambda)$-smooth. The above feature clipping technique is different from the DP-SGD algorithm [1] that only requires per-example gradient clipping. The major reason that we use data feature clipping (besides per-example gradient clipping), is for ensuring smoothness of the logistic regression loss function (by Proposition 5.1), which is a necessary condition for applying our privacy bound Theorem 3.3.

Per-example Clipping on Unregularized Gradient (for reducing gradient sensitivity without harming smoothness or strong convexity). Although feature clipping already bounds the gradient sensitivity by $2\sqrt{2(L^2+1)}$ (by Proposition 5.1), this bound grows with the feature clipping norm L. This in turn restricts the signal to noise ratio, and tends to give suboptimal privacy-utility trade-off in practical experiments. Therefore, we additionally perform per-example ℓ_2-clipping on the unregularized gradient (detailed pseudocode in Appendix F.1). Under per-example clipping on unregularized gradient, we prove in the following Proposition 5.2, that each gradient update in *regularized logistic regression* has finite gradient sensitivity, and preserves strong convexity and smoothness.

Proposition 5.2. *Let $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ be the logistic regression loss function defined in Equation* (8)*. Let $g_0(\theta; \mathbf{x}, \mathbf{y}) = \frac{\nabla \ell_0(\theta; \mathbf{x}, \mathbf{y})}{\|\nabla \ell_0(\theta; \mathbf{x}, \mathbf{y})\|_2} \cdot \min\{\|\nabla \ell_0(\theta; \mathbf{x}, \mathbf{y})\|_2, \frac{S_g}{2}\}$ be the clipped gradient of (unregularized) loss function $\ell_0(\theta; \mathbf{x}, \mathbf{y})$, under ℓ_2 clipping norm $\frac{S_g}{2}$. If $g(\theta; \mathbf{x}, \mathbf{y}) = g_0(\theta; \mathbf{x}, \mathbf{y}) + \lambda\theta$, and if the data vector $\mathbf{x}$ has bounded ℓ_2 norm, such that $\|\mathbf{x}\|_2 \leq L$, then $g(\theta; \mathbf{x}, y)$ has finite ℓ_2-sensitivity S_g, is continuous, and is almost everywhere differentiable with*

$$\lambda \cdot \mathbb{I}_{(d+1)\cdot c} \preceq \nabla_\theta g(\theta; \mathbf{x}, \mathbf{y}) \preceq (\beta + \lambda) \cdot \mathbb{I}_{(d+1)\cdot c} \tag{11}$$

for any $\theta, \theta' \in \mathbb{R}^{(d+1)\cdot c}$ and $\beta = \frac{L^2+1}{2}$.

We provide complete proof for this proposition in Appendix F.4. This construction of clipped unregularized gradient enables us to enjoy the benefits of gradient clipping (such as for speeding up convergence [41, 14]) while satsifying the necessary smoothness and strong convexity conditions for applying our privacy dynamics bound.

5.2 Composition-based privacy bound and privacy dynamics analysis for DP-SGD

Baseline composition-based privacy analysis. To the best of our knowledge, the moments accountant [1] combined with the privacy amplification bound for Subsampled Gaussian Mechanism [29] gives the strongest baseline composition-based privacy bound for DP-SGD. The original DP-SGD bound only holds for mini-batch sampling from a Poisson distribution. However, currently many privacy libraries [24, 40] still apply the DP-SGD analysis while implementing the "shuffle and partition" sampling scheme. Recent works [20, 21] further prove that the Rényi DP bound for DP-SGD under "shuffle and partition" matches that for DP-SGD under Poisson sampling in order. Therefore, we consider the moments accountant bound for subsampled Gaussian mechanisms [29] as a reasonable comparison baseline for our privacy bound of DP-SGD Algorithm 2 under "shuffle and partition".

Improved privacy dynamics analysis. We now compute our privacy dynamics bound Theorem 4.2 for the modified DP-SGD algorithm Algorithm 2 under "shuffle and partition". By plugging the

notation transformation in Appendix F.1 into Theorem 4.2, we prove the following privacy dynamics theorem for noisy mini-batch gradient descent on regularized logistic regression.

Corollary 5.3 (Privacy dynamics for noisy mini-batch gradient descent on regularized logistic regression)**.** *For the regularized logistic regression loss Equation* (7) *with regularization coefficient λ, if the data feature vector is clipped in ℓ_2-norm by L, and the unregularized gradient is clipped in ℓ_2 norm by $\frac{S_g}{2}$, then for $K \geq 1$ and $\frac{n}{b} \geq 2$, Algorithm 2 with stepsize $\eta < \frac{2}{(L^2+1)/2+2\lambda}$ and noise multiplier σ_{mul} satisfies $(\alpha, \varepsilon_{norm} + \varepsilon)$-Rényi Differential Privacy with*

$$\varepsilon \leq \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}} + \frac{1}{\alpha-1} \cdot \log\left(\underset{0 \leq j_0 < n/b}{Avg} e^{(\alpha-1)\varepsilon_0^{n/b-j_0}(\alpha)} \right) \quad (12)$$

where the terms $\varepsilon_0^j(\alpha)$ is upper-bounded for any $j = 1, \cdots, n/b$ as follows.

$$\varepsilon_0^j(\alpha) \leq \frac{2\alpha}{\sigma_{mul}^2} \cdot (1-\eta\lambda)^{2\cdot(j-1)} \cdot \frac{1}{\sum_{s=0}^{j-1}(1-\eta\lambda)^{2s}} \quad (13)$$

Proof Sketch. The proof is by applying Theorem 4.2 with gradient sensitivity S_g, smoothness constant $\frac{L^2+1}{2} + \lambda$, strong convexity constant λ (derived by Proposition 5.2)) and noise standard deviation $\sigma = \sqrt{\frac{\eta}{2}} \cdot \frac{1}{b} \cdot \sigma_{mul} \cdot \frac{S_g}{2}$ (explained in Appendix F.1). □

6 Conclusions and Discussion

We prove a novel converging last-iterate privacy bound for noisy stochastic mini-batch gradient descent on strongly convex smooth loss functions. Our bound substantially improves the prior privacy dynamics bound for noisy GD [15], by proving novel bounds for the additional privacy amplification (by randomized post-processing and sub-sampling) during training with stochastic mini-batches. Our results show that to obtain tighter privacy bound (thus achieving better privacy accuracy trade-off), differentially private learning algorithms needs to be evaluated by a last-iterate privacy bound, unless it has a very fast convergence (under which, due to the small number of epochs, the cost of composition bound is not significant).

Future Work and Other Related Work. For iteratively resampling a mini-batch of fixed size without replacement, there is room for improving our hidden-state privacy amplification bound Theorem 4.3 under smaller batch size b, because under *one* sampling without replacement step, the best known amplification rate is as small as $O(\frac{b^2}{n^2})$ [5, 37]. This suggests possibility for a better *hidden-state* privacy amplification bound (without using composition) for multiple steps of sub-sampling.

In our current analysis, strong convexity and smoothness of the loss functions are *necessary conditions* for obtaining a *converging* hidden-state Rényi DP bound. More specifically, strong convexity and smoothness ensure that log-Sobolev inequality with constant c (which is a condition required by Lemma 3.2) holds throughout the training process (i.e. after any number of epochs k and iterations j). A recent work Altschuler and Talwar [2] consider constrained optimization problem where θ is optimized over a bounded set with finite ℓ_2 diameter (rather than the unconstrained parameter space $\mathbb{R}^d$ considered in this paper). Under such setting, they prove that convexity (instead of strong convexity) and smoothness of the loss functions suffice to enable a time-independent Rényi DP bound for noisy SGD. However, it remains an important open problem to further relax the convexity and smoothness conditions for proving converging Rényi DP bound.

One of the most important motivation for studying hidden-state Rényi DP bound in this paper, is that most State-of-the-art privacy accuracy trade-off for differentially private learning are achieved via the last-iterate model (instead of average of iterates). However, in terms of theoretical privacy utility trade-off, prior works [7, 8] prove that the average of iterates achieves optimal privacy-utility trade-off in certain regimes. For convex stochastic optimization, by exploiting the privacy amplification of last-iterate, recent works [19, 15, 22] prove that last-iterate also achieves asymptotically optimal privacy-utility trade-off (for appropriate choice of step sizes or continuous-time algorithm). Therefore, it remains an interesting open problem as to whether last-iterate or average of iterate enables better (non-assymptotic) privacy accuracy trade-off.

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Appendix

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A Symbols

Algorithms and Definitions

Symbol	Meaning	Where
$\mathbf{x}$	A data vector	Algorithm 1
n	Size of a dataset	Algorithm 1
b	Size of a mini-batch	Algorithm 1
K	Total number of epochs in a learning algorithm	Algorithm 1
$\mathbb{I}_d$	Identity matrix with d rows and d columns	Algorithm 1
θ	Model parameters	Algorithm 1
$\ell(\theta, \mathbf{x})$	A loss function of θ parameterized by $\mathbf{x}$	Algorithm 1
D, D'	Neighboring datasets that differ in at most one record	Appendix B
S_g	ℓ_2-sensitivity of total gradient $g(\theta; D) = \sum_{\mathbf{x} \in D} \nabla \ell(\theta; \mathbf{x})$ with regard to neighboring datasets. More specifically, $S_g = \max_{D,D',\theta} \lVert g(\theta, D) - g(\theta, D') \rVert_2$	Algorithm 1
$\mathcal{N}(\mu, \sigma \cdot \mathbb{I}_d)$	Gaussian distribution over $\mathbb{R}^d$ with mean μ and covariance matrix $\sigma \cdot \mathbb{I}_d$	Algorithm 1
η	Stepsize for each iterative update in the learning algorithm	Algorithm 1
$\{0, 1, \ldots, n\}$	The set of all integers between 0 and n	Algorithm 1
$\mathbf{x}_i$	Data point i of dataset D, with indexing starting at 1	Algorithm 1
θ_k^j, B_k^j	The parameters or generated mini-batch after k epochs and j iterations of a learning algorithm, with indexing starting at 0	Algorithm 1
α	Rényi DP order	Appendix B
(ε, δ)	Differential Privacy parameters	Appendix B
(α, ε)	Rényi Differential Privacy parameters	Appendix B
c	A log-Sobolev inequality constant	Appendix B
λ	The strong convexity parameter for a loss function	Section 3
β	The smoothness parameter for a loss function	Section 3

Probability and Information Theory

Symbol	Meaning	Where
μ	A distribution over $\mathbb{R}^d$ with density $\mu(\theta)$	Section 3
$p(\theta)$	A probability density function over $\theta \in \mathbb{R}^d$	Section 2
$f_{\#}(\mu)$	The push forward distribution of μ under mapping f on the same domain	Section 3
$\mu * \nu$	The convolution of two distributions μ and ν	Section 3
$R_\alpha(\mu\|\nu)$	Rényi divergence between distributions μ and ν	Appendix B
$\mathbb{E}_{\theta\sim\mu}[f(\theta)]$	Expectation of $f(\theta)$ with respect to $\mu(\theta)$	Appendix B
$KL(\mu\|\nu)$	Kullback-Leibler divergence between distributions μ and ν	Appendix D

Calculus and Linear Algebra

Symbol	Meaning
t	A real scalar
$\lfloor t \rfloor$	The largest integer that is smaller than or equal than a real number t
$A_{i,j}$	Element i, j of matrix A
$\frac{dy}{dx}$	Derivative of y with respect to x
$\frac{\partial y}{\partial x}, \nabla_x y$	Partial derivative of y with respect to x
$\nabla_x^2 y$	Laplacian of y with respect to x
$\nabla_x y$	Gradient of y with respect to x
$x \cdot y$	Inner products between real vectors x and y
x	ℓ_2 norm of a vector x
$A \otimes B$	Tensor product between two real matrices A and B
x^T	Transpose of a real vector or matrix x
$f \circ g$	Composition of the functions or mappings f and g
$A \preceq B$	The matrix $B - A$ is semi-positive definite.

B Preliminaries

Definition B.1 (Differential Privacy [16, 17])**.** A randomized algorithm $\mathcal{A}$ is (ε, δ)-differentially private if for any neighboring datasets D, D', and for all possible event S in the output space of $\mathcal{A}$,

$$P\left(\mathcal{A}(D) \in S\right) \leq e^{\varepsilon} \cdot P\left(\mathcal{A}(D') \in S\right) + \delta \tag{14}$$

where we say D, D' are *neighboring* if they are of the *same size* and differ in at most one data record.

Definition B.2 ((α, ε)-Rényi DP [28])**.** A randomized algorithm $\mathcal{A}$ is said to satisfy (α, ε)-Rényi differential privacy (or (α, ε)-Rényi DP for short), if for any neighboring datasets D and D',

$$R_\alpha(\mathcal{A}(D)\|\mathcal{A}(D')) \leq \varepsilon, \text{ where } R_\alpha(\mu\|\nu) = \frac{1}{\alpha - 1} \log \mathbb{E}_{\theta\sim\nu}\left[\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^{\alpha}\right] \tag{15}$$

where $\mathcal{A}(D)$ ($\mathcal{A}(D')$) denote the distribution of output given input dataset D (D'), and $R_\alpha(\mu\|\nu)$ is the *Rényi divergence* [31] of order $\alpha > 1$ for two distributions with density $\mu(\theta)$ and $\nu(\theta)$ on $\mathbb{R}^d$.

Definition B.3 (log-Sobolev Inequality [35])**.** A distribution ν over $\mathbb{R}^d$ satisfies the log-Sobolev inequality (LSI) with constant c if for all smooth function $g : \mathbb{R}^d \to \mathbb{R}$ with $\mathbb{E}_{\theta\sim\nu}\left[g(\theta)^2\right] < \infty$,

$$\mathbb{E}_{\theta\sim\nu}\left[g(\theta)^2 \log\left(g(\theta)^2\right)\right] - \mathbb{E}_{\theta\sim\nu}\left[g(\theta)^2\right] \cdot \log \mathbb{E}_{\theta\sim\nu}\left[g(\theta)^2\right] \leq \frac{2}{c}\mathbb{E}_{\theta\sim\nu}\left[\|\nabla g(\theta)\|^2\right]. \tag{16}$$

C Discussion about the concurrent work [32, Corollary 3.3]

Chourasia et al. [15] prove privacy dynamics bound for noisy gradient descent, and Ryffel et al. [32] extend this bound to SGLD, by directly viewing each mini-batch update as gradient descent on a smaller dataset of size b (which is the size of a mini-batch). This approach is similar to our approach for deriving the naive privacy dynamics baseline Theorem 2.1, and the expression in [32, Corollary 3.3] is very similar to the expression in our Theorem 2.1, except for having n^2 (instead of b^2) in the bound denominator.

However, an inspection of the proof for [32, Lemma 3.4] shows that, this difference between [32, Corollary 3.3] and our Theorem 2.1 is caused because by a flawed assumption in [32, Lemma 3.3]. More specifically, [32, Lemma 3.3] wrongly assume that the LSI constant proved in [15, Lemma 5] (which only holds for a GD process) would also similarly hold for a SGLD process that takes the form of a more complex mixture distribution. Here, each mixture component is the conditional distribution of last-iterate parameters given a fixed sequence of mini-batches.

This assumption is wrong because a given mixture distribution generally satisfies a different LSI constant than each of its component distributions. Moreover, bounding the LSI constant for a mixture distribution is largely an open problem [42, 43, 36, 6, 13]. The current best bound for this problem, to the best of our knowledge, is [13, Theorem 1], which says that the LSI constant for a mixture distribution, depends on the LSI constant for the distribution of each component, and the *worst-case* χ^2 distance between *any* two components' distributions. Therefore, the actual LSI constant for SGLD process would be significantly smaller (related to the number of components in the mixture distribution) than the assumed LSI constant in [32, Lemma 3.4] (which only holds for the conditional parameter distribution given *a fixed mini-batch sequence*). After replacing the wrongly assumed LSI constant in [32, Lemma 3.4] with a correct LSI constant for SGLD process (that takes the form of mixture distribution), the privacy dynamics bounds in [32, Corollary 3.3] would be significantly worse (larger).

Due to this flawed assumption, in this paper, we do not compare our improved privacy dynamics theorem Theorem 3.3 with [32, Corollary 3.3]. Instead, we compare Theorem 3.3 with the naive privacy dynamics baseline Theorem 2.1 (which has similar expression as [32, Corollary 3.3]) in Figure 1.

D Proof for Section 3

We first establish a tool lemma for proving Lemma 3.1 and obtaining the partial differential inequality that bounds the growth of differential privacy loss.

Lemma D.1 (Lemma 5 in Vempala and Wibisono [35])**.** *Suppose ν is a distribution that satisfies log-Sobolev inequality with constant $c > 0$, and that has smooth density $\mu(\theta)$. Let $\alpha \geq 1$. For all measure μ that has smooth density $\nu(\theta)$,*

$$\frac{I_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} \geq \frac{2c}{\alpha^2} \cdot R_\alpha(\mu\|\nu) + \frac{2c}{\alpha^2} \cdot \alpha(\alpha-1)\frac{\partial R_\alpha(\mu\|\nu)}{\partial \alpha} \tag{17}$$

where $I_\alpha(\mu\|\nu) = \mathbb{E}_{\theta\sim\nu}\left[\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha \cdot \left\|\nabla \log \frac{\mu(\theta)}{\nu(\theta)}\right\|^2\right]$, *and* $E_\alpha(\mu\|\nu) = \mathbb{E}_{\theta\sim\nu}\left[\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha\right]$

Proof. This Lemma is initially proved in Vempala and Wibisono [35] Lemma 5. We give an alternative proof here, that only uses one step of inequality in Equation (23). We hope this alternative proof helps understand whether there is still room for improving this Lemma (e.g. by improving the inequality in Equation (23)).

We denote ρ to be another distribution with density $\rho(\theta) = \frac{1}{E_\alpha(\mu\|\nu)} \cdot \left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha \cdot \nu(\theta)$. By simple integration, we verify that $\int \rho(\theta)d\theta = \frac{E_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} = 1$. By definition,

$$\frac{I_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} = \frac{\mathbb{E}_{\theta\sim\nu}\left[\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha \cdot \left\|\nabla \log \frac{\mu(\theta)}{\nu(\theta)}\right\|^2\right]}{E_\alpha(\mu\|\nu)} \tag{18}$$

$$= \mathbb{E}_{\theta\sim\nu}\left[\frac{\rho(\theta)}{\nu(\theta)} \cdot \left\|\nabla \log \frac{\mu(\theta)}{\nu(\theta)}\right\|^2\right] \tag{19}$$

$$= \frac{1}{\alpha^2}\mathbb{E}_{\theta\sim\nu}\left[\frac{\rho(\theta)}{\nu(\theta)} \cdot \left\|\nabla \log \left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha\right\|^2\right] \tag{20}$$

$$\text{by } \nabla \log E_\alpha(\mu\|\nu) = 0, \quad = \frac{1}{\alpha^2}\mathbb{E}_{\theta\sim\nu}\left[\frac{\rho(\theta)}{\nu(\theta)} \cdot \left\|\nabla \log \left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha - \nabla \log E_\alpha(\mu\|\nu)\right\|^2\right] \tag{21}$$

$$= \frac{1}{\alpha^2}\mathbb{E}_{\theta\sim\nu}\left[\frac{\rho(\theta)}{\nu(\theta)} \cdot \left\|\nabla \log \frac{\rho(\theta)}{\nu(\theta)}\right\|^2\right] \tag{22}$$

By definition, $\mathbb{E}_{\theta\sim\nu}\left[\frac{\rho(\theta)}{\nu(\theta)} \cdot \left\|\nabla \log \frac{\rho(\theta)}{\nu(\theta)}\right\|^2\right]$ is the relative Fisher information $J(\rho\|\nu)$ of ρ with respect to ν. A celebrated equivalence result (e.g. Section 2.2 of Vempala and Wibisono [35]) says that, if and only if ν satsifies log-Sobolev inequality with constant c, the following relation between the KL divergence and relative Fisher information holds for all ρ:

$$KL(\rho\|\nu) \leq \frac{1}{2c}J(\rho\|\nu) \tag{23}$$

By plugging Equation (23) into Equation (22), we prove

$$\frac{I_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} \geq \frac{2c}{\alpha^2} \cdot KL(\rho\|\nu) \tag{24}$$

$$\text{by definition,} \quad = \frac{2c}{\alpha^2} \cdot \int_{\mathbb{R}^n} \rho(\theta) \log \frac{\rho(\theta)}{\nu(\theta)} d\theta \tag{25}$$

$$\text{by definition of } \rho, \quad = \frac{2c}{\alpha^2} \cdot \mathbb{E}_{\theta\sim\nu}\left[\frac{1}{E_\alpha(\mu\|\nu)} \cdot \left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha \cdot \left(\alpha \log \frac{\mu(\theta)}{\nu(\theta)} - \log E_\alpha(\mu\|\nu)\right)\right] \tag{26}$$

$$= \frac{2c}{\alpha} \cdot \frac{1}{E_\alpha(\mu\|\nu)} \cdot \mathbb{E}_{\theta\sim\nu}\left[\frac{\partial}{\partial\alpha}\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha\right] - \frac{2c}{\alpha^2} \cdot \frac{\log E_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} \cdot \mathbb{E}_{\theta\sim\nu}\left[\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha\right] \tag{27}$$

$$= \frac{2c}{\alpha} \cdot \frac{1}{E_\alpha(\mu\|\nu)} \cdot \mathbb{E}_{\theta\sim\nu}\left[\frac{\partial}{\partial\alpha}\left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha\right] - \frac{2c}{\alpha^2}\log E_\alpha(\mu\|\nu) \tag{28}$$

By exchanging the order of derivative and expectation (because $\mu(\theta)$ and $\nu(\theta)$ are smooth densities), we prove that Equation (28) is equivalent to the following inequality.

$$\frac{I_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} \geq \frac{2c}{\alpha} \cdot \frac{\partial}{\partial\alpha}\log \mathbb{E}_\alpha(\mu\|\nu) - \frac{2c}{\alpha^2}\log E_\alpha(\mu\|\nu) \tag{29}$$

By definition, we have $E_\alpha(\mu\|\nu) = e^{(\alpha-1)R_\alpha(\mu\|\nu)}$, therefore we prove

$$\frac{I_\alpha(\mu\|\nu)}{E_\alpha(\mu\|\nu)} \geq \frac{2c}{\alpha} \cdot \left(R_\alpha(\mu\|\nu) + (\alpha-1)\frac{\partial R_\alpha(\mu\|\nu)}{\partial\alpha}\right) - \frac{2c}{\alpha^2} \cdot (\alpha-1)R_\alpha(\mu\|\nu) \tag{30}$$

$$= \frac{2c}{\alpha^2} \cdot R_\alpha(\mu\|\nu) + \frac{2c}{\alpha^2} \cdot \alpha(\alpha-1)\frac{\partial R_\alpha(\mu\|\nu)}{\partial\alpha} \tag{31}$$

□

D.1 Proof for Lemma 3.1

Lemma 3.1. Let μ, ν be two distributions on $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping on $\mathbb{R}^d$. We denote $\mathcal{N}(0, 2t\sigma^2 \cdot \mathbb{I}_d)$ to be the standard Gaussian distribution on $\mathbb{R}^d$ with covariance

matrix $2t\sigma^2 \cdot \mathbb{I}_d$. We denote $p_t(\theta)$ and $p'_t(\theta)$ to be the probability density functions for the distributions $f_\#(\mu) * \mathcal{N}(0, 2t\sigma^2\mathbb{I}_d)$ and $f_\#(\nu) * \mathcal{N}(0, 2t\sigma^2\mathbb{I}_d)$ respectively, where $f_\#(\mu), f_\#(\nu)$ denote the push forward distributions of μ, ν under mapping f. Then if μ and ν satisfy log-Sobolev inequality with constant c, and if the mapping f is L-Lipschitz, then for any order $\alpha > 1$,

$$\frac{\partial}{\partial t} R_\alpha \left(p_t(\theta) \| p'_t(\theta)\right) \leq -c_t \cdot 2\sigma^2 \cdot \left(\frac{R_\alpha(p_t(\theta)\|p'_t(\theta))}{\alpha} + (\alpha - 1) \cdot \frac{\partial}{\partial \alpha} R_\alpha(p_t(\theta)\|p'_t(\theta)) \right), \quad (32)$$

where $c_t = \left(\frac{L^2}{c} + 2t\sigma^2\right)^{-1}$ is the log-Sobolev inequality constant for distributions $p_t(\theta)$ and $p'_t(\theta)$.

Proof. By definition, $p_t(\theta)$ and $p'_t(\theta)$ are probability density functions for distributions $f_\#(\mu) * \mathcal{N}(0, 2t\sigma^2\mathbb{I}_d)$ and $f_\#(\nu) * \mathcal{N}(0, 2t\sigma^2\mathbb{I}_d)$ respectively. Therefore $p_t(\theta)$ and $p'_t(\theta)$ satisfy the following Fokker-Planck equations.

$$\frac{\partial p_t(\theta)}{\partial t} = \sigma^2 \Delta p_t(\theta), \quad (33)$$

$$\frac{\partial p'_t(\theta)}{\partial t} = \sigma^2 \Delta p'_t(\theta) \quad (34)$$

We denote $E_\alpha(p_t(\theta)\|p'_t(\theta)) = \int \frac{p_t(\theta)^\alpha}{p'_t(\theta)^{\alpha-1}} d\theta$ to be the moment of the likelihood ratio function. Then by definition of Rényi divergence, we prove

$$R_\alpha(p_t(\theta)\|p'_t(\theta)) = \frac{1}{\alpha - 1} \log E_\alpha(p_t(\theta)\|p'_t(\theta)) \quad (35)$$

Therefore we compute the rate of Rényi divergence with regard to t as follows.

$$\frac{\partial}{\partial t} R_\alpha(p_t(\theta)\|p'_t(\theta)) = \frac{1}{\alpha - 1} \frac{\partial}{\partial t} \log E_\alpha(p_t(\theta)\|p'_t(\theta)) \quad (36)$$

$$= \frac{1}{(\alpha - 1) E_\alpha(p_t(\theta)\|p'_t(\theta))} \cdot \frac{\partial}{\partial t} E_\alpha(p_t(\theta)\|p'_t(\theta)) \quad (37)$$

By definition of $E_\alpha(p_t(\theta)\|p'_t(\theta))$, $$= \frac{1}{(\alpha - 1) E_\alpha(p_t(\theta)\|p'_t(\theta))} \cdot \frac{\partial}{\partial t} \left(\int \frac{p_t(\theta)^\alpha}{p'_t(\theta)^{\alpha-1}} d\theta \right) \quad (38)$$

By exchanging the order of derivative and integration, we prove

$$\frac{\partial}{\partial t} R_\alpha(p_t(\theta)\|p'_t(\theta)) = \frac{1}{(\alpha - 1) E_\alpha(p_t(\theta)\|p'_t(\theta))} \cdot \int \frac{\partial}{\partial t} \frac{p_t(\theta)^\alpha}{p'_t(\theta)^{\alpha-1}} d\theta \quad (39)$$

$$= \frac{1}{(\alpha - 1) E_\alpha(p_t(\theta)\|p'_t(\theta))} \cdot \int \left(\alpha \cdot \frac{p_t(\theta)^{\alpha-1}}{p'_t(\theta)^{\alpha-1}} \cdot \frac{\partial p_t(\theta)}{\partial t} - (\alpha - 1) \cdot \frac{p_t(\theta)^\alpha}{p'_t(\theta)^\alpha} \cdot \frac{\partial p'_t(\theta)}{\partial t} \right) d\theta \quad (40)$$

By the Fokker-Planck equations Equation (33) and Equation (34), we substitute the terms $\frac{\partial p_t(\theta)}{\partial t}$ and $\frac{\partial p'_t(\theta)}{\partial t}$ in the above equation as follows.

$$\frac{\partial}{\partial t} R_\alpha(p_t(\theta)\|p'_t(\theta)) \quad (41)$$

$$= \frac{1}{(\alpha - 1) E_\alpha(p_t(\theta)\|p'_t(\theta))} \cdot \int \left(\alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p'_t(\theta)^{\alpha-1}} \cdot \Delta p_t(\theta) - (\alpha - 1)\sigma^2 \cdot \frac{p_t(\theta)^\alpha}{p'_t(\theta)^\alpha} \cdot \Delta p'_t(\theta) \right) d\theta \quad (42)$$

By applying Green's first identity in Equation (42), the first intergration term in Equation (42) is changed to

$$\int \alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p'_t(\theta)^{\alpha-1}} \cdot \Delta p_t(\theta) d\theta = \lim_{r \to \infty} \int_{B_r} \alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p'_t(\theta)^{\alpha-1}} \cdot \Delta p_t(\theta) d\theta \quad (43)$$

$$= \lim_{r \to \infty} \int_{\partial B_r} \alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p'_t(\theta)^{\alpha-1}} \cdot \nabla p_t(\theta) \cdot d\mathbf{S} - \int \nabla \left(\alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p'_t(\theta)^{\alpha-1}} \right) \cdot \nabla p_t(\theta) d\theta, \quad (44)$$

where B_r is the unit ball centered around origin in d-dimensional Euclidean space with radius r. The limits in the first term of Equation (44) becomes zero given the smoothness and fast decay properties of $p_t(\theta)$, and the Lebesgue integrability of $\frac{p_t(\theta)^{\alpha-1}}{p_t'(\theta)^{\alpha-1}} \cdot p_t(\theta)$. Therefore we prove that

$$\int \alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p_t'(\theta)^{\alpha-1}} \cdot \Delta p_t(\theta) d\theta = -\int \nabla \left(\alpha\sigma^2 \cdot \frac{p_t(\theta)^{\alpha-1}}{p_t'(\theta)^{\alpha-1}} \right) \cdot \nabla p_t(\theta) d\theta, \tag{45}$$

Similarly by applying Green's first identity in Equation (42), the second intergration term in Equation (42) is changed to

$$\int -(\alpha-1)\sigma^2 \cdot \frac{p_t(\theta)^{\alpha}}{p_t'(\theta)^{\alpha}} \cdot \Delta p_t'(\theta) d\theta = \int \nabla \left((\alpha-1)\sigma^2 \cdot \frac{p_t(\theta)^{\alpha}}{p_t'(\theta)^{\alpha}} \right) \cdot \nabla p_t'(\theta) d\theta, \tag{46}$$

(This techinque for using Green's identity and bounding the rate of entropy change with relative Fisher information-like quantity, has been previously used for KL divergence in **?**], and Rényi divergence in Vempala and Wibisono [35], Chourasia et al. [15]. The result is also closely related to the well known *de Bruijn's indentity* in information theory literature.)

By plugging in Equation (45) and Equation (46) into Equation (42), we prove that

$$\frac{\partial}{\partial t} R_\alpha(p_t(\theta) \| p_t'(\theta)) \tag{47}$$

$$= \frac{\sigma^2}{(\alpha-1)E_\alpha(p_t(\theta)\|p_t'(\theta))} \cdot \int -\alpha \cdot \left\langle \nabla \left(\frac{p_t(\theta)^{\alpha-1}}{p_t'(\theta)^{\alpha-1}} \right), \nabla p_t(\theta) \right\rangle + (\alpha-1) \cdot \left\langle \nabla \left(\frac{p_t(\theta)^{\alpha}}{p_t'(\theta)^{\alpha}} \right), \nabla p_t'(\theta) \right\rangle d\theta \tag{48}$$

$$= \frac{\alpha(\alpha-1)\sigma^2}{(\alpha-1)E_\alpha(p_t(\theta)\|p_t'(\theta))} \cdot \int \frac{p_t(\theta)^{\alpha-2}}{p_t'(\theta)^{\alpha-2}} \cdot \left\langle \nabla \left(\frac{p_t(\theta)}{p_t'(\theta)} \right), -\nabla p_t(\theta) + \frac{p_t(\theta)}{p_t'(\theta)} \nabla p_t'(\theta) \right\rangle d\theta \tag{49}$$

$$= -\frac{\alpha\sigma^2}{E_\alpha(p_t(\theta)\|p_t'(\theta))} \cdot \int \frac{p_t(\theta)^{\alpha-2}}{p_t'(\theta)^{\alpha-2}} \cdot \left\langle \nabla \left(\frac{p_t(\theta)}{p_t'(\theta)} \right), \nabla \left(\frac{p_t(\theta)}{p_t'(\theta)} \right) \right\rangle \cdot p_t'(\theta) d\theta \tag{50}$$

$$= -\alpha\sigma^2 \cdot \frac{I_\alpha(p_t(\theta)\|p_t'(\theta))}{E_\alpha(p_t(\theta)\|p_t'(\theta))}, \tag{51}$$

where $I_\alpha(p_t(\theta)\|p_t'(\theta)) = \int \frac{p_t(\theta)^{\alpha-2}}{p_t'(\theta)^{\alpha-2}} \cdot \left\langle \nabla \left(\frac{p_t(\theta)}{p_t'(\theta)} \right), \nabla \left(\frac{p_t(\theta)}{p_t'(\theta)} \right) \right\rangle \cdot p_t'(\theta) d\theta$. Therefore, if $p_t(\theta)$ and $p_t'(\theta)$ satisfy log-Sobolev inequality with constant c_t, then by Lemma D.1, we obtain the following inequality.

$$\frac{I_\alpha(p_t(\theta)\|p_t'(\theta))}{E_\alpha(p_t(\theta)\|p_t'(\theta))} \geq \frac{2c_t}{\alpha^2} R_\alpha(p_t(\theta)\|p_t'(\theta)) + \frac{2c_t}{\alpha^2} \cdot \alpha(\alpha-1) \frac{\partial}{\partial\alpha} R_\alpha(p_t(\theta)\|p_t'(\theta)) \tag{52}$$

Meanwhile, by Lemma 16 in Vempala and Wibisono [35] and Lemma 17 in Vempala and Wibisono [35], the distributions (with densities $p_t(\theta)$ and $p_t'(\theta)$) indeed satisfy c_t-log-Sobolev inequality with

$$c_t = \left(\frac{L^2}{c} + 2t\sigma^2 \right)^{-1} \tag{53}$$

By plugging Equation (53) and Equation (52) into Equation (51), we prove the following bound for the rate of Rényi divergence.

$$\frac{\partial}{\partial t} R_\alpha(p_t(\theta)\|p_t'(\theta)) \leq -\alpha\sigma^2 \cdot \left(\frac{2c_t}{\alpha^2} R_\alpha(p_t(\theta)\|p_t'(\theta)) + \frac{2c_t}{\alpha^2} \cdot \alpha(\alpha-1) \cdot \frac{\partial}{\partial\alpha} R_\alpha(p_t(\theta)\|p_t'(\theta)) \right) \tag{54}$$

$$= -c_t \cdot 2\sigma^2 \cdot \left(\frac{R_\alpha(p_t(\theta)\|p_t'(\theta))}{\alpha} + (\alpha-1) \cdot \frac{\partial}{\partial\alpha} R_\alpha(p_t(\theta)\|p_t'(\theta)) \right) \tag{55}$$

where $c_t = \left(\frac{L^2}{c} + 2t\sigma^2 \right)^{-1}$ is the log-Sobolev inequality constant for distributions $p_t(\theta)$ and $p_t'(\theta)$. □

D.2 Proof for Lemma 3.2

Reduce analysis to point initialization:. Without loss of generality, in this paper, we only analyze recursive Rényi DP bounds for Algorithm 1 under an arbitrary point initialization for initial parameters θ_0^0. This is because, under an arbitrary initialization distribution, the last-iterate parameters θ_K^0 in Algorithm 1 follow a mixture distribution, with each component being the conditional output distribution given fixed initial parameters θ_0^0. Therefore, by the quasi-convexity of Rényi divergence, the largest (worst-case) Rényi DP bound for $R_\alpha(p(\theta_K^0|\theta_0^0)\|p({\theta'}_K^0|\theta_0^0))$ over all possible initial parameters θ_0^0, is also an upper bound for the Rényi privacy loss $R_\alpha(\theta_K^0\|{\theta'}_K^0)$ between (mixture) last-iterate parameters distributions for running Algorithm 1 on two neighboring datasets.

We now proceed to prove the recursive privacy bound Lemma 3.2.

Lemma 3.2. Let D, D' be an arbitrary pair of neighboring datasets that differ in the i_0-th data point (i.e. $x_{i_0} \neq x'_{i_0}$). Let B_k^j be a fixed mini-batch used (in iteration j of epoch k) in Algorithm 1, which contains b indices sampled from $\{1, \cdots, n\}$. We denote θ_k^j and ${\theta'}_k^j$ as the intermediate parameters in Algorithm 1 on input datasets D and D', respectively. If the distributions of θ_k^j and ${\theta'}_k^j$ satisfy log-Sobolev inequality with a constant c, and if the mini-batch GD mapping $f(\theta) = \theta - \eta \cdot \frac{1}{b} \cdot \sum_{i \in B_k^j} \ell(\theta; \mathbf{x}_i)$ is L-Lipschitz for parameters θ, then the following recursive bound for Rényi divergence holds.

$$\frac{R_\alpha(\theta_k^{j+1}\|{\theta'}_k^{j+1})}{\alpha} \leq \begin{cases} \frac{R_{\alpha'}(\theta_k^j\|{\theta'}_k^j)}{\alpha'} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} & \text{if } i_0 \notin B_k^j \\ \frac{R_\alpha(\theta_k^j\|{\theta'}_k^j)}{\alpha} + \frac{\eta S_g^2}{4\sigma^2 b^2} & \text{if } i_0 \in B_k^j \end{cases} \text{ with } \alpha' = \frac{\alpha - 1}{1 + \frac{c \cdot 2\eta\sigma^2}{L^2}} + 1. \tag{56}$$

Proof. 1. When $i_0 \notin B_k^j$, the noisy mini-batch gradient descent mapping under both dataset D and D' is written in the same way as $f(\theta) + \mathcal{N}(0, 2\eta\sigma^2\mathbb{I}_d)$, where $f(\theta) = \theta - \frac{\eta}{b} \cdot \sum_{i \in B_k^j} \nabla\ell(\theta; \mathbf{x}_i)$ (this is because $x_i = x'_i$ for $i \in B_k^j$, when $i_0 \notin B_k^j$). Therefore, we could use Lemma 3.1 and solve Equation (1) on the interval $t \in [0, \eta]$ to obtain the recursive privacy bound, where the Rényi divergence at $t = 0$ (i.e. before the update in iteration j of epoch k takes place) satisfies $R_\alpha(p_0(\theta)\|p'_0(\theta)) \leq \varepsilon_k^j(\alpha)$. We denote function $R(\alpha, t) = R_\alpha(p_t(\theta)\|p'_t(\theta))$. Then Equation (1) is equivalent to the following equation.

$$\begin{cases} \frac{\partial}{\partial t} R(\alpha, t) \leq -\left(\frac{L^2}{c} + 2t\sigma^2\right)^{-1} \cdot 2\sigma^2 \cdot \left(\frac{R(\alpha,t)}{\alpha} + (\alpha - 1) \cdot \frac{\partial}{\partial \alpha} R(\alpha, t)\right), \\ R(\alpha, 0) \leq \varepsilon_k^j(\alpha) \end{cases} \tag{57}$$

By substituting $u(\alpha, t) = \frac{R(\alpha,t)}{\alpha}$ into Equation (57), we prove that Equation (57) is equivalent to the following equation.

$$\begin{cases} \frac{\partial}{\partial t} u(\alpha, t) \leq -\left(\frac{L^2}{c} + 2t\sigma^2\right)^{-1} \cdot 2\sigma^2 \cdot \left(u(\alpha, t) + (\alpha - 1) \cdot \frac{\partial}{\partial \alpha} u(\alpha, t)\right), \\ u(\alpha, 0) \leq \frac{\varepsilon_k^j(\alpha)}{\alpha} \end{cases} \tag{58}$$

By change of variable $y = \ln(\alpha - 1)$, we prove that Equation (58) is equivalent to the following equation.

$$\begin{cases} \frac{\partial}{\partial t} u(y, t) \leq -\left(\frac{L^2}{c} + 2t\sigma^2\right)^{-1} \cdot 2\sigma^2 \cdot \left(u(y, t) + \frac{\partial}{\partial y} u(y, t)\right), \\ u(y, 0) \leq \frac{\varepsilon_k^j(e^y + 1)}{e^y + 1} \end{cases} \tag{59}$$

Now we do change of variable $\begin{cases} \tau = t \\ z = y - \int_0^t \left(\frac{L^2}{c} + 2t'\sigma^2\right)^{-1} \cdot 2\sigma^2 dt' + \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right) \end{cases}$ then by chain rule, we prove the following expressions for the partial derivatives using new

variables.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t} = \frac{\partial u}{\partial \tau} - \frac{\partial u}{\partial z} \cdot \left(\frac{L^2}{c} + 2t\sigma^2\right)^{-1} \cdot 2\sigma^2 \\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \tau} \cdot \frac{\partial \tau}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{\partial u}{\partial z} \end{cases} \tag{60}$$

By plugging Equation (60) into Equation (59), we prove that the partial differential inequality 59 is equivalent to the following inequality under new variables τ and z.

$$\begin{cases} \frac{\partial}{\partial \tau} u(z,\tau) \leq -\left(\frac{L^2}{c} + 2\tau\sigma^2\right)^{-1} \cdot 2\sigma^2 \cdot u(z,\tau), \\ u(z,0) \leq \frac{\varepsilon_k^j \left(e^{z - \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)} + 1\right)}{e^{z - \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)} + 1} \end{cases} \tag{61}$$

Now we observe that given any fixed z, Equation (61) is an ordinary differential equation with regard to τ, with a decay term that proportional to $-\left(\frac{L^2}{c} + 2\tau\sigma^2\right)^{-1} \cdot 2\sigma^2 \cdot u(z,\tau)$. Therefore, we directly solve Equation (61), and prove that

$$\ln(u(z,\tau)) - \ln(u(z,0)) \leq -\int_0^\tau \left(\frac{L^2}{c} + 2\tau'\sigma^2\right)^{-1} \cdot 2\sigma^2 d\tau' = -\ln\left(1 + \frac{c \cdot 2\tau\sigma^2}{L^2}\right) \tag{62}$$

We take $\tau = \eta$ in Equation (62), then we prove

$$\ln u(z,\eta) - \ln u(z,0) \leq -\ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right) \tag{63}$$

By plugging the initial condition for $u(z,0)$ in Equation (61) into Equation (63), we prove that the solution for u at the end of a set $\tau = \eta$ satisfies the following inequality.

$$u(z,\eta) \leq \frac{\varepsilon_k^j \left(e^{z - \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)} + 1\right)}{e^{z - \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)} + 1} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} \tag{64}$$

Now we translate the variables $z = z, \tau = \eta$ back to the old variables y and t by definitions, and prove that $t = \tau = \eta$ and $z = y - \int_0^t \left(\frac{L^2}{c} + 2t'\sigma^2\right)^{-1} \cdot 2\sigma^2 dt' + \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right) = y$. Therefore, under these variable substitutions, we prove that Equation (63) is equivalent to the following equation.

$$u(y,\eta) \leq \frac{\varepsilon_k^j \left(e^{y - \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)} + 1\right)}{e^{y - \ln\left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)} + 1} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} \tag{65}$$

Finally, we translate the variable y back to α, under the fixed variable $t = \eta$, by the definition $y = \log(\alpha - 1)$. Therefore, we prove that Equation (65) is equivalent to the following solution for $u(\alpha,\eta)$.

$$u(\alpha,\eta) \leq \frac{\varepsilon_k^j \left((\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1\right)}{(\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} \tag{66}$$

By the definition $u(\alpha,t) = \frac{R(\alpha,t)}{\alpha}$, we prove that Equation (66) is equivalent to the following solution for $R(\alpha,t)$.

$$\frac{R(\alpha,\eta)}{\alpha} \leq \frac{\varepsilon_k^j \left((\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1\right)}{(\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} \tag{67}$$

Therefore, by using the definition $R(\alpha, \eta) = \varepsilon_k^{j+1}(\alpha)$, we finish the proof for the Lemma statement, that when $i_0 \neq B_k^j$,

$$\frac{R_\alpha(\theta_k^{j+1} \| {\theta'}_k^{j+1})}{\alpha} \leq \frac{R_{\alpha'}(\theta_k^j \| {\theta'}_k^j)}{\alpha'} \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} \tag{68}$$

where $\alpha' = (\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1$.

2. When $i_0 \in B_k^j$, by composition theorem for Rényi differential privacy [28], and by the Rényi privacy bound for Gaussian mechanism [28] under ℓ_2-sensitivity S_g/b (for batch averaged gradient) and noise $\mathcal{N}(0, 2\eta\sigma^2 \mathbb{I}_d)$, we prove that

$$R_\alpha(\theta_k^{j+1} \| {\theta'}_k^{j+1}) \leq R_\alpha(\theta_k^j, \theta_k^{j+1} \| {\theta'}_k^j, {\theta'}_k^{j+1}) \tag{69}$$

$$\leq R_\alpha(\theta_k^j \| {\theta'}_k^j) + \frac{\eta^2 (S_g/b)^2}{2 \cdot 2\eta\sigma^2} = R_\alpha(\theta_k^j \| {\theta'}_k^j) + \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}. \tag{70}$$

□

D.3 Proof for LSI sequence for noisy mini-batch gradient descent

To apply the recurisve privacy bound Lemma 3.2 and prove a converging privacy dynamics, we first need to prove that the distributions of parameters θ_k^j in Algorithm 1 satisfy log-Sobolev inequality with certain constant c_k^j, that depends on k and j. In this section, we prove that noisy mini-batch gradient descent on convex smooth loss function, as well as strongly convex smooth loss functions,satusfies LSI with certain sequences of constants as follows.

Lemma D.2 (LSI constant sequence in Algorithm 1 for convex smooth loss)**.** *Suppose that the loss function $\ell(\theta; \mathbf{x})$ in Algorithm 1 is convex and β-smooth. If the step-size $\eta < \frac{2}{\beta}$, then for any $k = 0, \cdots, K-1$ and $j = 0, \cdots, n/b - 1$, the distribution of parameters θ_k^j in Algorithm 1 satisfies c_k^j-log-Sobolev inequality with*

$$c_k^j = \frac{1}{2\eta\sigma^2 \cdot (k \cdot n/b + j)}, \tag{71}$$

and we define $c_0^0 = \frac{1}{0} = +\infty$.

Proof. The mini-batch noisy gradient descent update could be written as $\theta_k^{j+1} = f(\theta_k^j) + \mathcal{N}(0, 2\eta\sigma^2 \mathbb{I}_d)$, where f is a deterministic mapping on $\mathbb{R}^d$ written as $f(\theta) = \theta - \eta \cdot \frac{\sum_{i \in B} \ell(\theta; \mathbf{x}_i)}{b}$, and B is a mini-batch of size b consisting of indices selected from $0, \cdots, n$.

Because the initialization is point distribution around θ_0, therefore θ_0^0 satisfies log-Sobolev inequality with constant $c_0 = \infty$.

By the β-smoothness of $\ell(\theta; \mathbf{x})$, and by $\eta < \frac{2}{\beta}$, we prove that the mini-batch gradient mapping $f(\theta)$ is 1-Lipschitz. Further using LSI under Lipchitz mapping (Lemma 16 in Vempala and Wibisono [35]) and under Gaussian convolution (Lemma 17 in Vempala and Wibisono [35]), we prove that

$$\frac{1}{c_k^j} = \frac{1}{c_k^{j-1}} + 2\eta\sigma^2 \tag{72}$$

$$= \frac{1}{c_0^0} + 2\eta\sigma^2 \cdot (k \cdot n/b + j) \tag{73}$$

$$\text{by } c_0^0 = +\infty, \quad = 2\eta\sigma^2 \cdot (k \cdot n/b + j) \tag{74}$$

$$\tag{75}$$

This suffices to prove the LSI sequence in Equation (71). □

Similarly, we prove another LSI sequence for noisy mini-batch gradient descent on strongly convex smooth loss function.

Lemma D.3 (LSI constant sequence in Algorithm 1 for strongly convex smooth loss)**.** *Suppose the loss function $\ell(\theta;\mathbf{x})$ in Algorithm 1 is λ-strongly convex and β-smooth. If the step-size $\eta < \frac{2}{\lambda+\beta}$, then for any $k = 0, \cdots, K-1$ and $j = 0, \cdots, n/b-1$, the distribution of parameters θ_k^j in Algorithm 1 satisfies c_k^j-log-Sobolev inequality with*

$$c_k^j = \frac{1}{2\eta\sigma^2} \cdot \frac{1}{\sum_{s=0}^{k\cdot n/b+j-1}(1-\eta\lambda)^{2s}} \tag{76}$$

Proof. The mini-batch noisy gradient descent update could be written as $\theta_k^{j+1} = f(\theta_k^j) + \mathcal{N}(0, 2\eta\sigma^2\mathbb{I}_d)$, where f is a deterministic mapping on $\mathbb{R}^d$ written as $f(\theta) = \theta - \eta \cdot \frac{\sum_{i\in B}\ell(\theta;\mathbf{x}_i)}{b}$, and B is a mini-batch of size b consisting of indices selected from $0, \cdots, n$.

Because the initialization is point distribution around θ_0, therefore θ_0^0 satisfies log-Sobolev inequality with constant $c_0 = \infty$.

By λ-strong convexity and β-smoothness of $\ell(\theta;\mathbf{x})$, and by $\eta < \frac{2}{\lambda+\beta}$, we prove that the mini-batch gradient mapping $f(\theta)$ is $1-\eta\lambda$-Lipschitz. Further using LSI under Lipchitz mapping (Lemma 16 in Vempala and Wibisono [35]) and under Gaussian convolution (Lemma 17 in Vempala and Wibisono [35]), we prove that

$$\frac{1}{c_k^j} = \frac{(1-\eta\lambda)^2}{c_k^{j-1}} + 2\eta\sigma^2 \tag{77}$$

$$= \frac{(1-\eta\lambda)^{2(k\cdot n/b+j)}}{c_0^0} + 2\eta\sigma^2 \cdot \sum_{s=0}^{k\cdot n/b+j-1}(1-\eta\lambda)^{2s} \tag{78}$$

$$\text{by } c_0^0 = +\infty, \quad = 2\eta\sigma^2 \cdot \sum_{s=0}^{k\cdot n/b+j-1}(1-\eta\lambda)^{2s} \tag{79}$$

This suffices to prove the LSI sequence in Equation (76). □

D.4 Equivalence Between Lemma 3.2 (Ours) And The Bound in Feldman et al. [18]

We now plug the LSI constant sequence derived in Lemma D.2 into the recursive privacy bound Lemma 3.2, and prove the following privacy dynamics theorems for noisy mini-batch gradient descent under smooth convex loss functions.

Theorem D.4 (Privacy dynamics under convex smooth loss)**.** *Under fixed mini-batches $B^0, \cdots, B^{n/b-1}$, if the loss function $\ell(\theta;x)$ is convex, β-smooth, and if its gradient has finite ℓ_2-sensitivity S_g, then Algorithm 1 with step-size $\eta < \frac{2}{\beta}$ satisfies (α,ε)-Rényi DP for data points in the batch B^{j_0}, with*

$$\varepsilon_j \le \frac{\alpha\eta S_g^2}{4\sigma^2 b^2} \cdot \frac{b}{n} \cdot (K-1) + \frac{\alpha\eta S_g^2}{4\sigma^2 b^2} \cdot \frac{1}{n/b - j_0} \tag{80}$$

$$\tag{81}$$

Proof. We first observe that for any batch index $j_0 = 0, \cdots, n/b-1$, the privacy bound for data points in batch B^{j_0} in Algorithm 1 (that has K epochs) is equivalent to the privacy bound for data points in the batch B^0 after $K-1$ epochs and $n/b - j_0$ iterations in Algorithm 1. That is, if $R_\alpha(\theta_k^j \| {\theta'}_k^j) \le \varepsilon_k^j(\alpha)$ is an upper bound for the Rényi divergence between distributions of parameters θ_k^j and ${\theta'}_k^j$ in running Algorithm 1 on neighboring datasets D and D', when the differing data point between D and D' is contained in the mini-batch B^0, then the privacy bound ε for data points in the mini-batch B^{j_0} of Algorithm 1 satisfies

$$\varepsilon \le \varepsilon_{K-1}^{n/b-j_0}(\alpha) \le \varepsilon_{K-1}^0(\alpha) + \varepsilon_0^{n/b-j_0}(\alpha) \tag{82}$$

$$\le (K-1)\cdot\varepsilon_0^{n/b}(\alpha) + \varepsilon_0^{n/b-j_0}(\alpha), \tag{83}$$

where the last two inequalities are by composition of Rényi DP guarantees.

Therefore, in the remaining proof, we only prove upper bounds for the terms $\varepsilon_0^j(\alpha)$ for $j = 1, \cdots, n/b$, that are required by Equation (83) for bounding ε.

By Lemma D.2, for any $k = 0, \cdots, K-1$ and $j = 0, \cdots, n/b-1$, the distribution of parameters θ_k^j satisfies log-Sobolev inequality with the following constant c_k^j.

$$c_k^j = \frac{1}{2\eta\sigma^2 \cdot (k \cdot n/b + j)} \tag{84}$$

By plugging the LSI constant sequence $\{c_k^j\}$ into Lemma 3.2, and by the noisy mini-batch gradient descent mapping, under convex smooth loss with stepsize $\eta < \frac{2}{\beta}$, is 1-Lipschitz, we prove the following recursive bound for $\varepsilon_k^j(\alpha)$, i.e. the Rényi DP bound for data points in the mini-batch B^0 of Algorithm 1. For any $k = 0, \cdots, K-1$ and any $j = 0, \cdots, n/b-1$,

$$\varepsilon_0^0(\alpha) = 0 \tag{85}$$

$$\frac{\varepsilon_k^{j+1}(\alpha)}{\alpha} \leq \begin{cases} \frac{\varepsilon_k^j(\alpha)}{\alpha} + \frac{\eta S_g^2}{4\sigma^2 b^2} & \text{if } j = 0 \\ \frac{\varepsilon_k^j(\alpha')}{\alpha'} \cdot \frac{k \cdot n/b + j}{k \cdot n/b + j + 1} & \text{if } j = 1, \cdots, n/b-1 \end{cases} \tag{86}$$

$$\text{where } \alpha' = (\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1 \tag{87}$$

$$\varepsilon_{k+1}^0(\alpha) = \varepsilon_k^{n/b}(\alpha) \tag{88}$$

By solving Equation (86) under $k = 0$, we prove that for any $j = 1, \cdots, n/b$, and for any $\alpha > 1$,

$$\frac{\varepsilon_0^j(\alpha)}{\alpha} \leq \frac{\eta S_g^2}{4\sigma^2 b^2} \cdot \prod_{j'=1}^{j-1} \frac{j'}{j'+1} \tag{89}$$

$$= \frac{\eta S_g^2}{4\sigma^2 b^2} \cdot \frac{1}{j} \tag{90}$$

By plugging Equation (90) into Equation (83), we prove the privacy bound Equation (80) in the theorem statement. □

Theorem D.4 is equivalent to Theorem 23 in Feldman et al. [18] for Algorithm 1 under single-epoch setting (where $K = 1$) with batch-size $b = 1$. For multi-epoch setting with $K = n$ and batch-size $b = 1$, Theorem D.4 is equivalent to Theorem 35 in Feldman et al. [18].

D.5 Proof for Theorem 3.3

We now plug the LSI constant sequence derived in Lemma D.3 into the recursive privacy bound Lemma 3.2, and prove the following privacy dynamics theorems for noisy mini-batch gradient descent under strongly convex smooth loss functions.

Theorem 3.3. Conditioned on a fixed sequence of partitioned mini-batches $B^0, \cdots, B^{n/b-1}$ in Line 3, if the loss function is λ-strongly convex, β-smooth and its gradient has ℓ_2-sensitivity S_g, then running Algorithm 1 for $K \geq 1$ epochs with step-size $\eta < \frac{2}{\lambda+\beta}$, satisfies (α, ε)-Rényi DP for data points in the batch B^{j_0}, with

$$\varepsilon \leq \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1 - (1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b - \lfloor \frac{n}{2b} \rfloor)}}{1 - (1-\eta\lambda)^{2\cdot(n/b - \lfloor \frac{n}{2b} \rfloor)}} + \varepsilon_0^{n/b - j_0}(\alpha) \tag{91}$$

where $\varepsilon_0^j(\alpha) = \frac{\alpha\eta S_g^2}{4\sigma^2 b^2} \cdot (1-\eta\lambda)^{2\cdot(j-1)} \cdot \frac{1}{\sum_{s=0}^{j-1}(1-\eta\lambda)^{2s}}$ for any $j = 1, \cdots, \frac{n}{b}$ (we assume $\frac{n}{b} \geq 2$).

Proof. We first observe that for any batch index $j_0 = 0, \cdots, n/b-1$, the privacy bound for data points in batch B^{j_0} in Algorithm 1 (that has K epochs) is equivalent to the privacy bound for data

points in the batch B^0 after $K-1$ epochs and $n/b - j_0$ iterations in Algorithm 1. That is, if $R_\alpha(\theta_k^j \| \theta'^j_k) \leq \varepsilon_k^j(\alpha)$ is an upper bound for the Rényi divergence between distributions of parameters θ_k^j and θ'^j_k in running Algorithm 1 on neighboring datasets D and D', when the differing data point between D and D' is contained in the mini-batch B^0, then the privacy bound ε, for running Algorithm 1 on neighboring datasets D and D' that differs in a point in the mini-batch B^{j_0}, satisfies

$$\varepsilon \leq \varepsilon_{K-1}^{n/b-j_0}(\alpha) \leq \varepsilon_{K-1}^0(\alpha) + \varepsilon_0^{n/b-j_0}(\alpha), \tag{92}$$

where the last inequality is by composition of Rényi DP guarantees.

Therefore, in the remaining proof, we only prove upper bounds for the terms $\varepsilon_0^{n/b-1-j_0}(\alpha)$ and $\varepsilon_{K-1}^0(\alpha)$, that are required by Equation (92) for bounding ε.

By Lemma D.3, for any $k = 0, \cdots, K-1$ and $j = 0, \cdots, n/b-1$, the distribution of parameters θ_k^j satisfies log-Sobolev inequality with the following constant c_k^j.

$$c_k^j = \frac{1}{2\eta\sigma^2} \cdot \frac{1}{\sum_{s=0}^{k\cdot n/b+j-1}(1-\eta\lambda)^{2s}} \tag{93}$$

By plugging the LSI constant sequence $\{c_k^j\}$ proved in Lemma D.3 into the proved recursive privacy bound Lemma 3.2, we prove that for running noisy mini-batch gradient descent on neighboring datasets that differ in a data point in the first mini-batch B^0, the following recursion for the Rényi divergence bound $\varepsilon_k^j(\alpha)$ holds: if the loss function is λ-strongly convex and β-smooth, and if the stepsize $\eta < \frac{2}{\lambda+\beta}$, then for any $k = 0, \cdots, K-1$, any $j = 0, \cdots, n/b-1$, and any $\alpha > 1$,

$$\varepsilon_0^0(\alpha) = 0 \tag{94}$$

$$\frac{\varepsilon_k^{j+1}(\alpha)}{\alpha} \leq \begin{cases} \frac{\varepsilon_k^j(\alpha)}{\alpha} + \frac{\eta S_g^2}{4\sigma^2 b^2} & \text{if } j = 0 \\ \frac{\varepsilon_k^j(\alpha')}{\alpha'} \cdot (1-\eta\lambda)^2 \cdot \frac{\sum_{s=0}^{k\cdot n/b+j-1}(1-\eta\lambda)^{2s}}{\sum_{s=0}^{k\cdot n/b+j}(1-\eta\lambda)^{2s}} & \text{if } j = 1, \cdots, n/b-1 \end{cases} \tag{95}$$

$$\text{where } \alpha' = (\alpha-1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1 \tag{96}$$

$$\varepsilon_{k+1}^0(\alpha) = \varepsilon_k^{n/b}(\alpha) \tag{97}$$

We now solve the above recursion, thus bounding the privacy loss ε of Algorithm 1 in Equation (92).

1. We first prove a bound for the term $\varepsilon_0^{n/b-j_0}(\alpha)$ in Equation (92). By solving Equation (95) under $k = 0$, we prove that for any $j = 1, \cdots, n/b$, and for any $\alpha > 1$,

$$\frac{\varepsilon_0^j(\alpha)}{\alpha} \leq \frac{\eta S_g^2}{4\sigma^2 b^2} \cdot \prod_{j'=1}^{j-1} \left((1-\eta\lambda)^2 \cdot \frac{\sum_{s=0}^{j'-1}(1-\eta\lambda)^{2s}}{\sum_{s=0}^{j'}(1-\eta\lambda)^{2s}} \right) \tag{98}$$

$$= \frac{\eta S_g^2}{4\sigma^2 b^2} \cdot (1-\eta\lambda)^{2\cdot(j-1)} \cdot \frac{1}{\sum_{s=0}^{j-1}(1-\eta\lambda)^{2s}} \tag{99}$$

2. We now prove a bound for the term $\varepsilon_{K-1}^0(\alpha)$ in Equation (92).

   The term $\varepsilon_k^j(\alpha)$ corresponds to the privacy bound for the data points in the batch B^0 for running Algorithm 1 with k epochs plus j iteration. This is equivalent to the composition $\mathcal{A}_2 \circ \mathcal{A}_1$ of two sub-mechanisms $\mathcal{A}_1$ and $\mathcal{A}_2$, where $\mathcal{A}_1$ corresponds to running Algorithm 1 with $K = k$ epochs, and $\mathcal{A}_2$ corresponds to running Algorithm 1 with only j iterations. Therefore, by the composition theorem for Rényi DP guarantees [28], we prove the following alternative recursive privacy bound for $\varepsilon_k^j(\alpha)$. For any $k = 0, \cdots, K-1$ and any $j = 0, \cdots, n/b-1$,

$$\varepsilon_k^{j+1}(\alpha) \leq \varepsilon_k^0(\alpha) + \varepsilon_0^{j+1}(\alpha) \tag{100}$$

We now prove an upper bound for $\varepsilon_k^j(\alpha)$, by carefully combining the original recursion Equation (95) and the new alternative recursion Equation (100) (obtained by composition theorem). We use the original recursion Equation (100) for recursively bounding $\varepsilon_k^{j+1}(\alpha)$ during the first half of one epoch, i.e., for $j = 0, \cdots, \lfloor \frac{n}{2b} \rfloor - 1$, and then we use the new alternative recursive bound Equation (95) for the second half of one epoch, i.e., for $j = \lfloor \frac{n}{2b} \rfloor, \cdots, n/b - 1$. [5] Via this combination, we obtain a new combined recursion for privacy bound as follows. For any $k = 0, \cdots, K-1$,

$$\frac{\varepsilon_k^{j+1}(\alpha)}{\alpha} \leq \begin{cases} \frac{\varepsilon_k^0(\alpha) + \varepsilon_0^{j+1}(\alpha)}{\alpha} & \text{if } j = 0, \cdots, \lfloor \frac{n}{2b} \rfloor - 1 \\ \frac{\varepsilon_k^j(\alpha')}{\alpha'} \cdot (1-\eta\lambda)^2 \cdot \frac{\sum_{s=0}^{k \cdot n/b + j - 1}(1-\eta\lambda)^{2s}}{\sum_{s=0}^{k \cdot n/b + j}(1-\eta\lambda)^{2s}} & \text{if } j = \lfloor \frac{n}{2b} \rfloor, \cdots, n/b - 1 \end{cases} \tag{101}$$

where $\alpha' = (\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1$.

We now solve this new recursion Equation (101) for one epoch, by accumulating $j = 0, \cdots, n/b - 1$. We prove that for any $k = 0, \cdots, K-1$,

$$\frac{\varepsilon_{k+1}^0(\alpha)}{\alpha} = \frac{\varepsilon_k^{n/b}(\alpha)}{\alpha} \tag{102}$$

$$\leq \frac{\varepsilon_k^0(\tilde{\alpha}) + \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\tilde{\alpha})}{\tilde{\alpha}} \cdot \prod_{j' = \lfloor \frac{n}{2b} \rfloor}^{n/b-1} (1-\eta\lambda)^2 \cdot \frac{\sum_{s=0}^{k \cdot n/b + j - 1}(1-\eta\lambda)^{2s}}{\sum_{s=0}^{k \cdot n/b + j}(1-\eta\lambda)^{2s}} \tag{103}$$

$$\leq \frac{\left(\varepsilon_k^0(\tilde{\alpha}) + \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\tilde{\alpha})\right)}{\tilde{\alpha}} \cdot (1-\eta\lambda)^{2 \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)} \tag{104}$$

where the RDP order $\tilde{\alpha} > 1$ is the $n/b - \lfloor \frac{n}{2b} \rfloor$ fold mapped value of α under repeated mappings $\alpha \leftarrow (\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1$.

We now further solve this new recursion Equation (101) for multiple epochs, by accumulating Equation (104) for $k = 0, 1, \cdots, K-1$. We prove that for any $k = 0, 1, \cdots, K-1$,

$$\frac{\varepsilon_{k+1}^0(\alpha)}{\alpha} \leq \frac{\varepsilon_0^0(\tilde{\alpha})}{\tilde{\alpha}} \cdot (1-\eta\lambda)^{2 \cdot (k+1) \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)} + \frac{\varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\tilde{\alpha})}{\tilde{\alpha}} \cdot \sum_{k'=1}^{k+1} (1-\eta\lambda)^{2 \cdot k' \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)} \tag{105}$$

for some RDP order $\tilde{\alpha} > 1$ that is the $(k+1) \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)$ fold mapped value of α under repeated mapping $\alpha \leftarrow (\alpha - 1) \cdot \left(1 + \frac{c \cdot 2\eta\sigma^2}{L^2}\right)^{-1} + 1$.

By further substituting $\varepsilon_0^0(\tilde{\alpha}) = 0$ at the initialization point θ_0 for any $\tilde{\alpha} > 1$ in the above equation, we prove that for any $k = 0, \cdots, K-1$

$$\frac{\varepsilon_{k+1}^0(\alpha)}{\alpha} = \frac{\varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\tilde{\alpha})}{\tilde{\alpha}} \cdot (1-\eta\lambda)^{2 \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)} \cdot \frac{1 - (1-\eta\lambda)^{2 \cdot (k+1) \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)}}{1 - (1-\eta\lambda)^{2 \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)}} \tag{106}$$

for some RDP order $\tilde{\alpha} > 1$.

By setting $k = K-2$ in Equation (106), we prove that for $K \geq 2$

$$\frac{\varepsilon_{K-1}^0(\alpha)}{\alpha} \leq \frac{\varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\tilde{\alpha})}{\tilde{\alpha}} \cdot \frac{1 - (1-\eta\lambda)^{2 \cdot (K-1) \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)}}{1 - (1-\eta\lambda)^{2 \cdot (n/b - \lfloor \frac{n}{2b} \rfloor)}} \tag{107}$$

for some RDP order $\tilde{\alpha} > 1$. By the format of $\varepsilon_0^j(\alpha)$ for any $\alpha > 1$ in our proof Equation (99) (i.e., $\frac{\varepsilon_0^j(\alpha)}{\alpha}$ is bounded by a constant for all $\alpha > 1$), we prove that Equation (107) is equivalent

[5] This seemingly artificial way of separating one epoch into two halves, and using the recursive bounds separately in each half, is for obtaining a small privacy bound at convergence.

to the following equation

$$\varepsilon_{K-1}^{0}(\alpha) \leq \varepsilon_{0}^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}} \tag{108}$$

For $K = 1$, by definition, we compute that $\varepsilon_{K-1}^{0}(\alpha) = 0$ =Right hand side of Equation (108), therefore Equation (108) also holds for $K = 1$.

We now plug our bound Equation (108) into Equation (92), and prove a bound for ε as follows.

$$\varepsilon \leq \varepsilon_{K-1}^{0}(\alpha) + \varepsilon_{0}^{n/b-j_0}(\alpha) \tag{109}$$

$$\leq \varepsilon_{0}^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor \frac{n}{2b} \rfloor)}} + \varepsilon_{0}^{n/b-j_0}(\alpha) \tag{110}$$

where the term $\varepsilon_0^j(\alpha)$ is upper bounded by Equation (99). This gives the privacy bound Equation (3) in the theorem statement. □

D.6 Explanations for the privacy bound derived from Balle et al. [5] in Figure 1

For convenience, we first translate [5, Theorem 5] into the symbols used in this paper, as well as under mini-batch with size $b > 1$, as follows.

Theorem D.5 (Balle et al. [5])**.** *Let $\ell(\theta, \mathbf{x})$ be an L-Lipschitz, β-smooth, λ-strongly convex loss function. If $\eta \leq \frac{2}{\beta+\lambda}$, then conditioned on a fixed sequence of partitioned mini-batches $B^0, \cdots, B^{n/b-1}$ in Line 3, Algorithm 1 satisfies (α, ε)-Rényi DP for data points in the mini-batch j_0, with*

$$\varepsilon = \begin{cases} \alpha \cdot \frac{2(L/b)^2}{2\eta\sigma^2} & j_0 = n/b-1 \\ \alpha \cdot \frac{2(L/b)^2}{(n/b-j_0-1)\cdot 2\eta\sigma^2}\left(1-\frac{2\eta\beta\lambda}{\beta+\lambda}\right)^{\frac{n/b-j_0}{2}} & j_0 = 0, \cdots, n/b-2 \end{cases} \tag{111}$$

The variables from Balle et al. [5] that we replaced are as follows: $L \to L/b$, this is because, in Algorithm 1, we additionally average each per-example gradient with the mini-batch size b; $i \to j_0 + 1$, this is because the index used in our paper starts from $j_0 = 0$, while the index used in Balle et al. [5] starts from $i = 1$; $n \to n/b$, this is because under mini-batch size $b > 1$, the number of iterations in one epoch in our paper is n/b, while the number of iteration in Balle et al. [5] equals the size of the datasets n; $\sigma^2 \to 2\eta\sigma^2$, this is because in our paper, we follow the noise scaling with variance $2\eta\sigma^2$ that is related to stepsize η, as in SGLD, while Balle et al. [5] use the standard Gaussian noise with variance σ^2.

By further substituting the Lipschitz requirement with its equivalent sensitivity assumption, i.e. $L \to \eta S_g/2$, and by using Rényi DP composition [28] over the epochs, we obtain that the privacy bound in Balle et al. [5, Theorem 5] is $\frac{\alpha\cdot\eta S_g^2}{4\cdot(n/b-1)\cdot b^2\sigma^2} \cdot \left(1-\frac{2\eta\beta\lambda}{\beta+\lambda}\right)^{\frac{n}{2b}} \cdot K$ when the differing data point is in the first batch, and the bound is $\frac{\alpha\cdot\eta S_g^2}{4\cdot(n/b-1)\cdot b^2\sigma^2} \cdot \left(1-\frac{2\eta\beta\lambda}{\beta+\lambda}\right)^{\frac{n}{2b}} \cdot (K-1) + \frac{\alpha\cdot\eta S_g^2}{4b^2\sigma^2}$ when the differing data point is in the last batch.

D.7 Revisiting noisy GD: A tighter bound than [15, Corollary 1]

In this section, we prove a new converging hidden-state privacy bound for the noisy GD algorithm, which is a special case of Algorithm 1 under $b = n$. We then show that our bound is slightly tighter than [15, Corollary 1] and also admits a conceptually simpler proof (our proof relies on the privacy amplification by randomized post-processing results in Section 3). We first state our privacy bound and its proof as follows.

Theorem D.6. *Let $\ell(\theta; \mathbf{x})$ be a λ-strongly convex, and β-smoooth loss function, with a finite total gradient sensitivity S_g, then the noisy gradient descent algorithm with step-size $\eta < \frac{1}{\beta}$, satisfies (α, ε) Rényi Differential Privacy with*

$$\varepsilon \leq \frac{\alpha\eta S_g^2}{2\sigma^2 n^2} \cdot \sum_{k=1}^{K}\left(1-\frac{\eta\lambda}{2}\right)^k. \tag{112}$$

Proof. We first offer a new perspective of viewing a noisy GD update as follows. Recall that one noisy GD update in Algorithm 1 is written as

$$\theta_{k+1}^0 = \theta_k^0 - \eta \cdot g(\theta_k^0; D) + \sqrt{2\eta\sigma^2} \cdot \mathcal{N}(0, \mathbb{I}_d) \text{ where } g(\theta_k^j; D) = \frac{1}{n}\sum_{i=1}^n \nabla \ell(\theta_k^j; \mathbf{x}_i) \tag{113}$$

Therefore, this update is equivalent to the following two steps:

$$\theta_k^{\frac{1}{2}} = \theta_k^0 - \eta \cdot g(\theta_k^0; D) + \sqrt{\eta\sigma^2} \cdot \mathcal{N}(0, \mathbb{I}_d) \text{ where } g(\theta_k^0; D) = \frac{1}{n}\sum_{i=1}^n \nabla \ell(\theta_k^0; \mathbf{x}_i) \tag{114}$$

$$\theta_{k+1}^0 = \theta_k^{\frac{1}{2}} + \sqrt{\eta\sigma^2} \cdot \mathcal{N}(0, \mathbb{I}_d) \tag{115}$$

That is, we view each noisy GD update as another noisy GD update with smaller noise scale followed by pure additive Gaussian noise.

Moreover, by Lemma D.3, we prove that distributions of θ_k^0 satisfies c_k^0-log Sobolev inequality with

$$c_k^0 = \frac{1}{2\eta\sigma^2} \cdot \frac{1}{\sum_{s=0}^{k-1}(1-\eta\lambda)^{2s}} \tag{116}$$

Therefore, by further using LSI under Lipschitz mapping ([35, Lemma 16]) and under Gaussian convolution ([35, Lemma 17]), we prove that the distribution of $\theta_k^{\frac{1}{2}}$ satisfies $c_k^{\frac{1}{2}}$-log Sobolev inequality with

$$\frac{1}{c_k^{\frac{1}{2}}} = \frac{(1-\eta\lambda)^2}{c_k^0} + \eta\sigma^2 \tag{117}$$

$$= 2\eta\sigma^2 \cdot \sum_{s=1}^k (1-\eta\lambda)^{2s} + \eta\sigma^2 \tag{118}$$

Therefore, by applying composition on the conceptual step from $\theta_k^0 \to \theta_k^{\frac{1}{2}}$, we prove that

$$\frac{R_\alpha(\theta_k^{\frac{1}{2}} \| {\theta'}_k^{\frac{1}{2}})}{\alpha} \leq \frac{R_\alpha(\theta_k^0 \| {\theta'}_k^0)}{\alpha} + \frac{\eta S_g^2}{2\sigma^2 n^2} \tag{119}$$

For the remaining conceptual step from $\theta_k^{\frac{1}{2}} \to \theta_{k+1}^0$, we note that pure additive Gaussian noise is equivalent to identity mapping convolved with Gaussian noise. Therefore, we apply $i_0 \notin B_k^j$ case of Lemma 3.2 with $L = 1$ (for identity mapping) and noise standard deviation $\frac{\sigma}{\sqrt{2}}$, and prove that

$$\frac{R_\alpha(\theta_{k+1}^0 \| {\theta'}_{k+1}^0)}{\alpha} \leq \frac{R_{\alpha'}(\theta_k^{\frac{1}{2}} \| {\theta'}_k^{\frac{1}{2}})}{\alpha'} \cdot \left(1 + c_k^{\frac{1}{2}} \cdot \eta\sigma^2\right)^{-1} \text{ with } \alpha' = \frac{\alpha - 1}{1 + c_k^{\frac{1}{2}} \cdot \eta\sigma^2} + 1 \tag{120}$$

Therefore, by combining (119) and (120), we prove the following recursive Rényi DP bound.

$$\frac{R_\alpha(\theta_{k+1}^0 \| {\theta'}_{k+1}^0)}{\alpha} \leq \left(\frac{R_{\alpha'}(\theta_k^0 \| {\theta'}_k^0)}{\alpha'} + \frac{\eta S_g^2}{2\sigma^2 n^2}\right) \cdot \left(1 + c_k^{\frac{1}{2}} \cdot \eta\sigma^2\right)^{-1} \text{ with } \alpha' = \frac{\alpha - 1}{1 + c_k^{\frac{1}{2}} \cdot \eta\sigma^2} + 1 \tag{121}$$

By further plugging in the LSI constant sequence (117), we simplify Equation (121) into the following inequality.

$$\frac{R_\alpha(\theta^0_{k+1}\|\theta'^0_{k+1})}{\alpha} \leq \left(\frac{R_{\alpha'}(\theta^0_k\|\theta'^0_k)}{\alpha'} + \frac{\eta S_g^2}{2\sigma^2 n^2}\right)\cdot\left(1 - \frac{1}{2\sum_{s=0}^{k}(1-\eta\lambda)^{2s}}\right) \tag{122}$$

$$\leq \left(\frac{R_{\alpha'}(\theta^0_k\|\theta'^0_k)}{\alpha'} + \frac{\eta S_g^2}{2\sigma^2 n^2}\right)\cdot\left(1 - \frac{1}{2\sum_{s=0}^{+\infty}(1-\eta\lambda)^{2s}}\right) \tag{123}$$

$$= \left(\frac{R_{\alpha'}(\theta^0_k\|\theta'^0_k)}{\alpha'} + \frac{\eta S_g^2}{2\sigma^2 n^2}\right)\cdot\left(1 - \frac{1-(1-\eta\lambda)^2}{2}\right) \tag{124}$$

$$\leq \left(\frac{R_{\alpha'}(\theta^0_k\|\theta'^0_k)}{\alpha'} + \frac{\eta S_g^2}{2\sigma^2 n^2}\right)\cdot\left(1 - \frac{\eta\lambda}{2}\right), \tag{125}$$

where $\alpha' = (\alpha-1)\cdot\left(1 - \frac{1}{2\sum_{s=0}^{k}(1-\eta\lambda)^{2s}}\right) + 1$, and the last inequality Equation (125) is because of the inequality $\eta\lambda < \frac{\lambda}{\beta} \leq 1$ that is ensured by the condition $\eta < \frac{1}{\beta}$.

Therefore, by solving the recursion Equation (125) from $k = 0, \cdots, K-1$, and by using $R_\alpha(\theta^0_0\|\theta'^0_0) = 0$ for any $\alpha > 1$, we prove the theorem statement in Theorem D.6.

$$\frac{R_\alpha(\theta^0_K\|\theta'^0_K)}{\alpha} \leq \frac{\eta S_g^2}{2\sigma^2 n^2}\cdot\sum_{k=1}^{K}\left(1 - \frac{\eta\lambda}{2}\right)^k \tag{126}$$

□

Comparison with [15, Corollary 1] We now compare our privacy bound Theorem D.6 with [15, Corollary 1] and show that our bound is tighter. For convenience, we now repeat the privacy bound for noisy gradient descent (i.e. when $b = n$ in Algorithm 1) that is proved in [15, Corollary 1] below.

Theorem D.7 (Corollary 1 in Chourasia et al. [15])**.** *Let $\ell(\theta;\mathbf{x})$ be a λ-strongly convex, and β-smoooth loss function, with a finite total gradient sensitivity S_g, then the noisy gradient descent algorithm with start parameter $\theta_0 \sim \mathcal{N}(0, \frac{2\sigma^2}{\lambda}\mathcal{I}_d)$, and step-size $\eta < \frac{1}{\beta}$, satisfies (α,ε) Rényi Differential Privacy with*

$$\varepsilon \leq \frac{\alpha S_g^2}{\lambda\sigma^2 n^2}(1 - e^{-\lambda\eta K/2}). \tag{127}$$

Because $1 - x < e^{-x}$ for $x \neq 0$, we are able to further relax the bound in Theorem D.6 as follows.

$$\varepsilon \leq \frac{\alpha\eta S_g^2}{2\sigma^2 n^2}\cdot\sum_{k=1}^{K}\left(1 - \frac{\eta\lambda}{2}\right)^k \tag{128}$$

$$< \frac{\alpha\eta S_g^2}{2\sigma^2 n^2}\cdot\left(1 - \frac{\eta\lambda}{2}\right)\cdot\sum_{k=0}^{K-1} e^{-\frac{\eta\lambda k}{2}} \tag{129}$$

$$= \frac{\alpha\eta S_g^2}{2\sigma^2 n^2}\cdot\left(1 - \frac{\eta\lambda}{2}\right)\cdot\frac{1 - e^{-\lambda\eta K}}{1 - e^{-\eta\lambda/2}} \tag{130}$$

Because $1 - e^{-x} > x\cdot e^{-x}$ for $x > 0$, we prove that $1 - e^{-\eta\lambda/2} \geq \frac{\lambda\eta}{2}\cdot e^{-\frac{\lambda\eta}{2}}$. By plugging this inequality to Equation (130), we prove that

$$\varepsilon < \frac{\alpha S_g^2}{\lambda\sigma^2 n^2}\cdot\frac{1 - \frac{\eta\lambda}{2}}{e^{-\frac{\lambda\eta}{2}}}\cdot(1 - e^{-\lambda\eta K}) \tag{131}$$

By again using $1 - x < e^{-x}$ for $x \neq 0$ into Equation (131), we prove that

$$\varepsilon < \frac{\alpha S_g^2}{\lambda \sigma^2 n^2} \cdot (1 - e^{-\lambda \eta K}) = \text{RHS of [15, Corollary 1]} \tag{132}$$

This shows that the privacy bound Theorem D.6 enabled by our analysis is strictly tighter than [15, Corollary 1].

E Proof for Section 4

Lemma 4.1. Let $\mu_1, \cdots, \mu_m$ and $\nu_1, \cdots, \nu_m$ be measures over R^d. Then for any $\alpha \geq 1$, and any $p_1, \cdots, p_m \geq 0$ that satisfies $p_1 + \cdots + p_m = 1$,

$$e^{(\alpha-1)\cdot R_\alpha(\sum_{j=1}^m p_j \mu_j \| \sum_{j=1}^m p_j \nu_j)} \leq \sum_{j=1}^m p_j \cdot e^{(\alpha-1)\cdot R_\alpha(\mu_j \| \nu_j)} \tag{133}$$

Proof. By definition of Rényi divergence,

$$e^{(\alpha-1)\cdot R_\alpha(\mu\|\nu)} = \int \left(\frac{\mu(\theta)}{\nu(\theta)}\right)^\alpha \cdot \mu(\theta) d\theta \tag{134}$$

Therefore by definition of f-divergence, $e^{(\alpha-1)\cdot R_\alpha(\mu\|\nu)}$ is f-divergence with $f(x) = x^\alpha$. By the convexity of f when $\alpha \geq 1$, and by applying Theorem 3.1 in Taneja and Kumar [33], we prove that the f-divergence $e^{(\alpha-1)\cdot R_\alpha(\mu\|\nu)}$ is jointly convex in arguments μ, ν. □

E.1 Proof for Theorem 4.2

Theorem 4.2 (Privacy dynamics for "shuffle and partition" mini-batch gradient descent). If the loss function $\ell(\theta; x)$ is λ-strongly convex, β-smooth, and if its gradient has finite ℓ_2-sensitivity S_g, then for $K \geq 1$ and $\frac{n}{b} \geq 2$, Algorithm 1 with stepsize $\eta < \frac{2}{\lambda+\eta}$ satisfies (α, ε)-Rényi DP for all data points with

$$\varepsilon \leq \varepsilon_0^{\lfloor \frac{n}{2b} \rfloor}(\alpha) \cdot \frac{1 - (1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b - \lfloor \frac{n}{2b} \rfloor)}}{1 - (1-\eta\lambda)^{2\cdot(n/b - \lfloor \frac{n}{2b} \rfloor)}} + \frac{1}{\alpha - 1} \cdot \log \left(\underset{0 \leq j_0 < n/b}{Avg} e^{(\alpha-1)\varepsilon_0^{n/b - j_0}(\alpha)} \right) \tag{135}$$

where the terms $\varepsilon_0^j(\alpha)$ is upper-bounded for any $j = 1, \cdots, n/b$ as follows.

$$\varepsilon_0^j(\alpha) \leq \frac{\alpha \eta S_g^2}{4\sigma^2 b^2} \cdot (1-\eta\lambda)^{2\cdot(j-1)} \cdot \frac{1}{\sum_{s=0}^{j-1}(1-\eta\lambda)^{2s}} \tag{136}$$

Proof. Our proof relies on the joint convexity of the scaled exponentiated Rényi divergence $e^{(\alpha-1)R_\alpha(\mu\|\nu)}$ in its arguments μ and ν). By further using the batch decomposition in Section 2, we prove

$$\varepsilon = \frac{1}{\alpha-1} \log e^{(\alpha-1)\cdot R_\alpha(\theta_K^0 \| {\theta'}_K^0)} \tag{137}$$

$$\leq \frac{1}{\alpha-1} \log \left(\sum_{B^0, \cdots, B^{n/b-1} \text{in Line 3}} p(B^0, \cdots, B^{n/b-1}) \cdot e^{(\alpha-1)\cdot R_\alpha\left(p(\theta_K^0 | B^0, \cdots, B^{n/b-1}) \| p({\theta'}_K^0 | B^0, \cdots, B^{n/b-1})\right)} \right) \tag{138}$$

Therefore, by plugging the privacy bound of Algorithm 1 proved in Theorem 3.3 into Equation (138), depending on which mini-batch contains the differing data point with any index i_0, we further prove

that

$$\varepsilon \leq \frac{1}{\alpha-1}\log\left(\sum_{j_0=0}^{n/b-1} p(i_0 \in B^{j_0}) \cdot e^{(\alpha-1)\cdot\left(\varepsilon_0^{\lfloor \frac{n}{2b}\rfloor}(\alpha)\cdot\frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}+\varepsilon_0^{n/b-j_0}(\alpha)\right)}\right) \tag{139}$$

$$= \frac{1}{\alpha-1}\log\left(\sum_{j_0=0}^{n/b-1} \frac{b}{n} \cdot e^{(\alpha-1)\cdot\left(\varepsilon_0^{\lfloor \frac{n}{2b}\rfloor}(\alpha)\cdot\frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}+\varepsilon_0^{n/b-j_0}(\alpha)\right)}\right) \tag{140}$$

□

Explanations for the terms in Theorem 4.2. The privacy bound Theorem 4.2 is strictly smaller than the composition-based privacy bound (derived from DP-SGD[1] and SGM [29] analysis) after $\frac{1}{\lambda\eta}+\frac{n}{b}$ epochs. More specifically, the first term in Equation (12) is strictly smaller than the composition-based privacy bound after $\frac{1}{\lambda\eta}$ epochs, and the second term in Equation (12) is strictly smaller than the composition-based privacy bound after n/b epochs (where $n/b \geq 2$). We explain the terms more specifically as follows.

1. The sensitivity for one mini-batch gradient update in Algorithm 1 is $\eta \cdot \frac{S_g}{b}$, the standard deviation of Gaussian noise added in one update is $\sqrt{2\eta\sigma^2}$, and the sampling probability for each data point is $\frac{b}{n}$. By Abadi et al. [1] and Mironov et al. [29], the composition-based privacy bound for one iteration is larger than $\frac{b^2}{n^2}\frac{\alpha\cdot\eta^2\cdot S_g^2/b^2}{2\cdot 2\eta\sigma^2} = \frac{b^2}{n^2}\frac{\alpha\cdot\eta S_g^2}{4\sigma^2 b^2}$. By Rényi DP composition over n/b iterations, the composition-based privacy bound for one epoch is larger than $\frac{b}{n}\cdot\frac{\alpha\cdot\eta S_g^2}{4\sigma^2 b^2}$.

2. The first term is upper bounded as follows, which is smaller than the composition-based privacy bound after $\frac{1}{\lambda\eta}$ epochs.

$$\varepsilon_0^{\lfloor\frac{n}{2b}\rfloor}(\alpha)\cdot\frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}} \tag{141}$$

$$\leq \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}\cdot(1-\eta\lambda)^{2\cdot(\lfloor\frac{n}{2b}\rfloor-1)}\cdot\frac{1}{\sum_{s=0}^{\lfloor\frac{n}{2b}\rfloor-1}(1-\eta\lambda)^{2s}}\cdot\frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}} \tag{142}$$

$$= \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}\cdot(1-\eta\lambda)^{2\cdot(\lfloor\frac{n}{2b}\rfloor-1)}\cdot\frac{1-(1-\eta\lambda)^2}{1-(1-\eta\lambda)^{2\cdot\lfloor\frac{n}{2b}\rfloor}}\cdot\frac{1-(1-\eta\lambda)^{2\cdot(K-1)\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}} \tag{143}$$

$$\leq \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}\cdot\frac{1}{1-(1-\eta\lambda)^{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)}} \quad \text{(by } (1-\eta\lambda)^x \text{ is monotonically decreasing for } x\in\mathbb{R}) \tag{144}$$

$$\leq \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}\cdot\frac{1}{2\cdot(n/b-\lfloor\frac{n}{2b}\rfloor)\cdot\lambda\eta} \quad \text{(by } (1-x)^a \leq 1-ax \text{ for } x>0 \text{ and } a\geq 1) \tag{145}$$

$$\leq \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}\cdot\frac{1}{n/b\cdot\lambda\eta} \quad \text{(by } \frac{n}{b}\geq 2) \tag{146}$$

3. The second term is upper bounded by $\varepsilon_0^1(\alpha) = \frac{\alpha\eta S_g^2}{4\sigma^2 b^2}$, which is smaller than the composition-based privacy bound after $\cdot\frac{n}{b}$ epochs.

E.2 Proof for Theorem 4.3

Theorem 4.3. If the loss function $\ell(\theta; x)$ is λ-strongly convex, β-smooth, and if its gradient has finite ℓ_2-sensitivity S_g, then Algorithm 1 with stepsize $\eta < \frac{2}{\lambda+\beta}$ satisfies (α, ε)-Rényi DP guarantee with

$$\varepsilon \leq \frac{1}{\alpha - 1} \log \left(S_K^0(\alpha) \right) \tag{147}$$

where the term $S_k^j(\alpha)$ is recursively defined by

$$S_0^0(\alpha) = 1 \tag{148}$$

$$S_k^{j+1}(\alpha) = \frac{b}{n} \cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}} \cdot S_k^j(\alpha) + (1 - \frac{b}{n}) \cdot S_k^j(\alpha)^{(1-\eta\lambda)^2}, \text{ for } k = 0, \cdots, K-1 \text{ and } j = 0, \cdots, n/b - 1 \tag{149}$$

$$S_{k+1}^0(\alpha) = S_k^{n/b}(\alpha) \text{ for } k = 0, \cdots, K-1 \tag{150}$$

$$\tag{151}$$

Proof. By definition,

$$e^{(\alpha-1)\varepsilon_k^{j+1}(\alpha)} = \int \left(\frac{p_k^{j+1}(\theta)}{p'^{j+1}_k(\theta)} \right)^\alpha p'^{j+1}_k(\theta) d\theta \tag{152}$$

$$= \int \left(\frac{\mathbb{E}_{B_0^0, \cdots, B_k^j} \left[p_k^{j+1}(\theta | B_0^0, \cdots, B_k^j) \right]}{\mathbb{E}_{B_0^0, \cdots, B_k^j} \left[p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j) \right]} \right)^\alpha \cdot \mathbb{E}_{B_0^0, \cdots, B_k^j} \left[p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j) \right] d\theta \tag{153}$$

By the joint convexity of the function $\frac{x^\alpha}{y^{\alpha-1}}$ on $x, y > 0$ (Lemma 20 Balle et al. [5]), and by the mini-batch decomposition for the distribution of the last iterate parameters in Section 2, we prove

$$e^{(\alpha-1)\varepsilon_k^{j+1}(\alpha)} \leq \mathbb{E}_{B_0^0, \cdots, B_k^j} \left[\int \left(\frac{p_k^{j+1}(\theta | B_0^0, \cdots, B_k^j)}{p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j)} \right)^\alpha \cdot p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j) d\theta \right] \tag{154}$$

We now derive a recursive scheme for computing the right hand side term denoted as $S_k^{j+1}(\alpha)$. By definition,

$$S_k^{j+1}(\alpha) = \mathbb{E}_{B_0^0, \cdots, B_k^j} \left[\int \left(\frac{p_k^{j+1}(\theta | B_0^0, \cdots, B_k^j)}{p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j)} \right)^\alpha \cdot p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j) d\theta \right] \tag{155}$$

The term inside expectation corresponds to Rényi privacy loss under fixed mini-batches $B_0^0, \cdots, B_k^j$. Therefore, by Lemma 3.2,

$$\int \left(\frac{p_k^{j+1}(\theta | B_0^0, \cdots, B_k^j)}{p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j)} \right)^\alpha \cdot p'^{j+1}_k(\theta | B_0^0, \cdots, B_k^j) d\theta \tag{156}$$

$$\leq \begin{cases} \int \left(\frac{p_k^j(\theta | B_0^0, \cdots, B_k^j)}{p'^j_k(\theta | B_0^0, \cdots, B_k^j)} \right)^\alpha \cdot p'^j_k(\theta | B_0^0, \cdots, B_k^j) d\theta \cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}} & \text{if } i_0 \in B_k^j \\ \left(\int \left(\frac{p_k^j(\theta | B_0^0, \cdots, B_k^j)}{p'^j_k(\theta | B_0^0, \cdots, B_k^j)} \right)^\alpha \cdot p'^j_k(\theta | B_0^0, \cdots, B_k^j) d\theta \right)^{(1-\eta\lambda)^2} & \text{if } i_0 \notin B_k^j \end{cases} \tag{157}$$

By plugging Equation (157), and by the definition of $S_k^{j+1}(\alpha)$, we prove

$$S_k^{j+1}(\alpha) = \mathbb{E}_{B_0^0,\cdots,B_k^j}\left[\int\left(\frac{p_k^{j+1}(\theta|B_0^0,\cdots,B_k^j)}{p'^{j+1}_k(\theta|B_0^0,\cdots,B_k^j)}\right)^\alpha \cdot p'^{j+1}_k(\theta|B_0^0,\cdots,B_k^j)d\theta\right] \tag{158}$$

$$= P(i_0 \in B_k^j)\cdot S_k^j \cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}} + P(i_0 \notin B_k)\cdot \tag{159}$$

$$\mathbb{E}_{B_0^0,\cdots,B_k^{j-1}}\left[\left(\int\left(\frac{p_k^j(\theta|B_0^0,\cdots,B_k^j)}{p'^j_k(\theta|B_0^0,\cdots,B_k^j)}\right)^\alpha \cdot p'^j_k(\theta|B_0^0,\cdots,B_k^j)d\theta\right)^{(1-\eta\lambda)^2}\right] \tag{160}$$

By further using the concavity of $x^{(1-\eta\lambda)^2}$, and the definition of $S_k^j(\alpha)$, we prove

$$S_k^{j+1}(\alpha) \le P(i_0 \in B_k^j)\cdot S_k^j \cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}} + P(i_0 \notin B_k)\cdot \tag{161}$$

$$\left(\mathbb{E}_{B_0^0,\cdots,B_k^{j-1}}\left[\int\left(\frac{p_k^j(\theta|B_0^0,\cdots,B_k^j)}{p'^j_k(\theta|B_0^0,\cdots,B_k^j)}\right)^\alpha \cdot p'^j_k(\theta|B_0^0,\cdots,B_k^j)d\theta\right]\right)^{(1-\eta\lambda)^2} \tag{162}$$

$$= \frac{b}{n}\cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}}\cdot S_k^j + \left(1-\frac{b}{n}\right)\cdot (S_k^j)^{(1-\eta\lambda)^2} \tag{163}$$

Therefore, the full recursive scheme for computing $S_k^j(\alpha)$ is as follows.

1. $S_0^0(\alpha) = 1$
2. $S_k^{j+1}(\alpha) = \frac{b}{n}\cdot e^{\frac{(\alpha-1)\alpha\eta S_g^2}{4\sigma^2 b^2}}\cdot S_k^j(\alpha) + (1-\frac{b}{n})\cdot S_k^j(\alpha)^{(1-\eta\lambda)^2}$, for $k = 0,\cdots,K-1$ and $j = 0,\cdots,n/b-1$.
3. $S_{k+1}^0(\alpha) = S_k^{n/b}(\alpha)$ for $k = 0,\cdots,K-1$.

And Algorithm 1 satisfies (α,ε)-Rényi DP guarantee with

$$\varepsilon \le \frac{1}{\alpha-1}\log\left(S_K^0(\alpha)\right) \tag{164}$$

□

F Proofs and Explanations for Section 5

F.1 Pseudocode for DP-SGD under notations in this paper

In Algorithm 2, we provide the pseudocode of an equivalent of DP-SGD algorithm under notations in our paper. For the convenience of implementation in existing privacy libaries, we introduce the noise multiplier σ_{mul} to substitute σ in Algorithm 1. We comment that the noisy gradient update in Algorithm 2 under noise multiplier σ_{mul}, is equivalent to a noisy gradient update in Algorithm 1 with noise standard deviation $\sigma = \sqrt{\frac{\eta}{2}}\cdot\frac{1}{b}\cdot\sigma_{mul}\cdot\frac{S_g}{2}$. By plugging this equivalent σ into Theorem 4.2, we prove the privacy dynamics bound in Corollary 5.3 for regularized logistic regression.

F.2 Proof for ensuring strong convexity

Regularized Logistic regression (for strong convexity). The loss function for regularized logistic regression in the multi-class setting (with per-class bias) is as follows.

$$\ell_\lambda(\theta;\mathbf{x},\mathbf{y}) = \ell_0(\theta;\mathbf{x},\mathbf{y}) + \frac{\lambda}{2}\|\theta\|_2^2 \tag{165}$$

Algorithm 2 $\mathcal{A}_{\text{implementation}}$: Noisy mini-batch Gradient Descent on regularized logistic regression loss function

Input: Data domain $\mathcal{X}$. Dataset $D = ((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2), \cdots, (\mathbf{x}_n, \mathbf{y}_n))$, where each data point consists of the feature vector $\mathbf{x}_i \in \mathbb{R}^d$ and the label vector $\mathbf{y}_i \in \{0,1\}^c$. The logistic regression loss function $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ defined as Equation (8) with parameter space $\mathbb{R}^{(d+1)\cdot c}$. Stepsize η, noise multiplier σ_{mul}, a (data-independent) parameter initialization distribution $p_0(\theta)$, mini-batch size b, feature clipping norm L, and (unregularized) gradient clipping norm $\frac{S_g}{2}$.

Feature Normalization: $\mathbf{x}_1, \cdots, \mathbf{x}_n \leftarrow$ normalize$(\mathbf{x}_1, \cdots, \mathbf{x}_n)$, where normalize() is an $(\alpha, \varepsilon_{norm})$-Rényi differentially private batch normalization or group normalization scheme described in Tramèr and Boneh [34] (Section 2.3 and Appendix B).

Feature Clipping: $\mathbf{x}_i \leftarrow \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|_2} \cdot \min\{\|\mathbf{x}_i\|_2, L\}$.

Initialization: Sample θ_0^0 from the initialization distribution $p_0(\theta)$.

Batch Generation: shuffle the indices set $\{1, \cdots, n\}$, and partition them into n/b sequential mini-batches $B^0, \cdots, B^{n/b-1}$ that are subsets of $\{1, \cdots, n\}$, each with size b.

for $k = 0, 1, \cdots, K-1$ **do**

  for $j = 0, 1, \cdots, n/b - 1$ **do**

    Gradient Clipping: $g_0(\theta_k^j, B^j) = \frac{1}{b} \sum_{x_i \in B^j} \frac{\nabla \ell(\theta_k^j; \mathbf{x}_i)}{\|\nabla \ell_0(\theta_k^j; \mathbf{x}_i)\|_2} \cdot \min\{\|\nabla \ell_0(\theta_k^j; \mathbf{x}_i)\|_2, \frac{S_g}{2}\}$

    Regularization: $g\left(\theta_k^j; B^j\right) = g_0(\theta_k^j; B^j) + \lambda \cdot \theta_k^j$

    $\theta_k^{j+1} = \theta_k^j - \eta \cdot g\left(\theta_k^j; B^j\right) + \eta \cdot \frac{1}{b} \cdot \sigma_{mul} \cdot \frac{S_g}{2} \cdot \mathcal{N}(0, \mathbb{I}_d)$

  $\theta_{k+1}^0 = \theta_k^{n/b}$

Output θ_K^0

where $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ is the following logistic regression loss function.

$$\ell_0(\theta; \mathbf{x}, \mathbf{y}) = -\mathbf{y}^1 \log\left(\frac{e^{\bar{x}^T \cdot \theta_1}}{e^{\bar{x}^T \cdot \theta_1} + \cdots + e^{\bar{x}^T \cdot \theta_c}}\right) - \cdots - \mathbf{y}^c \log\left(\frac{e^{\bar{x}^T \cdot \theta_c}}{e^{\bar{x}^T \cdot \theta_1} + \cdots + e^{\bar{x}^T \cdot \theta_c}}\right) \quad (166)$$

where $\bar{\mathbf{x}} = (\mathbf{x}, 1) \in \mathbb{R}^{d+1}$ denotes the concatenation of the data feature vector $\mathbf{x}$ and 1, and $\mathbf{y} = (\mathbf{y}^1, \cdots, \mathbf{y}^c)$ is the label vector. The parameter vector is $\theta = (\theta_1, \cdots, \theta_c) \in \mathbb{R}^{(d+1)\cdot c}$ that represents the weight and the per-class bias of the linear model. The logistic regression loss function is convex, and therefore the regularized logistic regression loss function is λ-strongly convex.

F.3 Proof for ensuring smoothness

Feature Normalization (for faster convergence). For a better convergence of the learning task, we follow Tramèr and Boneh [34], and use feature normalization (including both batch normalization and group normalization). Batch normalization first computes the per-channel mean and variance of the training dataset, in a differentially private way, and then normalizes each channel of each data point in the training dataset. Group normalization [38] separates the channels in a data feature vector into a number of different groups, and normalizes each channel of each data point with the per-point per-group mean and variance (which does not incur additional privacy cost). We refer the reader to [34] (their Section 2.2 and Appendix B) for more details.

Feature Clipping (for bounding the smoothness constant). To ensure that the condition of finite gradient sensitivity S_g is satisified in our experiments, we follow Feldman et al. [18] and normalize the data feature vector in ℓ_2 norm, such that $\|\mathbf{x}\|_2 \leq L$. Under this data feature clipping, we prove that the logistic regression loss function (8) is $(\frac{L^2+1}{2})$-smooth in the following Proposition 5.1.

Proposition F.1. *If the data feature vector* $\mathbf{x}$ *has bounded* ℓ_2 *norm, such that* $\|\mathbf{x}\|_2 \leq L$*, then the unregularized logistic regression loss function* $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ *Equation* (8) *is convex , L-Lipschitz and* β*-smooth with regard to parameters* θ*, for*

$$L = \sqrt{2(L^2+1)} \quad (167)$$

$$\beta = \frac{L^2+1}{2} \quad (168)$$

Proof. We compute the gradient and Hessian matrix for the logistic regression loss function as follows.

The gradient of the logistic regression loss function Equation (8) with respect to θ is:

$$\nabla_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) = -\mathbf{y}^1 \cdot \nabla_\theta \log\left(\frac{e^{\bar{\mathbf{x}}^T \cdot \theta_1}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}}\right) - \cdots - \mathbf{y}^c \cdot \nabla_\theta \log\left(\frac{e^{\bar{\mathbf{x}}^T \cdot \theta_c}}{e^{\bar{\mathbf{x}}^T \cdot \theta_c} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}}\right) \tag{169}$$

$$= \left(\left(\frac{e^{\bar{\mathbf{x}}^T \cdot \theta_1}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} - \mathbf{y}^1\right)\bar{\mathbf{x}}^T, \cdots, \left(\frac{e^{\bar{\mathbf{x}}^T \cdot \theta_1}}{e^{\bar{\mathbf{x}}^T \cdot \theta_c} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} - \mathbf{y}^c\right)\bar{\mathbf{x}}^T\right) \tag{170}$$

$$\tag{171}$$

The Hessian of loss function with respect to θ is:

$$\nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) = \begin{pmatrix} H_{11} & \cdots & H_{1c} \\ \vdots & \vdots & \vdots \\ H_{c1} & \cdots & H_{cc} \end{pmatrix} \tag{172}$$

where the submatrices are as follows.

$$\text{For } i = j, \quad H_{ii} = \frac{e^{\bar{\mathbf{x}}^T \cdot \theta_i}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} \cdot \left(1 - \frac{e^{\bar{\mathbf{x}}^T \cdot \theta_i}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}}\right) \bar{\mathbf{x}} \cdot \bar{\mathbf{x}}^T \tag{173}$$

$$\text{For } i \neq j, \quad H_{ij} = -\frac{e^{\bar{\mathbf{x}}^T \cdot \theta_i}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} \cdot \frac{e^{\bar{\mathbf{x}}^T \cdot \theta_j}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} \cdot \bar{\mathbf{x}} \cdot \bar{\mathbf{x}}^T \tag{174}$$

$$\tag{175}$$

Therefore, the Hessian matrix of the loss function with respect to θ equals the following Kronecker product.

$$\nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) = T \otimes (\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}^T) \tag{176}$$

$$\text{where } T_{ij} = \begin{cases} \frac{e^{\bar{\mathbf{x}}^T \cdot \theta_i}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} \cdot \left(1 - \frac{e^{\bar{\mathbf{x}}^T \cdot \theta_i}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}}\right) & i = j \\ -\frac{e^{\bar{\mathbf{x}}^T \cdot \theta_i}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} \cdot \frac{e^{\bar{\mathbf{x}}^T \cdot \theta_j}}{e^{\bar{\mathbf{x}}^T \cdot \theta_1} + \cdots + e^{\bar{\mathbf{x}}^T \cdot \theta_c}} & i \neq j \end{cases} \tag{177}$$

By Böhning [9], T is a positively semi-definite matrix that satisfies

$$T \preceq \frac{1}{2} \cdot \left(I_c - \frac{1}{c} \cdot \mathbf{1}_c \cdot \mathbf{1}_c^T\right), \text{ where } \mathbf{1}_c = (1, \cdots, 1)^T \in \mathbb{R}^c. \tag{178}$$

Therefore, the eigenvalues of T fall in the range $[0, \frac{1}{2}]$. Therefore, because the eigenvalues for the Kronecker product matrix $\nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) = T \otimes \left(\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}^T\right)$ consists of the product of eigenvalues for T and $\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}^T$, we prove that the eigenvalues for $\nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y})$ fall in the range of $[0, \frac{1}{2}\|\bar{\mathbf{x}}\|_2^2]$.

By $\|\mathbf{x}\|_2 \leq L$, we prove that $\|\bar{\mathbf{x}}\|_2 \leq \sqrt{L^2 + 1}$, therefore Equation (9) and Equation (10) (in the proposition statement) hold. □

These computations of these Lipschitz smoothness constants could also be cross-checked in the tensorflow privacy tutorial for logistic regression. `https://github.com/tensorflow/privacy/blob/master/tutorials/mnist_lr_tutorial.py`. The feature clipping technique is different from the DP-SGD [1] algorithm that only requires per-example gradient clipping. The major reason that we use data feature clipping (besides per-example gradient clipping), is for ensuring smoothness of the logistic regression loss function (by Proposition 5.1), which is a necessary condition for applying our privacy bound Theorem 3.3.

F.4 Proof for ensuring finite gradient sensitivity

Per-example Clipping on Unregularized Gradient (for reducing gradient sensitivity without harming smoothness or strong convexity). Although feature clipping already bounds the gradient sensitivity by $2\sqrt{2(L^2+1)}$ (by Proposition 5.1), this bound grows with the feature clipping norm L. This in turn restricts the signal to noise ratio, and does not give good empirical privacy-utility trade-off in our experiments. Therefore, we additionally perform per-example ℓ_2-clipping on the unregularized gradient (detailed pseudocode in Appendix F.1). Under per-example clipping on unregularized gradient, we prove in the following Proposition 5.2, that each gradient update in *regularized logistic regression* has finite gradient sensitivity, and preserves strong convexity and smoothness.

Proposition F.2. *Let $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ be the logistic regression loss function defined in Equation* (8)*. Let $g_0(\theta; \mathbf{x}, \mathbf{y}) = \frac{\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})}{\|\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})\|_2} \cdot \min\{\|\nabla \ell_0(\theta; \mathbf{x}, \mathbf{y})\|_2, \frac{S_g}{2}\}$ be the clipped gradient of (unregularized) loss function $\ell_0(\theta; \mathbf{x}, \mathbf{y})$, under ℓ_2 clipping norm $\frac{S_g}{2}$. If $g(\theta; \mathbf{x}, \mathbf{y}) = g_0(\theta; \mathbf{x}, \mathbf{y}) + \lambda\theta$, and if the data vector $\mathbf{x}$ has bounded ℓ_2 norm, such that $\|\mathbf{x}\|_2 \leq L$, then $g(\theta; \mathbf{x}, y)$ has finite ℓ_2-sensitivity S_g, is continuous, and is almost everywhere differentiable with*

$$\lambda \cdot \mathbb{I}_{(d+1)\cdot c} \preceq \nabla_\theta g(\theta; \mathbf{x}, \mathbf{y}) \preceq (\beta + \lambda) \cdot \mathbb{I}_{(d+1)\cdot c} \tag{179}$$

for any $\theta, \theta' \in \mathbb{R}^{(d+1)\cdot c}$ and $\beta = \frac{L^2+1}{2}$.

We provide complete proof for this proposition below. This construction of clipped unregularized gradient facilitates us to enjoy the benefits of gradient clipping (such as for speeding up convergence [41, 14]) while satsifying the necessary smoothness and strong convexity conditions for applying our privacy dynamics bound.

Proof. By definition Equation (8) for the logistic regression loss function, $\nabla \ell_0(\theta; \mathbf{x}, \mathbf{y})$ is twice continuously differentiable and convex. By Proposition 5.1, $\ell_\lambda(\theta; \mathbf{x}, \mathbf{y})$ is β-smooth with $\beta = \frac{L^2+1}{2}$. Therefore, the Hessian matrix of $\ell_0(\theta; \mathbf{x}, \mathbf{y})$ satisfies the following inequality.

$$0 \cdot \mathbb{I}_{(d+1)\cdot c} \preceq \nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) \preceq \beta \cdot \mathbb{I}_{(d+1)\cdot c}, \tag{180}$$

where d is the dimension of the input data feature vector $\mathbf{x}$, and c is the number of classes in the label vector $\mathbf{y}$. Moreover, the clipped (unregularized) gradient $g_0(\theta; \mathbf{x}, \mathbf{y})$ (under ℓ_2 clipping norm $\frac{S_g}{2}$) is continuous, and is almost everywhere differentiable as follows.

$$\nabla_\theta g_0(\theta; \mathbf{x}, \mathbf{y}) = \begin{cases} \nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) & \text{if } \|\nabla_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y})\|_2 < \frac{S_g}{2} \\ \frac{S_g}{2} \cdot \frac{1}{\|\nabla_\theta \ell_0(\theta;\mathbf{x},\mathbf{y})\|_2} \cdot M \cdot \nabla^2_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y}) & \text{if } \|\nabla_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y})\|_2 > \frac{S_g}{2} \end{cases} \tag{181}$$

where $M = \mathbb{I}_{(d+1)\cdot c} - \frac{\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})}{\|\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})\|_2} \cdot \left(\frac{\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})}{\|\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})\|_2}\right)^T$ is a symmetric matrix. Because $\frac{\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})}{\|\nabla \ell_0(\theta;\mathbf{x},\mathbf{y})\|_2}$ is a unit vector, we prove that M is positive semi-definite and satisfies $0 \cdot \mathbb{I}_{(d+1)\cdot c} \preceq M \preceq \mathbb{I}_{(d+1)\cdot c}$. Moreover, for the case where $\|\nabla_\theta \ell_0(\theta; \mathbf{x}, \mathbf{y})\|_2 > \frac{S_g}{2}$, we have that $\frac{S_g}{2} \cdot \frac{1}{\|\nabla_\theta \ell_0(\theta;\mathbf{x},\mathbf{y})\|_2} \leq 1$. By combining this ineuality with property of M and Equation (180), we prove that for any θ such that $\|\nabla_\theta \ell(\theta; \mathbf{x}, \mathbf{y})\|_2 \neq \frac{S_g}{2}$,

$$0 \cdot \mathbb{I}_{(d+1)\cdot c} \preceq \nabla_\theta g_0(\theta; \mathbf{x}, \mathbf{y}) \preceq \beta \cdot \mathbb{I}_{(d+1)\cdot c} \tag{182}$$

Therefore, by plugging this into the definition of $g(\theta; \mathbf{x}, \mathbf{y}) = g_0(\theta; \mathbf{x}, \mathbf{y}) + \lambda\theta$, we prove that $g(\theta; \mathbf{x}, \mathbf{y})$ is continous, and is almost everywhere differentiable with

$$\lambda \cdot \mathbb{I}_{(d+1)\cdot c} \preceq \nabla_\theta g(\theta; \mathbf{x}, \mathbf{y}) \preceq (\beta + \lambda) \cdot \mathbb{I}_{(d+1)\cdot c} \tag{183}$$

□