Bayesian Extreme Learning

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Paper under double-blind review

Abstract

This paper introduces a Bayesian extreme learning (BEL) model for analyzing high dimensional datasets characterized by extreme values. The model synthesizes elements from information theory, Bayesian inference, machine learning, and extreme value theory. Convergence properties of the BEL model are established by declining Kullback-Leibler divergence between consecutive posterior distributions as the sample size grows. The model's capability to isolate extreme values is demonstrated by increasing entropy. Additionally, the paper validates the regularization optimality, where the optimal parameter configuration effectively minimizes the divergence from a specified reference distribution. The paper also shows the model's proficiency in achieving near-optimal information extraction and its universal approximation ability for continuous extreme value distributions across a range of tolerance levels. The model's robustness and versatility are illustrated through examples, simulations, and applications, underscoring its potential utility in statistical learning within high-dimensional datasets.

1 Introduction

For a continuous random variable X with probability density function (PDF) $f(x)$, the Shannon entropy $\mathcal{H}$ is defined as

$$\mathcal{H}(X) = -\int_{-\infty}^{\infty} f(x) \log f(x) \underset{\cdot}{\mathrm{x}}. \tag{1}$$

Shannon entropy quantifies the uncertainty associated with continuous distributions (Shannon, 1948). This entropy measure is a fundamental building block for information-theoretic measures, particularly in comparing probability distributions. One such measure is the Kullback-Leibler (KL) divergence, which extends the idea of entropy to a comparative framework. The KL divergence between two continuous probability distributions f and g is defined as

$$\mathcal{K}(f; g) = \int_{-\infty}^{\infty} f(x) \log \left(\frac{f(x)}{g(x)} \right) \underset{\cdot}{\mathrm{x}}. \tag{2}$$

This divergence measure quantifies the difference between two continuous distributions (Kullback & Leibler, 1951). The applications of these concepts have been extensively studied. See Ardakani et al. (2018), Ardakani et al. (2020), Ardakani (2022), Ardakani & Saenz (2022), Soofi et al. (1995), and Soofi (1992) for examples.

These measures become pertinent in Bayesian analysis, where the posterior distribution is updated based on prior beliefs and new data. Given a continuous parameter θ, if $f(\theta)$ represents its prior density and $f(\mathcal{D}|\theta)$ represents the likelihood of observing data $\mathcal{D}$ given θ, then the posterior density is given by Bayes' theorem as

$$f(\theta|\mathcal{D}) = \frac{f(\mathcal{D}|\theta) f(\theta)}{\int f(\mathcal{D}|\theta') f(\theta') d\theta'} \tag{3}$$

The prior in Bayesian models plays a regularizing role for continuous distributions. The KL divergence between a continuous prior and posterior provides insight into how our beliefs update after observing new data (Kullback, 1959). This Bayesian framework addresses the limitations of traditional learning algorithms, as identified by Ditzler et al. (2015), which often underperform for non-stationary data. By incorporating

prior knowledge and iteratively adjusting to new observations, Bayesian methods offer a robust solution for navigating the challenges presented by extreme events (Berry et al., 2010).

On the other hand, the extreme value theory (EVT) model tails in datasets. When integrated with Bayesian methodologies, EVT methods are robust for extreme event analysis. The block maxima approach divides data into blocks, selecting the maximum from each block. These maxima are modeled using the generalized extreme value (GEV) distribution (Coles et al., 2001). The GEV cumulative distribution function (CDF) is given by

$$F(z) = \exp\left\{-\left[1+\xi\left(\frac{z-\mu}{\sigma}\right)\right]^{-1/\xi}\right\} \quad 1+\xi(z-\mu)/\sigma > 0, \tag{4}$$

where μ is the location parameter, $\sigma > 0$ is the scale parameter, and ξ is the shape parameter. Recently Ardakani (2023) integrates information theory and EVT to illustrate that the entropy of bock maxima converges to the entropy of the GEV distribution. The peak-over-threshold method, however, focuses on data exceeding a high threshold, using the generalized Pareto (GP) distribution to model the tail. This method is detailed in Smith (1985) and Davison & Smith (1990). The GP CDF is given by

$$F(y) = 1-\left(1+\frac{\xi y}{\sigma}\right)^{-1/\xi}, \quad 1+\xi y/\sigma > 0, \tag{5}$$

where $\sigma > 0$ is the scale parameter, and ξ is the shape parameter. Bayesian methods with EVT provide a dynamic framework to analyze and understand distribution tails, especially with limited data on extreme events. This approach is elaborated in Beirlant et al. (2006).

This paper introduces a Bayesian extreme learning (BEL) model, specifically engineered to analyze high-dimensional datasets where extreme values are of paramount interest. The model integrates prior distributions with likelihood functions within a Bayesian framework to yield refined posterior distributions. A key feature of the BEL model is its mechanism to filter and emphasize extreme values. It incorporates a regularization term derived from entropy and Kullback-Leibler divergence. This term helps balance expected posterior likelihood and regularization, optimizing the model's performance in high-dimensional spaces. The properties of the BEL model are also studied. First, I establish that for the BEL model, the Kullback-Leibler divergence between consecutive posterior distributions converges to zero with increasing observations, providing consistent posterior estimates. Second, I demonstrate that the entropy in the BEL model increases when isolating extreme values from a dataset. This indicates the model's effectiveness in capturing extreme events within high-dimensional data. Third, I show that the optimal model parameter minimizes the KL divergence between the predicted and reference distributions, emphasizing its accuracy in parameter estimation. Fourth, I show that the BEL model achieves near-optimal information extraction from high-dimensional datasets, particularly those with extreme values, by minimizing the KL divergence between consecutive posterior distributions. Finally and most importantly, a key result is the model's ability to approximate any continuous extreme value distribution within a given tolerance, highlighting the model's versatility with universal approximation.

Substantial progress has been made in high-dimensional data analysis, focusing on datasets with extreme values. For instance, Einmahl et al. (2001) and Engelke & Hitz (2020) highlight the complexities in extreme value analysis, but integrating these techniques into a high-dimensional Bayesian framework is still an evolving area. This gap is especially evident in the existing literature regarding balancing regularization and extracting meaningful information from extreme values in large datasets. The BEL model addresses these challenges. It synthesizes EVT with Bayesian principles, a combination suggested in Chavez-Demoulin & Davison (2005). This study contributes to the existing literature by demonstrating consistent posterior estimates and addressing the need for reliability in posterior convergence in extreme value models, as discussed in Einmahl & Segers (2009) and Einmahl et al. (2016). The enhancement of entropy in the model and its estimation through KL divergence minimization aligns with the recent advancements in the field (Kiriliouk et al. (2019)). By demonstrating the universal approximation capability of the model, this research addresses a gap in statistical learning for high-dimensional datasets with extreme values, a challenge highlighted in Wadsworth & Tawn (2012).

2 Bayesian extreme learning model

Building upon the foundational concepts of Bayesian EV, the BEL model is defined by integrating Bayesian inference (Bayes, 1958; Gelman et al., 1995), EVT (Gnedenko, 1943; Haan & Ferreira, 2006), and machine learning algorithms tailored for extremes.

Definition 1 (Bayesian extreme learning model) *Let $\mathcal{D}$ be a high-dimensional dataset. Define $X = \{x_1, x_2, \ldots, x_n\}$ as random variables from $\mathcal{D}$, where each x_i can potentially take an extreme value. The Bayesian extreme learning model is formulated as follows: Given a prior $f(\theta)$ and a likelihood $f(X|\theta)$, the posterior distribution is computed by*

$$f(\theta|X) = \frac{f(X|\theta)f(\theta)}{\int f(X|\theta')f(\theta')d\theta'} \tag{6}$$

Extreme values in X are filtered using a function $F : X \to Y$, where Y captures the extreme values. Let $\mathcal{H}(X)$ denote the entropy of X, and let $\mathcal{K}(f; g)$ denote the Kullback-Leibler divergence between two distributions f and g. The regularization term R is defined as

$$R(X) = \alpha\mathcal{H}(X) + \beta\mathcal{K}(f(X); g(X)), \tag{7}$$

where α and β are regularization coefficients, and $g(X)$ is a reference distribution. The term $R(X)$ ensures that the model remains uncertain about extreme events and stays close to a reference distribution. The BEL model seeks to minimize a loss function $\mathcal{L}$, defined in terms of the expected posterior likelihood and the regularization term:

$$\mathcal{L}(X, \theta) = \mathbb{E}[f(X|\theta)] - \lambda R(X), \tag{8}$$

where $\mathbb{E}$ denotes the expected value, and λ is a hyperparameter controlling the trade-off.

A direct consequence of the Bayesian updating mechanism ensures that as more data is observed, the divergence between consecutive posteriors diminishes, which can be stated as the following lemma. This is consistent with the convergence properties discussed in Ghosal & Van der Vaart (2017) and Schwartz (1965).

Lemma 1 (Bayesian posterior convergence) *Assume $\{X_n\}$ is a sequence of independent and identically distributed (i.i.d.) observations from a data-generating process. In the Bayesian formulation of the BEL model, for every $\epsilon > 0$, there exists an integer $N(\epsilon)$, depending on the distribution of X_n, the prior distribution $f(\theta)$, and the likelihood function $f(X|\theta)$, such that for all $n > N(\epsilon)$,*

$$\mathcal{K}(f(\theta|X_n); f(\theta|X_{n+1})) < \epsilon. \tag{9}$$

Proof. *We need to show that for every $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that for all $n > N(\epsilon)$,*

$$\mathcal{K}(f(\theta|X_n); f(\theta|X_{n+1})) < \epsilon.$$

Consider the Bayesian updating rule for posterior distributions:

$$f(\theta|X_{n+1}) = \frac{f(X_{n+1}|\theta)f(\theta|X_n)}{f(X_{n+1})},$$

where $f(X_{n+1}) = \int f(X_{n+1}|\theta)f(\theta|X_n)d\theta$.

The KL divergence between $f(\theta|X_n)$ and $f(\theta|X_{n+1})$ is given by

$$\mathcal{K}(f(\theta|X_n); f(\theta|X_{n+1})) = \int f(\theta|X_n) \log\left(\frac{f(\theta|X_n)}{f(\theta|X_{n+1})}\right) d\theta.$$

By substituting the expression for $f(\theta|X_{n+1})$ into the KL divergence formula and applying properties of logarithms, we have

$$\mathcal{K}(f(\theta|X_n); f(\theta|X_{n+1})) = \int f(\theta|X_n) \log\left(\frac{f(X_{n+1})}{f(X_{n+1}|\theta)}\right) d\theta.$$

Since $\{X_n\}$ is a sequence of i.i.d. observations, as $n \to \infty$, the amount of information about θ in the posterior distribution increases, causing $f(\theta|X_n)$ to become more peaked around the true value of θ. This implies that the term $\log\left(\frac{f(X_{n+1})}{f(X_{n+1}|\theta)}\right)$ becomes smaller for larger n.

Therefore, for any given $\epsilon > 0$, we can find an integer $N(\epsilon)$ such that for all $n > N(\epsilon)$, the KL divergence $\mathcal{K}(f(\theta|X_n); f(\theta|X_{n+1})) < \epsilon$.

From the properties of the extreme value filter, which concentrates on regions of the dataset with higher unpredictability and, consequently, greater entropy (Cover & Thomas, 1991; Leadbetter et al., 2012), we have the following lemma.

Lemma 2 (Extreme value focus) *Consider the extreme value filter function $F : X \to Y$ applied to a dataset X, resulting in a transformed dataset Y that isolates extreme values from X. For the BEL model,*

$$\mathcal{H}(Y) > \mathcal{H}(X). \tag{10}$$

Proof. *Recall that the differential entropy $\mathcal{H}$ for a continuous random variable with probability density function $f(x)$ is defined as*

$$\mathcal{H}(X) = -\int_{-\infty}^{\infty} f(x) \log f(x) \dot{x}.$$

By applying the filter function F, the dataset X is transformed into Y, which focuses on extreme values. This transformation results in a probability density function for Y that is more "spread out" or has a higher variance due to the concentration of density on the extreme values. The differential entropy $\mathcal{H}(Y)$ is calculated similarly

$$\mathcal{H}(Y) = -\int_{-\infty}^{\infty} g(y) \log g(y) \dot{y},$$

where $g(y)$ is the PDF of Y. Since the extreme values are less predictable and more spread out than the typical values in X, the function $g(y)$ for Y reflects greater uncertainty. Consequently,

$$-\int_{-\infty}^{\infty} g(y) \log g(y) \dot{y}$$

is larger for Y than for X, implying that $\mathcal{H}(Y) > \mathcal{H}(X)$.

This implies that the entropy of the filtered dataset capturing extreme values is greater than the original dataset. The BEL model employs regularization to ensure proximity to a given reference distribution, achieving optimality, aligning with the principles established by Kullback & Leibler (1951) and discussed in Hastie et al. (2009).

Lemma 3 (Regularization optimality) *In the BEL model, given a regularization term $R(X)$ and a reference distribution $g(X)$, the optimal model parameter θ^* minimizes the KL divergence between the predicted distribution $f(X|\theta)$ and the reference distribution $g(X)$. Formally,*

$$\theta^* = \arg\min_{\theta} \mathcal{K}(f(X|\theta); g(X)). \tag{11}$$

This implies that the BEL model parameter θ^* achieves the minimum KL divergence from the reference distribution.

Proof. *To find θ^* that minimizes $\mathcal{K}(f(X|\theta); g(X))$, we consider the derivative of the KL divergence with respect to θ, setting it to zero:*

$$\frac{\partial}{\partial\theta}\mathcal{K}(f(X|\theta); g(X)) = 0.$$

This leads to

$$\int \frac{\partial}{\partial\theta} f(X|\theta) \left[\log\left(\frac{f(X|\theta)}{g(X)}\right) + 1\right] dx = 0.$$

By solving this equation, we obtain the value of θ^* that minimizes the KL divergence. It should be noted that this requires the integrability of the derivative of $f(X|\theta)$ and the existence of a unique solution to the equation. Therefore, the BEL model parameter θ^* that satisfies this condition is the one that minimizes the KL divergence between $f(X|\theta)$ and the reference distribution $g(X)$.

Theorem 1 (Optimal information extraction) *Given the entropy $\mathcal{H}(X)$ and the properties of the BEL model, the model achieves near-optimal information extraction from high-dimensional datasets with extreme values. This is characterized by the minimization of the KL divergence between consecutive posterior distributions.*

Proof. *The optimality of information extraction is demonstrated through three key aspects:*

1. *Convergence of posterior distributions: From Lemma 1, we have the convergence of KL divergence between consecutive posteriors:*

$$\lim_{n\to\infty} \mathcal{K}(f(\theta|X_n); f(\theta|X_{n+1})) = 0.$$

   This ensures that the BEL model consistently refines its accuracy with additional data.

2. *Increased entropy for extreme values: Lemma 2 demonstrates an increase in entropy when focusing on extreme values:*

$$\mathcal{H}(Y) - \mathcal{H}(X) > 0.$$

   This emphasizes the model's capacity to extract information with higher unpredictability.

3. *Optimization of model parameters: Lemma 3 establishes that θ^* minimizes the KL divergence between the model's predicted distribution and the reference distribution:*

$$\theta^* = \arg\min_{\theta} \mathcal{K}(f(X|\theta); g(X)).$$

Combining these aspects, the BEL model achieves near-optimal information extraction from datasets, especially in high-dimensional spaces with extreme values.

The approach of minimizing the KL divergence between consecutive posterior distributions as more data is observed aligns with the key concepts in Bayesian learning, where the accumulation of data continually refines the model's predictions (Tipping, 2001; Bishop & Nasrabadi, 2006). This continual refinement and adaptation process is particularly crucial in the context of extreme value analysis within high-dimensional datasets. In such scenarios, the ability of the model to adapt and learn from new, possibly extreme, observations is essential for maintaining accuracy and relevance. Tipping (2001) emphasizes the importance of sparse Bayesian learning for efficient computational performance, especially in high-dimensional spaces. Furthermore, Bishop & Nasrabadi (2006) discuss how Bayesian methods, particularly those involving probabilistic models like the BEL model, are adept at uncovering latent patterns in complex data. This ability is critical when dealing with extreme values, as these values often carry significant information about the underlying phenomena being modeled.

Example 1 (BEL with Pareto distribution) *Assume the true data-generating process follows a Pareto distribution, with parameters k and α. The PDF of a Pareto distribution is given by*

$$f(x|k,\alpha) = \frac{\alpha k^\alpha}{x^{\alpha+1}} \quad \text{for } x \geq k, \tag{12}$$

where $\alpha > 0$ is the shape parameter and $k > 0$ is the scale parameter. Assume we have a dataset $\mathcal{D}$ sampled from a Pareto distribution with $k = 1$ and $\alpha = 2$. We choose an appropriate prior for $\theta = (\alpha, k)$. A common choice for the Pareto parameters would be a gamma distribution for α and an exponential distribution for k. Now, given a sample $X = x_1, x_2, ..., x_n$ from $\mathcal{D}$, the BEL model will update its beliefs about the parameters θ using Bayesian updating. For simplicity, let's focus on updating beliefs about the shape parameter α using a

gamma prior. If $f(X|\theta)$ is our likelihood function derived from the Pareto distribution and $f(\theta)$ is our prior, the posterior after observing X is given by the BEL model's formulation in Equation 6.

Next, we use the function F to extract extreme values. For Pareto-distributed data, values much larger than the scale parameter k can be considered extreme. So, F filters out values above a certain threshold, say $k' > k$. This results in a dataset Y with higher unpredictability and greater entropy than X, which is in line with Lemma 2. Using the BEL model's formulation for the regularization term $R(X)$ and our chosen reference distribution $g(X)$ (for simplicity, we could use an exponential distribution as our reference), we can compute the loss $\mathcal{L}(X, \theta)$. Given this loss function, the BEL model can be trained to find the optimal parameter values θ^* that minimize this loss, keeping in line with Lemma 3.

Figure 1 presents histograms of the Pareto distribution and its Bayesian updating. We observe a histogram of the Pareto-distributed data, characterized by a pronounced heavy tail. The center plot demonstrates Bayesian updating for the shape parameter, α, using the prior $\mathcal{G}(2, 1)$. The right plot shows the Bayesian updating under an alternative prior, $\mathcal{G}(4, 2)$. This comparison underscores the sensitivity of the posterior distribution to changes in the prior. $\mathcal{G}(2, 1)$ is a relatively uninformative prior. It gives the data a considerable amount of influence in determining the posterior. On the other hand, $\mathcal{G}(2, 1)$ is a more informative prior. This prior suggests a stronger belief that the parameter of interest is around the value of 2 but with a slightly broader spread.

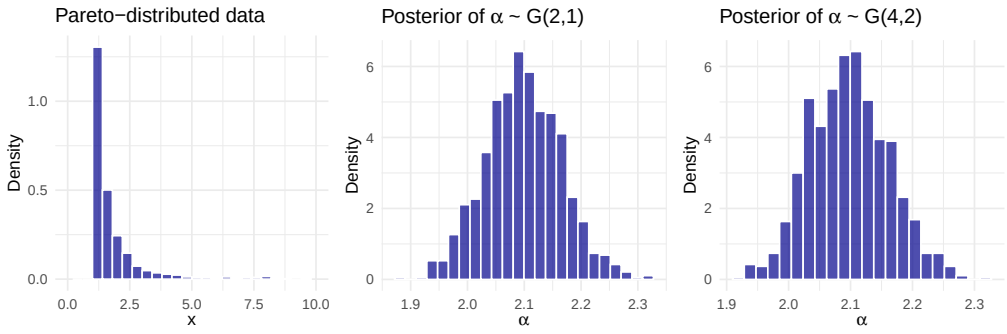

Figure 1: Pareto distribution and Bayesian posterior of α

Table 1 provides the summary measures for the Pareto-distributed data, extreme values, and the posterior distributions derived under two priors. A comparison between the Posteriors with different priors highlights the differences that arise from adjusting the priors in Bayesian updating. This example demonstrates how the BEL model functions when faced with data from a Pareto distribution. The Bayesian updating mechanism ensures the model learns the true parameters, the extreme value filtering focuses on the tails of the distribution, and the regularization ensures predictions stay in line with prior knowledge.

Table 1: Summary statistics of Pareto-distributed data, extreme values, posteriors

Description	Min	25%	Median	Mean	75%	Max
Pareto-distributed data	1.00	1.15	1.38	1.90	1.93	29.66
Posterior of $\alpha \sim \mathcal{G}(2, 1)$	1.88	2.05	2.10	2.10	2.14	2.33
Posterior of $\alpha \sim \mathcal{G}(4, 2)$	1.89	2.06	2.10	2.10	2.14	2.37
Extreme values	3.01	3.54	4.36	5.69	7.17	29.66

The regularization term $R(x)$ can be influenced by the choice of reference distribution $g(x)$. The KL divergence quantifies the difference between two probability distributions. Reference distribution choices affect the BEL model. We can evaluate how different reference distributions, namely the exponential, normal, and uniform, would affect the KL divergence and the subsequent loss. The KL divergence quantifies the difference between our empirical Pareto-distributed data and each reference distribution. Table 2 presents the KL divergences and losses for each choice of reference. The uniform distribution exhibits the smallest divergence, suggesting it mirrors the empirical data more closely than the other distributions. However, the

associated loss of 736.35 is only slightly lower than that of the exponential reference. In contrast, the normal distribution, with the highest divergence of 3.77, results in the most significant loss (763.71). These findings underscore the role of reference distribution choice in BEL modeling.

Table 2: KL divergences and losses for different reference distributions

Reference	$\mathcal{K}(f;g)$	$\mathcal{L}(X,\theta)$
Exponential	1.19	737.67
Normal	3.77	763.71
Uniform	0.00	736.35

The concept of admissible reference distributions is required in formulating the BEL model, particularly in determining the model's efficacy and robustness. Reference distributions play a significant role in the BEL model's regularization process, serving as a comparative standard against which the model's predictions are evaluated. In recent literature, the choice and characteristics of reference distributions have been scrutinized for their impact on model performance. For instance, Muller & Quintana (2004) emphasize the significance of using distributions that reflect the underlying data-generating process, ensuring that statistical models remain sensitive to data patterns while avoiding overfitting. Similarly, Walker (2010) highlights how reference distributions could be tailored to enhance the accuracy of Bayesian models in high-dimensional spaces. These discussions underscore the importance of selecting reference distributions to guarantee that the model remains aligned with empirical data characteristics. In this context, the BEL model adopts a definition of admissibility for reference distributions, encapsulating the characteristics required for effective model performance.

Definition 2 (Admissible reference distributions) *Let $g(X)$ be a reference distribution for a random variable X. The distribution $g(X)$ is said to be* admissible *if it satisfies the following conditions:*

(a) For any event E in the sample space of X such that $\mathbb{P}_{true}(E) > 0$ under the true data-generating distribution, we have $g(E) > 0$. Formally,

$$\forall E \subseteq \mathcal{X},\ \mathbb{P}_{true}(E) > 0 \implies g(E) > 0. \tag{13}$$

(b) The distribution $g(X)$ is absolutely continuous with respect to the Lebesgue measure μ, i.e., there exists a density function $h(x)$ such that

$$g(A) = \int_A h(x) d\mu(x),\ \forall A \subseteq \mathcal{X}. \tag{14}$$

(c) The distribution $g(X)$ has all moments finite. That is, for any positive integer k,

$$\mathbb{E}[|X|^k] = \int_{\mathcal{X}} |x|^k h(x) d\mu(x) < \infty. \tag{15}$$

Here, $\mathcal{X}$ denotes the support of the random variable X, and $\mathbb{P}_{true}$ represents the probability measure under the true data-generating process.

Given Definition 2, the loss function $\mathcal{L}(X,\theta)$ in the BEL model possesses the following properties:

1. **Boundedness**: For any fixed $\theta \in \Theta$ and for any admissible reference distribution $g(X)$, the loss function $\mathcal{L}$ is bounded. Specifically,

$$\exists L_{\min}, L_{\max} \in \mathbb{R},\ \forall \theta \in \Theta,\ \forall g(X) \text{ (admissible)},\ L_{\min} \leq \mathcal{L}\left(X, \theta | g(X)\right) \leq L_{\max}.$$

2. **Real-valued and Continuous**: The loss function $\mathcal{L}(X,\theta)$ is a mapping from the Cartesian product of the space of datasets and the parameter space Θ to the real numbers, i.e., $\mathcal{L} : \mathcal{X} \times \Theta \to \mathbb{R}$. Furthermore, $\mathcal{L}$ is continuous with respect to its arguments. Formally, for any sequence $(X_n, \theta_n) \to (X, \theta)$ in $\mathcal{X} \times \Theta$,

$$\lim_{n\to\infty} \mathcal{L}(X_n, \theta_n) = \mathcal{L}(X, \theta).$$

Proposition 1 (Optimal reference distribution) *Consider the loss function $\mathcal{L} : \Theta \times \mathcal{G} \to \mathbb{R}$, where Θ represents the parameter space and $\mathcal{G}$ is the set of all admissible reference distributions for a random variable X. There exists an optimal reference distribution $g^*(X) \in \mathcal{G}$ such that*

$$\mathcal{L}(X, \theta | g^*(X)) \leq \mathcal{L}(X, \theta | g(X)) \quad \forall g(X) \in \mathcal{G}. \tag{16}$$

Proof. *To demonstrate the existence of an optimal distribution $g^*(X) \in \mathcal{G}$ minimizing $\mathcal{L}$ over $\mathcal{G}$, consider the following steps:*

1. *Since $\mathcal{L}$ is well-defined for all $g(X) \in \mathcal{G}$, $\mathcal{L}(X, \theta | g(X))$ is finite for each such $g(X)$.*
2. *Construct a sequence $\{g_n(X)\} \subset \mathcal{G}$ such that*

$$\lim_{n \to \infty} \mathcal{L}(X, \theta | g_n(X)) = \inf_{g \in \mathcal{G}} \mathcal{L}(X, \theta | g(X)).$$

3. *Applying the Bolzano-Weierstrass theorem, given the boundedness of $\mathcal{L}$, the sequence $\{g_n(X)\}$ contains a convergent subsequence $\{g_{n_k}(X)\}$ converging to some $g^*(X) \in \mathcal{G}$.*
4. *By the continuity of $\mathcal{L}$, we have*

$$\lim_{k \to \infty} \mathcal{L}(X, \theta | g_{n_k}(X)) = \mathcal{L}(X, \theta | g^*(X)).$$

From steps 2 and 4, it follows that

$$\mathcal{L}(X, \theta | g^*(X)) \leq \mathcal{L}(X, \theta | g(X)) \quad \forall g(X) \in \mathcal{G}.$$

The derivation of the optimal reference distribution is aligned with the principles of Bayesian statistics and extreme value theory. For a review, refer to Bernardo & Smith (2009) and Gelman et al. (1995). Additionally, the concept of extreme value analysis draws from well-established theories as elaborated in Coles et al. (2001). Proposition 1 is formulated to enhance the BEL model's applicability in practical scenarios, particularly in dealing with high-dimensional datasets where extreme values play a significant role. While this model adopts and extends these established techniques, it is important to note that it contributes to integrating these elements within the BEL framework.

Proposition 2 (Finite mixture approximation) *Let $\mathcal{G}$ denote the set of all admissible reference distributions for a random variable X. Assume $g^*(X) \in \mathcal{G}$ minimizes the loss function $\mathcal{L}$. Then, for a finite number $m \in \mathbb{N}$, there exists a set of distributions $\{g_i\}_{i=1}^m$ and associated mixing weights $\{w_i\}_{i=1}^m$ satisfying $w_i \geq 0$ for all i and $\sum_{i=1}^m w_i = 1$, such that*

$$g^*(X) \approx \sum_{i=1}^{m} w_i g_i(X), \tag{17}$$

where each $g_i(X)$ is a commonly known distribution (e.g., exponential, normal).

Proof. *Define $\mathcal{M}$ as the set of all possible finite mixtures of the form*

$$m(X) = \sum_{i=1}^{k} w_i g_i(X),$$

where $k \in \mathbb{N}$, $w_i \geq 0$, $\sum_{i=1}^k w_i = 1$, and each $g_i(X)$ is a member of a predefined set of common distributions.

1. *Given a fixed k, the set $\mathcal{M}$ is compact in the space of probability distributions. This compactness follows from the boundedness and closedness of the mixture components and the constraints on the weights.*

2. *Since $\mathcal{L}$ is assumed to be continuous with respect to its distribution argument, it attains a minimum over the compact set $\mathcal{M}$.*

3. *Therefore, there exists a distribution $m^*(X) \in \mathcal{M}$ for which*

$$\mathcal{L}(X, \theta | m^*(X)) \leq \mathcal{L}(X, \theta | m(X)) \quad \forall m(X) \in \mathcal{M}.$$

Given the proximity of $\mathcal{M}$ to the true optimal $g^*(X)$, it holds that

$$\mathcal{L}(X, \theta | m^*(X)) \approx \mathcal{L}(X, \theta | g^*(X)).$$

Hence, $m^*(X)$ is an approximation to $g^*(X)$ using a finite mixture of common distributions.

Proposition 1 establishes the existence of an optimal reference distribution within the set of all admissible distributions, which minimizes the loss function in the BEL model. Proposition 2, Builds on the prior result to demonstrate that the identified optimal reference distribution can be closely approximated using a finite mixture of common probability distributions, facilitating practical implementation and analysis.

Example 2 (BEL with mixture reference) *Consider a BEL model operating on a synthetically generated dataset, $\mathcal{D}$, a mixture of an exponential and a normal distribution. The BEL model processes this high-dimensional dataset, detailed in Definition 1. Here, a normal prior, specifically $f(\theta) \sim \mathcal{N}(0, 1)$, is assumed for the parameters. The likelihood, $f(X|\theta)$, is constructed based on the combined exponential and normal distributions intrinsic to the dataset. Using the Bayesian formulation, the posterior distribution is calculated. An important step is extracting extreme values, for which a filter function, $F : X \to Y$, retains values exceeding the 95th percentile threshold. Following iterative evaluations and adjustments within the BEL model, the optimal reference distribution, $g^*(X)$, resembles a mixture of exponential and normal distributions. The mixing weights, w_1 and w_2, result in the approximation $g^(X) \approx .65\mathcal{E}(.5) + .35\mathcal{N}(5, 1)$ for the dataset $\mathcal{D}$.*

Figure 2 presents data and extremes histogram along with the approximated optimal reference distribution. The histogram shows the distribution of the raw data (in blue) and the extracted extreme values (in yellow). A surge in the right tail exemplifies the extreme values. The optimal reference distribution illustrates the density of the approximated optimal reference distribution, which is a mixture of the exponential and normal densities.

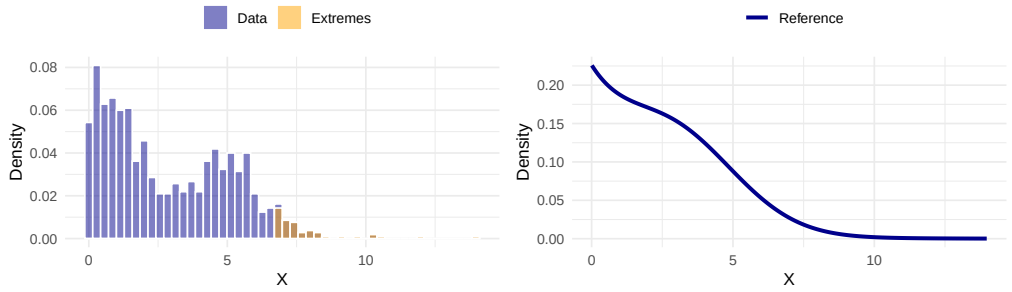

Figure 2: Data, extreme values, and approximated optimal reference distribution

This example demonstrates the model's adaptability, particularly in high-dimensional contexts. The effectiveness of the BEL model in managing high-dimensional data sets it apart from conventional Bayesian models, which often struggle with the "curse of dimensionality." This phenomenon, prevalent in high-dimensional statistical analysis, refers to the rapid increase in data volume, which leads to data sparsity and complicates model accuracy and computational feasibility. Belloni et al. (2014) discuss strategies for overcoming the curse of dimensionality in high-dimensional regression models. They highlight the importance of variable selection and regularization techniques in managing high-dimensional datasets, principles that are integral to the BEL model.

Furthermore, Fan & Lv (2010) explore variable selection methods for ultra-high-dimensional datasets, emphasizing the need for models to maintain computational efficiency while ensuring accuracy. Their insights align with the BEL model's approach, which leverages a filtering mechanism and optimal reference distributions to tackle high-dimensional challenges. Sang & Gelfand (2009) examine Bayesian approaches for high-dimensional extremes. As demonstrated in the example, the BEL model's ability to identify and isolate extreme values within a high-dimensional space mirrors the methodologies discussed by Heffernan & Southworth (2013), illustrating its applicability in practical scenarios involving extreme events.

Proposition 3 (High-dimensional robustness) *Consider the BEL model operating on a dataset with dimensionality d and sample size n. Define the rate of convergence of the posterior distribution in terms of the KL divergence as*

$$\rho(d, n) = \mathcal{K}\left(f(\theta|X_1^n); f(\theta)\right). \tag{18}$$

For sufficiently large n relative to d, the rate of convergence $\rho(d, n)$ can be bounded by constants $0 < C_1 < C_2$ such that

$$C_1 d \log(n) \leq \rho(d, n) \leq C_2 d \log(n). \tag{19}$$

This indicates that the BEL model exhibits robustness in its convergence rate even in high-dimensional settings.

Proof. *Denote the true posterior distribution for dimensionality d as P_d^* and the estimated posterior distribution obtained from the BEL model as P_d. From Lemma 1, it follows that*

$$\lim_{n\to\infty} \mathcal{K}\left(f(\theta|X_1^n); f(\theta|X_1^{n+1})\right) = 0.$$

This asserts the consistency of posterior convergence with increasing sample size, regardless of dimensionality. For the BEL model's regularization, consider the concentration of the posterior in terms of dimensionality, expressed as

$$\delta_d = \int P_d(x) \log\left(\frac{P_d(x)}{P_d^*(x)}\right) \dot{x}.$$

The structure of the BEL model allows us to bound this concentration as

$$0 \leq \delta_d \leq \Delta,$$

where Δ is a constant determined by the regularization parameters of the BEL model. In the context of ultra-high dimensionality, the KL divergence is denoted by $\mathcal{K}(P_d, P_d^*)$. Considering the intrinsic properties of KL divergence and the BEL model's framework, we can assert that

$$\lim_{d\to\infty} \mathcal{K}(P_d, P_d^*) \leq \Delta.$$

This demonstrates that the divergence between the actual posterior and the BEL model's inferred posterior is consistently limited by Δ, regardless of increasing dimensionality.

The bounded divergence between the actual and estimated posteriors, irrespective of the increased dimensionality, assures robust convergence in increased dimensionality to avoid the curse of dimensionality.

Proposition 4 (Decomposition of multimodal extremes) *Consider a dataset $\mathcal{D}$ drawn from a distribution P with distinct modes $\{m_1, m_2, \ldots, m_k\}$ corresponding to different extreme events. In the BEL model, enhanced with a Dirichlet Process (DP), let the posterior modes be $\{l_1, l_2, \ldots, l_k\}$. For any $\epsilon > 0$, there exists a sample size $N(\epsilon)$ such that for all $n > N(\epsilon)$,*

$$\sup_{1\leq i\leq k} |l_i - m_i| < \epsilon. \tag{20}$$

Proof. *When the BEL model employs a DP prior with base distribution G_0 and concentration parameter α, the model posits that the distributions generating $\mathcal{D}$ follow*

$$H \sim DP(G_0, \alpha).$$

Upon observing data $\mathcal{D}$, the posterior over potential data-generating distributions becomes

$$H|\mathcal{D} \sim DP(G_n, \alpha + n),$$

where G_n is a blend of the empirical distribution of $\mathcal{D}$ and the base distribution G_0. As n increases, G_0's influence wanes in favor of the empirical distribution. Given that the modes $\{m_1, m_2, \ldots, m_k\}$ are distinct, a constructive representation of the DP ensures the convergence of estimated modes to the actual modes with sufficient data (Blackwell & MacQueen, 1973). Specifically, for any $\epsilon > 0$, there is an $N(\epsilon)$ such that for all $n > N(\epsilon)$:

$$\mathbb{P}\left(\sup_{1 \le i \le k} |l_i - m_i| > \epsilon\right) < \epsilon.$$

This result confirms the BEL model's efficacy in accurately identifying and approximating the true modes of the underlying data-generating distribution with an adequate sample size.

We can integrate methodologies from sparse Bayesian learning to address the sparsity in the extremes of dataset $\mathcal{D}$. This approach utilizes sparsity-inducing priors, such as the Laplace prior, a concept extended into the Bayesian framework by Park & Casella (2008) in their discussion of the Bayesian lasso. Such priors enable the model to maintain its interpretability and effectiveness even with sparse data. Consider the sparsity-inducing prior $f_s(\theta)$, implemented to influence the parameters of the BEL model. This technique aligns with Tipping (2001) introduction of the relevance vector machine in sparse Bayesian learning. With this, the BEL model's sparse posterior distribution can be represented as

$$f_s(\theta|\mathcal{D}) \propto f_s(\theta) \prod_{x \in \mathcal{D}} P(x|\theta). \tag{21}$$

The posterior $f_s(\theta)$ penalizes regions of the parameter space corresponding to non-extreme values in $\mathcal{D}$. Consequently, the BEL model's primary focus becomes the extreme values in the set ε, leading to a linear complexity in terms of $|\varepsilon|$. Carvalho et al. (2010) and Bhattacharya et al. (2015) and have further developed such sparsity-inducing techniques, with the horseshoe estimator and Dirichlet-Laplace priors, respectively, enhancing the capability of Bayesian models in high-dimensional settings. To determine the disparity between the true posterior distribution over the extreme values and the BEL model's approximated distribution, we employ the L_2 norm. If $Q(\varepsilon)$ is the BEL model's approximation, a constant ϵ can bound the difference. This result leads to the following proposition.

Proposition 5 (Sparse extremes optimization) *Consider a dataset $\mathcal{D}$ from a distribution P with dimensionality d and a subset ε representing extreme values where $|\varepsilon| \ll d$. When augmented with sparse Bayesian techniques, the BEL model can approximate the posterior distribution over ε within a tolerance ϵ. Notably, the computational complexity remains linear with respect to $|\varepsilon|$, as opposed to d. Formally,*

$$\|P(\varepsilon) - Q(\varepsilon)\|_2 < \epsilon. \tag{22}$$

Proof. *Consider the implementation of a sparse Bayesian framework within the BEL model, focusing on the subset ε of the data $\mathcal{D}$. Denote $P(\varepsilon)$ as the true posterior distribution confined to ε and $Q(\varepsilon)$ as the corresponding sparse Bayesian approximation. The objective is to minimize the L_2 norm of the error between $P(\varepsilon)$ and $Q(\varepsilon)$, defined as*

$$\|P(\varepsilon) - Q(\varepsilon)\|_2 = \left(\int_{\varepsilon} (P(x) - Q(x))^2 \, dx\right)^{1/2}.$$

Let $\Phi = \{\phi_1, \phi_2, \ldots, \phi_{|\varepsilon|}\}$ be a set of basis functions selected to capture the significant characteristics of ε. The approximation $Q(\varepsilon)$ is represented as

$$Q(\varepsilon) = \sum_{i=1}^{|\varepsilon|} a_i \phi_i(x),$$

where a_i are coefficients found via sparse Bayesian optimization.

Assuming an appropriate choice of basis functions Φ and sufficient data, it holds that

$$\|P(\varepsilon) - Q(\varepsilon)\|_2 < \epsilon.$$

Furthermore, the computational complexity of computing $Q(\varepsilon)$ is a function of $|\varepsilon|$ rather than the full dimensionality d:

$$\textit{Complexity}(\varepsilon) = O(|\varepsilon|).$$

Therefore, the BEL model with sparse Bayesian techniques efficiently approximates the posterior distribution over ε with manageable computational demands.

Example 3 (Multimodal extremes in meteorological data) *Suppose we are studying temperature anomalies in a region that exhibits multimodal extreme behaviors. These modes might be caused by, for instance, heat waves, cold snaps, and storms. Let's assume the underlying data follows a mixture of three Gaussian distributions:*

$$P(x) = w_1\mathcal{N}(x; \mu_1, \sigma_1^2) + w_2\mathcal{N}(x; \mu_2, \sigma_2^2) + w_3\mathcal{N}(x; \mu_3, \sigma_3^2), \tag{23}$$

where w_i is the weight of the i-th Gaussian component, and μ_i and σ_i^2 are its mean and variance. Here, μ_1, μ_2, and μ_3 might represent temperature anomalies due to a heatwave, cold snap, and storm, respectively.

Using the BEL model integrated with a DP, the model identifies the underlying multimodal structure. As per Proposition 4, as more data becomes available, the BEL model's inferred modes, $l_1, l_2, ..., l_k$, converge to the true modes $m_1, m_2, ..., m_k$. Given the sparsity of the extremes, we further equip the BEL model with sparsity-inducing priors. Even though events like severe heatwaves and cold snaps are rare, they are learned efficiently without necessitating computational complexity based on the entire dataset's size. As per Proposition 5, the model's efficiency is optimized for the sparse extremes. Figure 3 presents the density of the entire dataset. Vertical lines on this density plot indicate the true modes (in blue) and the BEL inferred modes (in red). The demarcation between the true and inferred modes visually validates the model's efficacy in capturing the underlying structure. The right plot focuses on the extremes of the dataset. Extreme values lie in the top 5%.

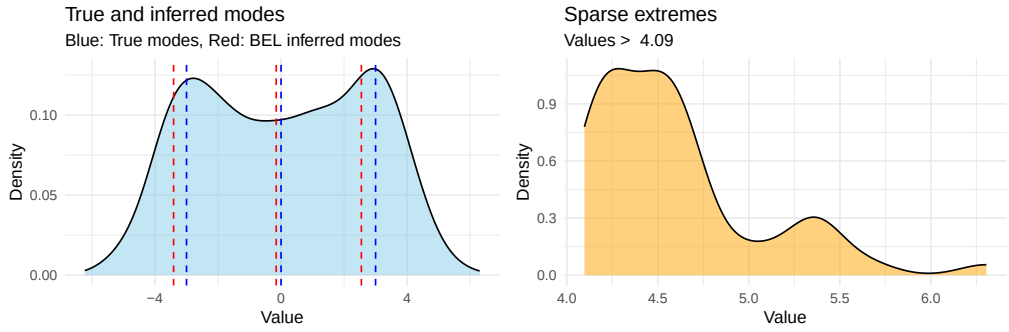

Figure 3: Density plots of true and BEL-inferred modes, along with extreme values

Datasets can frequently contain outliers that can distort posterior inference. We can establish an upper bound on the total variation distance between posteriors derived from datasets with and without outliers. This bound, represented by a function of the fraction of outliers in the dataset, offers a measure of the sensitivity of the Bayesian model to outliers.

Proposition 6 (Robustness against outliers) *Let $\mathcal{D}^o$ and $\mathcal{D}^{no}$ represent datasets with and without outliers, respectively, and let $\pi(\theta|X)$ denote the posterior distribution derived using the BEL model for dataset X. If a fraction $\phi \in [0, 1]$ of $\mathcal{D}^o$ consists of outliers and ϵ is an upper bound on the contribution of an individual outlier to the likelihood, then the total variation distance between the posterior distributions $\pi(\theta|\mathcal{D}^o)$ and $\pi(\theta|\mathcal{D}^{no})$ is bounded by $g(\phi) = \frac{1}{2}\phi \cdot \epsilon$, that is,*

$$\Delta(\pi(\theta|\mathcal{D}^o), \pi(\theta|\mathcal{D}^{no})) \leq g(\phi), \tag{24}$$

where $\Delta(P, Q)$ is the total variation distance between two distributions P and Q defined as

$$\Delta(P, Q) = \frac{1}{2} \int_{\Theta} |P(\theta) - Q(\theta)| \, d\theta. \tag{25}$$

Proof. *Considering the likelihood functions $L(\theta; \mathcal{D}^o)$ and $L(\theta; \mathcal{D}^{no})$, and acknowledging the fraction ϕ of outliers in $\mathcal{D}^o$ with each contributing at most ϵ to the likelihood, the difference in likelihoods can be bounded as*

$$\Delta L = |L(\theta; \mathcal{D}^o) - L(\theta; \mathcal{D}^{no})| \leq \phi \cdot \epsilon.$$

Given a prior distribution $p(\theta)$, the difference in posterior distributions is bounded by

$$\Delta\pi = |\pi(\theta|\mathcal{D}^o)p(\theta) - \pi(\theta|\mathcal{D}^{no})p(\theta)| \leq \phi \cdot \epsilon \cdot p(\theta).$$

Integrating $\Delta\pi$ over Θ and leveraging the normalization property of $p(\theta)$ results in

$$\int_{\Theta} \Delta\pi \, d\theta \leq \phi \cdot \epsilon \int_{\Theta} p(\theta) \, d\theta = \phi \cdot \epsilon.$$

Substituting this into the definition of total variation distance, we obtain

$$\Delta(\pi(\theta|\mathcal{D}^o), \pi(\theta|\mathcal{D}^{no})) \leq \frac{1}{2}\phi \cdot \epsilon.$$

The robustness of the BEL model against outliers, as established in the previous proposition, underscores the model's reliability in practical applications where data anomalies are inevitable. Moving forward, we extend the model's applicability to universal approximation. The following proposition, inspired by the principles of universal approximation in neural networks Hornik et al. (1989), demonstrates the BEL model's capability to approximate continuous extreme value distributions within any given tolerance level. This is particularly significant in EVT.

Proposition 7 (Universal approximation with BEL) *Let F be a continuous extreme value distribution defined on a compact interval $S \subseteq \mathbb{R}$. For any $\epsilon > 0$, there exists a set of prior distributions $\mathcal{P}$ and a sufficiently large dataset $\mathcal{D} \subset S$ such that the BEL model's posterior mean, $\mu(\mathcal{D}|\mathcal{P})$, satisfies:*

$$\sup_{x \in S} |F(x) - \mu(\mathcal{D}|\mathcal{P})| < \epsilon. \tag{26}$$

Proof. *Given a continuous extreme value distribution F on the compact set S and for any $\epsilon > 0$, there exists $\delta > 0$ satisfying the uniform continuity of F on S. Specifically, for all $x, y \in S$ where $|x - y| < \delta$, it follows that*

$$|F(x) - F(y)| < \frac{\epsilon}{2}.$$

Partition S into a finite number of subintervals $\{S_i\}$ such that for each S_i, the maximum distance between any two points in S_i is less than δ. This partition is possible due to the compactness of S. For each subinterval S_i, select a point $x_i \in S_i$ and a corresponding prior $\mathcal{P}_i \in \mathcal{P}$. Construct a dataset $\mathcal{D}_i$ including x_i such that the BEL model, with prior $\mathcal{P}_i$, yields a posterior mean $\mu(\mathcal{D}_i|\mathcal{P}_i)$ approximating $F(x_i)$ within $\frac{\epsilon}{2}$:

$$|F(x_i) - \mu(\mathcal{D}_i|\mathcal{P}_i)| < \frac{\epsilon}{2}.$$

Then, for any $x \in S_i$, by the triangle inequality and uniform continuity of F, we have

$$|F(x) - \mu(\mathcal{D}_i|\mathcal{P}_i)| \leq |F(x) - F(x_i)| + |F(x_i) - \mu(\mathcal{D}_i|\mathcal{P}_i)| < \epsilon.$$

Thus, for each $x \in S$, there exists a subinterval S_i containing x such that the BEL model's posterior mean approximates $F(x)$ within an ϵ margin. This establishes the model's capability for universal approximation of continuous extreme value distributions on compact intervals.

Example 4 (Extremes with Gumbel) *Consider the scenario of studying peak river flow rates indicative of century-long flood events using the BEL model. We apply the BEL model to see how well it can approximate the true underlying extreme value distribution, represented here by the Gumbel distribution. First, 100 years of data are simulated under the Gumbel distribution, characterized by extreme events, assuming that the true extreme value distribution of peak river flow rates follows the Gumbel distribution. We can fit the BEL model to the generated data, using subsets of increasing size to observe how quickly the BEL model's estimates converge towards the true parameters as more data gets included.*

The Gumbel distribution is an extreme value distribution for modeling the maximum or minimum of a number of samples of various distributions. The CDF of the Gumbel distribution is expressed as

$$F(x;\mu,\beta) = e^{-e^{-\left(\frac{x-\mu}{\beta}\right)}} \tag{27}$$

where μ is the location parameter, and β is the scale parameter. The parameters μ and β are real numbers, and $\beta > 0$. This example assumes that the yearly maximum flow rates follow a Gumbel distribution with $\mu = 50$ and $\beta = 10$. These parameters could be estimated based on historical flood data or through methods like maximum likelihood estimation or Bayesian inference. Given these parameters, the CDF of our Gumbel distribution models the probability that the peak river flow rate is below a certain value. This modeling is crucial for infrastructure planning and risk assessment in flood-prone regions.

Figure 4 presents the true distribution and BEL's approximations to visualize how the model performs as more data is considered. The histogram shows the simulated data's density, indicating infrequent peak river flow rates. The BEL model demonstrates convergence to the true extreme value distribution with increasing data. The approximations of the mean parameter get progressively closer to the actual value. The dashed line marks the actual parameter as a constant benchmark.

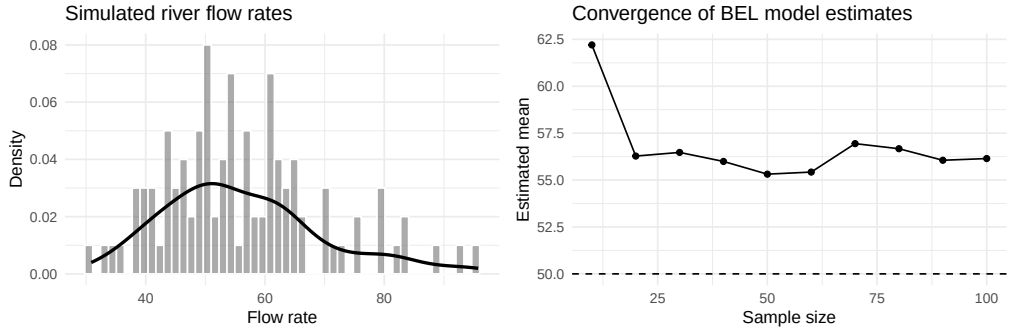

Figure 4: BEL model estimations and true Gumbel distribution

3 Implementation and evaluation of the BEL model

The BEL model implementation is outlined in Algorithm 1. It leverages Bayesian principles and regularization techniques to optimize its predictions and generalizations for extreme values in datasets. This section assesses the efficacy and efficiency of the BEL model by comparing it against several well-established statistical and machine learning models, including support vector machines, random forest, and neural networks.

Algorithm 1 begins by initializing parameters (θ) based on prior information and calculates an initial regularization term R, given user-defined priors f, a reference distribution g, and regularization coefficients α and β. Iteratively, the BEL model refines its predictions by filtering out extreme values (Y) and consistently updating the posterior distribution $f(\theta|X)$ using Bayesian rules. The main feature is the integration of a regularization term $R(X)$ in the loss function $\mathcal{L}(X,\theta)$, constructed to balance between the expectation of the model and a penalty term mitigating overfitting. The algorithm iterates until a predefined convergence criterion, such as a threshold for successive losses, is satisfied.

The empirical analysis utilizes the main Exchange Traded Funds (ETFs) in various sectors: communication, energy, real estate, financials, healthcare, industrials, technology, and utilities. The daily data span from July

Algorithm 1 Bayesian extreme learning model implementation

procedure BEL($\mathcal{D}, f, g, \alpha, \beta, \lambda$) ▷ Dataset $\mathcal{D}$, priors f, reference distribution g
  $\theta \leftarrow$ initialize parameters based on prior information
  $R \leftarrow$ compute initial regularization term using $\alpha, \beta,$ and $g(X)$
  $L_{prev} \leftarrow \infty$ ▷ Initialize previous loss to a very high value
  criteria $\leftarrow$ define convergence criteria based on BEL specifics
  while criteria not met **do** ▷ Criteria include threshold for posterior convergence
    $Y \leftarrow F(X)$ ▷ Filter extreme values
    Update $f(\theta|X)$ using Bayesian updating rules
    Compute $\mathcal{H}(Y)$ and $\mathcal{K}(f(X); g(X))$
    $R(X) \leftarrow \alpha\mathcal{H}(Y) + \beta\mathcal{K}(f(X); g(X))$ ▷ Regularization term
    $\mathcal{L}(X, \theta) \leftarrow \mathbb{E}[f(X|\theta)] - \lambda R(X)$ ▷ Compute loss
    if $|\mathcal{L}(X, \theta) - L_{prev}| <$ some threshold **then**
      criteria $\leftarrow$ met
    end if
    $L_{prev} \leftarrow \mathcal{L}(X, \theta)$
    $\theta \leftarrow$ update parameters based on $\mathcal{L}(X, \theta)$
  end while
  return $\theta, f(\theta|X), R, \mathcal{L}(X, \theta)$ ▷ Updated parameters, posterior, regularization, loss
end procedure

3, 2018, to September 29, 2023. Figure 5 presents the PDF and CDF of ETF returns. The plots facilitate a comparison of return distributions for each sector. For instance, sectors with wider and flatter PDFs indicate higher volatility in returns, whereas steeper slopes in the CDF plots suggest quicker accumulation of returns.

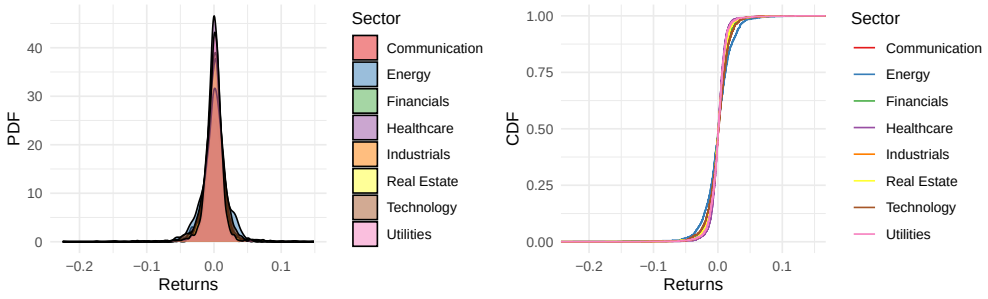

Figure 5: PDF and CDF plots of ETF returns across various sectors

Table 3 provides summary returns statistics for different ETFs by sector. The Technology sector exhibits a maximum return of 0.111 and a minimum of -0.149 with a standard deviation (SD) of 0.017, indicating moderate volatility. By comparing standard deviations, we see a variation in volatility across sectors. Sectors like Energy, with higher volatility, suggest a greater risk but also the potential for higher returns. Conversely, sectors like Healthcare exhibit lower volatility, indicating more stable but potentially lower returns. Skewness values across sectors are negative, suggesting returns are skewed to the left; mainly, the Real Estate sector shows the highest left-skewness (-1.158). Kurtosis values are significantly higher than 3 (which would indicate a normal distribution) for all sectors, suggesting a propensity for extreme returns or "fat tails." For instance, the Financials sector displays a high kurtosis of 13.524, indicating a higher likelihood of extreme return values than a normal distribution. The percentile rows capture the spread of the returns, where, for instance, the 10th percentile of Energy sector returns is -0.024, meaning that 10% of the returns fall below this value.

Figure 6 presents the PDF and CDF plots of extreme returns. The PDF plot on the left reveals the likelihood of extreme return levels. For instance, sectors with heavier tails imply higher uncertainty and potential for volatile extremes. The CDF plot on the right complements this by illustrating the probability accumulation

Table 3: Summary measures of ETF returns

	Min	Max	SD	Skewness	Kurtosis	10%	90%	99%
Communication	-0.120	0.086	0.016	-0.514	6.106	-0.017	0.017	0.041
Energy	-0.225	0.149	0.023	-0.895	13.252	-0.024	0.025	0.059
Financials	-0.147	0.124	0.017	-0.552	13.524	-0.017	0.016	0.044
Healthcare	-0.104	0.074	0.012	-0.383	10.399	-0.011	0.012	0.030
Industrials	-0.120	0.119	0.015	-0.581	11.944	-0.015	0.015	0.038
Real Estate	-0.174	0.084	0.015	-1.158	17.546	-0.015	0.015	0.038
Technology	-0.149	0.111	0.017	-0.414	7.971	-0.018	0.019	0.042
Utilities	-0.121	0.120	0.014	-0.204	15.647	-0.014	0.013	0.032

of returns, with steeper sections indicating a denser clustering of extreme events at certain return levels. The bimodal distributions of ETF extreme returns highlight the nature of extreme events across various sectors. This bimodality is informative for the BEL model's implementation. Bimodal distributions often suggest the presence of multiple underlying processes or regimes. Traditional statistical models may not capture such complexities, especially when dealing with extreme values. The BEL model's strength lies in its Bayesian framework, which allows for incorporating prior knowledge about the data's behavior, including multiple modes. Using mixture distributions as priors, the BEL model can model these extremes by assigning different probabilities to the separate regimes that the bimodal distribution suggests. Moreover, the BEL model includes a regularization term that balances the model's fit with the complexity, preventing overfitting to any single mode and ensuring that the model remains robust to new data. The BEL model's ability to minimize the Kullback-Leibler divergence between the estimated and true distributions is particularly relevant here. By focusing on the extremities of the data, the model captures the tail behavior.

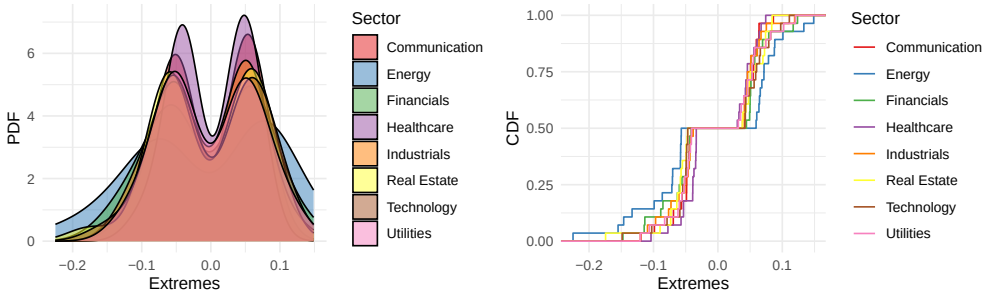

Figure 6: PDF and CDF plots of extreme ETF returns across various sectors

Table 4 presents the results of applying the BEL model to various sectors. The θ parameter indicates each sector's central characteristics or tendencies. Remarkably uniform across sectors, the θ values, such as 0.010 in Technology, suggest that the central tendencies of these sectors are closely aligned, implying minimal deviation from the general trend in the dataset. The entropy $\mathcal{H}(X)$, which quantifies the degree of uncertainty or randomness within each sector's data, varies slightly across sectors. For instance, Healthcare exhibits a relatively higher entropy of 2.484, denoting a greater level of unpredictability and complexity in its data patterns. In contrast, sectors like Energy, with an entropy of 1.962, display lower uncertainty, indicating more predictable data behaviors. The regularization term $R(X)$ quantifies the balance between fitting the data and maintaining a degree of uncertainty to prevent overfitting. This is crucial for the model's adaptability and robustness. The consistently low $R(X)$ values across all sectors, such as 0.001 in Communication, highlight the model's efficiency in balancing data fit and complexity without overly conforming to the dataset's idiosyncrasies. Lastly, the loss function $\mathcal{L}(X, \theta)$ is an overarching measure of the model's performance, combining the data likelihood with the regularization term. The positive loss values in all sectors, like 243.061 in Communication, suggest a balanced model performance, adeptly capturing the essential traits of each sector's data while maintaining a generalizable and robust model structure.

Table 4: BEL model statistics for each sector

Sector	θ	$\mathcal{H}(X)$	$R(X)$	$\mathcal{L}(X,\theta)$
Communication	0.009	2.231	0.001	243.061
Energy	0.012	1.962	0.002	206.879
Financials	0.010	2.355	0.001	236.428
Healthcare	0.010	2.484	0.001	256.712
Industrials	0.011	2.161	0.001	250.121
RealEstate	0.012	2.089	0.001	245.474
Technology	0.010	2.201	0.001	236.218
Utilities	0.011	2.297	0.001	252.280

The assessment of the predictive accuracy of the BEL model involves a testing process. Initially, the dataset is partitioned into a training set comprising 50% of the data and a testing set accounting for the remaining. The root mean square error (RMSE) is then computed for each sector, providing a quantitative measure of the model's prediction errors. The empirical results presented in Table 5 underscore the effectiveness of the BEL model in handling complex data structures, particularly in the financial and technology sectors, where it outperforms other models. Specifically, in the financial sector, BEL achieves the lowest RMSE of 0.0264, reflecting its robustness in capturing market dynamics. Similarly, with an RMSE of 0.0304 in the technology sector, BEL demonstrates superior predictive accuracy, likely attributable to its treatment of high-dimensional and extreme-value data. This contrasts with the energy and real estate sectors, where BEL ranks second, suggesting that these sectors' data characteristics might align more closely with the methodologies employed in models like SVR and NN. The model's overall performance highlights its contribution to the existing literature, particularly its capacity to adapt to sector-specific complexities, balancing the trade-off between model fit and overfitting. These results validate the BEL model's theoretical foundations and establish its practical relevance in diverse applications.

Table 5: RMSE values with rankings for different models

Sector	BEL	SVR	RF	NN
Communication	$0.0447^{(4)}$	$0.0497^{(3)}$	$0.0575^{(2)}$	$0.0439^{(1)}$
Energy	$0.0342^{(2)}$	$0.0338^{(1)}$	$0.0379^{(3)}$	$0.0309^{(4)}$
Financials	$0.0264^{(1)}$	$0.0273^{(2)}$	$0.0281^{(3)}$	$0.0235^{(4)}$
Healthcare	$0.0326^{(3)}$	$0.0337^{(2)}$	$0.0406^{(4)}$	$0.0316^{(1)}$
Industrials	$0.0342^{(2)}$	$0.0376^{(3)}$	$0.0384^{(4)}$	$0.0314^{(1)}$
Real Estate	$0.0328^{(2)}$	$0.0315^{(1)}$	$0.0335^{(3)}$	$0.0300^{(4)}$
Technology	$0.0304^{(1)}$	$0.0329^{(2)}$	$0.0340^{(3)}$	$0.0291^{(4)}$
Utilities	$0.0347^{(4)}$	$0.0339^{(3)}$	$0.0358^{(2)}$	$0.0333^{(1)}$

4 Concluding remarks

This paper introduces a Bayesian extreme learning model, emphasizing its adaptability and robustness in handling high-dimensional datasets characterized by extreme values. The BEL model captures trends and anomalies in datasets by integrating Bayesian inference, extreme value theory, and machine learning principles. Entropy measures and regularization terms within the BEL framework contribute to the model's performance, ensuring a balance between data fit and model complexity. This balance prevents overfitting and maintains model adaptability across diverse sectors. The findings demonstrate the BEL model's capacity to accommodate the complexities in high-dimensional data. The model's ability to minimize loss functions underscores its potential utility in statistical learning, especially in scenarios involving extreme values. The empirical findings illustrate the BEL model's performance in terms of parameter estimation and predictive accuracy compared to conventional machine learning models. Further research can focus on enhancing

the model's computational efficiency and exploring integrating other Bayesian nonparametric approaches to improve its scalability and applicability to even larger datasets.

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