Chromatic Correlation Clustering, Revisited

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Abstract

Chromatic Correlation Clustering (CCC) (introduced by Bonchi et al. [6]) is a natural generalization of the celebrated Correlation Clustering (CC) problem. It models objects with categorical pairwise relationships by an edge-colored graph, and has many applications in data mining, social networks and bioinformatics. We show that there exists a 2.5-approximation to the CCC problem based on a Linear Programming (LP) approach, thus improving the best-known approximation ratio of 3 achieved by Klodt et al. [25]. We also present an efficient heuristic algorithm for CCC leveraging a greedy clustering strategy, and conduct extensive experiments to demonstrate the effectiveness and efficiency of our proposed algorithm.

1 Introduction

Clustering is the task of partitioning a set of objects into groups according to their relationships. One of the most important clustering problems is Correlation Clustering (CC), which has attracted great interests in recent years (e.g., [2, 10]). In the CC problem, the input instance is an undirected complete graph with each edge labeled as either similar or dissimilar, and the goal is to partition the graph nodes into an arbitrary number of clusters such that the number of disagreements is minimized, where a "disagreement" occurs if a dissimilar edge links two nodes in the same cluster or a similar edge links two nodes in different clusters. However, CC only models binary relationships, which is insufficient in many applications. For example, the relationships in a social network could have multiple types such as colleague, schoolmate, and family. To address this issue, Bonchi et al. [6] generalize the CC problem to the Chromatic Correlation Clustering (CCC) problem, where an edge-colored graph is used to model the multi-type relationships in practice, i.e., each edge is associated with one color and each color represents a type of relationship. The goal of CCC is to partition the graph nodes into several clusters and assign a color to each cluster to minimize the number of disagreements, where a disagreement happens in any case other than that two non-adjacent nodes belong to different clusters

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or that two adjacent nodes belong to the same cluster with the cluster's color being identical to the color of the edge between these nodes. Due to its generality, CCC has wide applications including link classification, entity de-duplication, mining protein complexes in protein-protein interaction networks, and so on [3, 6, 25].

Related work. The original Correlation Clustering (CC) problem has been extensively studied since the seminal work of Ben-Dor et al. [5]. Bansal et al. [4] proved its NP-hardness and gave the first constant-factor approximation, which was improved to 4 in [9] by rounding the solution to a linear programming relaxation. Ailon et al. [2] proposed a simple linear-time algorithm dubbed Pivot that achieves 3-approximation, and also proposed an LP-rounding based algorithm with an improved approximation ratio of 2.5. Chawla et al. [10] further improved this ratio to 2.06 by using a more sophisticated LP-rounding technique, which nearly matches the known integrality gap of 2 [9] for the LP formulation of CC. Some studies also proposed fast heuristic algorithms for CC [16, 17, 29, 30, 34], but without providing any performance guarantee. Due to the importance of CC, a lot of its variants and related problems have been investigated in the literature [2, 7, 8, 9, 10, 13, 14, 18, 21, 22, 24, 26, 27, 31, 32, 33].

Bonchi et al. [6] initiated the studies on the Chromatic Correlation Clustering (CCC) problem and proposed a heuristic for it without a provable approximation ratio. Anava et al. [3] presented a 4-approximation algorithm for CCC based on LP-rounding, and also provided a more efficient 11-approximation algorithm dubbed Reduce and Cluster (RC) and a heuristic dubbed Deep Cluster (DC). Recently, Klodt et al. [25] showed that the classical Pivot algorithm proposed in Ailon et al. [2] for the CC problem also has an approximation ratio of 3 for CCC, and showed that the RC algorithm proposed in [3] actually has an approximation ratio of 5 for CCC. Although Pivot has the best-known approximation ratio for CCC, it works in a color-blind manner and hence has unsatisfying practical performance. Therefore, Klodt et al. [25] further proposed a heuristic dubbed Greedy Expansion (GE) which has better practical performance but without a performance guarantee. Therefore, it still remains as an open problem to find an approximation ratio for CCC tighter than the 3-approximation achieved by Pivot. CC (or CCC) also bears some similarities to the clique partitioning, multicut and k-center/k-median problems, which have been studied in some excellent proposals (e.g., [11, 15, 19, 20]). However, it is highly non-trivial (if possible) to adapt the solutions of these studies to the CCC problem.

Our contributions. In this paper, we propose a 2.5-approximation to the CCC problem based on an LP approach, improving the best-known ratio of 3 achieved by Klodt et al. [25] and being very close to previously known integrality gap of 2 [9]. We achieve this ratio by using a simple yet effective algorithm based on rounding the optimal solution to a linear programming relaxation of CCC, which is very different from the rounding algorithm proposed in [3]. More specifically, we first classify all vertices of the graph into several groups, with the vertices in each group colored the same, and then cluster the nodes in each group using a procedure similar to the canonical Pivot algorithm. Both the classifying phase and the clustering phase described above use the LP-solution to CCC to ensure guaranteed clustering quality. Although our LP-based clustering algorithm is randomized, it can be de-randomized to achieve a deterministc 2.5-approximation. Due to this novel clustering framework, our performance analysis is more involved than previous work to address the difficulty of counting the number of disagreements in a color-sensitive manner.

We notice that the algorithms proposed by prior studies for CCC with the best empirical performance are heuristics without performance guarantees. Therefore, we also propose a fast heuristic dubbed GreedyVote that adopts a novel greedy strategy roughly explained as follows. GreedyVote creates a cluster by firstly picking two neighboring "seed vertices", and then growing the seed set by letting the seed vertices vote for every unclustered vertex and greedily adding vertices with the largest vote counts into the seed set. We conduct extensive experiments using real-world datasets and the results strongly demonstrate the superiorities of GreedyVote in terms of both clustering quality and efficiency, compared to the state-of-the-art algorithms with or without performance guarantees.

2 Preliminaries

In the Chromatic Correlation Clustering (CCC) problem, an input instance consists of a finite set L of colors (representing positive relationships), a special color $\gamma \notin L$ (representing negative relationship or no relationship), and an undirected complete graph $G = (V, E)$ with a color function $\phi\colon E \mapsto$

$L \cup \{\gamma\}$. The objective is to find a colored partition of the vertex set V that minimizes the number of disagreements. More formally, a solution to this problem is a clustering $S = (\mathcal{C}, \Phi)$ consisting of a partition of the vertices $\mathcal{C}\colon V \mapsto \{C_1, C_2, \dots\}$ and a cluster-coloring function $\Phi\colon range(\mathcal{C}) \mapsto L$. The number of disagreements between G and S can be written as $\sum_{uv \in E} d_{G,S}(uv)$, where

$$d_{G,S}(uv) = \begin{cases} 0 & \phi(uv) = \gamma \wedge \mathcal{C}(u) \neq \mathcal{C}(v), \\ 0 & \phi(uv) \in L \wedge \mathcal{C}(u) = \mathcal{C}(v) \wedge \Phi(\mathcal{C}(u)) = \phi(uv), \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

For simplicity, we use $d(uv)$ as a shorthand for $d_{G,S}(uv)$. There also exists a Linear Programming (LP) relaxation of the chromatic correlation clustering problem [3], i.e., **[CCC-LP]** listed below. Let us consider an integral solution satisfying $x^c_{uv}, x^c_u \in \{0, 1\}$ for all $u \in V$, $uv \in E$, $c \in L$ to **[CCC-LP]**. The variable x^c_u indicates whether vertex u is in a cluster of color $c \in L$ or not (i.e., if $x^c_u = 0$ then u is in the cluster colored by c). Similarly, the variable x^c_{uv} indicates whether two vertices u and v are clustered into the same cluster colored by $c \in L$ (i.e., if $x^c_{uv} = 0$ then u and v belong to the same cluster colored by c). Constraint (4) guarantees that each node belongs to one colored cluster. Constraint (2) implies that the edge uv can be in the cluster colored by $c \in L$ only if both vertex u and v are in this cluster. Constraint (3) ensures that if uv and vw are in the same cluster then the edge wu must also be in the same cluster. According to the definition of the variable x^c_{uv}, we know that an edge $uv \in E$ with color $c \in L$ does not cause a disagreement only if $x^c_{uv} = 0$, and that an edge uv with color γ does not cause a disagreement only if $x^c_{uv} = 1$ for any color $c \in L$. As such, the objective function of **[CCC-LP]** consists of two parts: the first part counts the number of disagreements caused by the edges colored by $c \in L$, and the second part counts the number of disagreements caused by the edges colored by γ. Note that we can omit the variable x^c_u when $|L| = 1$ and then it becomes the same LP-formulation of CC [2], and hence the integrality gap of **[CCC-LP]** is at least 2 [9].

$$\min \sum_{uv \in E\colon \phi(uv) \in L} x^{\phi(uv)}_{uv} + \sum_{uv \in E\colon \phi(uv) = \gamma} \sum_{c \in L} (1 - x^c_{uv}) \qquad \textbf{[CCC-LP]}$$

$$\text{s.t.} \quad x^c_{uv} \geq x^c_u, x^c_v \qquad \forall u, v \in V, \forall c \in L \tag{2}$$

$$x^c_{uv} + x^c_{vw} \geq x^c_{wu} \qquad \forall u, v, w \in V, \forall c \in L \tag{3}$$

$$\sum_{c \in L} x^c_v = |L| - 1 \qquad \forall v \in V \tag{4}$$

$$x^c_{uv} = x^c_{vu} \qquad \forall u, v \in V, \forall c \in L \tag{5}$$

$$x^c_{uv}, x^c_u \in [0, 1] \qquad \forall u, v \in V, \forall c \in L \tag{6}$$

3 An LP-based Approximation Algorithm

In this section, we present an LP-based approximation algorithm as shown in Algorithm 1, namely LP-Clustering. It is randomized and we will show how to de-randomize it in Section C. Intuitively, we aim to round the LP solution x^c_v for all $v \in V, c \in L$ and x^c_{uv} for all $uv \in E, c \in L$ to integers being feasible to the LP problem such that the objective value of the rounded integer solution is close to that of the optimal fractional solution as much as possible. In a nutshell, our LP-Clustering algorithm consists of two phases, including a vertex partitioning phase by rounding x^c_v for all $v \in V, c \in L$ and a clustering phase for each partition by rounding x^c_{uv} for all $uv \in E, c \in L$. The detailed procedures of the two phases are given as follows.

Phase 1 **Partitioning** (Lines 2–4): the vertex set V is partitioned into $|L| + 1$ disjoint subsets such that $S_c = \{v \in V\colon x^c_v < \frac{1}{2}\}$ for each $c \in L$ and $O = V \setminus \left(\bigcup_{c \in L} S_c\right)$. Note that S_c may be an empty set for $c \in L$. And all S_c are determined.

Phase 2 **Clustering** (Lines 5–12): (i) for the partition O, each *outlier* $v \in O$ is assigned to a singleton cluster with an arbitrary color (Lines 5–6); (ii) and for each partition S_c, the algorithm iteratively chooses an unclustered vertex from S_c uniformly at random as the center of a new cluster C_k with color c, and adds each unclustered vertex $u \in S_c$ to C_k with a probability of $(1 - x^c_{uv})$ independently (Lines 7–12).

In the following, we elaborate the intuition. Recall that the variables x^c_v indicates the *impossibility* of vertex v being in a cluster of color c, while x^c_{uv} represents the *distance* between vertices u and v with

Algorithm 1: LP-Clustering

Input: An undirected complete graph $G = (V, E)$ with a color function $\phi\colon E \mapsto L \cup \{\gamma\}$, the LP solution x, i.e., $\{x_v^c \colon \forall v \in V, \forall c \in L\}$ and $\{x_{uv}^c \colon \forall uv \in E, \forall c \in L\}$

Output: A clustering $\{C_1, C_2, \dots\}$ and a cluster-coloring function Φ

Initialize $k \leftarrow 0$ and $O \leftarrow V$;
foreach $c \in L$ **do**
  Let S_c be the subset of nodes in O with value $x_v^c < \frac{1}{2}$, i.e., $S_c \leftarrow \{v \in O \colon x_v^c < \frac{1}{2}\}$;
  Remove S_c from O;
foreach $v \in O$ **do**
  Create a new cluster $C_k = \{v\}$ by increasing k by 1, and set $\Phi(C_k)$ as an arbitrary color;
foreach $c \in L$ **do**
  while $S_c \neq \emptyset$ **do**
    Pick a vertex $v \in S_c$ uniformly at random;
    Create a new cluster $C_k \leftarrow \{v\}$ by increasing k by 1, and set $\Phi(C_k) \leftarrow c$;
    foreach $u \in S_c \setminus \{v\}$ **do** add u to C_k with probability $1 - x_{uv}^c$;
    Remove C_k from S_c;
return $\{C_1, C_2, \dots, C_k\}, \Phi$

respect to color c. Based on this observation, our LP-Clustering algorithm first determines the *vertex color* for each vertex v by rounding x_v^c, i.e., the color of the cluster to which vertex v belongs with the maximum likelihood. That is, v will be assigned to a cluster with color c if $x_v^c < 1/2$, which ensures that $x_v^c \leq x_v^{\bar{c}}$ for any $\bar{c} \in L$ implied by Observation 4.1. This is equivalent to round x_v^c to 0 and $x_v^{\bar{c}}$ to 1 for every $\bar{c} \neq c$. Meanwhile, we call a vertex v *outlier* if $x_v^c \geq 1/2$ for every $c \in L$, which is far away from any other vertex u as to every color c due to the fact that $x_{uv}^c \geq x_v^c \geq 1/2$. Then, for each outlier v, we just arbitrarily choose a color c (e.g., the one with the smallest value of $x_v^{\bar{c}}$) to round x_v^c to 0 and $x_v^{\bar{c}}$ to 1 for every $\bar{c} \neq c$. It is trivial to see that the rounding of x_v^c actually utilizes a greedy strategy such that for each v, the smallest x_v^c is rounded to 0 while the others are rounded to 1.

Next, we finalize the clustering by rounding x_{uv}^c with respect to the rounded value of x_v^c for each $v \in V$ and $c \in L$. Specifically, for each outlier $v \in O$, assigning it to a singleton cluster, i.e., a cluster containing this node only, is equivalent to round x_{uv}^c to 1 for every $u \in V \setminus \{v\}$ and $c \in L$. Since $x_{uv}^c \geq x_v^c \geq 1/2$, such a rounding ensures the corresponding cost of the integer solution within twice of that of the fractional solution. Moreover, for the subset S_c of V containing vertices with color c, we iteratively select an unclustered vertex v from S_c uniformally at random and create a new cluster $C_k = \{v\}$, until every vertex in S_c is included in some cluster. Then, we add each unclustered vertex $u \in S_c$ to C_k with a probability of $(1 - x_{uv}^c)$, i.e., rounding x_{uv}^c to 0 with a probability of $(1 - x_{uv}^c)$ and to 1 with the other probability so that the rounded x_{uv}^c in expectation is exactly x_{uv}^c. Such a rounding again ensures the corresponding cost of the integer solution within a constant factor of that of the fractional solution.

In the following theorem, we give our main result that the LP-Clustering algorithm achieves a 2.5-approximation in expectation for the Chromatic Correlation Clustering (CCC) problem, which improves the best-known factor of 3 achieved by Klodt et al. [25].

Theorem 3.1. *The LP-Clustering algorithm achieves an approximation of* 2.5 *in expectation for the CCC problem.*

Next in Section 4, we prove our main theoretical result in Theorem 3.1, where the full proof of a core lemma (i.e., Lemma 4.3) requires a non-trivial analysis and can be found in Appendix A.

4 Performance Analysis

4.1 Overview

For our randomized rounding approach based on the LP solution, the expected total number of disagreements is $\mathbb{E}[\sum_{uv \in E} d(uv)]$, where $d(uv)$ is defined in (1). Similarly, the objective function in

[CCC-LP] can be written as $\sum_{uv\in E} lp(uv)$, where $lp(uv)$ denotes the cost on edge uv, i.e.,

$$lp(uv) := \begin{cases} x_{uv}^{\phi(uv)}, & \text{if } \phi(uv) \in L, \\ \sum_{c\in L}(1 - x_{uv}^{c}) & \text{if } \phi(uv) = \gamma. \end{cases} \tag{7}$$

Therefore, to prove Theorem 3.1, it suffices to show that

$$\mathbb{E}\Big[\sum_{uv\in E} d(uv)\Big] \le \frac{5}{2} \cdot \sum_{uv\in E} lp(uv). \tag{8}$$

Since the cost is characterized by each edge uv, according to the partitioning of $V = O \cup \big(\bigcup_{c\in L} S_c\big)$, we partition the edge set E into $|L| + 2$ disjoint subsets E_o, E_i and E_c for $c \in L$, i.e.,

$$\begin{aligned} E_o &:= \{uv \in E \colon v \in O\}, \\ E_i &:= \{uv \in E \colon u \in S_c, v \in S_{\bar{c}}, c \neq \bar{c}\}, \\ E_c &:= \{uv \in E \colon u \in S_c, v \in S_c\},\ \forall c \in L. \end{aligned}$$

Clearly, $E = E_o \cup E_i \cup (\bigcup_{c\in L} E_c)$. In particular, E_o is the set of edges incident on outliers in O; E_i is the set of edges between non-outliers cross two vertex partitions S_c and $S_{\bar{c}}$; and E_c is the set of edges between vertices within the same vertex partition S_c. In what follows, we establish the relation between LP-Clustering and LP (i.e., the objective function of **[CCC-LP]**) in terms of the cost on different category of edge.

For each edge $uv \in E_o$, it is an inter-cluster edge. By (1), we can get that $d(uv) = 1$ if $\phi(uv) \in L$ and $d(uv) = 0$ otherwise. Meanwhile, constraint (2) ensures that $x_{uv}^{c} \ge x_{v}^{c} \ge \frac{1}{2}$ for every $u \neq v$ and $c \in L$ when v is an outlier. This implies that $lp(uv) = x_{uv}^{\phi(uv)} \ge \frac{1}{2}$ if $\phi(uv) \in L$ and $lp(uv) \ge 0$ otherwise. We immediately have $d(uv) \le 2 \cdot lp(uv)$.

For each edge $uv \in E_i$, it is again an inter-cluster edge, indicating that $d(uv) = 1$ if $\phi(uv) \in L$ and $d(uv) = 0$ otherwise. To derive $lp(uv)$, we first present a useful property of the fractional solution.

Observation 4.1. *Given a vertex $v \in V$ and any two distinct colors $c_1, c_2 \in L$ then $x_v^{c_1} + x_v^{c_2} \ge 1$.*

In fact, Observation 4.1 can be easily obtained by contradiction, since otherwise $\sum_{c\in L} x_v^c < |L| - 1$, which contradicts constraint (4). Denote by c_u the vertex color of u such that $x_u^{c_u} < \frac{1}{2}$. Then, by Observation 4.1, we have $x_u^c \ge \frac{1}{2}$ for any $c \neq c_u$. As a consequence, $lp(uv) = x_{uv}^{\phi(uv)} \ge x_u^{\phi(uv)} \ge \frac{1}{2}$ if $\phi(uv) \in L \setminus \{c_u\}$, $lp(uv) = x_{uv}^{\phi(uv)} \ge x_v^{c(u)} \ge \frac{1}{2}$ if $\phi(uv) = c_u \neq c_v$, and $lp(uv) \ge 0$ otherwise. This concludes that $d(uv) \le 2 \cdot lp(uv)$.

Finally, if we show that $\mathbb{E}[\sum_{uv\in E_c} d(uv)] \le \alpha \cdot \sum_{uv\in E_c} lp(uv)$ for every $c \in L$ and some $\alpha \ge 2$, combing with the above analysis, we immediately derive an upper bound on the expected cost of our output clustering, i.e.,

$$\begin{aligned} \mathbb{E}\Big[\sum_{uv\in E} d(uv)\Big] &= \sum_{uv\in E_o} d(uv) + \sum_{uv\in E_i} d(uv) + \sum_{c\in L} \mathbb{E}\Big[\sum_{uv\in E_c} d(uv)\Big] \\ &\le 2 \cdot \sum_{uv\in E_o} lp(uv) + 2 \cdot \sum_{uv\in E_i} lp(uv) + \alpha \cdot \sum_{c\in L} \sum_{uv\in E_c} lp(uv) \\ &\le \alpha \cdot \sum_{uv\in E} lp(uv). \end{aligned}$$

We thus obtain the following lemma.

Lemma 4.1. *Given $\alpha \ge 2$, if the clustering returned by LP-Clustering satisfies for every $c \in L$,*

$$\mathbb{E}\Big[\sum_{uv\in E_c} d(uv)\Big] \le \alpha \cdot \sum_{uv\in E_c} lp(uv),$$

then LP-Clustering achieves an approximation guarantee of α for chromatic correlation clustering.

Therefore, to get the approximation ratio of LP-Clustering, by Lemma 4.1, we just need to show the relation between $\mathbb{E}[\sum_{uv\in E_c} d(uv)]$ and $\sum_{uv\in E_c} lp(uv)$ for each color $c \in L$.

4.2 More Detailed Analysis

In what follows, we focus on the elementary analysis over E_c for every $c \in L$. For each given S_c, LP-Clustering iteratively chooses some vertices to create a new cluster. Denote by S_c^t the unclustered vertex set at the beginning of iteration t, by E_c^t the induced edge set by S_c^t containing all the edges with both endpoints in S_c^t, and by C_c^t the cluster created in this iteration. With removing C_c^t from S_c^t, the edges incident on vertices in C_c^t, i.e., $E_c^t \setminus E_c^{t+1} = \{uv \in E_c \colon u, v \in S_c^t, (u \in C_c^t \vee v \in C_c^t)\}$, are also removed from E_c. Then, the number of disagreements of these edges (i.e., $E_c^t \setminus E_c^{t+1}$), denoted by D_t, can be derived by

$$D_t = \sum_{uv \in E_c^t \colon \phi(uv)=c} \mathbb{1}(\mathcal{E}_{u-v}^t) + \sum_{uv \in E_c^t \colon \phi(uv) \in L \setminus \{c\}} \big(\mathbb{1}(\mathcal{E}_{u-v}^t) + \mathbb{1}(\mathcal{E}_{u+v}^t)\big) + \sum_{uv \in E_c^t \colon \phi(uv)=\gamma} \mathbb{1}(\mathcal{E}_{u+v}^t),$$

where $\mathbb{1}(\mathcal{E})$ denotes the indicator function of the event $\mathcal{E}$, $\mathcal{E}_{u-v}^t$ denotes the event that only one of u and v is in C_c^t, and $\mathcal{E}_{u+v}^t$ denotes the event that both u and v are in C_c^t. Similarly, the LP cost on E_c^t, denoted by L_t, can be derived by

$$L_t = \sum_{uv \in E_c^t \colon \phi(uv) \in L} \big(\mathbb{1}(\mathcal{E}_{u-v}^t) + \mathbb{1}(\mathcal{E}_{u+v}^t)\big) x_{uv}^{\phi(uv)} + \sum_{uv \in E_c^t \colon \phi(uv)=\gamma} \big(\mathbb{1}(\mathcal{E}_{u-v}^t) + \mathbb{1}(\mathcal{E}_{u+v}^t)\big) \sum_{i \in L} (1 - x_{uv}^i).$$

Let T be the stopping time such that all vertices in S_c are clustered after T iterations. Then, we know that $\sum_{t=0}^T D_t = \sum_{uv \in E_c} d(uv)$ and $\sum_{t=0}^T L_t = \sum_{uv \in E_c} lp(uv)$, since each edge in E_c can only be removed exactly once. If we show that $\mathbb{E}[D_t] \le \alpha \mathbb{E}[L_t]$ for all t, then we know that $Z_s = \sum_{t=0}^s (\alpha L_t - D_t)$ is a submartingale (i.e., $\mathbb{E}[Z_{s+1} \mid Z_s] \ge Z_s$) and hence $\mathbb{E}[Z_T] \ge \mathbb{E}[Z_0]$, which yields that

$$\alpha \sum_{uv \in E_c} lp(uv) - \mathbb{E}\Big[\sum_{uv \in E_c} d(uv)\Big] = \mathbb{E}[Z_T] \ge \mathbb{E}[Z_0] = \alpha \mathbb{E}[L_0] - \mathbb{E}[D_0] \ge 0.$$

To get $\mathbb{E}[D_t] \le \alpha \mathbb{E}[L_t]$, it suffices to show its conditional version that $\mathbb{E}[D_t \mid S_c^t] \le \alpha \mathbb{E}[L_t \mid S_c^t]$ for all $S_c^t \subset S_c$. Given S_c^t, each vertex $w \in S_c^t$ is picked as the center of the cluster C_c^t with a probability of $\frac{1}{|S_c^t|}$. Thus, given an event $\mathcal{E}$, we have

$$\mathbb{E}[\mathbb{1}(\mathcal{E}) \mid S_c^t] = \Pr[\mathcal{E} \mid S_c^t] = \frac{1}{|S_c^t|} \sum_{w \in S_c^t} \Pr[\mathcal{E} \mid center = w, S_c^t].$$

For simplicity, let

$$d(uv \mid w) := \begin{cases} \Pr[\mathcal{E}_{u-v}^t \mid center = w, S_c^t], & \text{if } \phi(uv) = c, \\ \Pr[\mathcal{E}_{u-v}^t \mid center = w, S_c^t] + \Pr[\mathcal{E}_{u+v}^t \mid center = w, S_c^t], & \text{if } \phi(uv) \in L \setminus \{c\}, \\ \Pr[\mathcal{E}_{u+v}^t \mid center = w, S_c^t], & \text{if } \phi(uv) = \gamma. \end{cases}$$

Moreover, according to the definitions of $\mathcal{E}_{u-v}^t$ and $\mathcal{E}_{u+v}^t$, we have

$$\Pr[\mathcal{E}_{u-v}^t \mid center = w, S_c^t] = (1 - x_{vw}^c) x_{uw}^c + (1 - x_{uw}^c) x_{vw}^c,$$
$$\Pr[\mathcal{E}_{u+v}^t \mid center = w, S_c^t] = (1 - x_{uw}^c)(1 - x_{vw}^c).$$

Note that by setting $x_{uu}^c = 0$ for each $u \in S_c^t$, the above equations also hold when $w = u$ or $w = v$. As a consequence, we have

$$d(uv \mid w) = \begin{cases} (1 - x_{vw}^c) x_{uw}^c + (1 - x_{uw}^c) x_{vw}^c, & \text{if } \phi(uv) = c, \\ 1 - x_{uw}^c x_{vw}^c, & \text{if } \phi(uv) \in L \setminus \{c\}, \\ (1 - x_{uw}^c)(1 - x_{vw}^c), & \text{if } \phi(uv) = \gamma. \end{cases}$$

Putting it together, we can express $\mathbb{E}[D_t \mid S_c^t]$ as

$$\mathbb{E}[D_t \mid S_c^t] = \frac{1}{|S_c^t|} \sum_{w \in S_c^t} \sum_{uv \in E_c^t} d(uv \mid w) = \frac{1}{2|S_c^t|} \sum_{u,v,w \in S_c^t} d(uv \mid w),$$

where $d(uu \mid w) := 0$ for all $u, w \in S_c^t$. Similarly, let

$$lp(uv \mid w) := \begin{cases} (1 - x_{uw}^c x_{vw}^c) x_{uv}^{\phi(uv)}, & \text{if } \phi(uv) \in L, \\ (1 - x_{uw}^c x_{vw}^c) \sum_{i \in L} (1 - x_{uv}^i), & \text{if } \phi(uv) = \gamma. \end{cases}$$

Algorithm 2: GreedyVote

Input: An undirected graph $G = (V, E)$ with a color function $\phi : E \to L$, parameters ϵ and m.
Output: A clustering $\{S_1, S_2, \ldots, S_k\}$ and a cluster-coloring function Φ.

```
Initialize k = 0;
while E ≠ ∅ do
    Let M be a set of m edges sampled with replacement from E;
    Let uv be the edge in M that maximizes |N_φ(uv)(u) ∩ N_φ(uv)(v)| / |N(u) ∪ N(v)|;
    Create a new cluster S_k = {u, v} by increasing k by one, and remove S_k from V;
    while V ≠ ∅ do
        Let each vertex in S_k vote for every vertex in V such that vote 0.5 when there is an edge
          between the two vertices, and an extra 0.5 if the edge color is φ(uv);
        Let w ∈ V be the vertex with the largest vote count and let ℓ be the vote count of w;
        if ℓ ≥ ε|S_k| then Add w to S_k and remove w from V;
        else Break;
    Let c be the most frequent color of the edges with both endpoints in S_k, and set Φ(S_k) = c;
    Remove all edges in {uv ∈ E | u ∈ S_k} from E;
foreach v ∈ V do
    Create a new cluster S_k = {v} by increasing k by one;
    Let Φ(S_k) = c where c is an arbitrary color in L;
return {S_1, S_2, ..., S_k}, Φ
```

Then, $\mathbb{E}[L_t \mid S_c^t]$ can be expressed as

$$\mathbb{E}[L_t \mid S_c^t] = \frac{1}{|S_c^t|} \sum_{w \in S_c^t} \sum_{uv \in E_c^t} lp(uv \mid w) = \frac{1}{2|S_c^t|} \sum_{u,v,w \in S_c^t} lp(uv \mid w),$$

where $lp(uu \mid w) := 0$ for all $u, w \in S_c^t$. Observing that $d(uv \mid w) \leq \alpha \cdot lp(uv \mid w)$ does not always hold, thanks to the symmetry, we consider $D(uvw) := d(uv \mid w) + d(vw \mid u) + d(wu \mid v)$ as a whole. That is,

$$\mathbb{E}[D_t \mid S_c^t] = \frac{1}{6|S_c^t|} \sum_{u,v,w \in S_c^t} \big(d(uv \mid w) + d(vw \mid u) + d(wu \mid v)\big) = \frac{1}{6|S_c^t|} \sum_{u,v,w \in S_c^t} D(uvw).$$

Similarly, letting $L(uvw) := lp(uv \mid w) + lp(vw \mid u) + lp(wu \mid v)$, we have

$$\mathbb{E}[L_t \mid S_c^t] = \frac{1}{6|S_c^t|} \sum_{u,v,w \in S_c^t} \big(lp(uv \mid w) + lp(vw \mid u) + lp(wu \mid v)\big) = \frac{1}{6|S_c^t|} \sum_{u,v,w \in S_c^t} L(uvw).$$

Therefore, it suffices to show that $D(uvw) \leq \alpha L(uvw)$. We thus obtain the following lemma.

Lemma 4.2. *Given a vertex set S_c, if it always holds that $D(uvw) \leq \alpha L(uvw)$ for all $u, v, w \in S_c^t$, we have $\mathbb{E}[\sum_{uv \in E_c} d(uv)] \leq \alpha \cdot \sum_{uv \in E_c} lp(uv)$*

Finally, the following lemma gives such a relation.

Lemma 4.3. *For all $c \in L$ and $u, v, w \in S_c^t$, we have $D(uvw) \leq 2.5L(uvw)$.*

The proof of Lemma 4.3 is based on a nontrivial case analysis with respect to the the edge color of $\phi(uv)$, $\phi(vw)$ and $\phi(wu)$, which can be found in Appendix A. Now, putting it together of Lemmas 4.1–4.3 concludes Theorem 3.1.

5 A Greedy Clustering Heuristic

For the CCC problem, prior studies have shown that some heuristics without any performance guarantees (e.g., the Deep Cluster algorithm in [3] and the Greedy Expansion algorithm in [25]) can empirically perform much better than the existing algorithms with provable approximation ratios including the Pivot algorithm with the best-known ratio of 3. This motivates us to design a fast heuristic with better performance in practice, namely the GreedyVote algorithm shown in Algorithm 2. Roughly speaking, GreedyVote repeats in iterations and creates a cluster S_k in the kth iteration according to the following two phases of operations:

Phase 1 **Seed selection** (Lines 3–5): Randomly select m candidate edges and pick one of them (denoted by uv) to create a "seed set" $S_k = \{u, v\}$;

Phase 2 **Vote to grow** (Lines 6–10): Each vertex in the seed set S_k votes for (or scores) every currently unclustered vertex, and the vertex w with the largest vote count is added into S_k if satisfying the condition in Line 9. Repeat this process until there are no qualified vertices.

In the sequel, we provide more detailed explanations on the two phases mentioned above. The purpose of the "seed selection" phase is to identify two neighboring nodes that are "most likely" to be in the same cluster of a good solution. To achieve this, we use a metric inspired by the well-known Jaccard Similarity to measure the goodness of u and v being clustered together, i.e., $\frac{|N_{\phi(uv)}(u) \cap N_{\phi(uv)}(v)|}{|N(u) \cup N(v)|}$ in Line 4, where $N(u)$ denotes the set of all neighbors of u and $N_c(u) : \forall c \in L$ denotes the set of vertices adjacent to u by c-colored edges. Instead of checking every pair of neighboring nodes for this metric, we randomly check m edges and pick the best one among them to save running time.

Given the initial seed set constructed by Phase 1, the purpose of the "vote to grow" phase is to identify some qualified vertices and add them into S_k one by one. The rational for judging whether a candidate vertex w should be added into S_k or not is roughly explained as follows. Let $|E_c(S_k, w)|$ denote the number of edges with color c ($\forall c \in L$) linking w and any vertex in S_k, and let $error(S_k)$ denote the number of disagreements caused by edges with both endpoints in S_k. Let u and v be the initial two seed vertices in S_k found by Line 4, and suppose that S_k is assigned the color of $\phi(uv)$. Then, not adding w into S_k would cause at least $Er_1 = error(S_k) + \sum_{c \in L} |E_c(S_k, w)|$ disagreements, while adding w into S_k would bring $Er_2 = error(S_k) + |S_k| - |E_{\phi(uv)}(S_k, w)|$ disagreements. Clearly, adding w into S_k would be beneficial only when $Er_2 - Er_1 = |S_k| - |E_{\phi(uv)}(S_k, w)| - \sum_{c \in L} |E_c(S_k, w)| \leq 0$, or equivalently $|E_{\phi(uv)}(S_k, w)| + \sum_{c \in L} |E_c(S_k, w)| \geq |S_k|$. Actually, it is not hard to verify that the voting mechanism in Line 7 results in the vote count of any candidate vertex w being equal to $0.5|E_{\phi(uv)}(S_k, w)| + 0.5 \sum_{c \in L} |E_c(S_k, w)|$, while we replace $|S_k|$ by $\epsilon|S_k|$ in the if-condition of Line 9 to offset the inaccuracy of counting the number of disagreements by considering every vertex individually.

When Phase 1 and Phase 2 described above finishes, no qualified vertices can be added into the current clusters. So GreedyVote creates a cluster for each leftover vertex and assign it an arbitrary color in L (Lines 13-15). It can be seen that GreedyVote can be implemented to run in $\mathcal{O}(|E| + \Delta|V|)$ time for any constant m, where Δ is the maximum degree of G (see Appendix B for a formal proof).

6 Performance Evaluation

In this section, we compare our GreedyVote (GV) algorithm (i.e., Algorithm 2) with six state-of-the-art algorithms for the CCC problem, including: (1) The Pivot Algorithm [2]; (2) The Reduce and Cluster (RC) Algorithm [3]; (3) The Deep Cluster (DC) algorithm [3]; (4) The Chromatic Ball (CB) algorithm [6]; (5) The Greedy Expansion (GE) Algorithm [25]; and (6) The modified greedy expansion (GER) Algorithm [25]. Among these algorithms, Pivot and RC have approximation ratios of 3 and 5, respectively, and all other algorithms are heuristics. In our experiments, we set $\epsilon = 0.55$ and $m = 10$ for the GV algorithm, and follow all the parameter settings (if any) of other algorithms according to their proposals. We implement the evaluated algorithms using Python and also use the public code of Klodt et al. [25] for implementation. All our experiments are conducted on an Intel(R) Xeon(R) Gold 6126 CPU @ 2.60GHz with 128GB of RAM. Each algorithm is run for 50 times and the average result is reported.

We use ten real-world datasets from different domains in our experiments, as listed in Table 1. Among these datasets, Facebook1, Facebook2, Lastfm and Twitter are social network graphs downloaded from [28], where vertices represent users and edges represent friendships. Each user has several properties representing the "social circles" interested by the user. Following the setting in Anava et al. [3], we assign each social circle a color and set the color of each edge uv as the color of u and v's common social circle (if there are multiple common circles then choose an arbitrary one). The DAWN dataset [1][25] models a drug abuse warning network where vertices represent drugs and edges indicate that they were used together before an emergency room visit. Following [25] , the color of the edge represents the most common outcome of the visit. The Cooking dataset [23][25] models an ingredient co-occur network where vertices represent ingredients and edges indicate that they co-occur in some recipes. Following [25], the color of the edge represents the most common

Table 1: Characteristics of the datasets used in the experiments.

Datasets	$\|V\|$	$\|E\|$	$\|L\|$
Facebook1	2871	62334	193
Facebook2	3978	78011	46
Lastfm	7624	27806	10
Twitter	74402	1003951	99
DAWN	2109	96047	10
Cooking	6714	479921	21
String1	17458	419190	4
String2	18152	401582	4
DBLP	73624	835414	100
MAG	80198	237261	11

Table 2: Compare all algorithms on the Number of Disagreements (NOD), where the reported data of all algorithms except GV are normalized by those of GV. Lowest values are highlighted in bold.

Datasets	**Pivot** [2]	**RC** [3]	**DC** [3]	**CB** [6]	**GE** [25]	**GER** [25]	**GV (ours)**
Facebook1	1.486	1.283	1.218	1.246	1.015	1.022	**38,773**
Facebook2	1.419	1.305	1.236	1.220	1.015	1.025	**52,164**
Lastfm	1.442	1.267	1.267	1.056	1.013	1.034	**23,046**
Twitter	1.449	1.138	1.134	1.050	1.004	1.007	**925,028**
DAWN	1.460	1.236	1.253	1.115	**0.995**	1.035	84,492
Cooking	1.306	1.110	1.092	1.039	1.004	1.001	**456435**
String1	1.535	1.219	1.229	1.072	1.018	1.038	**359,202**
String2	1.424	1.409	1.214	1.429	1.018	1.017	**112,340**
DBLP	1.563	1.143	1.103	1.141	1.014	1.040	**540,683**
MAG	1.568	1.294	1.286	1.078	1.021	1.040	**127,383**

cuisine. String1 and String2 [12] are networks containing protein-protein interactions, where vertices represent proteins and edges represent pairs of interacting proteins. Following [6][25], we assign each protein-interaction a color and choose an arbitrary color for an edge if there are multiple interactions between two proteins. Finally, DBLP and MAG [25] are co-authorship networks where vertices represent authors and each edge is labeled (colored) by the most frequent journal or conference that two authors co-authored.

In Table 2, we list the Number of Disagreements (NOD) of all the evaluated algorithms, where the reported data of all algorithms except GV are normalized by those of GV. It can be seen that Pivot performs the worst, as it works in a color-blind way. Although RC has a worse approximation ratio than Pivot, it performs better than Pivot but worse than CB and DC, while CB and DC have similar performance on NOD. The GER algorithm outperforms the Pivot, RC, CB and DC algorithms on NOD, while GE even performs better than GER. However, it can be seen that our GV algorithm outperforms the best baseline GE in the literature on all the datasets except DAWN, while the gap between GV and GE is no more than 0.5% on the DAWN dataset. This demonstrates the superiority of our GV algorithm on the metric of NOD.

In Table 3, we compare the running time of the implemented algorithms. It can be seen that Pivot and GE generally incur the shortest and longest running time, respectively. Compared to GE, GV runs faster on all datasets except String2, MAG and Cooking, with the performance gain ranging from 11.0% to 78.8%, while the average performance gain is 27%. This can be explained by the fact that GV has a better time complexity of $\mathcal{O}(|E| + \Delta|V|)$ than the $\mathcal{O}(\Delta_L|E| + \Delta|V|)$ time complexity of GE, where Δ_L is the maximum number of distinct colors of the edges incident to any vertex.

In summary, the experimental results demonstrate that, our GreedyVote algorithm outperforms all the implemented state-of-the-art algorithms (with or without performance guarantee) in terms of the objective function value of chromatic correlation clustering, and it also runs significantly faster than GE in general, where GE is the heuristic proposed by Klodt et al. [25] with the best-known empirical performance on clustering quality.

Table 3: Compare all algorithms on the running time (seconds).

Datasets	**Pivot** [2]	**RC** [3]	**DC** [3]	**CB** [6]	**GE** [25]	**GER** [25]	**GV (ours)**
Facebook1	0.06	0.43	0.64	0.08	0.86	0.80	0.54
Facebook2	0.07	0.55	0.89	0.10	0.93	1.04	0.66
Lastfm	0.03	0.25	0.34	0.06	0.54	0.52	0.41
Twitter	0.87	6.71	8.78	2.16	17.27	10.75	15.00
DAWN	0.06	0.57	1.16	0.12	1.31	1.01	1.18
Cooking	0.24	3.28	4.92	1.17	8.73	4.27	9.54
String1	0.23	2.77	4.72	0.78	8.99	5.90	5.85
String2	0.57	3.10	3.35	0.60	2.97	5.36	3.37
DBLP	0.97	6.78	7.03	2.15	18.81	9.74	10.52
MAG	0.48	2.54	2.90	0.66	3.96	4.53	3.99

7 Conclusion

We have revisited the Chromatic Correlation Clustering (CCC) problem, which generalizes the correlation clustering problem and has many important applications in machine learning and data mining. For this problem, prior studies have proposed both approximation algorithms with provable performance ratios and heuristics with better practical performance. We have proposed a 2.5-approximation to CCC based on rounding the solution to a linear programming relaxation of CCC, and also a heuristic algorithm based on a "greedy voting" strategy. Our theoretical analysis and extensive experiments on real-world datasets demonstrate that our approach outperforms the existing ones both on the theoretical approximation ratio and on the practical performance in terms of time efficiency and clustering quality.

Acknowledgement

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] See the Abstract and Section 1
   (b) Did you describe the limitations of your work? [Yes] See the Abstract and Section 6
   (c) Did you discuss any potential negative societal impacts of your work? [No]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See section 2
   (b) Did you include complete proofs of all theoretical results? [Yes] See section 4 and the supplemental material
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See the supplemental material
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] see Section 6
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [Yes]
(b) Did you mention the license of the assets? [Yes]
(c) Did you include any new assets either in the supplemental material or as a URL? [Yes]
(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [Yes]
(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [Yes] We used public datasets

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

Appendix

A Proof of Lemma 4.3

We show that $D(uvw) \leq 2.5L(uvw)$ for all $c \in L$ and $u, v, w \in S_c^t$ by enumerating all possible combinations of $\big(\phi(uv), \phi(vw), \phi(wu)\big)$. Note that due to symmetry, we do not consider the order of $\big(\phi(uv), \phi(vw), \phi(wu)\big)$.

Without loss of generality, we first consider the special case of $u = v$. Observing that $D(uuu) = L(uuu) = 0$, so we consider $w \neq u$. By definition, it is easy to obtain that

$$D(uuw) = 2d(uw \mid u) = \begin{cases} 2x_{uw}^c, & \text{if } \phi(uw) = c, \\ 2, & \text{if } \phi(uv) \in L \setminus \{c\}, \\ 2(1 - x_{uw}^c), & \text{if } \phi(uw) = \gamma, \end{cases}$$

$$L(uuw) = 2lp(uw \mid u) = \begin{cases} 2x_{uv}^{\phi(uw)}, & \text{if } \phi(uw) \in L, \\ 2\sum_{i \in L}(1 - x_{uw}^i), & \text{if } \phi(uw) = \gamma. \end{cases}$$

Thus, it is trivial to see that $D(uuw) \leq L(uuw)$ if $\phi(uw) = c$ or $\phi(uw) = \gamma$. If $\phi(uw) \in L \setminus \{c\}$, then by Observation 4.1 we know that $x_{uw}^{\phi(uw)} \geq \frac{1}{2}$, which implies $D(uuw) \leq 2L(uuw)$.

Now, it remains to consider the general case that u, v, w are distinct vertices. In the following, we give a useful property of the fractional solution to **[CCC-LP]**.

Observation A.1. *Given three distinct vertices $u, v, w \in S_c$ and a color $\bar{c} \in L \setminus \{c\}$, then $x_{uv}^{\bar{c}} \geq \max\{\frac{1}{2}, 1 - x_{uv}^c, 1 - x_{vw}^c, 1 - x_{wu}^c\}$.*

Proof. By the definition of S_c, Observation 4.1 and constraint (2), we have

$$\begin{aligned} &x_{uv}^{\bar{c}} \geq x_u^{\bar{c}} \geq 1 - x_u^c > \frac{1}{2}, \\ &x_{uv}^{\bar{c}} + x_{uv}^c \geq x_u^{\bar{c}} + x_u^c \geq 1, \\ &x_{uv}^{\bar{c}} + x_{vw}^c \geq x_v^{\bar{c}} + x_v^c \geq 1, \\ &x_{uv}^{\bar{c}} + x_{wu}^c \geq x_u^{\bar{c}} + x_u^c \geq 1. \end{aligned}$$

Rearranging them completes the proof. □

For brevity, we use $x_1 = x_{uv}^c, x_2 = x_{vw}^c, x_3 = x_{wu}^c$ and $m = \max\{\frac{1}{2}, 1 - x_1, 1 - x_2, 1 - x_3\}$, and refer to the triangle $\big(\phi(uv), \phi(vw), \phi(wu)\big)$ as (c_1, c_2, c_3) where c_i is one of the colors in $\{c, \bar{c}, \gamma\}$ and $\bar{c}$ is any color in $L \setminus \{c\}$. In what follows, we investigate all combinations of (c_1, c_2, c_3).

A.1 (γ, γ, γ) Triangles

By definition,

$$D(uvw) = (1-x_1)(1-x_2) + (1-x_2)(1-x_3) + (1-x_3)(1-x_1).$$

Meanwhile,

$$\begin{aligned}
L(uvw) &= (1-x_1x_2)\sum_{i\in L}(1-x^i_{wu}) + (1-x_2x_3)\sum_{i\in L}(1-x^i_{uv}) + (1-x_1x_3)\sum_{i\in L}(1-x^i_{vw}) \\
&\geq (1-x_1x_2)(1-x_3) + (1-x_2x_3)(1-x_1) + (1-x_1x_3)(1-x_2) \\
&\geq (1-x_1)(1-x_3) + (1-x_2)(1-x_1) + (1-x_3)(1-x_2) \\
&= D(uvw).
\end{aligned}$$

A.2 (γ, γ, c) Triangles

We have

$$\begin{aligned}
D(uvw) &= (1-x_2)(1-x_3) + (1-x_1)(1-x_3) + (1-x_1)x_2 + (1-x_2)x_1, \\
\text{and} \quad L(uvw) &= (1-x_2x_3)\sum_{i\in L}(1-x^i_{wu}) + (1-x_1x_3)\sum_{i\in L}(1-x^i_{uv}) + (1-x_1x_2)x_3 \\
&\geq (1-x_2x_3)(1-x_1) + (1-x_1x_3)(1-x_2) + (1-x_1x_2)x_3, \\
&\geq \tfrac{1}{2}\big((1-x_3)(1-x_1) + (1-x_3)(1-x_2) + (1-x_1+1-x_2)(1+x_3)\big).
\end{aligned}$$

Thus,

$$2L(uvw) - D(uvw) \geq (1-x_1+1-x_2) - (1-x_1)x_2 - (1-x_2)x_1 = 2(1-x_1)(1-x_2) \geq 0.$$

A.3 $(\gamma, \gamma, \bar{c})$ Triangles

We have

$$\begin{aligned}
D(uvw) &= (1-x_2)(1-x_3) + (1-x_1)(1-x_3) + 1 - x_1x_2, \\
\text{and} \quad L(uvw) &= (1-x_2x_3)\sum_{i\in L}(1-x^i_{wu}) + (1-x_1x_3)\sum_{i\in L}(1-x^i_{uv}) + (1-x_1x_2)x^{\bar{c}}_{wu} \\
&\geq (1-x_2x_3)(1-x_1) + (1-x_1x_3)(1-x_2) + (1-x_1x_2)m \\
&\geq (1-x_3)(1-x_1) + (1-x_3)(1-x_2) + \tfrac{1}{2}(1-x_1x_2).
\end{aligned}$$

Thus, $2L(uvw) - D(uvw) \geq 0$.

A.4 (γ, c, c) Triangles

We have

$$\begin{aligned}
D(uvw) &= (1-x_2)(1-x_3) + (x_1+x_3-2x_1x_3) + (x_1+x_2-2x_1x_2) \\
&= 1 + x_2x_3 + 2x_1(1-x_2-x_3), \\
\text{and} \quad L(uvw) &= (1-x_2x_3)\sum_{i\in L}(1-x^i_{wu}) + (1-x_1x_3)x_2 + (1-x_1x_2)x_3 \\
&\geq (1-x_2x_3)(1-x_1) + (1-x_1x_3)x_2 + (1-x_1x_2)x_3 \\
&= 1 - x_1 + x_2 + x_3 - x_2x_3 - x_1x_2x_3.
\end{aligned}$$

Thus,

$$\frac{5}{2}\cdot L(uvw) - D(uvw) \geq \frac{3}{2} + \frac{5(x_2+x_3)}{2} - \frac{7x_2x_3}{2} - x_1\Big(\frac{9}{2} - 2x_2 - 2x_3 + \frac{5x_2x_3}{2}\Big) \triangleq F.$$

Obviously,

$$\frac{\partial F}{\partial x_1} = -\frac{9}{2} + 2x_2 + 2x_3 - \frac{5x_2x_3}{2} < 0.$$

This implies that F decreases along with x_1. Recall that by constraint (3), we have $x_1 \le x_2 + x_3$. Thus, when $x_2 + x_3 \ge 1$, F achieves its minimum at $x_1 = 1$ such that

$$\begin{aligned} F &\ge \frac{3}{2} + \frac{5(x_2+x_3)}{2} - \frac{7x_2x_3}{2} - \Big(\frac{9}{2} - 2x_2 - 2x_3 + \frac{5x_2x_3}{2}\Big) \\ &= -3 + \frac{9(x_2+x_3)}{2} - 6x_2x_3 \\ &= -\frac{3}{2}(x_2+x_3-1)(x_2+x_3-2) + \frac{3}{2}(x_2-x_3)^2 \\ &\ge 0, \end{aligned}$$

where the last inequality is because $1 \le x_2 + x_3 \le 2$.

On the other hand, when $x_2 + x_3 \le 1$, F achieves its minimum at $x_1 = x_2 + x_3$ such that

$$\begin{aligned} F &\ge \frac{3}{2} + \frac{5(x_2+x_3)}{2} - \frac{7x_2x_3}{2} - (x_2+x_3)\Big(\frac{9}{2} - 2x_2 - 2x_3 + \frac{5x_2x_3}{2}\Big) \\ &= \frac{3}{2} - 2(x_2+x_3) - \frac{7x_2x_3}{2} + 2(x_2+x_3)^2 - \frac{5x_2x_3(x_2+x_3)}{2} \triangleq F^*. \end{aligned}$$

Due to symmetry, without loss of generality, let $x_2 = y + a$ and $x_3 = y - a$ with $0 \le a \le y \le \frac{1}{2}$. Then, we can rewrite F^* as

$$\begin{aligned} F^* &= \frac{3}{2} - 4y - \frac{7(y^2-a^2)}{2} + 8y^2 - 5y(y^2-a^2) \\ &\ge -5y^3 + \frac{9y^2}{2} - 4y + \frac{3}{2} \\ &= -5y^2\Big(y - \frac{1}{2}\Big) + (2y-3)\Big(y - \frac{1}{2}\Big) \ge 0. \end{aligned}$$

Putting it together yields $\frac{5}{2} \cdot L(uvw) - D(uvw) \ge 0$.

A.5 $(\gamma, c, \bar{c})$ Triangles

We have

$$\begin{aligned} D(uvw) &= (1-x_2)(1-x_3) + (x_1 + x_3 - 2x_1x_3) + (1 - x_1x_2) \\ &= 2 - x_2 + x_2x_3 + x_1(1 - x_2 - 2x_3), \\ \text{and} \quad L(uvw) &= (1 - x_2x_3)\sum_{i \in L}(1 - x_{wu}^i) + (1 - x_1x_3)x_2 + (1 - x_1x_2)x_{wu}^{\bar{c}} \\ &\ge (1 - x_2x_3)(1 - x_1) + (1 - x_1x_3)x_2 + (1 - x_1x_2)m \\ &= 1 - x_1 + x_2 + m - x_2x_3 - x_1x_2m. \end{aligned}$$

Thus,

$$\frac{5}{2} \cdot L(uvw) - D(uvw) \ge \frac{1+5m}{2} + \frac{7x_2}{2} - \frac{7x_2x_3}{2} - x_1\Big(\frac{7}{2} + \frac{(5m-2)x_2}{2} - 2x_3\Big) \triangleq F.$$

Again, when m is independent of x_1 and $m \ge 1/2$, we have

$$\frac{\partial F}{\partial x_1} = -\frac{7}{2} - \frac{(5m-2)x_2}{2} + 2x_3 < 0.$$

This implies that F decreases along with x_1. When $x_2 + x_3 \ge 1$, F achieves its minimum at $x_1 = 1$ such that

$$\begin{aligned} F &\ge \frac{1+5m}{2} + \frac{7x_2}{2} - \frac{7x_2x_3}{2} - \Big(\frac{7}{2} + \frac{(5m-2)x_2}{2} - 2x_3\Big) \\ &= \frac{5m(1-x_2)}{2} - 3 + \frac{9x_2}{2} + 2x_3 - \frac{7x_2x_3}{2} \triangleq F^*. \end{aligned}$$

If $x_2 \ge 1/2$, observing that $5m = m + 4m \ge 1 - x_3 + 4/2 = 3 - x_3$, we have

$$F^* \ge \frac{(3-x_3)(1-x_2)}{2} - 3 + \frac{9x_2}{2} + 2x_3 - \frac{7x_2x_3}{2} = \frac{3(2x_2-1)(1-x_3)}{2} \ge 0.$$

If $x_2 < 1/2$, observing that $5m \geq 5(1-x_2)$, we have

$$F^* \geq \frac{5(1-x_2)^2}{2} - 3 + \frac{9x_2}{2} + \frac{4-7x_2}{2} \cdot (1-x_2) = \frac{3(2x_2-1)^2}{2} \geq 0,$$

where the first inequality is because $4 - 7x_2 \geq 0$ and $x_3 \geq 1 - x_2$.

On the other hand, when $x_2 + x_3 \leq 1$, F achieves its minimum at $x_1 = x_2 + x_3$ such that

$$\begin{aligned} F &\geq \frac{1+5m}{2} + \frac{7x_2}{2} - \frac{7x_2x_3}{2} - (x_2+x_3)\Big(\frac{7}{2} + \frac{(5m-2)x_2}{2} - 2x_3\Big) \\ &= \frac{5m(1-x_2(x_2+x_3))}{2} + \frac{1}{2} - \frac{7x_3}{2} - \frac{x_2x_3}{2} + x_2^2 + 2x_3^2 \triangleq F^*. \end{aligned}$$

If $x_2 \geq 1/2$, observing that $5m \geq 5/2$, we have

$$\begin{aligned} F^* &\geq \frac{5(1-x_2(x_2+x_3))}{4} + \frac{1}{2} - \frac{7x_3}{2} - \frac{x_2x_3}{2} + x_2^2 + 2x_3^2 \\ &= \frac{7(1-x_3)^2 - x_2^2 - 7x_2x_3 + x_3^2}{4} \\ &\geq \frac{7x_2^2 - x_2^2 - 7x_2x_3 + x_3^2}{4} \\ &= \frac{(6x_2 - x_3)(x_2 - x_3)}{4} \geq 0, \end{aligned}$$

where the second inequality is by $1 - x_3 \geq x_2$, and the last inequality is by $x_3 \leq 1 - x_2 \leq 1/2 \leq x_2$.

If $1/8 \leq x_2 < 1/2$, observing that $5m \geq 5(1-x_2)$, we have

$$\begin{aligned} F^* &\geq \frac{5(1-x_2)(1-x_2(x_2+x_3))}{2} + \frac{1}{2} - \frac{7x_3}{2} - \frac{x_2x_3}{2} + x_2^2 + 2x_3^2 \\ &\geq \frac{5(1-x_2)^2}{2} + x_2^2 + 2\Big(x_3 - \frac{7}{8}\Big)^2 - \frac{x_2x_3}{2} - \frac{33}{32}, \end{aligned}$$

which is increasing in x_3 since $x_3 \leq 1 - x_2 \leq 7/8$. Thus,

$$\begin{aligned} F^* &\geq \frac{5(1-x_2)^2}{2} + x_2^2 + 2\Big(1 - x_2 - \frac{7}{8}\Big)^2 - \frac{x_2(1-x_2)}{2} - \frac{33}{32} \\ &= \frac{3(2x_2-1)^2}{2} \\ &\geq 0. \end{aligned}$$

If $x_2 < 1/8$, similarly, we have

$$\begin{aligned} F^* &\geq \frac{5(1-x_2)^2}{2} + x_2^2 + 2\Big(x_3 - \frac{7}{8}\Big)^2 - \frac{x_2x_3}{2} - \frac{33}{32} \\ &\geq \frac{5(1-1/8)^2}{2} - \frac{1}{16} - \frac{33}{32} \\ &\geq 0. \end{aligned}$$

Putting it together yields $\frac{5}{2} \cdot L(uvw) - D(uvw) \geq 0$.

A.6 $(\gamma, \bar{c}, \bar{c})$ Triangles

We have

$$\begin{aligned} D(uvw) &= (1-x_2)(1-x_3) + (1-x_1x_3) + (1-x_1x_2), \\ \text{and} \quad L(uvw) &= (1-x_2x_3)\sum_{i \in L}(1-x_{wu}^i) + (1-x_1x_3)x_{vw}^{\bar{c}} + (1-x_1x_2)x_{wu}^{\bar{c}} \\ &\geq (1-x_2x_3)(1-x_1) + (1-x_1x_3)m + (1-x_1x_2)m. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{5}{2}\cdot L(uvw) &\geq 2m\cdot\big((1-x_1x_3)+(1-x_1x_2)\big)+\frac{(1-x_1x_3)m}{2}+\frac{(1-x_1x_2)m}{2}\\ &\geq \big((1-x_1x_3)+(1-x_1x_2)\big)+\frac{(1-x_1x_3)(1-x_2)}{2}+\frac{(1-x_1x_2)(1-x_3)}{2}\\ &\geq \big((1-x_1x_3)+(1-x_1x_2)\big)+\frac{(1-x_3)(1-x_2)}{2}+\frac{(1-x_2)(1-x_3)}{2}\\ &= D(uvw).\end{aligned}$$

A.7 (c,c,c) Triangles

We have

$$\begin{aligned}D(uvw) &= (x_2+x_3-2x_2x_3)+(x_1+x_3-2x_1x_3)+(x_1+x_2-2x_1x_2)\\ &= 2(x_1+x_2+x_3-x_1x_2-x_2x_3-x_1x_3),\\ \text{and}\quad L(uvw) &= (1-x_2x_3)x_1+(1-x_1x_3)x_2+(1-x_1x_2)x_3\\ &= x_1+x_2+x_3-3x_1x_2x_3.\end{aligned}$$

Thus, $2L(uvw)-D(uvw)=2x_1x_2(1-x_3)+2x_2x_3(1-x_1)+2x_1x_3(1-x_2)\geq 0$.

A.8 $(c,c,\bar{c})$ Triangles

We have

$$\begin{aligned}D(uvw) &= (x_2+x_3-2x_2x_3)+(x_1+x_3-2x_1x_3)+(1-x_1x_2),\\ \text{and}\quad L(uvw) &= (1-x_2x_3)x_1+(1-x_1x_3)x_2+(1-x_1x_2)x_{wu}^{\bar{c}}\\ &\geq (1-x_2x_3)x_1+(1-x_1x_3)x_2+\frac{1}{2}\cdot(1-x_1x_2).\end{aligned}$$

Thus,

$$\frac{5}{2}\cdot L(uvw)-D(uvw)\geq\frac{(3+4x_3)(x_1+x_2)}{2}-2x_3+\frac{1-x_1x_2}{4}-5x_1x_2x_3\triangleq F.$$

Due to symmetry, without loss of generality, let $x_1=y+a$ and $x_2=y-a$ with $0\leq a\leq y\leq 1$. Then, we have

$$F=-\Big(5x_3+\frac{1}{4}\Big)(y^2-a^2)+(3+4x_3)y-2x_3+\frac{1}{4}\geq-\Big(5x_3+\frac{1}{4}\Big)y^2+(3+4x_3)y-2x_3+\frac{1}{4}.$$

It is easy to verify that $-5y^2+4y-2\leq 0$, which indicates $(-5y^2+4y-2)x_3$ decreases along with x_3. Thus, if $y\geq 1/2$, we can get that

$$F\geq -5x_3y^2+(3+4x_3)y-2x_3\geq -5y^2+7y-2\geq\min\{-5+7-2,-5/4+7/2-2\}=0.$$

Meanwhile, if $y\leq 1/2$, using $x_3\leq x_1+x_2=2y$ (by constraint 3), we can get that

$$F\geq -10y^3+\frac{31}{4}y^2-y+\frac{1}{4}=y^2(-10y+5)+\Big(y-\frac{1}{2}\Big)^2+\frac{7}{4}y^2\geq 0.$$

Putting it together yields $\frac{5}{2}\cdot L(uvw)-D(uvw)\geq 0$.

A.9 $(c,\bar{c},\bar{c})$ Triangles

We have

$$\begin{aligned}D(uvw) &= (x_2+x_3-2x_2x_3)+(1-x_1x_3)+(1-x_1x_2),\\ \text{and}\quad L(uvw) &= (1-x_2x_3)x_1+(1-x_1x_3)x_{vw}^{\bar{c}}+(1-x_1x_2)x_{wu}^{\bar{c}}\\ &\geq (1-x_2x_3)x_1+\frac{1}{2}\cdot(1-x_1x_3)+\frac{1}{2}\cdot(1-x_1x_2).\end{aligned}$$

Thus,

$$\frac{5}{2} \cdot L(uvw) - D(uvw) \geq \frac{5(1-x_2x_3)x_1}{2} - (x_2 + x_3 - 2x_2x_3) + \frac{2 - x_1x_2 - x_1x_3}{4} \triangleq F.$$

Observing that F is linear in x_1, we first assume that F decreases along with x_1. Then, as $x_1 \leq 1$, we have

$$F \geq \frac{5(1-x_2x_3)}{2} - (x_2 + x_3 - 2x_2x_3) + \frac{2 - x_3 - x_2}{4} = \frac{1 - x_2x_3}{2} + \frac{5(2 - x_2 - x_3)}{4} \geq 0.$$

Next, we consider that F increases along with x_1. Due to symmetry, without loss of generality, suppose that $x_2 \geq x_3$. Using $x_1 \geq x_2 - x_3$ (by constraint 3), we have

$$F \geq \frac{5(1-x_2x_3)(x_2-x_3)}{2} - (x_2 + x_3 - 2x_2x_3) + \frac{2 - (x_2+x_3)(x_2-x_3)}{4} \triangleq F^*.$$

Then, we can get that

$$\frac{\partial F^*}{x_2} = \frac{5(1 - 2x_2x_3 + x_3^2)}{2} - (1 - 2x_3) - \frac{x_2}{2} \geq \frac{5x_3^2}{2} - 3x_3 + 1 = \frac{5}{2}\Big(x_3 - \frac{3}{5}\Big)^2 + \frac{1}{10} \geq 0,$$

where the second inequality is because $x_2 \leq 1$. Thus, F^* increases along with x_2, which gives rise to

$$F^* \geq \frac{5(1-x_3x_3)(x_3-x_3)}{2} - (x_3 + x_3 - 2x_3x_3) + \frac{2 - (x_3+x_3)(x_3-x_3)}{4} = 2\Big(x_3 - \frac{1}{2}\Big)^2 \geq 0.$$

Putting it together yields $\frac{5}{2} \cdot L(uvw) - D(uvw) \geq 0$.

A.10 $(\bar{c}, \bar{c}, \bar{c})$ Triangles

We have

$$\begin{aligned} D(uvw) &= (1 - x_2x_3) + (1 - x_1x_3) + (1 - x_1x_2), \\ \text{and} \quad L(uvw) &= (1 - x_2x_3)x_{uv}^{\bar{c}} + (1 - x_1x_3)x_{vw}^{\bar{c}} + (1 - x_1x_2)x_{wu}^{\bar{c}} \\ &\geq \frac{1}{2}\big((1 - x_2x_3) + (1 - x_1x_3) + (1 - x_1x_2)\big). \end{aligned}$$

Thus, $2L(uvw) - D(uvw) \geq 0$.

This completes the case-by-case analysis for proving $D(uvw) \leq 2.5L(uvw)$.

B Time Complexity Analysis of the GreedyVote Algorithm

Note that computing $\frac{|N_{\phi(uv)}(u) \cap N_{\phi(uv)}(v)|}{|N(u) \cup N(v)|}$ for any $uv \in E$ can be implemented in $O(\Delta)$ time given the adjacent vertex lists of u, v and the adjacent matrix of G. Besides, the while-loop in Lines 2–12 iterates for at most $\mathcal{O}(|V|)$ times due to the reason that at least two vertices are removed from V in each iteration. Therefore, Lines 3–4 cost $\mathcal{O}(\Delta|V|)$ running time in total, considering that m is a predefined constant.

In Lines 6–10 of GreedyVote, when a new vertex w is added into S_k, only the vote counts of the vertices in $C = \{v \mid wv \in E \wedge v \notin S_k \cup \{w\}\}$ would be changed. Therefore, we can check each edge in $\{wv \mid v \in C\}$ to determine the next vertex w' to be added into S_k, while determining the new best color of $S_k \cup \{w\}$ (i.e., the most frequent color of the edges with both endpoints in $S_k \cup \{w\}$) at the same time. This implies that the operations in Lines 2–12 except Lines 3–4 can be implemented in $\mathcal{O}(|E|)$ time in total.

Synthesizing the above analysis, we know that the time complexity of the GreedyVote algorithm is $\mathcal{O}(|E| + \Delta|V|)$ for any constant m.

C Derandomization

In this section, we show that our approximation algorithm 1 can be de-randomized using some tricks inspired by Chawla et al. [10]. The resultant deterministic algorithm also guarantees a 2.5-approximation.

It is noted that phase 1 (Lines 2–4) is a deterministic procedure according to Observation 4.1. Lines 5–6 can also be derandomized by a fixed coloring method easily. Therefore, to get a deterministic algorithm, it is sufficient to derandomize the clustering process on subgraph S_c.

According to the definitions of $d(uv \mid w)$ and $lp(uv \mid w)$, we can rewrite them as follows:

$$d(uv \mid w) = \begin{cases} (1-p^c_{vw})p^c_{uw} + (1-p^c_{uw})p^c_{vw}, & \text{if } \phi(uv) = c, \\ 1 - p^c_{uw}p^c_{vw}, & \text{if } \phi(uv) \in L \setminus \{c\}, \\ (1-p^c_{uw})(1-p^c_{vw}), & \text{if } \phi(uv) = \gamma; \end{cases}$$

and

$$lp(uv \mid w) = \begin{cases} (1-p^c_{uw}p^c_{vw})x^{\phi(uv)}_{uv}, & \text{if } \phi(uv) \in L, \\ (1-p^c_{uw}p^c_{vw})\sum_{i\in L}(1-x^i_{uv}), & \text{if } \phi(uv) = \gamma. \end{cases}$$

Here $1 - p^c_{uv}$ denotes the probability of the event that u and v are in the same cluster colored by c, used in line 11. Now we attempt to change the value of each p^c_{uv} to either 0 or 1 to maximize the function:

$$2.5\mathbb{E}[L_t \mid S^t_c] - \mathbb{E}[D_t \mid S^t_c]. \tag{9}$$

This can be done as follows: for each edge $uv \in E^t_c$, we pick a value from $\{0, 1\}$ to assign to p^c_{uv} that maximizes the function (9). After all p^c_{uv} for $uv \in E^t_c$ are integers (e.g. 0 or 1), the algorithm picks the vertex w to maximize $2.5\mathbb{E}[L_t \mid S^t_c, p_t = w] - \mathbb{E}[D_t \mid S^t_c, p_t = w]$ where $p_t = w$ denotes the event that the center of the new cluster at iteration t is vertex w. Algorithm gets the deterministic cluster in iteration t because the center of the cluster is deterministic and the probability of the event that each vertex is add to the cluster is either 0 or 1.

Note that all terms of $L(uvw)$ and $D(uvw)$ are linear functions of p^c_{uv}, p^c_{vw}, and p^c_{wu} according to the definitions. Then, $2.5\mathbb{E}[L_t \mid S^t_c] - \mathbb{E}[D_t \mid S^t_c] = \frac{1}{6|S^t_c|}\sum_{u,v,w\in S^t_c}(2.5L(uvw) - D(uvw))$ are linear in each p^c_{uv} for $uv \in E^t_c$. Therefore, the value of (9) may only increase when we greedily change p^c_{uv} to 0 or 1. Combined with our previous analysis, function (9), which is equal to $\frac{1}{6|S^t_c|}\sum_{w\in S^t_c}\sum_{uv\in E^t_c}(2.5lp(uv \mid w) - d(uv \mid w))$, is nonnegative after each p^c_{uv} is replaced with 0 or 1. Hence, we can find some vertex $w^\star \in S^c_t$ to make $\sum_{uv\in E^t_c}(2.5lp(uv \mid w^\star) - d(uv \mid w^\star)) \geq 0$. It means that the number of disagreements is upper bounded by 2.5 times the LP cost among the edges removed at iteration t, and we get $D_t \leq 2.5L_t$. Therefore, this deterministic algorithm achieves an approximation ratio of 2.5.

D Addtional Experiments

In this section, we present more experimental results under the same experimental settings in Sec. 6. Each implemented algorithm is run for 50 times for each dataset.

First, we compare the implemented algorithms on their variances, with the results shown by Table 4. As GV, GE, GER are the three algorithms with the best performance on the average Number of Disagreements (NOD), we also plot the distribution of their NOD in a visualized way in Figure A. From Table 4 and Figure A, it can be seen that the variances of GV are smaller than those of GE on all 10 datasets except DAWN. Compared to GER, the variances of GV are smaller on 6 out of 10 datasets, but GV outperforms GER in terms of average NOD on all 10 datasets (see Sec. 2) . Moreover, Figure A shows that GV achieves smaller NOD than GE and GER in *every* of the 50 runs on the four datasets including Lastfm, DBLP, Twitter and MAG. Finally, among all the seven implemented algorithms, GV achieves the smallest variance on 6 out of 10 datasets.

Next, we evaluate the impact of the parameter m of GV on the number of disagreements (NOD) and running time (RT), shown in Figure B. As expected, the running time of GV increases with m, while the NOD of GV decreases when m increases. This reveals the tradeoff on efficiency and effectiveness for selecting the value of m. However, the overall time complexity of GV (i.e., $\mathcal{O}(|E| + \Delta|V|)$) is independent on m.

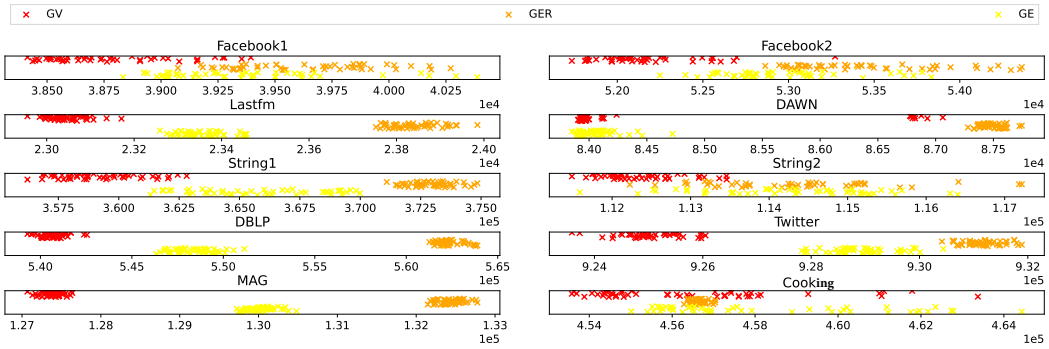

Figure A: The distribution of NOD for GE (yellow), GER (orange), and GV (red)

Table 4: Compare all algorithms on the variance of NOD

Datasets	**Pivot** [2]	**RC** [3]	**DC** [3]	**CB** [6]	**GE** [25]	**GER** [25]	**GV (ours)**
Facebook1	2.17e7	3.47e6	4.78e6	3.21e6	1.34e5	9.11e4	**7.36e4**
Facebook2	1.48e7	8.68e6	6.92e6	3.75e6	1.34e5	1.76e5	**8.19e4**
Lastfm	7.45e6	9.42e5	1.13e6	6.91e3	2.62e3	2.82e3	**1.69e3**
Twitter	7.83e7	8.07e6	1.69e7	8.55e4	3.19e4	2.21e4	**1.56e4**
DAWN	1.68e9	2.21e8	1.44e8	1.12e6	2.75e4	1.02e4	1.23e6
Cooking	7.05e9	2.06e9	3.63e8	1.81e7	6.50e6	3.16e4	5.38e6
String1	4.54e9	2.49e8	1.03e9	1.23e7	6.81e6	7.79e5	2.61e6
String2	3.17e7	4.27e7	4.87e7	3.90e6	1.12e6	1.11e6	**2.30e5**
DBLP	2.16e10	1.50e8	2.15e8	4.26e6	3.42e5	1.26e5	2.84e5
MAG	3.24e8	1.08e7	9.11e6	2.67e6	1.23e6	4.52e5	**4.31e5**

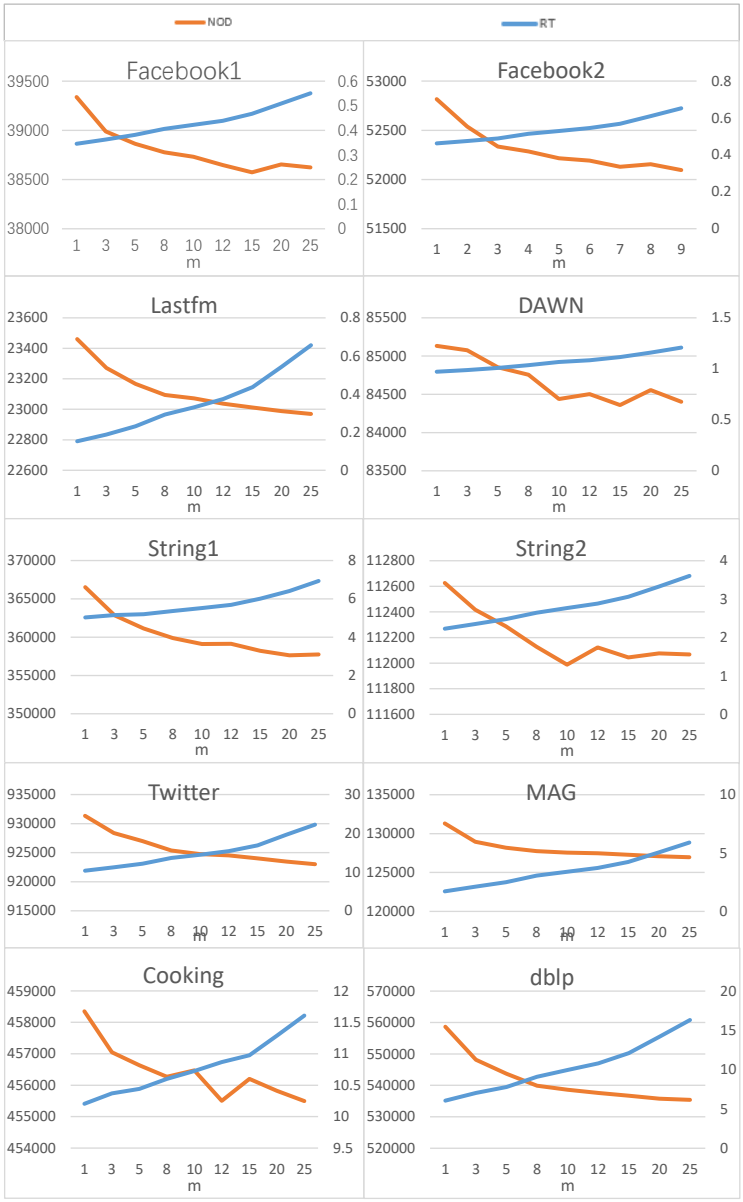

Figure B: Impact of GV's parameter m on NOD (the number of disagreements, plotted by orange lines) and RT (running time, plotted by blue lines) on all 10 datasets. For each dataset, the left Y-axis denotes the values of NOD, and the right Y-axis denotes the values of RT (seconds).