PROVABLE ADAPTIVITY IN ADAM

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ABSTRACT

Adaptive Moment Estimation (Adam) has been observed to converge faster than stochastic gradient descent (SGD) in practice. However, such an advantage has not been theoretically characterized – the existing convergence rate of Adam is no better than SGD. We attribute this mismatch between theory and practice to a commonly used assumption: the smoothness is globally upper bounded by some constant L (called *L-smooth condition*). Specifically, compared to SGD, Adam adaptively chooses a learning rate better suited to the *local smoothness*. This advantage becomes prominent when the local smoothness varies drastically across the domain, which however is hided under L-smooth condition. In this paper, we analyze the convergence of Adam under a condition called (L_0, L_1)-smooth condition, which allows the local smoothness to grow with the gradient norm. This condition has been empirically verified to be more realistic for deep neural networks (Zhang et al., 2019a) than the L-smooth condition. Under (L_0, L_1)-smooth condition, we establish the convergence for Adam with practical hyperparameters. As such, we argue that Adam can adapt to the local smoothness, justifying Adam's *adaptivity*. In contrast, SGD can be arbitrarily slow under this condition. Hence, we theoretically characterize the advantage of Adam over SGD.

1 INTRODUCTION

Machine learning tasks are often formulated as solving the following finite-sum problem.

$$\min_{\boldsymbol{w}\in\mathbb{R}^d} f(\boldsymbol{w}) = \sum_{i=0}^{n-1} f_i(\boldsymbol{w}), \tag{1}$$

where $\{f_i(\boldsymbol{w})\}_{i=0}^{n-1}$ is lower bounded, n denotes the number of samples or mini-batches, and $\boldsymbol{w}$ denotes the trainable parameters. Among various gradient-based optimizers, Stochastic Gradient Descent (SGD) is a simple and popular method to solve Eq. (1). However, Adaptive gradient methods including Adaptive Moment estimation (Adam) (Kingma & Ba, 2014) is recently observed to outperform SGD in modern deep neural tasks including GANs (Brock et al., 2018), BERTs (Kenton & Toutanova, 2019), GPTs (Brown et al., 2020) and ViTs (Dosovitskiy et al., 2020). For instance, as reported in Figure 1. (a): SGD converges much slower than Adam during the training of Transformers. Similar phenomena are also reported in BERT training (Zhang et al., 2019b).

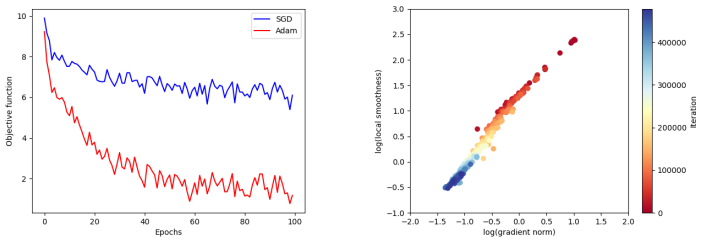
The empirical success of Adam comes from its special update rules. Firstly, it uses the heavy-ball momentum mechanism controlled by a hyperparameter β_1. Second, it uses an adaptive learning rate strategy. In particular, the learning rate of Adam contains exponential moving averages of past squared gradients, which is weighted by β_2. Larger β_1 and β_2 will bring more history information into the update. The update rule of Adam is given in Eq. (2) (presented later in Section 3).

Despite its practical success, the theoretical understanding of Adam is limited. For instance, the existing convergence rates of Adam are no better than that of SGD (Zhang et al., 2022; Shi et al., 2021; Défossez et al., 2020; Zou et al., 2019; De et al., 2018; Guo et al., 2021). As such, there is a mismatch between Adam's superior empirical performance and its theoretical understanding.

To close the gap between theory and practice, we revisit the existing analyses for Adam. Current setups for the analysis of Adam fail to model the real-world applications: all of the convergence analyses of Adam are based on L-smooth condition, i.e., the Lipschitz coefficient of the gradient is globally bounded. However, it has been recently observed that L-smooth condition does not hold

in many deep learning tasks (Zhang et al., 2019a; Crawshaw et al., 2022). This gap in setting can obscure Adam's superiority: different local Lipschitz coefficients of the gradient (local smoothness) may require different optimal learning rates in terms of convergence. However, the learning rate of SGD is ignorant of the local smoothness along the training trajectory. If the local smoothness does not change sharply along the trajectory, the optimal learning rate does not change much. Then SGD can work well by selecting a learning rate through grid search. However, when the local smoothness varies dramatically, a learning rate fits some points well may fit other points along the trajectory arbitrarily badly, which indicates SGD may converge arbitrarily slow (detailed discussions can be found in [Theorem 4, (Zhang et al., 2019a)] and Section 4.3 in this paper). In contrast, Adam adapts the update according to the local information and does not suffer from such an issue.

Following the above methodology, we analyze the convergence of Adam under (L_0, L_1)-smooth condition, which assumes the local smoothness (the spectral norm of the local Hesssian) to be upper bounded by $L_1 \cdot (\text{local gradient norm}) + L_0$ (Assumption 1). As the gradient norm can be unbounded, (L_0, L_1)-smooth condition allows the local smoothness to grow fiercely with the gradient norm. Moreover, it has been demonstrated by (Zhang et al., 2019a; 2020a; Crawshaw et al., 2022) that for many practical neural networks, (L_0, L_1)-smooth condition more closely characterizes the optimization landscape (as illustrated by Figure 1 (b)) along the optimization trajectories.



(a) Training loss (b) Gradient norm vs. smoothness

Figure 1: For Transformer (Vaswani et al., 2017) on the WMT 2014 dataset, we plot (a) the training loss of SGD and Adam and (b) the gradient norm vs. the local smoothness on the training trajectory. It can be observed that the training process nearly satisfies local smoothness $= e \times$ gradient norm, which thus obeys (L_0, L_1)-smooth condition.

Under (L_0, L_1)-smooth condition, we successfully establish the convergence of Adam. Meanwhile, in contrast, under the same assumption, it is proved that SGD can be arbitrarily slow in (Zhang et al., 2019a) or even diverge (see Section 4.3). Therefore, our theory demonstrates that Adam can provably converge faster than SGD. The main contribution of this paper is summarized as follows.

We derive the first convergence result of Adam under (L_0, L_1)-smooth condition (Theorem 1). First, our convergence result is established under the mildest assumption so far:

- (L_0, L_1)-smooth condition is strictly weaker than L-smooth condition. Relaxing the smoothness condition is important to characterize the advantage of Adam over SGD.
- Our result does not require the bounded gradient assumption (i.e. $\|\nabla f(x)\| \leq C$). Removing such a condition is necessary, as otherwise (L_0, L_1)-smooth condition will degenerate to L-smooth condition. Our result does not need other strong assumptions like bounded adaptor or large ε (see Eq. (2)), either.

Furthermore, the conclusion of our convergence result is among the strongest.

- Our convergence result holds for every possible trajectory. This is much stronger than the common results of "convergence in expectation" and is technically challenging.
- In our convergence results, the setting of hyperparameters (β_1, β_2) is close to practice. Specifically, our result holds for any β_1 and any β_2 close to 1, which matches the practical settings (for example, 0.9 and 0.999 in deep learning libraries).

We also provide counter-examples where SGD diverges under the same setting as Theorem 1. Therefore, our results can shed new light on why Adam outperforms SGD in deep learning tasks.

The rest of this paper is organized as follows: Section 2 summarizes the related works; Section 3, introduces notations and assumptions; Section 4 presents our convergence result of Adam along with the proof ideas; finally, Section 5 presents experiments to support our theoretical claims.

2 RELATED WORKS

Convergence analysis for Adam. When Adam is firstly proposed in (Kingma & Ba, 2015), the authors provide a convergence proof. However, Reddi et al. (2019) point out that the proof in (Kingma & Ba, 2015) has flaws. They further provide simple counterexamples with which Adam diverges. Ever since then, there have been many attempts to modify the update rules of Adam to ensure convergence: e.g., AMSGrad (Reddi et al., 2019) and AdaBound (Luo et al., 2019). Due to the limited space, we introduce more details of the Adam-variants in Appendix A.

On the other hand, vanilla Adam works well in practice, and the divergence is not reported unless for carefully constructed examples. This empirical phenomenon motivates researchers to rethink the counterexamples. The counterexamples states "for every $\beta_1 < \sqrt{\beta_2}$, there exists a problem that Adam diverges". That is to say, the divergence statement requires picking (β_1, β_2) before fixing the problem, while in practice, the algorithmic parameters are often picked according to the problem. Based on this observation, a recent work (Zhang et al., 2022) proves that Adam can converge with (β_1, β_2) picked after the problem is given.

Similar analyses on vanilla Adam and RMSProp are also shown under various conditions in (Zaheer et al., 2018b; Zou et al., 2019; Défossez et al., 2020; De et al., 2018; Guo et al., 2021; Shi et al., 2021). We list and compare these results in Table 1 for convenience. In summary, we emphasize that all the above works (including those for Adam-variants) require L-smooth condition. In addition, most of them require stronger assumptions such as bounded gradient or large ϵ.

	$(\mathbf{L_0}, \mathbf{L_1})$**-Smoothness**	Allow $\beta_1 \neq 0^{(a)}$	Allow $\varepsilon = 0^{(b)}$	Allow unbounded gradient	Trajectory-wise convergence
Zaheer et al. (2018b)	×	×	×	×	×
De et al. (2018)$^{(c)}$	×	×	×	×	×
Défossez et al. (2020)	×	✓	×	×	×
Guo et al. (2021)$^{(d)}$	×	✓	×	×	×
Huang et al. (2021)$^{(d)}$	×	✓	×	×	×
Shi et al. (2021)	×	×	✓	✓	✓
Zhang et al. (2022)	×	✓	✓	✓	×
This work	✓	✓	✓	✓	✓

Table 1: Comparison of existing convergence result of Adam. All the literature requires L-smooth condition, which is strictly stronger than (L_0, L_1)-smooth condition. We provide some explanation on the upper footmarks: (a): When $\beta_1 = 0$, Adam is reduced to RMSProp (Hinton et al., 2012), the analysis of which is essentially simpler due to the lack of the momentum term; (b): ε is the stability hyperparameter in Adam (see Eq. (2)). In practice, ε is often a small number such as 10^{-8}. In theory, ε should be arbitrarily small, including 0. (c): Zaheer et al. (2018b) further requires the signs of the gradients to remain the same along the trajectory; (d): Guo et al. (2021); Huang et al. (2021) require $\sqrt{\boldsymbol{\nu}_{k,i} + \varepsilon}$ to be lower bounded, which is equivalent to requiring $\varepsilon > 0$.

In this work, we focus on the convergence of vanilla Adam. In particular, our result is the first to relax L-smooth condition for Adam. Moreover our convergence analysis of Adam is proved under the mildest assumptions so far and with the strongest conclusions, which can be easily extended to other Adam-variants as well.

Generalized smoothness assumption. There are several attempts on relaxing L-smooth condition. Zhang et al. (2019a) propose (L_0, L_1)-smooth condition to theoretically explain the acceleration effect of clipped SGD over SGD. Similar results are also extended to clipped SGD with momentum (Zhang et al., 2020a) and differentially-private SGD (DP SGD) (Yang et al., 2022). Through extensive experiments, these works empirically showed that (L_0, L_1)-smooth condition holds when Adam outperforms SGD. However, they did not theoretically analyze Adam in this setting.

Theoretical comparison between adaptive gradient methods and SGD. The comparison between adaptive gradient methods and SGD is a popular topic. There are several theoretical works from different perspectives. Ward et al. (2020); Défossez et al. (2020); Faw et al. (2022) show that AdaGrad can converge with any constant learning rate. They argue that "tuning-free" is one advantage of AdaGrad over SGD. This line of works has the following differences with us. First, they analyze

AdaGrad while we focus on Adam. Second, they require L-smooth, while we do not have this condition. Third, their proposed convergence rate is no better than SGD, while we show Adam can converge faster.

Xie et al. (2022) study the escaping rate from saddle points of the continuous dynamics approximation of Adam and SGD. They show that adaptive learning rate can help escape saddle points efficiently. This work is orthogonal to ours: they compare the behavior of Adam and SGD *after* a certain stationary point is reached, while we focus on comparing the iteration complexity of approaching the stationary points.

Zhang et al. (2020b) prove that the (adaptive) clipped gradient method converges faster than SGD when the stochastic noises are heavy-tailed. As such, they argue that the heavy-tailed noise is one cause of SGD's poor performance. However, a recent work (Chen et al., 2021b) shows that (S)GD still performs poorly even when the stochastic noise is removed in the training process (by using the full-batch gradient), which shows that the effect of the stochastic noise may not be crucial. There are also recent works (Zhou et al., 2020; Zou et al., 2021) trying to compare the generalization behaviors of adaptive gradient methods and SGD. These works are orthogonal to the topic of this work. More detailed discussion is deferred to Appendix A.

3 PRELIMINARIES

This section introduces notations, definitions, and assumptions that are used throughout this work.

Notations. We list the notations that are used in the formal definition of the randomly-shuffled Adam and its convergence analysis.

- (Vector) We define $\boldsymbol{a} \odot \boldsymbol{b}$ as the Hadamard product (i.e., component-wise product) between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with the same dimension. We also define $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ as the ℓ^2 inner product between $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\|\boldsymbol{a}\|_p$ as the ℓ^p norm of $\boldsymbol{a}$ (specifically, we abbreviate the ℓ^2 norm of $\boldsymbol{a}$ as $\|\boldsymbol{a}\|$). We define $\mathbb{1}_d$ as an all-one vector with dimension d.
- (Derivative) For a function $f(\boldsymbol{w}) : \mathbb{R}^d \to \mathbb{R}$, we define $\nabla f(\boldsymbol{w})$ as the gradient of f at point $\boldsymbol{w}$, and $\partial_l f(\boldsymbol{w})$ as the l-th partial derivative of f at point $\boldsymbol{w}$, i.e., $\partial_l f(\boldsymbol{w}) = (\nabla f(\boldsymbol{w}))_l$.
- (Array) We define $[m_1, m_2] \triangleq \{m_1, \cdots, m_2\}, \forall m_1, m_2 \in \mathbb{N}, m_1 \leq m_2$. Specifically, we use $[m] \triangleq \{1, \cdots, m\}$.

Formal Definition of Adam. Based on the n-sum problem Eq. (1), we provide the update rule of Adam as follows. We initialize $\boldsymbol{w}_{1,0}, \boldsymbol{m}_{1,-1}$, and $\boldsymbol{\nu}_{1,-1}$ as any point in $\mathbb{R}^d$. At the beginning of every outer loop $k \in \mathbb{N}^+$, we sample $\{\tau_{k,0}, \cdots, \tau_{k,n-1}\}$ as a random permutation of $\{0, 1, \cdots, n-1\}$. Then, in each inner loop $i \in [0, n-1]$, we respectively calculate the $1st$-order momentum $\boldsymbol{m}_{k,i}$, the $2st$-order momentum $\boldsymbol{\nu}_{k,i}$, and the parameter $\boldsymbol{w}_{k,i+1}$ as

$$\begin{cases} \boldsymbol{m}_{k,i} = \beta_1 \boldsymbol{m}_{k,i-1} + (1-\beta_1) \nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}), \\ \boldsymbol{\nu}_{k,i} = \beta_2 \boldsymbol{\nu}_{k,i-1} + (1-\beta_2) \nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}) \odot \nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}), , \\ \boldsymbol{w}_{k,i+1} = \boldsymbol{w}_{k,i} - \frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,i}} + \varepsilon \mathbb{1}_d} \odot \boldsymbol{m}_{k,i}. \end{cases} \tag{2}$$

In the end of the outer loop k, Adam updates $\boldsymbol{w}_{k+1,0} = \boldsymbol{w}_{k,n}, \boldsymbol{\nu}_{k+1,-1} = \boldsymbol{\nu}_{k,n-1}, \boldsymbol{m}_{k+1,-1} = \boldsymbol{m}_{k,n-1}$ as a preparation for the next outer loop. $\boldsymbol{m}_{k,i}$ and $\boldsymbol{\nu}_{k,i}$ are weighted averages with hyper-paramter $\beta_1 \in [0, 1)$ and $\beta_2 \in [0, 1)$, respectively. We choose η_k to be diminishing learning rate $\eta_k = \frac{\eta_1}{\sqrt{k}}$. In practice, ε is adopted for numerical stability and it is often chosen to be 10^{-8} in practice. In our theory, we allow ε to be an arbitrary non-negative constant including 0. We further emphasize that that random shuffling is not a key factor in our analysis. The proof also works for other settings.

We make two mild assumptions on the objective function (Eq. (1)).

Assumption 1 ((L_0, L_1)-smooth condition)**.** *$f_i(\boldsymbol{w})$ satisfies (L_0, L_1)-smooth condition, i.e., there exists positive constants (L_0, L_1), such that, $\forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{R}^d$ satisfying $\|\boldsymbol{w}_1 - \boldsymbol{w}_2\| \leq \frac{1}{L_1}$,*

$$\|\nabla f_i(\boldsymbol{w}_1) - \nabla f_i(\boldsymbol{w}_2)\| \leq (L_0 + L_1 \|\nabla f_i(\boldsymbol{w}_1)\|) \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|. \tag{3}$$

Eq. (3) is firstly introduced by Zhang et al. (2019a). It is strictly weaker than the classical L-smooth condition (i.e., $L_1 = 0$ in Assumption 3, see [Remark 2.3, (Zhang et al., 2020a)] for

details). For instance, Eq. (3) holds for a wide range of polynomials and even the exponential functions, while L-smooth condition does not hold for polynomials with degree larger than 2. More importantly, empirical observation in (Zhang et al., 2019a; 2020a) demonstrates that Eq. (3) is a better characterization of the loss landscape of neural networks than L-smooth condition, especially in the tasks where Adam significantly outperforms SGD.

Assumption 2 (Affine Noise Variance)**.** *$\forall \boldsymbol{w} \in \mathbb{R}^d$, the gradients of $\{f_i(\boldsymbol{w})\}_{i=0}^{n-1}$ has the following connection with the gradient of $f(\boldsymbol{w})$:*

$$\sum_{i=0}^{n-1} \|\nabla f_i(\boldsymbol{w})\|^2 \leq D_1 \|\nabla f(\boldsymbol{w})\|^2 + D_0.$$

Assumption 2 is quite general. When $D_1 = 1/n$, it becomes the "bounded variance" with constant D_0/n. The "bounded variance" condition is common in both SGD and Adam analysis (e.g. (Ghadimi et al., 2016; Zaheer et al., 2018a; Huang et al., 2021)). Assumption 2 allows more flexible choices of $D_1 \neq n$ and thus it is strictly weaker than bounded variance.

When $D_0 > 0$, the objective function is often called "non-realizable" (Shi et al., 2021; Zhang et al., 2022). In this case, adaptive gradient methods, even with diminishing learning rate, cannot reach the exact stationary points. Instead, they only converge to a bounded region containing stationary points (Zaheer et al., 2018a; Shi et al., 2021; Zhang et al., 2022). This phenomenon is also observed empirically (see Figure 2).

When $D_0 = 0$, Assumption 2 becomes $\sum_{j=0}^{n-1} \|\nabla f_j(x)\|_2^2 \leq D_1 \|\nabla f(x)\|_2^2$. This is often called "strong growth condition" (SGC) (Schmidt & Roux, 2013; Vaswani et al., 2019). SGC is increasingly popular recently as it is known to be reasonable in the overparameterized regime where neural networks can interpolate all data points (Vaswani et al., 2019). We will show that Adam can converge to stationary points if $D_0 = 0$.

4 MAIN RESULT

4.1 ADAM CONVERGES WITH (L_0, L_1)-SMOOTH CONDITION

We formally present our main result as follows:

Theorem 1. *Consider Adam defined as Eq. (2) with diminishing learning rate $\eta_k = \frac{\eta_1}{\sqrt{k}}$. Let Assumptions 1 and 2 hold. Suppose that β_1 and β_2 satisfies $\beta_1^2 < \beta_2$ and*

$$\begin{aligned}\delta(\beta_2) \triangleq & \sqrt{d} \max\left\{1 - \frac{1}{\sqrt{\beta_2^{n-1} + 8n\frac{1-\beta_2^{n-1}}{\beta_2^n}}}, \sqrt{\frac{\beta_2}{1 - \frac{2(1-\beta_2)n}{\beta_2^n}}} - 1, \beta_2^{-n/2} - 1, 1 - \sqrt{\beta_2}\right\} \frac{n}{\beta_2^{\frac{\sqrt{n}}{2}}} \\ \leq & \frac{1}{2(4+\sqrt{2})\sqrt{D_1}\left(n - 1 + \frac{1+\beta_1}{1-\beta_1}\right)}. \end{aligned} \tag{4}$$

Then, we have that

$$\min_{k \in [1,T]} \left\{ \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\sqrt{D_0}} \right\} \leq \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right) + \mathcal{O}(\sqrt{D_0} \min\{\delta(\beta_2), \delta(\beta_2)^2\}). \tag{5}$$

Theorem 1 shows that Adam converges to the neighborhood of stationary points under (L_0, L_1)-smooth condition. In Section 4.2, we present a proof sketch to illustrate our proof idea.

Remark 1 (The choice of hyperparameters (β_1, β_2))**.** *In Theorem 1, the feasible range of (β_1, β_2) subject to the two constraints: $\beta_1^2 < \beta_2$ and Inequality (4). We emphasize that the intersection of these two constraints is not empty. One can easily observe that $\delta(\beta_2) \to 0$ as $\beta_2 \to 1$, and thus the constraints hold when β_1 is fixed and β_2 approaches 1. Therefore, Theorem 1 indicates that Adam can work when β_2 close enough to 1. This agrees with β_2 selection in deep learning libraries (e.g., 0.999 in Pytorch).*

We discuss a bit more on β_1. First, The requirement $\beta_1^2 < \beta_2$ is a standard condition in Adam-family literature (Reddi et al., 2018; Zou et al., 2019; Défossez et al., 2020; Shi et al., 2021). Since β_2 is suggested to be large, this condition covers flexible choice of $\beta_1 \in [0, 1)$. Second, although the dependency between β_1 and β_2 in Inequality (4) seems complicated, the effect of β_1 on Inequality (4) can be omitted when adopting the practical value such as $\beta_1 = 0.9$. This is because the term "$\frac{1+\beta_1}{1-\beta_1}$" is way smaller than the term "$(n-1)$" in the right-hand-side of Inequality (4). For example, when $\beta_1 = 0.9$ (the default setting in deep learning libraries), $\frac{1+\beta_1}{1-\beta_1} = 19$. This number is much smaller than n on real datasets (for instance, on CIFAR-10 (He et al., 2016), $n = 50k/128 \approx 390$; on ImageNet (You et al., 2017), $n = 1.2m/8k = 150$). As a result, constraint (4) is inactive for practical choice of β_1.

Remark 2 (Convergence to the neighborhood of stationary points)**.** *When $D_0 \neq 0$, our theorem can only ensure that Adam converges to a neighborhood of stationary points. As discussed under Assumption 2, convergence to the bounded region is common for Adam analysis: even with L-smooth condition, Zhang et al. (2022) can only show that Adam with diminishing learning rates converges to a bounded region. This is because even with diminishing η_k, the effective learning rate $\frac{\eta_k}{\varepsilon \mathbb{1}_d + \sqrt{\boldsymbol{\nu}_{k,i}}}$ may not decay. We further conduct experiments and observe that Adam cannot reach the stationary point over a simple quadratic function with $D_0 \neq 0$ (see Figure 2). The good news is that the neighborhood shrinks with $\beta_2 \to 1$. This also explains why large β_2 is preferred in practice.*

We further comment on a bit on the size of the region. We refine the bounded region $\{\boldsymbol{w} : \min\{\frac{\|\nabla f(\boldsymbol{w}))\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w})\|^2}{\sqrt{D_0}}\} \leq \mathcal{O}(\sqrt{D_0}\delta(\beta_2))\}$ in existing literature (Shi et al., 2021; Zhang et al., 2022) to $\{\boldsymbol{w} : \min\{\frac{\|\nabla f(\boldsymbol{w}))\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w})\|^2}{\sqrt{D_0}}\} \leq \mathcal{O}(\sqrt{D_0}\min\{\delta(\beta_2), \delta(\beta_2)^2\})\}$ through a more careful analysis. We emphasize that this change contributes to a sharp improvement since $\delta(\beta_2)$ is required to be close to 0.

When $D_0 = 0$ (i.e., under strong growth condition), Theorem 1 further indicates Adam will converge to stationary points. We present this result in Corollary 1.

Corollary 1. *Let all the conditions in Theorem 1 hold and further assume $D_0 = 0$, we have*

$$\min_{k \in [1,T]} \|\nabla f(\boldsymbol{w}_{k,0})\| \leq G_2(L_0, L_1, \beta_1, \beta_2)\frac{\log T}{\sqrt{T}}.$$

4.2 Proof ideas

Here, we briefly explain our proof idea for the convergence result. Specifically, we identify the challenges and solve them one by one. Similar to other analyses of gradient-based optimizers, we start with the following descent lemma:

$$f(\boldsymbol{w}_{k+1,0}) - f(\boldsymbol{w}_{k,0}) \leq \underbrace{\langle \boldsymbol{w}_{k+1,0} - \boldsymbol{w}_{k,0}, \nabla f(\boldsymbol{w}_{k,0})\rangle}_{\text{First Order}} + \underbrace{\frac{L_{loc}}{2}\|\boldsymbol{w}_{k+1,0} - \boldsymbol{w}_{k,0}\|^2}_{\text{Second Order}}, \tag{6}$$

where L_{loc} is the local smoothness, and is bounded by $(L_0 + L_1\|\nabla f(\boldsymbol{w}_{k,0})\|)$ due to (L_0, L_1)-smooth condition (Assumption 1).

We would like to show that (i) the first order term is negative and that (ii) the first order term dominates the change of loss $f(\boldsymbol{w}_{k+1,0}) - f(\boldsymbol{w}_{k,0})$. Proving (i) is equivalent to proving the alignment between the update $\boldsymbol{w}_{k+1,0} - \boldsymbol{w}_{k,0} = -\eta_k \sum_i \frac{1}{\varepsilon \mathbb{1}_d + \sqrt{\boldsymbol{\nu}_{k,i}}} \odot \boldsymbol{m}_{k,i}$ and the negative gradient $-\nabla f(\boldsymbol{w}_{k,0})$.

As the momentum $\boldsymbol{m}_{k,i}$ and the adaptor $\frac{1}{\varepsilon \mathbb{1}_d + \sqrt{\boldsymbol{\nu}_{k,i}}}$ are both stochastic and they are highly statistically correlated, we first decouple the correlation between the momentum and the adaptor following the routine approach: we control the change of $\boldsymbol{\nu}_{k,i}$ within one epoch by setting β_2 close to 1. Specifically, as $\boldsymbol{\nu}_{k,i} = \beta_2 \boldsymbol{\nu}_{k,i-1} + (1-\beta_2)\nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}) \odot \nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})$, setting β_2 close enough to 1 leads to $\boldsymbol{\nu}_{k,i} \approx \boldsymbol{\nu}_{k,i-1} \approx \cdots \approx \boldsymbol{\nu}_{k,0}$. Consequently, Adam with β_2 close enough to 1 can then be viewed as momentum SGD with coordinate-wise but constant learning rate $\frac{1}{\varepsilon \mathbb{1}_d + \sqrt{\boldsymbol{\nu}_{k,0}}}$ within the k-th epoch (with close-enough-to-1 β_2), i.e.,

$$\boldsymbol{w}_{k,i+1} \approx \boldsymbol{w}_{k,i} - \eta_k \frac{\boldsymbol{m}_{k,i}}{\sqrt{\boldsymbol{\nu}_{k,0}} + \varepsilon}, \boldsymbol{w}_{k+1,0} \approx \boldsymbol{w}_{k,0} - \eta_k \frac{\sum_i \boldsymbol{m}_{k,i}}{\sqrt{\boldsymbol{\nu}_{k,0}} + \varepsilon}. \tag{7}$$

It remains to show $-\eta_k\langle\frac{\sum_i \boldsymbol{m}_{k,i}}{\sqrt{\boldsymbol{\nu}_{k,0}}+\varepsilon}, -\nabla f(\boldsymbol{w}_{k,0})\rangle > 0$. Nevertheless, such a claim is not necessarily correct, as the momentum $\boldsymbol{m}_{k,i}$ contains a heavy historical signal, and may push the update away from the negative gradient direction. We consider this problem as our first technical challenge.

Challenge I: How to deal with the historical information in the momentum? Even after we disentangle the adaptor and the momentum, the historical information in the momentum prevents the alignment between $-\eta_k\frac{\sum_i \boldsymbol{m}_{k,i}}{\sqrt{\boldsymbol{\nu}_{k,0}}+\varepsilon}$ and $-\nabla f(\boldsymbol{w}_{k,0})$, and thus the first order term may be non-negative.

Insight I: A new potential function. We overcome this challenge by proposing a new potential function and using a new descent lemma to replace Eq. (6). Our construction of the potential function is based on the following observation: we revisit the update rule in Eq. (2) and rewrite it in the following manner:

$$\frac{\boldsymbol{m}_{k,i}-\beta_1\boldsymbol{m}_{k,i-1}}{1-\beta_1} = \nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}).$$

Notice that the right-hand-side of the above equation contains no historical gradients but only the gradient of the current step! We then get $1/\sqrt{\boldsymbol{\nu}_{k,i}}$ into the play, leading to

$$\begin{aligned}\frac{\boldsymbol{w}_{k,i+1}-\beta_1\boldsymbol{w}_{k,i}}{1-\beta_1} - \frac{\boldsymbol{w}_{k,i}-\beta_1\boldsymbol{w}_{k,i-1}}{1-\beta_1} &= \frac{\boldsymbol{w}_{k,i+1}-\boldsymbol{w}_{k,i}-\beta_1(\boldsymbol{w}_{k,i}-\boldsymbol{w}_{k,i-1})}{1-\beta_1}\\ \approx -\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,0}}+\varepsilon\mathbb{1}_d}\odot\frac{\boldsymbol{m}_{k,i}-\beta_1\boldsymbol{m}_{k,i-1}}{1-\beta_1} &= -\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,0}}+\varepsilon\mathbb{1}_d}\odot\nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}).\end{aligned}$$

One can see that the sequence $\{\boldsymbol{u}_{k,i}\triangleq\frac{\boldsymbol{w}_{k,i}-\beta_1\boldsymbol{w}_{k,i-1}}{1-\beta_1}\}$ are (approximately) doing SGD within one epoch (with coordinate-wise but constant learning rate $\boldsymbol{\nu}_{k,i}$)! Further notice that $\boldsymbol{u}_{k,i}=\boldsymbol{w}_{k,i}+\beta_1\frac{\boldsymbol{w}_{k,i}-\boldsymbol{w}_{k,i-1}}{1-\beta_1}$ is close to $\boldsymbol{w}_{k,i}$. Therefore, we choose our potential function as $f(\boldsymbol{u}_{k,i})$. The Taylor's expansion of f at $\boldsymbol{u}_{k,0}$ then provides a new descent lemma, i.e.,

$$f(\boldsymbol{u}_{k+1,0})-f(\boldsymbol{u}_{k,0}) \le \underbrace{\langle\boldsymbol{u}_{k+1,0}-\boldsymbol{u}_{k,0},\nabla f(\boldsymbol{u}_{k,0})\rangle}_{\text{First Order}} + \underbrace{\frac{L_0+L_1\|\nabla f(\boldsymbol{w}_{k,0})\|}{2}\|\boldsymbol{w}_{k+1,0}-\boldsymbol{w}_{k,0}\|^2}_{\text{Second Order}}, \tag{8}$$

By noticing $\boldsymbol{w}_{k,i}\approx\boldsymbol{u}_{k,i}\approx\boldsymbol{u}_{k,0}$, the first order term can be further approximated as follows,

$$\langle\boldsymbol{u}_{k+1,0}-\boldsymbol{u}_{k,0},\nabla f(\boldsymbol{u}_{k,0})\rangle \approx -\langle\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,0}}+\varepsilon\mathbb{1}_d}\odot\nabla f(\boldsymbol{w}_{k,0}),\nabla f(\boldsymbol{w}_{k,0})\rangle, \tag{9}$$

which is negative, and the challenge is addressed.

Remark 3. *Zhang et al. (2022) also encounter this challenge. They overcome this challenge by additionally assuming that the random permutation to generate $\{\tau_{k,0},\cdots,\tau_{k,n-1}\}$ is uniformly sampled from all possible permutation orders. Based on this additional assumption, the expectation of the momentum can be shown close to the gradient and thus they derive an in-expectation result. In contrast, we do not need such an assumption. Further, their proof is more complex than ours as they need to deal with the historical gradient signal in the momentum. Our potential function allows us to offer a trajectory-wise convergence result with simplified proof.*

To complete the proof, we move on to show that the second order term in Eq. (8) is minor compared to the first order term, which we identify as the second challenge. We note that such a property does not hold for SGD under (L_0,L_1)-smooth condition: the first order term in Eq. (6) is (approximately) $-\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2$ for SGD, while the corresponding second order term is

$$\text{(SGD)}\quad \frac{L_0+L_1\|\nabla f(\boldsymbol{w}_{k,0})\|}{2}\|\boldsymbol{w}_{k+1,0}-\boldsymbol{w}_{k,0}\|^2 = \mathcal{O}(\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|^3).$$

In other words, the second order term has one more $\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|$ factor than the first order. Without any boundedness condition on the gradient, this factor could be unboundedly large and thus makes the second order term dominate the dynamics of the objective function f. This eventually leads to slow convergence or even divergence for SGD under (L_0,L_1)-smooth condition (see counter-examples in Section 4.3).

Challenge II: How to show the second order term in Eq. (8) is minor for Adam under (L_0, L_1)-smooth condition?

Insight II: Adaptivity in Adam. We overcome this challenge by leveraging the adaptivity in Adam. We first revisit the above discussion for SGD, and find the additional $\|\nabla f(\boldsymbol{w}_{k,0})\|$ is because the update of SGD has the dependency on gradient with degree 1. Viewing the first order and the second order term in Eq. (6) respectively as linear and quadratic functions of $\boldsymbol{w}_{k+1,0} - \boldsymbol{w}_{k,0}$, their coefficients also contain dependency on the gradient with order 1. This leads to a higher-degree dependency on the gradient of the second order term $(1 + 2 \times 1)$ than the first order $(1 + 1)$. On the other hand, Adam does not suffer from this problem: both the momentum and the adaptor have the same unit as the gradient, and thus the update (equals momentum/adaptor) has a dependency on the gradient with degree 0, which leads to the same degree of dependency on the gradient for the first order $(1 + 2 \times 0)$ and second order term $(1 + 2 \times 0)$.

Formally, for the first order term, a calculation for Eq. (9) yields

$$-\left\langle \frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,0}} + \varepsilon \mathbb{1}_d} \odot \nabla f(\boldsymbol{w}_{k,0}), \nabla f(\boldsymbol{w}_{k,0}) \right\rangle = \Omega\left(\eta_k \min\left\{ \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\sqrt{D_0}} \right\}\right).$$

Meanwhile, due to the hyperparameter setting $\beta_1 < \sqrt{\beta_2}$, $\|\boldsymbol{m}_{k,i}/\boldsymbol{\nu}_{k,i}\|$ is bounded, and thus the second order term can be bounded by

$$\frac{L_0 + L_1\|\nabla f(\boldsymbol{w}_{k,0})\|}{2}\|\boldsymbol{w}_{k+1,0} - \boldsymbol{w}_{k,0}\|^2 = \mathcal{O}(\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|) = \mathcal{O}\left(\eta_k^2 \min\left\{ \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\sqrt{D_0}} \right\}\right).$$

Here the last equality is due to the mean value inequality $\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\| \le \frac{1}{2}\eta_k^2 + \frac{1}{2}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0}\|^2$ and we drop the minor term $\mathcal{O}(\eta_k^2) = \mathcal{O}(1/k)$. One can now easily observe that the first order term is larger than the second order by order of $1/\eta_k$. This addresses the second challenge.

4.3 SGD MAY CONVERGE ARBITRARILY SLOW WITH (L_0, L_1)-SMOOTH CONDITION

As the (L_0, L_1)-smooth condition appears to be a more precise characterization of the landscape of neural networks than L-smooth condition, one may wonder how SGD performs under such a circumstance. Here, we find examples that satisfy (L_0, L_1)-smooth condition, but GD and SGD diverges. This implies that under this realistic smoothness assumption, (S)GD can be arbitrarily slow.

GD may converge arbitrarily slow with (L_0, L_1)-smooth condition. To begin with, we notice that Zhang et al. (2019a) have already provided counterexamples over which GD converges arbitrarily slowly. However, their examples use constant learning rate GD, which, unfortunately, does NOT match the setting of Theorem 1 (i.e., diminishing learning rate), and thus does not suffice to provide a fair comparison between GD and (full-batch) Adam. To fill this gap in setting, we provide the following example:

Example 1. *There exists a domain $\mathcal{X}$ and a target function $f : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$, such that for every initial learning rate η_1, there exists a region $\mathcal{E} \subset \mathcal{X}$ with infinite measure, such that GD with diminishing learning rate $\eta_t = \frac{\eta_1}{\sqrt{t}}$ and initialized in $\mathcal{E}$ will drive $f(\boldsymbol{w}_t)$ and $\|\nabla f(\boldsymbol{w}_t)\|$ to infinity.*

We defer a concrete construction of the example to Appendix B.1. The intuition of this example is that with a fixed initial learning rate, a bad initialization may lead to arbitrarily worse local smoothness due to globally unbounded smoothness, and make the first-step stepsize relatively large. This leads to an even worse (larger) local smoothness at the next step. Also, the increase of the local smoothness can be rapid enough to offset the decrease in learning rate, and eventually drive the loss to divergence. On the contrary, the update norm of Adam is bounded regardless of the initialization, and thus Adam does not suffer from the same problem and will converge.

One may notice that the divergence issue in the above example can be equivalently viewed as "given initialization, a large learning rate will bring divergence". A natural question is that whether reducing the learning rate will lead to convergence. The answer could be yes, but it is true only if some additional assumption is provided (for example, Assumption 4 in (Zhang et al., 2019a), which assumes the gradient is bounded over a loss sub-level set). However, in this case, the feasible initial learning rate depends on the local smoothness over the trajectory. Since the initial gradient can be arbitrarily large, the initial learning rate needs to be arbitrarily small, which thus leads to a slow

convergence rate if bad initialization is picked. This is exactly the intuition of the lower bound in (Zhang et al., 2019a).

Similar negative results also apply to SGD. Actually, the situation is even worse: SGD still suffer from divergence for learning rate $1/L$, which can lead to convergence for GD.

Example 2. *There exists a domain $\mathcal{X}$, a initial point $\boldsymbol{w}_0 \in \mathcal{X}$ and a target function $f = \sum_{i=0}^{n-1} f_i$, over which SGD with initial learning rate $\eta_1 = \frac{1}{L}$ and diminishing learning rate $\eta_t = \frac{\eta_1}{\sqrt{t}}$ diverges while GD with the same setting converges, where L is the smoothness upper bound of the loss sub-level set $\{\boldsymbol{w} \in \mathcal{X} : f(\boldsymbol{w}) \leq f(\boldsymbol{w}_0)\}$.*

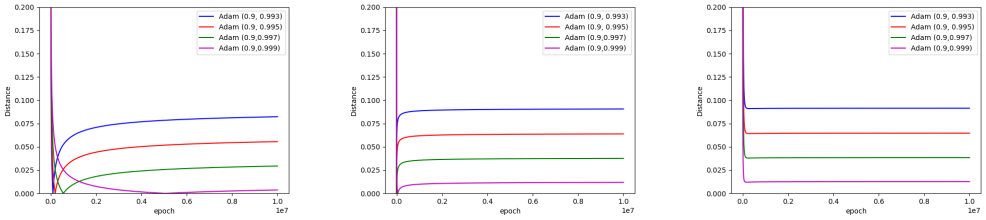
We defer the concrete construction to Appendix B.2. Note that GD converges with the initial learning rate $\frac{1}{L}$, while SGD diverges in this setting. The intuition of this example is that even with small learning rate, a non-full-batch update can step out the loss sub-level set and increase the local smoothness coefficient.

5 Experiments

We conduct numerical experiments to support our theoretical findings. We illustrate through the experiment that with $D_0 \neq 0$, Adam may not converge to the stationary point. Specifically, we run Adam on a synthetic optimization target which is given as follows:

$$\begin{cases} f_0(x) = (x-3)^2 + \dfrac{1}{10} \\ f_i(x) = -\dfrac{1}{10}\left(x - \dfrac{10}{3}\right)^2 + \dfrac{1}{10},\ i \in [1, 9] \end{cases},\ f(x) = \sum_{i=0}^{9} f_i(x) = \frac{1}{10}x^2.$$

Such an example is proposed by Shi et al. (2021). One can easily observe that $f(x)$ and $\{f_i(x)\}_{i=0}^{9}$ satisfies Assumption 1 and Assumption 2 with non-zero $D_0 \neq 0$ ($D_1 = 208$,$D_0 = 75\frac{5}{9}$), thus Theorem 1 applies. With diminishing learning rates $\eta_t = \frac{10}{\sqrt{t}}$, $\eta_t = \frac{1}{\sqrt{t}}$, and $\eta_t = \frac{0.1}{\sqrt{t}}$, we plot the distance between Adam's trajectory and the unique minimum 0 of f in Figure 2. It can be observed that Adam does not converge to the unique minimum 0, but gets closer to the minimum if choosing β_2 closer to 1, which supports Theorem 1.



(a) Learning rate $\eta_t = \frac{10}{\sqrt{t}}$ (b) Learning rate $\eta_t = \frac{1}{\sqrt{t}}$ (c) Learning rate $\eta_t = \frac{0.1}{\sqrt{t}}$

Figure 2: Performance of Adam on a synthetic objective function satisfying (L_0, L_1)-smooth condition. Adam doesn't converge to the unique stationary point, but gets closer to the stationary point as $\beta_2 \to 1$.

6 Conclusion

In this paper, we take the first step to theoretically understand the adaptivity in Adam. Specifically, we prove that *Adam can converge arbitrarily faster than SGD*. Our assumption is realistic and close to practical settings. On the other hand, there is still a lot to explore for Adam's performance. For example, it is interesting to analyze how momentum helps in Adam's optimization.

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A ADDITIONAL RELATED WORKS

In this section, we provide discussions on more related works.

New variants of Adam. Ever since Reddi et al. (2019) pointed out the non-convergence issue of Adam, many new variants of Adam are designed. For instance, Zou et al. (2019); Gadat & Gavra (2020); Chen et al. (2018b; 2021a) replaced the constant hyperparameters by iterate-dependent ones e.g. β_{1t} or β_{2t}. AMSGrad (Reddi et al., 2019) and AdaFom (Chen et al., 2018b) enforced $\{v_t\}$ to be an non-decreasing. Similarly, AdaBound (Luo et al., 2019) imposed constraints $v_t \in [C_l, C_u]$ to prevent the learning rate from vanishing or exploding. Similarly, Zhou et al. (2018b) adopted a new estimate of v_t to correct the bias. In addition, there are attempts to combine Adam with Nesterov momentum (Dozat, 2016) as well as warm-up techniques (Liu et al., 2020). There are also some works providing theoretical analysis on the variants of Adam. For instance, Zhou et al. (2018a) studied the convergence of AdaGrad and AMSGrad. Gadat & Gavra (2020) studied the asymptotic behavior of a subclass of adaptive gradient methods from landscape point of view. Their analysis applies to Adam-variants with $\beta_1 = 0$ and β_2 increasing along the iterates. In summary, all these works require L-smooth condition. Though the new variants of Adam can probably converge, the convergence rate is no better than SGD.

Generalization ability of Adaptive gradient methods. The generalization ability of Adam is a hot debate topic. For instance, Wang et al. (2021) studied the implicit bias of adaptive optimization algorithms on homogeneous neural networks. They proved that the convergent direction of Adam and RMSProp is the same as SGD. Zhou et al. (2020); Xie et al. (2022); Zou et al. (2021) argue that Adam preferred sharp local-min while GD prefers the wide ones. As such, they argue that Adam generalizes worse than SGD. Zou et al. (2021) prove that Adam generalizes worse than SGD over a specific model. There are also several attempts to improve the generalization ability of Adam. For instance, Padam (Chen et al., 2018a) introduced a partial adaptive parameter to improve the generalization performance. AdamW (Loshchilov & Hutter, 2017) improved regularization in Adam by decoupling the weight decay from the gradient-based update.

B DIVERGING EXAMPLE FOR (S)GD

In this section, we discuss the divergence issue for (S)GD under the (L_0, L_1)-smooth condition. This section is organized as follows: we will first consider the simple full-batch case, i.e., $n = 1$ and $f(\boldsymbol{w}) = f_0(\boldsymbol{w})$ (therefore there is no random reshuffling and no randomness in the process of training), in which case we will show GD may suffer from divergence under the (L_0, L_1) assumption while Adam converges to the stationary point. We then step further to the random reshuffling case and show that the randomness may brought additional divergence issues.

B.1 COMPARE GD AND FULL-BATCH ADAM

To start with, we notice there is an example in Zhang et al. (2019a) over which GD will converge arbitrarily slow. We re-state their result as follows:

Property 1 (Theorem 4, Zhang et al. (2019a))**.** *For every fixed learning rate $\eta \in \mathbb{R}^+$ and tolerate error bound $\varepsilon \in \mathbb{R}^+$, there exists a domain $\mathcal{X}$, an optimization problem $f : \mathcal{X} \to \mathbb{R}$, and an initialization point $\boldsymbol{w}_1 \in \mathcal{X}$ satisfying f is lower with (L_0, L_1) smoothness and satisfying $M \triangleq \sup\{\|\nabla f(\boldsymbol{w})\| | \boldsymbol{w} \text{ such that } f(\boldsymbol{w}) \leq f(\boldsymbol{w}_0)\} < \infty$, such that GD over f starting from $\boldsymbol{w}_1$ with constant learning rate η, such that any $t \in \mathbb{Z}^+$ smaller than $\frac{L_1 M(f(\boldsymbol{w}_1) - \min_{\boldsymbol{w} \in \mathcal{X}} f(\boldsymbol{w}) - 5\varepsilon/8)}{8\varepsilon^2(\log M + 1)}$ satisfies $\|\nabla f(\boldsymbol{w}_t)\| > \varepsilon$.*

By Property 1, we know that the convergence of GD can be arbitrarily slow depending on the gradient upper bound M of the loss sub-level set $\{\boldsymbol{w} : f(\boldsymbol{w}) \leq f(\boldsymbol{w}_1)\}$, which further depends on the initialization. In other words, the performance of GD sensitively relies on the initialization.

However, Property 1 does not suffice to show Adam performs better than GD, as in Property 1 the learning rate is constant while in Theorem 1, it is decaying. To fill this gap in setting, we provide the following example satisfying Assumptions 2 and 1, on which GD with decaying learning rate diverges if initialized badly.

Example 3 (Example 1, restated)**.** *Let $n = 1$ and $p = 2$ with $f_0((w^1, w^2)) = e^{-\frac{\sqrt{3}}{2}w^1 - \frac{1}{2}w^2} + e^{\frac{\sqrt{3}}{2}w^1 - \frac{1}{2}w^2}$ and $f((w^1, w^2)) = f_0((w^1, w^2))$. Then, f satisfies Assumptions 1 and 2 . However, for every initial learning rate η_1, there exists a region $\mathcal{E}$ with infinite measure over $\mathbb{R}^2$ such that, with GD with decaying learning rate $\eta_k = \frac{\eta_1}{\sqrt{k}}$ and start point in $\mathcal{E}$, f and ∇f diverge to infinity.*

Proof. **Notations.** To simplify the notation, we define $\boldsymbol{x}_1 = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $\boldsymbol{x}_2 = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$. It can be observed that $\|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1$ and $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = -\frac{1}{2}$. We also denote the parameter given by the k-th iteration of GD as $\boldsymbol{w}_k$. By definition, we have

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k + \eta_k e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_k \rangle} \boldsymbol{x}_1 + \eta_k e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_k \rangle} \boldsymbol{x}_2.$$

We further define $g(x) \triangleq \eta_1 e^{\frac{3}{8}x} - 17x - 16\eta_1 e^{-\frac{15}{16}x}$. As $g(x) \to \infty$ as $x \to \infty$, there exists a large enough positive constant $C > 1$, such that for every $x > C$, $g(x) > 0$.

Verify the assumptions. Assumption 2 immediately follows as $n = 1$. We then check Assumption 1. We have

$$\nabla f(\boldsymbol{w}) = -e^{-\langle \boldsymbol{w}, \boldsymbol{x}_1 \rangle} \boldsymbol{x}_1 - e^{-\langle \boldsymbol{w}, \boldsymbol{x}_2 \rangle} \boldsymbol{x}_2, \nabla^2 f(\boldsymbol{w}) = \boldsymbol{x}_1^\top \boldsymbol{x}_1 e^{-\langle \boldsymbol{w}, \boldsymbol{x}_1 \rangle} + \boldsymbol{x}_2^\top \boldsymbol{x}_2 e^{-\langle \boldsymbol{w}, \boldsymbol{x}_2 \rangle}.$$

Therefore, $\|\nabla^2 f_0(\boldsymbol{w})\|_2 \leq e^{-\langle \boldsymbol{w}, \boldsymbol{x}_1 \rangle} + e^{-\langle \boldsymbol{w}, \boldsymbol{x}_2 \rangle}$, while

$$\|\nabla f(\boldsymbol{w})\| \geq \langle -e^{-\langle \boldsymbol{w}, \boldsymbol{x}_1 \rangle} \boldsymbol{x}_1 - e^{-\langle \boldsymbol{w}, \boldsymbol{x}_2 \rangle} \boldsymbol{x}_2, (0, 1) \rangle \geq \frac{1}{2} e^{-\langle \boldsymbol{w}, \boldsymbol{x}_1 \rangle} + \frac{1}{2} e^{-\langle \boldsymbol{w}, \boldsymbol{x}_2 \rangle}.$$

As a conclusion, Assumption 1 is satisfied with $L_0 = 0$ and $L_1 = 2$.

Initialization. We define region $\mathcal{E}$ as

$$\mathcal{E} \triangleq \{a_1 \boldsymbol{x}_1 + \boldsymbol{b_1} : a_1 > C, a_1 > 16\|\boldsymbol{b_1}\|\} \cup \{a_1 \boldsymbol{x}_2 + \boldsymbol{b_1} : a_1 > C, a_1 > 16\|\boldsymbol{b_1}\|\}.$$

It should be noticed that $\mathcal{E}$ has infinite measure.

Iteration. As $\{a_1 \boldsymbol{x}_1 + \boldsymbol{b_0} : a_1 > C, a_1 > 16\|\boldsymbol{b_1}\|\}$ is symmetric with $\{a_1 \boldsymbol{x}_2 + \boldsymbol{b_1} : a_1 > C, a_1 > 16\|\boldsymbol{b_1}\|\}$, we analyze the case when $\boldsymbol{w}_1$ is in the former set without loss of generality, i.e., $\boldsymbol{w}_1 = a_1 \boldsymbol{x}_1 + \boldsymbol{b_1}$ with $a_1 > 0$ and $a_1 > 16\|\boldsymbol{b_1}\|$.

We will prove by induction that $\forall k \geq 1$, $\boldsymbol{w}_k = a_k \boldsymbol{x}_{i_k} + \boldsymbol{b_k}$, where i_k equals to 1 if k is odd and 2 if k is even. a_k and $\boldsymbol{b_k}$ satisfies $a_k \geq 16\|\boldsymbol{b_k}\|$ and $a_k \geq 17^{k-1} C$.

For $k = 1$, the proof directly follows by the initialization. For $k \neq 1$, suppose that the above property holds for all the iterations before the k-th iteration. Suppose k is even. Then,

$$\boldsymbol{w}_{k-1} = a_{k-1} \boldsymbol{x}_1 + \boldsymbol{b_{k-1}},$$

which leads to

$$\|\boldsymbol{w}_{k-1}\| \leq a_{k-1} + \|\boldsymbol{b_{k-1}}\| \leq \frac{17}{16} a_{k-1},$$

$$\langle \boldsymbol{w}_{k-1}, \boldsymbol{x}_1 \rangle \geq a_{k-1} - \|\boldsymbol{b_{k-1}}\| \geq \frac{15}{16} a_{k-1},$$

and

$$\langle \boldsymbol{w}_{k-1}, \boldsymbol{x}_2 \rangle \leq -\frac{1}{2} a_{k-1} + \|\boldsymbol{b_{k-1}}\| \leq -\frac{7}{16} a_{k-1}.$$

Consequently,

$$\begin{aligned} \boldsymbol{w}_k =& \boldsymbol{w}_{k-1} + \eta_{k-1} e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1} \rangle} \boldsymbol{x}_1 + \eta_{k-1} e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_{k-1} \rangle} \boldsymbol{x}_2 \\ \leq& \eta_{k-1} e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_{k-1} \rangle} \boldsymbol{x}_2 + \boldsymbol{w}_{k-1} + \eta_{k-1} e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1} \rangle} \boldsymbol{x}_1. \end{aligned}$$

The norm of $\boldsymbol{w}_{k-1} + \eta_{k-1} e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1} \rangle} \boldsymbol{x}_1$ can be bounded as

$$\|\boldsymbol{w}_{k-1} + \eta_{k-1} e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1} \rangle} \boldsymbol{x}_1\| \leq \|\boldsymbol{w}_{k-1}\| + \eta_{k-1} e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1} \rangle} \leq \frac{17}{16} a_{k-1} + \eta_{k-1} e^{-\frac{15}{16} a_{k-1}},$$

while the coefficient of $\boldsymbol{x}_2$ can be lower bounded as

$$\eta_{k-1}e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_{k-1}\rangle} \geq \eta_{k-1}e^{\frac{7}{16}a_{k-1}} \geq \eta_1 e^{\frac{7}{16}a_{k-1}-\frac{1}{2}\log k-1} \geq \eta_1 e^{\frac{3}{8}a_{k-1}+\frac{1}{16}17^{k-1}C-\frac{1}{2}(k-2)} \geq \eta_1 e^{\frac{3}{8}a_{k-1}}.$$

Therefore,

$$\eta_{k-1}e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_{k-1}\rangle} - 16\|\boldsymbol{w}_{k-1} + \eta_{k-1}e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1}\rangle}\boldsymbol{x}_1\| \geq \eta_1 e^{\frac{3}{8}a_{k-1}} - 17a_{k-1} + 16\eta_{k-1}e^{-\frac{15}{16}a_{k-1}}$$
$$\geq g(a_{k-1}).$$

As by the induction condition $a_{k-1} \geq 17^{k-1}C \geq C$, we have that $g(a_{k-1}) \geq 0$. Therefore, choosing $a_k = \eta_{k-1}e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_{k-1}\rangle}$ and $\boldsymbol{b_k} = \|\boldsymbol{w}_{k-1} + \eta_{k-1}e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_{k-1}\rangle}\boldsymbol{x}_1\|$ completes the proof for k.

The proof of the case that k is odd follows the same routine by simply exchanging $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in the above proof.

Consequently, for $\|\nabla f(\boldsymbol{w}_k)\|$, we have that when k is even

$$\|\nabla f(\boldsymbol{w}_k)\| \geq \|\boldsymbol{x}_1 e^{-\langle \boldsymbol{x}_2, \boldsymbol{w}_k\rangle}\| - \|\boldsymbol{x}_2 e^{-\langle \boldsymbol{x}_1, \boldsymbol{w}_k\rangle}\| \geq e^{\frac{5}{16}a_k} - e^{-\frac{15}{16}a_k} \to \infty.$$

Similar claim holds for the case k is odd. We then complete the proof that $\|\nabla f(\boldsymbol{w}_k)\| \to \infty$ as $k \to \infty$. The claim that $f(\boldsymbol{w}_k) \to \infty$ as $k \to \infty$ can be derived following the same routine.

The proof is completed. □

On the other hand, as f in Example 1 satisfies Assumption 1 and Assumption 2 (with $D_0 = 0$), Theorem 1 applies and full batch Adam will converge to a stationary point regardless of the initialization. Example 1 indicates that even for the full-batch case, Adam appears to be less sensitive to the initialization compared with GD under (L_0, L_1)-smooth condition.

However, under the same setting and fixing the initialization point, tuning down the initial learning rate η_1 may help GD converge. Specifically, when gradient norm of the loss sub-level set $\{\boldsymbol{w} : f(\boldsymbol{w}) \leq f(\boldsymbol{w}_0)\}$ is upper bounded, if we choose the learning rate to be sufficiently small (for example, smaller than $2/(L_0 + L_1 M)$[1] where M is the gradient norm upper bound of the loss sub-level set), we can make the loss keeps decreasing as GD iterates, and the parameter stays in the loss sub-level set. However, such a learning rate can be rather small when M is large and leads to a slow convergence of GD, which is exactly the intuition behind Property 1.

Furthermore, when $n \geq 2$ and SGD is adopted, tuning down the learning rate may not help any more. We will discuss such an issue in the following section.

B.2 Compare SGD and Adam

First of all, Example 3 can be simply extended to provide a divergent example for SGD.

Example 4 (Example 1, restated)**.** *Let $n = 2$ and $p = 2$ with $f_0((w^1, w^2)) = e^{-\frac{\sqrt{3}}{2}w^1 - \frac{1}{2}w^2}$, $f_1(x) = e^{\frac{\sqrt{3}}{2}w^1 - \frac{1}{2}w^2}$, and $f((w^1, w^2)) = f_0((w^1, w^2))$. Then, f satisfies Assumptions 1 and 2 (with $D_0 = 0$). However, for every initial learning rate η_1, there exists a region $\mathcal{E}$ with infinite measure over $\mathbb{R}^2$ such that, with SGD with decaying learning rate $\eta_k = \frac{\eta_1}{\sqrt{k}}$ and start point in $\mathcal{E}$, f and ∇f diverge to infinity.*

The proof follows the similar routine as Example 3 and we omit it here. Note as $D_0 = 0$ in this case, Adam still converges to the stationary point.

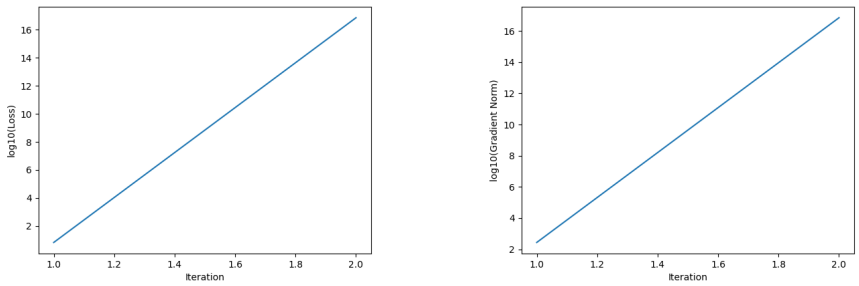
On the other hand, as discussed in the previous section, for SGD over (L_0, L_1)-smooth condition, a small learning rate may no longer help and SGD will still diverge. We illustrate this idea by consider the following example.

Example 5 (Diverging Example for SGD with Small Learning Rate)**.** *Specifically, we rescale the $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in Example 3 as $\tilde{\boldsymbol{x}}_1 = \boldsymbol{x}_1$ and $\tilde{\boldsymbol{x}}_2 = 50\boldsymbol{x}_2$. We choose initialization point $\boldsymbol{w}_1 = \tilde{\boldsymbol{x}}_1$, and choose learning rate as $\eta_1 = 1/L$ with decaying learning rate $\eta_t = \frac{\eta_1}{\sqrt{t}}$, where L is defined as the smoothness upper bound of the loss sub-level set, i.e.,*

$$L \triangleq \arg\max \left\{\|\nabla^2 f(\boldsymbol{w})\|_2 : f(\boldsymbol{w}) \leq f(\boldsymbol{w}_1)\right\}.$$

[1] $2/(L_0 + L_1 M)$ serves as the largest reasonable setting of the learning rate for GD, as f over the loss sub-level set has smoothness upper bound $(L_0 + L_1 M)$.

By the definition of the previous section, GD on such an example will converge to the stationary point, as the iterates will stay in the sub-level set, the smoothness will be small, and the loss will decrease. However, we run SGD on this example and find that the loss and gradient norm will explode in three steps (Figure 3).



(a) Training Loss (b) Gradient Norm

Figure 3: Example on which SGD with decaying learning rate and small initial learning rate diverges. We only plot the first two steps because the values exceed the math range in python in the third step.

The intuition behind Example 5 is that the update calculate from a batch in SGD does not essentially reduce the loss and may no longer keeps the iterates stay in the loss sub-level set. This can eventually make the loss diverging, or keep the parameter away from the saddle point in the training of neural networks.

C Proof of Theorem 1

This appendix provides the formal proof of Theorem 1. Specifically, we will first make some preparations by ① showing notations and ② proving lemmas which characterize several basic properties of the Adam optimizer, and then prove Theorem 1 based on the lemmas.

Remark 4. *In the remaining proof of this paper, we assume **without the loss of generality that** η_1 **is small enough**, such that the following requirements are fulfilled (with notations explained latter):*

- $2C_2\sqrt{d}\eta_1 \leq \frac{1}{L_1}$. *This will latter ensure that we can directly apply the definition of (L_0, L_1)-smooth condition (Assumption 1) to parameter sequence $\{\boldsymbol{w}_{k,i}\}_{k,i}$;*
- $\frac{1}{4(2\sqrt{2}+1)} \geq \sqrt{D_1}C_{11}\eta_1$. *This will latter ensure the second-order term is smaller than the first-order term at the end of the proof.*

The proof can be easily extended to general cases (while certainly more cumbersome) by selecting large enough K and using the epoch K as a new start point and derive the results after epoch K (this is due to η_k is decaying, and K is finite, the epochs before epoch k can be uniformly bounded and we then derive the desired result for all epochs).

Without the loss of generality, we also take the following initialization: $\boldsymbol{w}_{1,0} = \boldsymbol{w}_0$, $\boldsymbol{m}_{1,-1} = \nabla f_{\tau_{1,-1}}(\boldsymbol{w}_0)$ ($\tau_{1,-1}$ can be any integer in $[0, n-1]$), and $\boldsymbol{\nu}_{l,1,-1} = \max_j\{\partial_l f_j(\boldsymbol{w}_0)^2\}$, $\forall l$ (here the maximum is taken component-wisely). We take the initialization to have a more concise proof, while the proof can be easily extended to all the initialization as the information of the initialization in the exponentially decayed average of Adam (both in $\boldsymbol{m}_{k,i}$ and $\boldsymbol{\nu}_{k,i}$) decays rapidly with k increasing.

C.1 Preliminaries

C.1.1 Notations

Here we provide a complete list of notations used in the appendix for a clear reference.

- We use $(k_1, i_1) \leq (<)(k_2, i_2)$ for $\forall k_1, k_2 \in \mathbb{N}^+$ and $i_1, i_2 \in \{0, \cdots, n-1\}$, if either $k_1 < k_2$ or $k_1 = k_2$ and $i_1 \leq (<) i_2$
- We define function $g(x) : [0, 1] \to \mathbb{R}^{/-}$ as

$$g(\beta_2) \triangleq \max\left\{\frac{1}{\sqrt{\beta_2^{n-1}}} - 1, 1 - \frac{1}{\sqrt{\beta_2^{n-1} + 8n\frac{1-\beta_2^{n-1}}{\beta_2^n}}}, 1 - \sqrt{\beta_2}, \sqrt{\frac{\beta_2}{\left(1 - (1-\beta_2)\frac{2n}{\beta_2^n}\right)}} - 1\right\}.$$

- We define constants $\{C_i\}_{i=1}^{10}$ as follows:

$$C_1 \triangleq \frac{(1-\beta_1)^2}{1-\beta_2}\frac{1}{1-\frac{\beta_1^2}{\beta_2}} + 1,$$

$$C_2 \triangleq nC_1 + \frac{\beta_1}{1-\beta_1}C_1\left(1+\sqrt{2}\right),$$

$$C_3 \triangleq C_1\left(n(L_0 + L_1\sqrt{D_0}) + 2\sqrt{2}(L_0 + L_1\sqrt{D_0})\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\frac{\sqrt{\beta_2}}{1-\sqrt{\beta_2}} + 8\sqrt{2n}L_0\frac{1}{1-\beta_2^n}\right),$$

$$C_4 \triangleq 4L_1C_1\sqrt{D_1}\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}$$

$$C_5 \triangleq n^2(1 + n\sqrt{d}C_1\eta_1 L_1\sqrt{n}\sqrt{D_1})\left(C_4 + \frac{dC_4\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right),$$

$$C_6 \triangleq \left(dC_3 + \frac{C_4 n\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right)\eta_1^2,$$

$$C_7 \triangleq 3n\left(C_4 + \frac{dC_4}{1-\sqrt{\beta_2^n}}\right)\left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n^2\sqrt{d}C_1\eta_1^3 + \left(dC_3 + \frac{C_2C_4n\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right)\eta_1^2,$$

$$C_8 \triangleq \sqrt{\frac{2n^2}{\beta_2^n}}L_1\sqrt{D_1}n\sqrt{n} + dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\left(1+\frac{1}{1-\beta_2^n}\right)(n + n^{\frac{5}{2}}\sqrt{d}C_1\eta_1L_1\sqrt{D_1}) + 2\frac{\beta_1}{(1-\beta_1)\eta_1}\sqrt{d}C_1,$$

$$C_9 \triangleq \sqrt{\frac{2n^2}{\beta_2^n}}d(n^2L_0 + n\sqrt{n}L_1\sqrt{D_0})C_1\eta_1^2 + g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0 + L_1\sqrt{D_0})d\sqrt{d}\eta_1^2,$$

$$C_{10} \triangleq 3dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\left(1+\frac{1}{1-\beta_2^n}\right)n\left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n\sqrt{d}C_1\eta_1^3 + C_9,$$

$$C_{11} \triangleq (\frac{1}{2} + C_2)C_5 + C_8 + \frac{3L_1\sqrt{n}\sqrt{D_1}C_2^2d}{2},$$

$$C_{12} \triangleq (\frac{1}{2} + C_2)C_6 + C_9 + \frac{nL_0 + L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_1^2,$$

$$C_{13} \triangleq (\frac{1}{2} + C_2)C_7 + C_{10} + \frac{nL_0 + L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_1^2.$$

C.1.2 Auxiliary Lemmas

Here we provide auxiliary lemmas describing basic properties of Adam and the descent lemma under (L_0, L_1)-smooth condition.

Basic Properties of Adam

The following lemma characterizes the update norm of Adam.

Lemma 1 (Bounded Update)**.** *If $\beta_1 < \sqrt{\beta_2}$, we have $\forall k \in \mathbb{N}^+$, $i \in \{0, \cdots, n-1\}$,*

$$\frac{|\boldsymbol{m}_{l,k,i}|}{\sqrt{\boldsymbol{\nu}_{l,k,i}} + \varepsilon} \leq C_1,$$

where

$$C_1 \triangleq \frac{(1-\beta_1)^2}{1-\beta_2}\frac{1}{1-\frac{\beta_1^2}{\beta_2}} + 1.$$

Furthermore, we have $|\boldsymbol{w}_{l,k,i+1} - \boldsymbol{w}_{l,k,i}| \leq C_1\eta_k$, and thus $\|\boldsymbol{w}_{k,i+1} - \boldsymbol{w}_{k,i}\| \leq C_1\eta_k\sqrt{d}$.

Proof. By the definition of $\boldsymbol{m}_{k,i}$, we have

$$
\begin{aligned}
&(\boldsymbol{m}_{l,k,i})^2 \\
=&\Bigg((1-\beta_1)\sum_{j=0}^{i}\beta_1^{(k-1)n+i-((k-1)n+j)}\partial_l f_{\tau_{k,j}}(\boldsymbol{w}_{k,j}) \\
&+(1-\beta_1)\sum_{m=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{(k-1)n+i-((m-1)n+j)}\partial_l f_{\tau_{m,j}}(\boldsymbol{w}_{m,j})+\beta_1^{(k-1)n+i+1}\partial_l f_{\tau_{1,-1}}(\boldsymbol{w}_{1,0})\Bigg)^2 \\
\leq&\Bigg((1-\beta_1)\sum_{j=0}^{i}\beta_1^{(k-1)n+i-((k-1)n+j)}|\partial_l f_{\tau_{k,j}}(\boldsymbol{w}_{k,j})| \\
&+(1-\beta_1)\sum_{m=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{(k-1)n+i-((m-1)n+j)}|\partial_l f_{\tau_{m,j}}(\boldsymbol{w}_{m,j})|+\beta_1^{(k-1)n+i+1}\max_{s\in[n]}|\partial_l f_s(\boldsymbol{w}_{1,0})|\Bigg)^2 \\
\overset{(\star)}{\leq}&\Bigg((1-\beta_2)\sum_{j=0}^{i}\beta_2^{(k-1)n+i-((k-1)n+j)}|\partial_l f_{\tau_{k,j}}(\boldsymbol{w}_{k,j})|^2 \\
&+(1-\beta_2)\sum_{m=1}^{k-1}\sum_{j=0}^{n-1}\beta_2^{(k-1)n+i-((m-1)n+j)}|\partial_l f_{\tau_{m,j}}(\boldsymbol{w}_{m,j})|^2+\beta_2^{(k-1)n+i+1}\max_{s\in[n]}|\partial_l f_s(\boldsymbol{w}_{1,0})|^2\Bigg) \\
&\cdot\left(\frac{(1-\beta_1)^2}{1-\beta_2}\sum_{j=0}^{(k-1)n+i}\left(\frac{\beta_1^2}{\beta_2}\right)^j+\left(\frac{\beta_1^2}{\beta_2}\right)^{(k-1)n+i+1}\right) \\
=&\left(\frac{(1-\beta_1)^2}{1-\beta_2}\sum_{j=0}^{(k-1)n+i}\left(\frac{\beta_1^2}{\beta_2}\right)^j+\left(\frac{\beta_1^2}{\beta_2}\right)^{(k-1)n+i+1}\right)\boldsymbol{\nu}_{l,k,i} \\
\leq&\left(\frac{(1-\beta_1)^2}{1-\beta_2}\frac{1}{1-\frac{\beta_1^2}{\beta_2}}+1\right)\boldsymbol{\nu}_{l,k,i}=C_1\boldsymbol{\nu}_{l,k,i},
\end{aligned}
$$

where Eq. $(\star)$ is due to the Cauchy-Schwartz's Inequality. We complete the proof of the first claim. The second claim then follows directly from the update rule

$$
\boldsymbol{w}_{l,k,i+1}-\boldsymbol{w}_{l,k,i}=\eta_k\frac{\boldsymbol{m}_{l,k,i}}{\sqrt{\boldsymbol{\nu}_{l,k,i}}+\varepsilon}.
$$

The proof is completed. □

Based on Lemma 1, we then provide estimations for the norms of the momentum and the adaptor.

Lemma 2 (Estimation of the norm of the momentum)**.** *We have for all* $l\in[d], k\in\mathbb{Z}^+, i\in[n]$,

$$
\begin{aligned}
|\boldsymbol{m}_{l,k,i}|\leq&\max_{i'\in[n]}|\partial_l f_{i'}(\boldsymbol{w}_{k,0})|+\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})\sqrt{d}\eta_k+L_1C_1\sqrt{D_1}\eta_k\sum_{j=0}^{i-1}\|\nabla f(\boldsymbol{w}_{k,j})\| \\
&+L_1C_1\sqrt{D_1}\sum_{t=1}^{k-1}\eta_{k-t}\sum_{j=0}^{n-1}\beta_1^{tn+i-j}\|\nabla f(\boldsymbol{w}_{k-t,j})\|.
\end{aligned}
$$

Similarly, $l\in[d], k\in\mathbb{Z}^+/\{1\}$,

$$
|\boldsymbol{m}_{l,k-1,n-1}|\leq\max_{i'\in[n]}|\partial_l f_{i'}(\boldsymbol{w}_{k,0})|+\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}C_1\eta_{k-t}\sqrt{d}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,j})\|+\frac{2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k}{1-\beta_1}.
$$

Proof. To begin with, for any $t \in [k-1]$ and any $j \in [0, n-1]$, we have the following estimation for $\partial_l f_i(\boldsymbol{w}_{k-t,j})$:

$$
\begin{aligned}
&|\partial_l f_i(\boldsymbol{w}_{k-t,j})| \\
\leq &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} |\partial_l f_i(\boldsymbol{w}_{k-t,p}) - \partial_l f_i(\boldsymbol{w}_{k-t,p+1})| + \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} |\partial_l f_i(\boldsymbol{w}_{k-r,p}) - \partial_l f_i(\boldsymbol{w}_{k-r,p+1})| \\
\leq &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} (L_0 + L_1\|\nabla f_i(\boldsymbol{w}_{k-t,p})\|)\|\boldsymbol{w}_{k-t,p} - \boldsymbol{w}_{k-t,p+1}\| \\
&+ \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} (L_0 + L_1\|\nabla f_i(\boldsymbol{w}_{k-r,p})\|)\|\boldsymbol{w}_{k-r,p} - \boldsymbol{w}_{k-r,p+1}\| \\
\leq &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} (L_0 + L_1\|\nabla f_i(\boldsymbol{w}_{k-t,p})\|) C_1\eta_{k-t}\sqrt{d} \\
&+ \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} (L_0 + L_1\|\nabla f_i(\boldsymbol{w}_{k-r,p})\|) C_1\eta_{k-r}\sqrt{d} \\
\leq &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} \left(L_0 + L_1\sqrt{\sum_{i'\in[n]} \|\nabla f_{i'}(\boldsymbol{w}_{k-t,p})\|^2}\right) C_1\eta_{k-t}\sqrt{d} \\
&+ \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} \left(L_0 + L_1\sqrt{\sum_{i'\in[n]} \|\nabla f_{i'}(\boldsymbol{w}_{k-r,p})\|^2}\right) C_1\eta_{k-r}\sqrt{d} \\
\leq &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} \left(L_0 + L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p})\| + L_1\sqrt{D_0}\right) C_1\eta_{k-t}\sqrt{d} \\
&+ \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} \left(L_0 + L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| + L_1\sqrt{D_0}\right) C_1\eta_{k-r}\sqrt{d} \\
\overset{(*)}{\leq} &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p}) C_1\eta_{k-t}\sqrt{d} + \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| C_1\eta_{k-r}\sqrt{d} \\
&+ 2(L_0 + L_1\sqrt{D_0}) C_1\sqrt{d}\eta_{k-1}(tn-j) \\
\leq &|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=j}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p}) C_1\eta_{k-t}\sqrt{d} + \sum_{r=1}^{t-1}\sum_{p=0}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| C_1\eta_{k-r}\sqrt{d} \\
&+ 2\sqrt{2}(L_0 + L_1\sqrt{D_0}) C_1\sqrt{d}\eta_k(tn-j).
\end{aligned}
$$

where Inequality $(*)$ is due to $\forall a, b \in \mathbb{N}^+, a > b$, $\sum_{i=0}^{b} \frac{1}{\sqrt{a-i}} \leq 2\frac{b+1}{a}$.

Similarly, we have that for any $j \in [0, n-1]$,

$$
\begin{aligned}
|\partial_l f_i(\boldsymbol{w}_{k,j})| &\leq |\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=0}^{j-1} |\partial_l f_i(\boldsymbol{w}_{k,p+1}) - \partial_l f_i(\boldsymbol{w}_{k,p})| \\
&\leq |\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=0}^{j-1} \left(L_0 + L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,p})\| + L_1\sqrt{D_0}\right) C_1\eta_k\sqrt{d} \\
&= |\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{p=0}^{j-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,p})\| C_1\eta_k\sqrt{d} + j(L_0 + L_1\sqrt{D_0}) C_1\sqrt{d}\eta_k.
\end{aligned}
$$

Therefore, the norm of $\boldsymbol{m}_{l,k,i}$ can be bounded as

$$
\begin{aligned}
&|\boldsymbol{m}_{l,k,i}| \\
\leq&(1-\beta_1)\sum_{j=0}^{i}\beta_1^{(k-1)n+i-((k-1)n+j)}|\partial_l f_{\tau_{k,j}}(\boldsymbol{w}_{k,j})| + (1-\beta_1)\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn+i-j}|\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j})| \\
&+ \beta_1^{(k-1)n+i+1}|\partial_l f_{\tau_{1,0}}(\boldsymbol{w}_{1,0})| \\
\leq&(1-\beta_1)\sum_{j=0}^{i}\beta_1^{(k-1)n+i-((k-1)n+j)}|\partial_l f_{\tau_{k,j}}(\boldsymbol{w}_{k,0})| + (1-\beta_1)\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn+i-j}|\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})| \\
&+ \beta_1^{(k-1)n+i+1}|\partial_l f_{\tau_{1,0}}(\boldsymbol{w}_{k,0})| \\
&+ (1-\beta_1)\sum_{j=0}^{i}\beta_1^{(k-1)n+i-((k-1)n+j)}\left(\sum_{p=0}^{j-1}C_1\eta_k\sqrt{d}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,p})\| + (L_0+L_1\sqrt{D_0})C_1\eta_k\sqrt{d}j\right) \\
&+ (1-\beta_1)\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn+i-j}\left(\sum_{p=j}^{n-1}C_1\eta_{k-t}\sqrt{d}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p})\|\right. \\
&\left.+\sum_{r=1}^{t-1}\sum_{p=0}^{n-1}C_1\eta_{k-r}\sqrt{d}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(tn-j)\right) \\
&+ \beta_1^{(k-1)n+i+1}\left(\sum_{t=1}^{k-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(k-1)n\right) \\
\leq&\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)\sqrt{d}C_1(L_0+L_1\sqrt{D_0})\eta_k + L_1C_1\sqrt{D_1}\eta_k\sum_{j=0}^{i-1}\|\nabla f(\boldsymbol{w}_{k,j})\| \\
&+ L_1C_1\sqrt{D_1}\sum_{t=1}^{k-1}\eta_{k-t}\sum_{j=0}^{n-1}\beta_1^{tn+i-j}\|\nabla f(\boldsymbol{w}_{k-t,j})\|.
\end{aligned}
$$

Following the same routine, we have

$$
\begin{aligned}
&|\boldsymbol{m}_{l,k,-1}| \\
\leq&(1-\beta_1)\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}|\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j})| + \beta_1^{(k-1)n}|\partial_l f_{\tau_{1,0}}(\boldsymbol{w}_{1,0})| \\
\leq&(1-\beta_1)\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}|\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})| + \beta_1^{(k-1)n}|\partial_l f_{\tau_{1,0}}(\boldsymbol{w}_{k,0})| \\
&+ (1-\beta_1)\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}C_1\sqrt{d}\left(\sum_{p=j}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p})\|\eta_{k-t} + \sum_{r=1}^{t-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|\eta_{k-r}\right. \\
&\left.+ 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(tn-j)\right) \\
&+ \beta_1^{(k-1)n}\left(\sum_{t=1}^{k-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(k-1)n\right) \\
\leq&\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}C_1\eta_{k-t}\sqrt{d}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,j})\| \\
&+ \frac{2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k}{1-\beta_1}.
\end{aligned}
$$

The proof is completed. □

Lemma 3 (Estimation of the norm of the adaptor)**.** *We have for all $l \in [d], k \in \mathbb{Z}^+$,*

$$|\boldsymbol{\nu}_{l,k,0}| \geq \beta_2^n \frac{1-\beta_2}{1-\beta_2^n} \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 - \sqrt{\sum_{i\in[n]} |\partial_l f_i(\boldsymbol{w}_{k,0})^2|} \left(8\sqrt{2n}\eta_k C_1 L_0 \frac{1-\beta_2}{(1-\beta_2^n)^2}\beta_2^n \right.$$
$$\left. + 4L_1 C_1 \frac{1-\beta_2}{1-\beta_2^n} \frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}} \left(\sum_{t=1}^{k-1} \beta_2^n \sqrt{\beta_2}^{(r-1)n} \eta_{k-t} \sum_{j=0}^{n-1} (\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,j})\| + \sqrt{D_0}) \right) \right),$$

and

$$|\boldsymbol{\nu}_{l,k,0}| \leq 2 \max_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 + 2 \left(2\sqrt{2}\eta_k C_1 (L_0 + L_1\sqrt{D_0}) \frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}} \frac{\sqrt{\beta_2}}{1-\sqrt{\beta_2}} \right.$$
$$\left. + L_1 C_1 \sqrt{D_1} \sum_{t=1}^{k-1} \eta_{k-t} \frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}} \sum_{j=0}^{n-1} \sqrt{\beta_2}^{(t-1)n} \|\nabla f(\boldsymbol{w}_{k-t,j})\| \right)^2.$$

Proof. By the definition of $\boldsymbol{\nu}_{l,k,0}$, we have

$$\begin{aligned}
&\boldsymbol{\nu}_{l,k,0} \\
=&(1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2 + \sum_{t=1}^{k-1}\sum_{j=0}^{n-1} (1-\beta_2)\beta_2^{tn-j} \partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j})^2 \\
&+ \beta_2^{(k-1)n+1} \max_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{1,0})^2 \\
\geq&(1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2 + \sum_{t=1}^{k-1}\sum_{j=0}^{n-1} (1-\beta_2)\beta_2^{tn} \partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j})^2 \\
&+ \beta_2^{(k-1)n+1} \frac{1}{n} \sum_{i=1}^{n} \partial_l f_i(\boldsymbol{w}_{1,0})^2 \\
=&(1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2 + \sum_{t=1}^{k-1}\sum_{j=0}^{n-1} (1-\beta_2)\beta_2^{tn} (\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0}) + \partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j}) - \partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0}))^2 \\
&+ \beta_2^{(k-1)n+1} \frac{1}{n} \sum_{i=1}^{n} (\partial_l f_i(\boldsymbol{w}_{k,0}) + \partial_l f_i(\boldsymbol{w}_{1,0}) - \partial_l f_i(\boldsymbol{w}_{k,0}))^2 \\
\geq&(1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2 + \sum_{t=1}^{k-1}\sum_{j=0}^{n-1} (1-\beta_2)\beta_2^{tn} \partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})^2 \\
&+ \beta_2^{(k-1)n+1} \frac{1}{n} \sum_{i=1}^{n} \partial_l f_i(\boldsymbol{w}_{k,0})^2 \\
&- \sum_{t=1}^{k-1}\sum_{j=0}^{n-1} (1-\beta_2)\beta_2^{tn} |\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})||\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0}) - \partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j})| \\
&- \beta_2^{(k-1)n+1} \frac{1}{n} \sum_{i=1}^{n} |\partial_l f_i(\boldsymbol{w}_{k,0})||\partial_l f_i(\boldsymbol{w}_{k,0}) - \partial_l f_i(\boldsymbol{w}_{1,0})| \\
\overset{(\star)}{\geq}& \left(\beta_2^n \frac{1-\beta_2^{(k-1)n}}{1-\beta_2^n}(1-\beta_2) + \frac{\beta_2^{(k-1)n+1}}{n} \right) \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 \\
&- \sum_{t=1}^{k-1}\sum_{j=0}^{n-1} (1-\beta_2)\beta_2^{tn} |\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})| \left(\sum_{r=1}^{t}\sum_{p=0}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| C_1 \eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0 + L_1\sqrt{D_0}) C_1 \sqrt{d}\eta_k tn \right) \\
&- \beta_2^{(k-1)n+1} \frac{1}{n} \sum_{i=1}^{n} |\partial_l f_i(\boldsymbol{w}_{k,0})| \left(\sum_{r=1}^{k-1}\sum_{p=0}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| C_1 \eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0 + L_1\sqrt{D_0}) C_1 \sqrt{d}\eta_k (k-1)n \right)
\end{aligned}$$

$$
\begin{aligned}
\geq & \beta_2^n \frac{1-\beta_2}{1-\beta_2^n} \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 \\
& - \sum_{t=1}^{k-1}\sum_{j=0}^{n-1}(1-\beta_2)\beta_2^{tn}|\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})| \left(\sum_{r=1}^{t}\sum_{p=0}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k tn\right) \\
& - \beta_2^{(k-1)n+1}\frac{1}{n}\sum_{i=1}^{n}|\partial_l f_i(\boldsymbol{w}_{k,0})| \left(\sum_{r=1}^{k-1}\sum_{p=0}^{n-1} L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\| C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k (k-1)n\right) \\
\geq & \beta_2^n \frac{1-\beta_2}{1-\beta_2^n} \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 - 8\sqrt{2}\eta_k C_1 L_0 \frac{1-\beta_2}{(1-\beta_2^n)^2}\beta_2^n \sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| \\
& - 4L_1C_1\frac{1-\beta_2}{1-\beta_2^n}\sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| \left(\sum_{r=1}^{k-1}\beta_2^{rn}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f_i(\boldsymbol{w}_{k-r,j})\|\right) \\
\geq & \beta_2^n \frac{1-\beta_2}{1-\beta_2^n} \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 - 8\sqrt{2}\eta_k C_1 L_0 \frac{1-\beta_2}{(1-\beta_2^n)^2}\beta_2^n \sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| \\
& - 4L_1C_1\frac{1-\beta_2}{1-\beta_2^n}\|\nabla f_i(\boldsymbol{w}_{k,0})\| \left(\sum_{r=1}^{k-1}\beta_2^{rn}\eta_{k-r}\sum_{j=0}^{n-1}(\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,j})\| + \sqrt{D_0})\right) \\
\geq & \beta_2^n \frac{1-\beta_2}{1-\beta_2^n} \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 - 8\sqrt{2n}\eta_k C_1 L_0 \frac{1-\beta_2}{(1-\beta_2^n)^2}\beta_2^n \sqrt{\sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})^2|} \\
& - 4L_1C_1\frac{1-\beta_2}{1-\beta_2^n}\sqrt{\sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})^2|} \left(\sum_{r=1}^{k-1}\beta_2^{rn}\eta_{k-r}\sum_{j=0}^{n-1}(\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,j})\| + \sqrt{D_0})\right) \\
\geq & \beta_2^n \frac{1-\beta_2}{1-\beta_2^n} \sum_{i\in[n]} \partial_l f_i(\boldsymbol{w}_{k,0})^2 - 8\sqrt{2n}\eta_k C_1 L_0 \frac{1-\beta_2}{(1-\beta_2^n)^2}\beta_2^n \sqrt{\sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})^2|} \\
& - 4L_1C_1\frac{1-\beta_2}{1-\beta_2^n}\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\sqrt{\sum_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})^2|} \left(\sum_{r=1}^{k-1}\beta_2^n\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}(\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,j})\| + \sqrt{D_0})\right).
\end{aligned}
$$

The first claim is proved.

As for the upper bound, we have

$$
\begin{aligned}
&\boldsymbol{\nu}_{l,k,0} \\
=&(1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2 + \sum_{t=1}^{k-1}\sum_{j=0}^{n-1}(1-\beta_2)\beta_2^{tn-j}\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k-t,j})^2 \\
&+ \beta_2^{(k-1)n+1}\max_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{1,0})^2 \\
\leq&2(1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2 + 2\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}(1-\beta_2)\beta_2^{tn-j}\partial_l f_{\tau_{k-t,j}}(\boldsymbol{w}_{k,0})^2 \\
&+ 2\beta_2^{(k-1)n+1}\max_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{k,0})^2 \\
&+ 2\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}(1-\beta_2)\beta_2^{tn-j}\left(\sum_{p=j}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p})C_1\eta_{k-t}\sqrt{d}\right. \\
&\left.+\sum_{r=1}^{t-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(tn-j)\right)^2 \\
&+ 2\beta_2^{(k-1)n+1}\left(\sum_{r=1}^{k-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(k-1)n\right)^2 \\
\leq&2\max_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{k,0})^2 + 2\left(\sum_{t=1}^{k-1}\sum_{j=0}^{n-1}\sqrt{1-\beta_2}\sqrt{\beta_2}^{tn-j}\left(\sum_{p=j}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-t,p})C_1\eta_{k-t}\sqrt{d}\right.\right. \\
&\left.+\sum_{r=1}^{t-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(tn-j)\right) \\
&\left.+\sqrt{\beta_2}^{(k-1)n+1}\left(\sum_{r=1}^{k-1}\sum_{p=0}^{n-1}L_1\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,p})\|C_1\eta_{k-r}\sqrt{d} + 2\sqrt{2}(L_0+L_1\sqrt{D_0})C_1\sqrt{d}\eta_k(k-1)n\right)\right)^2 \\
\leq&2\max_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{k,0})^2 + 2\left(2\sqrt{2}\eta_k C_1(L_0+L_1\sqrt{D_0})\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\frac{\sqrt{\beta_2}}{1-\sqrt{\beta_2}}\right. \\
&\left.+ L_1C_1\sqrt{D_1}\sum_{t=1}^{k-1}\eta_{k-t}\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\sum_{j=0}^{n-1}\sqrt{\beta_2}^{tn-j}\|\nabla f(\boldsymbol{w}_{k-t,j})\|\right)^2 \\
\leq&2\max_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{k,0})^2 + 2\left(2\sqrt{2}\eta_k C_1(L_0+L_1\sqrt{D_0})\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\frac{\sqrt{\beta_2}}{1-\sqrt{\beta_2}}\right. \\
&\left.+ L_1C_1\sqrt{D_1}\sum_{t=1}^{k-1}\eta_{k-t}\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\sum_{j=0}^{n-1}\sqrt{\beta_2}^{(t-1)n}\|\nabla f(\boldsymbol{w}_{k-t,j})\|\right)^2.
\end{aligned}
$$

The proof is completed. □

We then immediately have the following corollary when $\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|$ is large enough.

Corollary 2. *If*

$$
\begin{aligned}
\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| \geq& 4L_1C_1\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\left(\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}(\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k-r,j})\| + \sqrt{D_0})\right) \\
&+ 2\sqrt{2}\eta_k C_1(L_0+L_1\sqrt{D_0})\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\frac{\sqrt{\beta_2}}{1-\sqrt{\beta_2}} + 8\sqrt{2n}\eta_k C_1 L_0\frac{1}{1-\beta_2^n} \\
&+ \eta_k C_1\left(n(L_0+L_1\sqrt{D_0}) + L_1\sqrt{D_1}\left(\sum_{p=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,p})\|\right)\right), \qquad (10)
\end{aligned}
$$

then

$$
\frac{\beta_2^n}{2}\frac{1}{n}\sum_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{k,0})^2 \leq \boldsymbol{\nu}_{l,k,0} \leq 4\max_{i\in[n]}\partial_l f_i(\boldsymbol{w}_{k,0})^2.
$$

Furthermore, if Eq. (10) holds, we have $\forall i \in \{0, \cdots, n-1\}$,

$$\beta_2^{n-1}\boldsymbol{\nu}_{l,k,0} \leq \boldsymbol{\nu}_{l,k,i} \leq \left(\beta_2^{n-1} + 8n\frac{1-\beta_2^{n-1}}{\beta_2^n}\right)\boldsymbol{\nu}_{l,k,0},$$

and

$$\frac{1}{\beta_2}\left(1-(1-\beta_2)\frac{2n}{\beta_2^n}\right)\boldsymbol{\nu}_{l,k,0} \leq \boldsymbol{\nu}_{l,k,-1} \leq \frac{1}{\beta_2}\boldsymbol{\nu}_{l,k,0},$$

Proof. The first claim is derived by directly applying the range of $\max_{i\in[n]} |\partial_l f_i(\boldsymbol{w}_{k,0})|$ into Lemma 3.

As for the second claim, we have

$$\boldsymbol{\nu}_{l,k,i} = \beta_2^i\boldsymbol{\nu}_{l,k,0} + (1-\beta_2)(\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})^2 + \cdots + \beta_2^{i-1}\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,1})^2).$$

On the other hand, since $\forall j \in \{0, \cdots, n-1\}$

$$\begin{aligned}
|\partial_l f_i(\boldsymbol{w}_{k,j})| \leq& \max_{p\in[n]} |\partial_l f_p(\boldsymbol{w}_{k,0})| + \eta_k C_1\left(j(L_0 + L_1\sqrt{D_0}) + L_1\sqrt{D_1}\left(\sum_{p=0}^{j-1}\|\nabla f(\boldsymbol{w}_{k,p})\|\right)\right)\\
\leq& \max_{p\in[n]} |\partial_l f_p(\boldsymbol{w}_{k,0})| + \eta_k C_1\left(n(L_0 + L_1\sqrt{D_0}) + L_1\sqrt{D_1}\left(\sum_{p=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,p})\|\right)\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\beta_2^{n-1}\boldsymbol{\nu}_{l,k,0} \leq \boldsymbol{\nu}_{l,k,i}\\
\leq&\beta_2^i\boldsymbol{\nu}_{l,k,0} + 2(1-\beta_2)\max_{p\in[n]}\partial_l f_p(\boldsymbol{w}_{k,0})^2(1+\cdots+\beta_2^{i-1})\\
&+ 2(1-\beta_2)(1+\cdots+\beta_2^{i-1})\eta_k^2 C_1^2\left(n(L_0 + L_1\sqrt{D_0}) + L_1\sqrt{D_1}\left(\sum_{p=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,p})\|\right)\right)^2\\
=&\beta_2^i\boldsymbol{\nu}_{l,k,0} + 2(1-\beta_2^i)\max_{p\in[n]}\partial_l f_p(\boldsymbol{w}_{k,0})^2 + 2(1-\beta_2^i)\eta_k^2 C_1^2\left(n(L_0 + L_1\sqrt{D_0}) + L_1\sqrt{D_1}\left(\sum_{p=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,p})\|\right)\right)^2.
\end{aligned}$$

Therefore, if Eq. (10) holds, we then have

$$\begin{aligned}
\boldsymbol{\nu}_{l,k,i} \leq&\beta_2^i\boldsymbol{\nu}_{l,k,0} + 4(1-\beta_2^i)\max_{p\in[n]}\partial_l f_p(\boldsymbol{w}_{k,0})^2\\
\leq&\beta_2^i\boldsymbol{\nu}_{l,k,0} + 4\frac{n}{n}(1-\beta_2^i)\sum_{p\in[n]}\partial_l f_p(\boldsymbol{w}_{k,0})^2 \leq \left(\beta_2^i + 8n\frac{1-\beta_2^i}{\beta_2^n}\right)\boldsymbol{\nu}_{l,k,0}\\
\leq&\left(\beta_2^{n-1} + 8n\frac{1-\beta_2^{n-1}}{\beta_2^n}\right)\boldsymbol{\nu}_{l,k,0}.
\end{aligned}$$

Following the same routine, we have

$$\beta_2\boldsymbol{\nu}_{l,k,-1} \leq \boldsymbol{\nu}_{l,k,0},$$

and if Eq. (10) holds,

$$\begin{aligned}
\boldsymbol{\nu}_{l,k,-1} =&\frac{1}{\beta_2}\left(\boldsymbol{\nu}_{l,k,0} - (1-\beta_2)\partial_l f_{\tau_{k,0}}(\boldsymbol{w}_{k,0})^2\right)\\
\geq&\frac{1}{\beta_2}\left(\boldsymbol{\nu}_{l,k,0} - (1-\beta_2)\max_p\partial_l f_p(\boldsymbol{w}_{k,0})^2\right)\\
\geq&\boldsymbol{\nu}_{l,k,0}\frac{1}{\beta_2}\left(1-(1-\beta_2)\frac{2n}{\beta_2^n}\right).
\end{aligned}$$

The proof of the second claim is completed. □

Remark 5. *For brevity, we denote*

$$C_3 \triangleq C_1 \left(n(L_0 + L_1\sqrt{D_0}) + 2\sqrt{2}(L_0 + L_1\sqrt{D_0})\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}\frac{\sqrt{\beta_2}}{1-\sqrt{\beta_2}} + 8\sqrt{2n}L_0\frac{1}{1-\beta_2^n} \right),$$

$$C_4 \triangleq 4L_1C_1\sqrt{D_1}\frac{\sqrt{1-\beta_2}}{1-\sqrt{\beta_2}}.$$

The right-hand-size of Eq. (10) is smaller than

$$\begin{aligned}\max_{i\in[n]} |\partial_l f_i(\boldsymbol{w}_{k,0})| \geq & C_3\eta_k + C_4\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\| \\ & + C_4 n\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r} + \eta_k C_4\left(\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|\right). \end{aligned} \tag{11}$$

Furthermore, we define $g(\beta_2)$ as

$$g(\beta_2) \triangleq \max\left\{ \frac{1}{\sqrt{\beta_2}^{n-1}} - 1, 1 - \frac{1}{\sqrt{\beta_2^{n-1} + 8n\frac{1-\beta_2^{n-1}}{\beta_2^n}}}, 1-\sqrt{\beta_2}, \sqrt{\frac{\beta_2}{\left(1-(1-\beta_2)\frac{2n}{\beta_2^n}\right)}} - 1 \right\},$$

and the conclusion of Corollary 2 can be translated into that if Eq. (11) holds,

$$\left| \frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,i}}} - \frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}} \right| \leq g(\beta_2)\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}},$$

and

$$\left| \frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,-1}}} - \frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}} \right| \leq g(\beta_2)\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}.$$

In the end of this section, we draw the connection of the gradients across the epoch.

Lemma 4. $\forall k \in \mathbb{N}^+, i \in \{0, \cdots, n-1\}$,

$$\|\nabla f(\boldsymbol{w}_{k,i})\| \leq (1 + n\sqrt{d}C_1\eta_1 L_1\sqrt{n}\sqrt{D_1})\|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n\sqrt{d}C_1\eta_k.$$

Proof. By Assumption 1, we have

$$\begin{aligned} & \|\nabla f(\boldsymbol{w}_{k,i})\| \\ \leq & \|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sum_{i=1}^{n}\|\nabla f_i(\boldsymbol{w}_{k,0})\|\right)\|\boldsymbol{w}_{k,i} - \boldsymbol{w}_{k,0}\| \\ \leq & \|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sum_{i=1}^{n}\|\nabla f_i(\boldsymbol{w}_{k,0})\|\right) i\sqrt{d}C_1\eta_k \\ \leq & \|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sqrt{n}\sqrt{\sum_{i=1}^{n}\|\nabla f_i(\boldsymbol{w}_{k,0})\|^2}\right) i\sqrt{d}C_1\eta_k \\ \leq & \|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sqrt{n}\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,0})\| + L_1\sqrt{n}\sqrt{D_0}\right) i\sqrt{d}C_1\eta_k \\ \leq & (1 + n\sqrt{d}C_1\eta_1 L_1\sqrt{n}\sqrt{D_1})\|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right) n\sqrt{d}C_1\eta_k. \end{aligned}$$

The proof is completed. □

Descent Lemma Under (L_0, L_1)-smooth condition We need a descent lemma assuming (L_0, L_1)-smooth condition similar as the case assuming L smoothness. Specifically, for a function $h : \mathcal{X} \to \mathbb{R}$

satisfying L-smooth condition and two points $\boldsymbol{w}$ and $\boldsymbol{v}$ in the domain $\mathcal{X}$, by Taylor's expansion, we have

$$h(\boldsymbol{w}) \leq h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v} \rangle + \frac{L}{2}\|\boldsymbol{w} - \boldsymbol{v}\|^2.$$

This is called "Descent Lemma" by existing literature Sra (2014), as it guarantees that the loss decreases with proper parameter update. Parallel to the above inequality, we have the following Descent Lemma under (L_0, L_1) smoothness.

Lemma 5. *Assume that function $h : \mathcal{X} \to \mathbb{R}$ satisfies (L_0, L_1)-smooth condition, i.e., $\forall \boldsymbol{w}, \boldsymbol{v} \in \mathcal{X}$ satisfying $\|\boldsymbol{w} - \boldsymbol{v}\| \leq \frac{1}{L_1}$,*

$$\|\nabla h(\boldsymbol{w}) - \nabla h(\boldsymbol{v})\| \leq (L_0 + L_1\|\nabla h(\boldsymbol{v})\|)\|\boldsymbol{w} - \boldsymbol{v}\|.$$

Then, for any three points $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} \in \mathcal{X}$ satisfying $\|\boldsymbol{w} - \boldsymbol{u}\| \leq \frac{1}{L_1}$ and $\|\boldsymbol{v} - \boldsymbol{u}\| \leq \frac{1}{L_1}$. Then,

$$h(\boldsymbol{w}) \leq h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle + \frac{1}{2}(L_0 + L_1\|\nabla h(\boldsymbol{u})\|)(\|\boldsymbol{v} - \boldsymbol{u}\| + \|\boldsymbol{w} - \boldsymbol{u}\|)\|\boldsymbol{w} - \boldsymbol{v}\|.$$

Proof. By the Fundamental theorem of calculus, we have

$$\begin{aligned}
h(\boldsymbol{w}) =& h(\boldsymbol{v}) + \int_0^1 \langle \nabla h(\boldsymbol{v} + a(\boldsymbol{w} - \boldsymbol{v})), \boldsymbol{w} - \boldsymbol{v} \rangle \mathrm{d}a \\
=& h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle + \int_0^1 \langle \nabla h(\boldsymbol{v} + a(\boldsymbol{w} - \boldsymbol{v})) - \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle \mathrm{d}a \\
\leq& h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle + \int_0^1 \|\nabla h(\boldsymbol{v} + a(\boldsymbol{w} - \boldsymbol{v})) - \nabla h(\boldsymbol{u})\|\|\boldsymbol{w} - \boldsymbol{v}\| \mathrm{d}a \\
\overset{(\star)}{\leq}& h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle + \int_0^1 (L_0 + L_1\|\nabla h(\boldsymbol{u})\|)\|\boldsymbol{v} + a(\boldsymbol{w} - \boldsymbol{v}) - \boldsymbol{u}\|\|\boldsymbol{w} - \boldsymbol{v}\| \mathrm{d}a \\
\leq& h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle + \int_0^1 (L_0 + L_1\|\nabla h(\boldsymbol{u})\|)((1-a)\|\boldsymbol{v} - \boldsymbol{u}\| + a\|\boldsymbol{w} - \boldsymbol{u}\|)\|\boldsymbol{w} - \boldsymbol{v}\| \mathrm{d}a \\
\leq& h(\boldsymbol{v}) + \langle \nabla h(\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{v} \rangle + \frac{1}{2}(L_0 + L_1\|\nabla h(\boldsymbol{u})\|)(\|\boldsymbol{v} - \boldsymbol{u}\| + \|\boldsymbol{w} - \boldsymbol{u}\|)\|\boldsymbol{w} - \boldsymbol{v}\|,
\end{aligned}$$

where Inequality $(\star)$ is due to

$$\|\boldsymbol{v} + a(\boldsymbol{w} - \boldsymbol{v}) - \boldsymbol{u}\| = \|(1-a)(\boldsymbol{v} - \boldsymbol{u}) + a(\boldsymbol{w} - \boldsymbol{u})\| \leq (1-a)\|\boldsymbol{v} - \boldsymbol{u}\| + a\|\boldsymbol{w} - \boldsymbol{u}\| \leq \frac{1}{L_1},$$

and thus the definition of (L_0, L_1)-smooth condition can be applied.

The proof is completed. □

C.2 Proof of Adam's Convergence

Proof of Theorem 1. We define $\boldsymbol{u}_k \triangleq \frac{\boldsymbol{w}_{k,0} - \beta_1 \boldsymbol{w}_{k,-1}}{1-\beta_1}$ (with $\boldsymbol{w}_{1,-1} \triangleq \boldsymbol{w}_{1,0} = \boldsymbol{w}_0$), and let $\boldsymbol{u}_{l,k}$ be the i-th component of $\boldsymbol{u}_k$, $\forall k \in \mathbb{N}^+$, $l \in [d]$. Then, by Lemma 1, we immediately have $\forall l \in [d]$, $|\boldsymbol{u}_{l,k} - \boldsymbol{w}_{l,k,0}|$ is bounded as

$$\begin{aligned}
|\boldsymbol{u}_{l,k} - \boldsymbol{w}_{l,k,0}| =& \left| \frac{\boldsymbol{w}_{l,k,0} - \beta_1 \boldsymbol{w}_{l,k,-1}}{1-\beta_1} - \boldsymbol{w}_{l,k,0} \right| \\
=& \frac{\beta_1}{1-\beta_1} |\boldsymbol{w}_{l,k,0} - \boldsymbol{w}_{l,k,-1}| \leq \frac{\beta_1}{1-\beta_1} C_1 \eta_1 \frac{1}{\sqrt{k-1}} \qquad (12) \\
\leq& \frac{\sqrt{2}\beta_1}{1-\beta_1} C_1 \eta_1 \frac{1}{\sqrt{k}} \leq \frac{\sqrt{2}\beta_1}{1-\beta_1} C_1 \eta_k \leq C_2 \eta_k, \qquad (13)
\end{aligned}$$

and

$$
\begin{aligned}
&|\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k}| \\
=&\left|\frac{\boldsymbol{w}_{l,k+1,0}-\beta_1\boldsymbol{w}_{l,k+1,-1}}{1-\beta_1}-\frac{\boldsymbol{w}_{l,k,0}-\beta_1\boldsymbol{w}_{l,k,-1}}{1-\beta_1}\right| \\
=&\left|(\boldsymbol{w}_{l,k+1,0}-\boldsymbol{w}_{l,k,0})+\frac{\beta_1}{1-\beta_1}(\boldsymbol{w}_{l,k+1,0}-\boldsymbol{w}_{l,k+1,-1})-\frac{\beta_1}{1-\beta_1}(\boldsymbol{w}_{l,k,0}-\boldsymbol{w}_{l,k,-1})\right| \\
\leq&\left|(\boldsymbol{w}_{l,k+1,0}-\boldsymbol{w}_{l,k,0})+\frac{\beta_1}{1-\beta_1}(\boldsymbol{w}_{l,k+1,0}-\boldsymbol{w}_{l,k+1,-1})-\frac{\beta_1}{1-\beta_1}(\boldsymbol{w}_{l,k,0}-\boldsymbol{w}_{l,k,-1})\right| \\
\leq& nC_1\eta_1\frac{1}{\sqrt{k}}+\frac{\beta_1}{1-\beta_1}C_1\eta_1\left(\frac{1}{\sqrt{k}}+\frac{\sqrt{2}}{\sqrt{k}}\right)=C_2\eta_1\frac{1}{\sqrt{k}}=C_2\eta_k, \qquad (14)
\end{aligned}
$$

where C_2 is defined as $C_2 \triangleq nC_1+\frac{\beta_1}{1-\beta_1}C_1\left(1+\sqrt{2}\right)$.

We then analyze the change of the Lyapunov function $f(\boldsymbol{u}_k)$ along the iterations. Specifically, by Lemma 5, we have

$$
\begin{aligned}
&f(\boldsymbol{u}_{k+1}) \\
\leq& f(\boldsymbol{u}_k)+\langle\nabla f(\boldsymbol{w}_{k,0}),\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\rangle+\frac{nL_0+L_1\sum_{i\in[n]}\|\nabla f_i(\boldsymbol{w}_{k,0})\|}{2}(\|\boldsymbol{w}_{k,0}-\boldsymbol{u}_k\|+\|\boldsymbol{w}_{k,0}-\boldsymbol{u}_{k+1}\|)\|\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\| \\
\leq& f(\boldsymbol{u}_k)+\langle\nabla f(\boldsymbol{w}_{k,0}),\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\rangle+\frac{nL_0+L_1\sum_{i\in[n]}\|\nabla f_i(\boldsymbol{w}_{k,0})\|}{2}3C_2^2d\eta_k^2 \\
\leq& f(\boldsymbol{u}_k)+\langle\nabla f(\boldsymbol{w}_{k,0}),\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\rangle+\frac{nL_0+L_1\sqrt{n}\sqrt{\sum_{i\in[n]}\|\nabla f_i(\boldsymbol{w}_{k,0})\|^2}}{2}3C_2^2d\eta_k^2 \\
\leq& f(\boldsymbol{u}_k)+\langle\nabla f(\boldsymbol{w}_{k,0}),\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\rangle+\frac{nL_0+L_1\sqrt{n}\sqrt{D_0+D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2}}{2}3C_2^2d\eta_k^2 \\
\leq& f(\boldsymbol{u}_k)+\langle\nabla f(\boldsymbol{w}_{k,0}),\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\rangle+\frac{nL_0+L_1\sqrt{n}(\sqrt{D_0}+\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,0})\|)}{2}3C_2^2d\eta_k^2 \\
\overset{(*)}{=}& f(\boldsymbol{u}_k)+\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})+\sum_{l\in\mathbb{L}_{small}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k}) \\
&+\frac{nL_0+L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_k^2+\frac{3L_1\sqrt{n}\sqrt{D_1}C_2^2d\eta_k^2}{2}\|\nabla f(\boldsymbol{w}_{k,0})\|.
\end{aligned}
$$

Here in Eq. $(*)$, $\mathbb{L}_{large}$ and $\mathbb{L}_{small}$ are respectively defined as

$$
\begin{aligned}
\mathbb{L}_{large}^k &= \{l : l\in[d], \text{s.t. Eq. (11) holds}\}, \\
\mathbb{L}_{small}^k &= \{l : l\in[d], \text{s.t. Eq. (11) doesn't hold}\}.
\end{aligned}
$$

Apparently, $\mathbb{L}_{large}\cup\mathbb{L}_{small}=[d]$. We than tackle $\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})$ and $\sum_{l\in\mathbb{L}_{small}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})$ respectively.

①Analysis for $\sum_{l\in\mathbb{L}_{small}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})$: By directly applying the range of $\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|$, we have

$$
\begin{aligned}
&\frac{1}{n}\left|\sum_{l\in\mathbb{L}_{small}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})\right| \\
\leq& dC_2\eta_k\left(C_3\eta_k+C_4\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\|+C_4n\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}+\eta_kC_4\left(\sum_{p=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,p})\|\right)\right).
\end{aligned}
$$

Summing over k from 1 to t then leads to

$$
\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{T}\left|\sum_{l\in\mathbb{L}_{small}^{k}}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})\right| \\
\leq&\sum_{k=1}^{T} dC_2C_3\eta_k^2 + dC_2C_4\sum_{k=1}^{T}\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\| + C_2C_4 n\sum_{k=1}^{T}\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r} + C_2C_4\sum_{k=1}^{T}\eta_k^2\sum_{p=0}^{n-1}\|\nabla \\
\leq&\sum_{k=1}^{T} dC_2C_3\eta_k^2 + \frac{dC_2C_4}{1-\sqrt{\beta_2^n}}\sum_{k=1}^{T-1}\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\| + \frac{C_2C_4 n}{1-\sqrt{\beta_2^n}}\sum_{k=1}^{T-1}\eta_k^2 + C_2C_4\sum_{k=1}^{T}\eta_k^2\sum_{p=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,p})\| \\
\leq&\left(dC_2C_3 + \frac{C_2C_4 n}{1-\sqrt{\beta_2^n}}\right)\eta_1^2(1+\ln T) + \left(C_2C_4 + \frac{dC_2C_4}{1-\sqrt{\beta_2^n}}\right)\sum_{k=1}^{T}\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|,
\end{aligned}
$$

which by Lemma 4 further leads to

$$
\begin{aligned}
&\sum_{k=1}^{T}\left|\sum_{l\in\mathbb{L}_{small}^{k}}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})\right| \\
\leq& n\left(C_2C_4 + \frac{dC_2C_4}{1-\sqrt{\beta_2^n}}\right)\sum_{k=1}^{T}\eta_k^2\sum_{j=0}^{n-1}\left((1+n\sqrt{d}C_1\eta_1L_1\sqrt{n})\|\nabla f(\boldsymbol{w}_{k,0})\| + \left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n\sqrt{d}C_1\eta_k\right) \\
&+ n\left(dC_2C_3 + \frac{C_2C_4 n}{1-\sqrt{\beta_2^n}}\right)\eta_1^2(1+\ln T) \\
\leq& n^2(1+n\sqrt{d}C_1\eta_1L_1\sqrt{n})\left(C_2C_4 + \frac{dC_2C_4}{1-\sqrt{\beta_2^n}}\right)\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\| + \left(dC_2C_3 + \frac{C_2C_4 n}{1-\sqrt{\beta_2^n}}\right)\eta_1^2(1+\ln T) \\
&+ n\left(C_2C_4 + \frac{dC_2C_4}{1-\sqrt{\beta_2^n}}\right)\left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n^2\sqrt{d}C_1\sum_{k=1}^{T}\eta_k^3 \\
\leq& n^2(1+n\sqrt{d}C_1\eta_1L_1\sqrt{n}\sqrt{D_1})\left(C_2C_4 + \frac{dC_2C_4\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right)\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\| + \left(dC_2C_3 + \frac{C_2C_4 n\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right)\eta_1^2(1+\ln T) \\
&+ 3n\left(C_2C_4 + \frac{dC_2C_4}{1-\sqrt{\beta_2^n}}\right)\left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n^2\sqrt{d}C_1\eta_1^3. \qquad (15)
\end{aligned}
$$

We further define

$$
\begin{aligned}
C_5 &\triangleq n^2(1+n\sqrt{d}C_1\eta_1L_1\sqrt{n}\sqrt{D_1})\left(C_4 + \frac{dC_4\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right), \\
C_6 &\triangleq \left(dC_3 + \frac{C_4 n\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right)\eta_1^2, \\
C_7 &\triangleq 3n\left(C_4 + \frac{dC_4}{1-\sqrt{\beta_2^n}}\right)\left(nL_0 + L_1\sqrt{n}\sqrt{D_0}\right)n^2\sqrt{d}C_1\eta_1^3 + \left(dC_3 + \frac{C_2C_4 n\sqrt{D_1}}{1-\sqrt{\beta_2^n}}\right)\eta_1^2,
\end{aligned}
$$

and thus

$$
\sum_{k=1}^{T}\left|\sum_{l\in\mathbb{L}_{small}^{k}}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})\right| \leq C_2\left(C_5\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\| + C_6\ln T + C_7\right).
$$

②Analysis for $\sum_{l\in\mathbb{L}_{large}^{k}}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})$: This term requires a more sophisticated analysis. To begin with, we provide a decomposition of $\boldsymbol{u}_{k+1}-\boldsymbol{u}_k$. According to the definition of $\boldsymbol{u}_k$, we

have

$$
\begin{aligned}
&\boldsymbol{u}_{k+1}-\boldsymbol{u}_k \\
=&\frac{(\boldsymbol{w}_{k+1,0}-\beta_1\boldsymbol{w}_{k+1,-1})-(\boldsymbol{w}_{k,0}-\beta_1\boldsymbol{w}_{k,-1})}{1-\beta_1} \\
=&\frac{(\boldsymbol{w}_{k+1,0}-\boldsymbol{w}_{k,0})-\beta_1(\boldsymbol{w}_{k+1,-1}-\boldsymbol{w}_{k,-1})}{1-\beta_1} \\
=&\frac{\sum_{i=0}^{n-1}(\boldsymbol{w}_{k,i+1}-\boldsymbol{w}_{k,i})-\beta_1\sum_{i=0}^{n-1}(\boldsymbol{w}_{k,i}-\boldsymbol{w}_{k,i-1})}{1-\beta_1} \\
=&\frac{(\boldsymbol{w}_{k+1,0}-\boldsymbol{w}_{k+1,-1})+(1-\beta_1)\sum_{i=0}^{n-2}(\boldsymbol{w}_{k,i+1}-\boldsymbol{w}_{k,i})-\beta_1(\boldsymbol{w}_{k,0}-\boldsymbol{w}_{k,-1})}{1-\beta_1} \\
\overset{(\star)}{=}&-\frac{\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,n-1}}}\odot\boldsymbol{m}_{k,n-1}+(1-\beta_1)\sum_{i=0}^{n-2}\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,i}}}\odot\boldsymbol{m}_{k,i}-\beta_1\frac{\eta_{k-1}}{\sqrt{\boldsymbol{\nu}_{k-1,n-1}}}\odot\boldsymbol{m}_{k-1,n-1}}{1-\beta_1} \\
=&-\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{k,0}}}\odot\frac{\boldsymbol{m}_{k,n-1}+(1-\beta_1)\sum_{i=0}^{n-2}\boldsymbol{m}_{k,i}-\beta_1\boldsymbol{m}_{k-1,n-1}}{1-\beta_1}-\eta_k\left(\left(\frac{1}{\sqrt{\boldsymbol{\nu}_{k,n-1}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{k,0}}}\right)\odot\frac{\boldsymbol{m}_{k,n-1}}{1-\beta_1}\right. \\
&\left.+\sum_{i=0}^{n-2}\left(\frac{1}{\sqrt{\boldsymbol{\nu}_{k,i}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{k,0}}}\right)\odot\boldsymbol{m}_{k,i}-\frac{\beta_1}{1-\beta_1}\left(\frac{1}{\sqrt{\boldsymbol{\nu}_{k-1,n-1}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{k,0}}}\right)\odot\boldsymbol{m}_{k-1,n-1}\right)-\frac{\beta_1}{1-\beta_1}(\eta_{k-1}-\eta_k)\frac{1}{\sqrt{\boldsymbol{\nu}_{k-1,n-1}}}\odot\boldsymbol{m}_{k-}
\end{aligned}
\tag{16}
$$

Here equation $(\star)$ is due to a direct application of the update rule of $\boldsymbol{w}_{k,i}$.

We then analyze the above three terms respectively, namely, we define

$$
\begin{aligned}
a_l^1 &\triangleq -\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\frac{\boldsymbol{m}_{l,k,n-1}+(1-\beta_1)\sum_{i=0}^{n-2}\boldsymbol{m}_{l,k,i}-\beta_1\boldsymbol{m}_{l,k-1,n-1}}{1-\beta_1}=-\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\sum_{i=0}^{n-1}\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,i}), \\
a_l^2 &\triangleq -\eta_k\left(\left(\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,n-1}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\right)\frac{\boldsymbol{m}_{l,k,n-1}}{1-\beta_1}+\sum_{i=0}^{n-2}\left(\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,i}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\right)\boldsymbol{m}_{l,k,i}\right. \\
&\qquad\left.-\frac{\beta_1}{1-\beta_1}\left(\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k-1,n-1}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\right)\boldsymbol{m}_{l,k-1,n-1}\right), \\
a_l^3 &\triangleq -\frac{\beta_1}{1-\beta_1}(\eta_{k-1}-\eta_k)\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k-1,n-1}}}\boldsymbol{m}_{l,k-1,n-1}.
\end{aligned}
$$

One can then easily observe that by Eq. (16),

$$
\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1}-\boldsymbol{u}_{l,k})=\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^1+\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^2+\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^3.
$$

②. (A) Tackling Term $\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^1$:

We have

$$
\begin{aligned}
&\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^1 \\
=&-\sum_{l\in\mathbb{L}_{large}^k}\partial_l\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\partial_l f(\boldsymbol{w}_{k,0})\left(\sum_{i=0}^{n-1}\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,0})\right)-\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\partial_l f(\boldsymbol{w}_{k,0})\left(\sum_{i=0}^{n-1}(\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})-\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,0}))\right) \\
=&-\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\partial_l f(\boldsymbol{w}_{k,0})^2-\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\partial_l f(\boldsymbol{w}_{k,0})\left(\sum_{i=0}^{n-1}(\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})-\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,0}))\right) \\
\overset{(\star)}{=}&-\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\partial_l f(\boldsymbol{w}_{k,0})^2+\mathcal{O}\left(\eta_k^2\right)+\mathcal{O}\left(\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|\right),
\end{aligned}
$$

where Eq. $(\star)$ is due to

$$
\begin{aligned}
&\left|\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\partial_l f(\boldsymbol{w}_{k,0})\left(\sum_{i=0}^{n-1}(\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})-\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,0}))\right)\right| \\
\overset{(*)}{\leq}&\eta_k\sqrt{\frac{2n^2}{\beta_2^n}}\left(\sum_{l\in\mathbb{L}_{large}^k}\sum_{i=0}^{n-1}|\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})-\partial_l f_{\tau_{k,i}}(\boldsymbol{w}_{k,0})|\right) \\
\leq&\eta_k\sqrt{\frac{2n^2}{\beta_2^n}}\left(\sqrt{d}\sum_{i=0}^{n-1}\|\nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,i})-\nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,0})\|\right) \\
\overset{(\circ)}{\leq}&\eta_k\sqrt{\frac{2n^2}{\beta_2^n}}\sqrt{d}\sum_{i=0}^{n-1}(L_0+L_1\|\nabla f_{\tau_{k,i}}(\boldsymbol{w}_{k,0})\|)\|\boldsymbol{w}_{k,i}-\boldsymbol{w}_{k,0}\| \\
\leq&\eta_k\sqrt{\frac{2n^2}{\beta_2^n}}\sqrt{d}(nL_0+L_1\sqrt{D_1}\sqrt{n}\|\nabla f(\boldsymbol{w}_{k,0})\|+\sqrt{n}L_1\sqrt{D_0})n\sqrt{d}C_1\eta_k \\
\overset{(\bullet)}{\leq}&\sqrt{\frac{2n^2}{\beta_2^n}}d(n^2L_0+n\sqrt{n}L_1\sqrt{D_0})C_1\eta_k^2+\eta_k^2 d\sqrt{\frac{2n^2}{\beta_2^n}}L_1\sqrt{D_1}n\sqrt{n}\|\nabla f(\boldsymbol{w}_{k,0})\|.
\end{aligned}
$$

Here Eq. $(*)$ is due to Corollary 2, Eq. $(\circ)$ is due to f_i is (L_0, L_1)-smooth, $\forall i$, and Eq. $(\bullet)$ is due to Lemma 1.

②. (B) Tackling Term $\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^2$:

We have for any $l\in\mathbb{L}_{max}$,

$$
\begin{aligned}
&|\partial_l f(\boldsymbol{w}_{k,0})a_l^2| \\
\leq&\eta_k|\partial_l f(\boldsymbol{w}_{k,0})|\left(\left|\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,n-1}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\right|\frac{|\boldsymbol{m}_{l,k,n-1}|}{1-\beta_1}+\sum_{i=0}^{n-2}\left|\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,i}}}-\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\right||\boldsymbol{m}_{l,k,i}|\right. \\
&\left.-\frac{\beta_1}{1-\beta_1}\left|\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k-1,n-1}}}+\frac{1}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\right||\boldsymbol{m}_{l,k-1,n-1}|\right) \\
\overset{(\star)}{\leq}&\eta_k g(\beta_2)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\left(\frac{|\boldsymbol{m}_{l,k,n-1}|}{1-\beta_1}+\sum_{i=0}^{n-2}|\boldsymbol{m}_{l,k,i}|+\frac{\beta_1}{1-\beta_1}|\boldsymbol{m}_{l,k-1,n-1}|\right) \\
\overset{(*)}{\leq}&\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
&+\eta_k^2 g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})\sqrt{d} \\
&+\eta_k^2 g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\| \\
&+\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\sum_{t=1}^{k-1}\eta_{k-t}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}\|\nabla f(\boldsymbol{w}_{k-t,j})\|,
\end{aligned}
$$

where Inequality $(\star)$ is due to Corollary 2, and $g(\beta_2)$ is defined in Lemma 5 , and Inequality $(*)$ is due to Lemma 2, by which we have $\forall i\in\{-1,\cdots,n-1\}$

$$
\begin{aligned}
|\boldsymbol{m}_{l,k,i}|\leq&\max_{i'\in[n]}|\partial_l f_{i'}(\boldsymbol{w}_{k,0})|+\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})\sqrt{d}\eta_k+L_1C_1\sqrt{D_1}\eta_k\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\| \\
&+L_1C_1\sqrt{D_1}\sum_{t=1}^{k-1}\eta_{k-t}\sum_{j=0}^{n-1}\beta_1^{tn-1-j}\|\nabla f(\boldsymbol{w}_{k-t,j})\|.
\end{aligned}
$$

Therefore, summing over $\mathbb{L}_{large}^k$ and k leads to

$$
\begin{aligned}
&\sum_{k=1}^{T}\left|\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^2\right| \\
\leq&\sum_{k=1}^{T}\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
&+\sum_{k=1}^{T}\eta_k^2 g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})d\sqrt{d} \\
&+dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\sum_{k=1}^{T}\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\| \\
&+dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\sum_{k=1}^{T}\eta_k\sum_{t=1}^{k-1}\eta_{k-t}\sum_{j=0}^{n-1}\beta_1^{(t-1)n}\|\nabla f(\boldsymbol{w}_{k-t,j})\| \\
\leq&\sum_{k=1}^{T}\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
&+g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})d\sqrt{d}\eta_1(1+\ln T) \\
&+dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\left(1+\frac{1}{1-\beta_2^n}\right)\sum_{k=1}^{T}\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\| \\
\overset{(\star)}{\leq}&\sum_{k=1}^{T}\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
&+g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})d\sqrt{d}\eta_1(1+\ln T) \\
&+dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\left(1+\frac{1}{1-\beta_2^n}\right) \\
&\cdot\sum_{k=1}^{T}\eta_k^2\sum_{j=0}^{n-1}\left((1+n\sqrt{d}C_1\eta_1L_1\sqrt{n}\sqrt{D_1})\|\nabla f(\boldsymbol{w}_{k,0})\|+\left(nL_0+L_1\sqrt{n}\sqrt{D_0}\right)n\sqrt{d}C_1\eta_k\right) \\
\leq&\sum_{k=1}^{T}\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
&+g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}\left(n+\frac{2\sqrt{2}\beta_1}{1-\beta_1}\right)C_1(L_0+L_1\sqrt{D_0})d\sqrt{d}\eta_1^2(1+\ln T) \\
&+dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\left(1+\frac{1}{1-\beta_2^n}\right)(n+n^{\frac{5}{2}}\sqrt{d}C_1\eta_1L_1\sqrt{D_1})\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\| \\
&+3dg(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}}L_1C_1\sqrt{D_1}\left(1+\frac{1}{1-\beta_2^n}\right)n\left(nL_0+L_1\sqrt{n}\sqrt{D_0}\right)n\sqrt{d}C_1\eta_1^3.
\end{aligned}
$$

where Inequality $(\star)$ is due to Lemma 4.

②. (C) Tackling Term $\sum_{l\in\mathbb{L}_{large}^k}\partial_l f(\boldsymbol{w}_{k,0})a_l^3$:

For any $l \in \mathbb{L}_{large}^k$,

$$
\begin{aligned}
&|\partial_l f(\boldsymbol{w}_{k,0}) a_l^3| \\
\leq & \frac{\beta_1}{1-\beta_1} |\eta_{k-1} - \eta_k| \frac{1}{\sqrt{\boldsymbol{\nu}_{l,k-1,n-1}}} |\boldsymbol{m}_{l,k-1,n-1}| |\partial_l f(\boldsymbol{w}_{k,0})| \\
\leq & \frac{\beta_1 \eta_1}{(1-\beta_1)} \frac{1}{\sqrt{k}\sqrt{k-1}(\sqrt{k}+\sqrt{k-1})} C_1 |\partial_l f(\boldsymbol{w}_{k,0})| \\
= & \frac{\beta_1 \eta_k}{(1-\beta_1)} \frac{1}{\sqrt{k-1}(\sqrt{k}+\sqrt{k-1})} C_1 |\partial_l f(\boldsymbol{w}_{k,0})|.
\end{aligned}
$$

Summing over k and $\mathbb{L}_{large}^k$ then leads to

$$
\begin{aligned}
&\sum_{k=1}^{T} \sum_{l \in \mathbb{L}_{large}^k} |\partial_l f(\boldsymbol{w}_{k,0}) a_l^3| \\
\leq & \frac{\beta_1}{(1-\beta_1)} \sum_{k=1}^{T} \sum_{l \in \mathbb{L}_{large}^k} \frac{\eta_k}{\sqrt{k-1}(\sqrt{k}+\sqrt{k-1})} C_1 |\partial_l f(\boldsymbol{w}_{k,0})| \\
\leq & 2 \frac{\beta_1}{(1-\beta_1)\eta_1} \sqrt{d} C_1 \sum_{k=1}^{T} \eta_k^2 \|\nabla f(\boldsymbol{w}_{k,0})\|.
\end{aligned}
$$

Put ②.(A), ②.(B), and ②.(C) together. We have

$$
\begin{aligned}
&\sum_{k=1}^{T} \sum_{l \in \mathbb{L}_{large}^k} \partial_l f(\boldsymbol{w}_{k,0})(\boldsymbol{u}_{l,k+1} - \boldsymbol{u}_{l,k}) \\
\leq & -\sum_{k=1}^{T} \sum_{l \in \mathbb{L}_{large}^k} \frac{\eta_k}{\sqrt{\boldsymbol{\nu}_{l,k,0}}} \partial_l f(\boldsymbol{w}_{k,0})^2 + \sum_{k=1}^{T} \sum_{l \in \mathbb{L}_{large}^k} \eta_k g(\beta_2) \left(n - 1 + \frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}} \left(\max_{i \in [n]} |\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
& + C_8 \sum_{k=1}^{T} \eta_k^2 \|\nabla f(\boldsymbol{w}_{k,0})\| + C_9 \ln T + C_{10}, \qquad (17)
\end{aligned}
$$

where C_5, C_6, and C_7 are constants defined as

$$
C_8 \triangleq \sqrt{\frac{2n^2}{\beta_2^n}} L_1 \sqrt{D_1} n \sqrt{n} + d g(\beta_2) \left(n - 1 + \frac{1+\beta_1}{1-\beta_1}\right) \frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}} L_1 C_1 \sqrt{D_1} \left(1 + \frac{1}{1-\beta_2^n}\right) (n + n^{\frac{5}{2}} \sqrt{d} C_1 \eta_1 L_1 \sqrt{D_1}) + 2 \frac{\beta_1}{(1-\beta_1)\eta_1} \sqrt{d}
$$

$$
C_9 \triangleq \sqrt{\frac{2n^2}{\beta_2^n}} d(n^2 L_0 + n\sqrt{n} L_1 \sqrt{D_0}) C_1 \eta_1^2 + g(\beta_2) \left(n - 1 + \frac{1+\beta_1}{1-\beta_1}\right) \frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}} \left(n + \frac{2\sqrt{2}\beta_1}{1-\beta_1}\right) C_1 (L_0 + L_1 \sqrt{D_0}) d\sqrt{d} \eta_1^2,
$$

$$
C_{10} \triangleq 3 d g(\beta_2) \left(n - 1 + \frac{1+\beta_1}{1-\beta_1}\right) \frac{\sqrt{2}n}{\beta_2^{\frac{n}{2}}} L_1 C_1 \sqrt{D_1} \left(1 + \frac{1}{1-\beta_2^n}\right) n \left(n L_0 + L_1 \sqrt{n} \sqrt{D_0}\right) n \sqrt{d} C_1 \eta_1^3 + C_9.
$$

We then analyze the first two terms in Eq. (17) here. Specifically, we have

$$
\begin{aligned}
&\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{\sqrt{\boldsymbol{\nu}_{l,k,0}}+\varepsilon}-\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\overset{(\star)}{\geq}&\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{\sqrt{\boldsymbol{\nu}_{l,k,0}}+\varepsilon}-\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\geq&\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}-\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\overset{(\circ)}{=}&\sum_{l\in[d]}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}-\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
&+\mathcal{O}\left(\eta_k^2+\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\|+\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}+\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|\right)\\
\geq&\sum_{l\in[d]}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}-\sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
&+\mathcal{O}\left(\eta_k^2+\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\|+\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}+\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|\right),
\end{aligned}
$$

where Inequality $(\star)$ is due to Corollary 2 and Equality $(\circ)$ is due to

$$
\begin{aligned}
&\sum_{l\in\mathbb{L}_{small}^k}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\leq\sum_{l\in\mathbb{L}_{small}^k}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\leq\frac{n}{2}\eta_k\sum_{l\in\mathbb{L}_{small}^k}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\\
\leq&\frac{nd\eta_k}{2}\left(C_3\eta_k+C_4\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\|+C_4 n\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}+\eta_k C_4\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|\right).
\end{aligned}
$$

Parallel to Eq. (15) and summing the right-hand-side of the above inequality over k from 1 to t, we have

$$
\begin{aligned}
&\sum_{t=1}^{T}\frac{nd\eta_k}{2}\left(C_3\eta_k+C_4\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\|+C_4 n\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}+\eta_k C_4\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|\right)\\
\leq&\frac{1}{2}\left(C_5\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|+C_6\ln T+C_7\right).
\end{aligned}
$$

Suppose now there does not exist an iteration $k\in[T]$, such that

$$
\|\nabla f(\boldsymbol{w}_{k,0})\|\leq 2\sqrt{d}(2\sqrt{2}+1)\sqrt{D_0}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}},
$$

since otherwise, the proof has been completed. By Lemma 6, we then have

$$
\begin{aligned}
&\sum_{l\in\mathbb{L}_{large}^k}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{\sqrt{\boldsymbol{\nu}_{l,k,0}}+\varepsilon}-\sum_{l\in\mathbb{L}_{large}^k}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\boldsymbol{\nu}_{l,k,0}}+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\geq&\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}\\
&+\mathcal{O}\left(\eta_k^2+\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k-r,j})\|+\eta_k\sum_{r=1}^{k-1}\sqrt{\beta_2}^{(r-1)n}\eta_{k-r}+\eta_k^2\sum_{j=0}^{n-1}\|\nabla f(\boldsymbol{w}_{k,j})\|\right).
\end{aligned}
$$

Putting ① and ② together and summing over k, we have

$$
\begin{aligned}
&f(\boldsymbol{u}_{T+1})-f(\boldsymbol{u}_1)\\
\leq&-\sum_{k=1}^{T}\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+\left((\frac{1}{2}+C_2)C_5+C_8\right)\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|\\
&+\left((\frac{1}{2}+C_2)C_6+C_9\right)\ln T+\left((\frac{1}{2}+C_2)C_7+C_{10}\right)+\sum_{k=1}^{T}\frac{nL_0+L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2 d\eta_k^2+\sum_{k=1}^{T}\frac{3L_1\sqrt{n}\sqrt{D_1}C_2^2 d\eta_k^2}{2}\|\nabla f(\boldsymbol{w}_{k,0})\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\sum_{k=1}^{T}\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+\left((\frac{1}{2}+C_2)C_5+C_8+\frac{3L_1\sqrt{n}\sqrt{D_1}C_2^2d}{2}\right)\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\| \\
&\quad+\left((\frac{1}{2}+C_2)C_6+C_9+\frac{nL_0+L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_1^2\right)\ln T+\left((\frac{1}{2}+C_2)C_7+C_{10}+\frac{nL_0+L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_1^2\right) \\
&\leq\sum_{k=1}^{T}\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+C_{11}\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|+C_{12}\ln T+C_{13},
\end{aligned}
$$

where C_{11}, C_{12}, and C_{13} is defined as

$$
\begin{aligned}
C_{11}&\triangleq(\frac{1}{2}+C_2)C_5+C_8+\frac{3L_1\sqrt{n}\sqrt{D_1}C_2^2d}{2}, \\
C_{12}&\triangleq(\frac{1}{2}+C_2)C_6+C_9+\frac{nL_0+L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_1^2, \\
C_{13}&\triangleq(\frac{1}{2}+C_2)C_7+C_{10}+\frac{nL_0+L_1\sqrt{n}\sqrt{D_0}}{2}3C_2^2d\eta_1^2.
\end{aligned}
$$

On the other hand, as for $\forall k\in[T]$,

$$
\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|\leq\frac{1}{4}\frac{\sqrt{D_0}+\varepsilon}{\sqrt{D_1}}\eta_k^2+\frac{\sqrt{D_1}}{\sqrt{D_0}+\varepsilon}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|^2,
$$

we have that

$$
\begin{aligned}
\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|\leq&\frac{1}{4}\frac{\sqrt{D_0}+\varepsilon}{\sqrt{D_1}}\eta_k^2+\eta_k^2\min\left\{\|\nabla f(\boldsymbol{w}_{k,0})\|,\frac{\sqrt{D_1}}{\sqrt{D_0}+\varepsilon}\|\nabla f(\boldsymbol{w}_{k,0})\|^2\right\} \\
=&\frac{1}{4}\frac{\sqrt{D_0}+\varepsilon}{\sqrt{D_1}}\eta_k^2+\sqrt{D_1}\eta_k^2\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\sqrt{D_0}+\varepsilon}\right\},
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&f(\boldsymbol{u}_{T+1})-f(\boldsymbol{u}_1) \\
&\leq-\sum_{k=1}^{T}\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+C_{11}\sum_{k=1}^{T}\eta_k^2\|\nabla f(\boldsymbol{w}_{k,0})\|+C_{12}\ln T+C_{13} \\
&\leq-\sum_{k=1}^{T}\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+\frac{\sqrt{D_0}+\varepsilon}{4\sqrt{D_1}}C_{11}\sum_{k=1}^{T}\eta_k^2+C_{12}\ln T+C_{13} \\
&\quad+\sqrt{D_1}C_{11}\sum_{k=1}^{T}\eta_k^2\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\} \\
&\leq-\sum_{k=1}^{T}\eta_k\left(\frac{1}{2(2\sqrt{2}+1)}-\sqrt{D_1}C_{11}\eta_k\right)\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+\left(C_{12}+\frac{\sqrt{D_0}+\varepsilon}{4\sqrt{D_1}}C_{11}\eta_1^2\right)\ln T \\
&\quad+\left(C_{13}+\frac{\sqrt{D_0}+\varepsilon}{4\sqrt{D_1}}C_{11}\eta_1^2\right) \\
&\leq-\sum_{k=1}^{T}\eta_k\frac{1}{4(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}+\left(C_{12}+\frac{\sqrt{D_0}+\varepsilon}{4\sqrt{D_1}}C_{11}\eta_1^2\right)\ln T \\
&\quad+\left(C_{13}+\frac{\sqrt{D_0}+\varepsilon}{4\sqrt{D_1}}C_{11}\eta_1^2\right).
\end{aligned}
$$

The proof is completed. □

Lemma 6. *Let Assumptions 1 and 2 hold. Let $\beta_1^2<\beta_2$ and Eq. (4) hold. Then, either there exists a iteration $k\in[T]$, such that either*

$$
\|\nabla f(\boldsymbol{w}_{k,0})\|\leq2\sqrt{d}(2\sqrt{2}+1)\sqrt{D_0}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}},
$$

or for all iteration $k \in [1, T]$, we have that

$$\sum_{l\in[d]} \frac{\eta_k \partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]} |\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)$$
$$\geq \eta_k \frac{1}{2(2\sqrt{2}+1)} \min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon + \sqrt{D_0}}\right\}.$$

Proof. To begin with, we have

$$\sum_{l\in[d]} \frac{\eta_k \partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]} |\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)$$
$$\overset{(\star)}{\geq} \sum_{l\in[d]} \frac{\eta_k \partial_l f(\boldsymbol{w}_{k,0})^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \varepsilon} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)$$
$$= \frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \varepsilon} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right),$$

where Inequality $(\star)$ is due to that

$$\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| = \sqrt{\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|^2}$$
$$\leq \sqrt{\sum_{i\in[n]}\sum_{l'=1}^{d}|\partial_{l'} f_i(\boldsymbol{w}_{k,0})|^2} = \sqrt{\sum_{i\in[n]}\|\nabla f_i(\boldsymbol{w}_{k,0})\|^2} \leq \sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0}.$$

We respectively consider the case $\varepsilon \leq \sqrt{D_0}$ and $\varepsilon > \sqrt{D_0}$.

Case I: $\varepsilon \leq \sqrt{D_0}$. In this case, we have that

$$\frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \varepsilon} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})| + \varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)$$
$$\geq \frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \sqrt{D_0}} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right) \frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)$$
$$= \frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \sqrt{D_0}} - \sum_{l\in[d]} \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}|\partial_l f(\boldsymbol{w}_{k,0})|$$
$$\geq \frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \sqrt{D_0}} - \sqrt{d}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\|\nabla f(\boldsymbol{w}_{k,0})\|.$$

We further discuss the case depending on whether $\|\nabla f(\boldsymbol{w}_{k,0})\|^2 \leq \frac{D_0}{D_1}$ or not.

Case I.1: $\|\nabla f(\boldsymbol{w}_{k,0})\|^2 \leq \frac{D_0}{D_1}$. In this case, the last line of the above equations can be further lower bounded by

$$\frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 + D_0} + \sqrt{D_0}} - \sqrt{d}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\|\nabla f(\boldsymbol{w}_{k,0})\|$$
$$\geq \frac{\eta_k \|\nabla f(\boldsymbol{w}_{k,0})\|^2}{(2\sqrt{2}+1)\sqrt{D_0}} - \sqrt{d}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\|\nabla f(\boldsymbol{w}_{k,0})\|$$
$$= \eta_k\left(\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{(2\sqrt{2}+1)\sqrt{D_0}} - \sqrt{d}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\right)\|\nabla f(\boldsymbol{w}_{k,0})\|$$

Case I.2: $\|\nabla f(\boldsymbol{w}_{k,0})\|^2 > \frac{D_0}{D_1}$.

$$
\begin{aligned}
&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+D_0}+\sqrt{D_0}} - \sqrt{d}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\|\nabla f(\boldsymbol{w}_{k,0})\| \\
\geq&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{(2\sqrt{2}+1)\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,0})\|} - \sqrt{d}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\|\nabla f(\boldsymbol{w}_{k,0})\| \\
=&\eta_k\left(\frac{1}{(2\sqrt{2}+1)\sqrt{D_1}} - \sqrt{d}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\right)\|\nabla f(\boldsymbol{w}_{k,0})\| \\
\overset{(*)}{\geq}&\eta_k\frac{1}{2(2\sqrt{2}+1)\sqrt{D_1}}\|\nabla f(\boldsymbol{w}_{k,0})\|,
\end{aligned}
$$

where Inequality $(*)$ is due to the constraint on β_2.

Therefore, we have either (1). there exists a iteration $k \in [T]$, such that

$$
\|\nabla f(\boldsymbol{w}_{k,0})\| \leq 2\sqrt{d}(2\sqrt{2}+1)\sqrt{D_0}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}},
$$

or (2).for all $k \in [1, T]$,

$$
\begin{aligned}
&\sum_{l\in[d]}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
\geq&\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}}, \frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\sqrt{D_0}}\right\}.
\end{aligned}
$$

Case II: $\varepsilon > \sqrt{D_0}$. In this case, we have that

$$
\begin{aligned}
&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+D_0}+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
\geq&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+\varepsilon^2}+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right).
\end{aligned}
$$

Similar as **Case I**, we further divides the case regarding the value of $\|\nabla f(\boldsymbol{w}_{k,0})\|$.

Case II.1: $D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 \leq \varepsilon^2$. In this case, we have

$$
\begin{aligned}
&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+\varepsilon^2}+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
\geq&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{(2\sqrt{2}+1)\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right) \\
\geq&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{(2\sqrt{2}+1)\varepsilon} - \eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\varepsilon}\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+D_0} \\
=&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|}{\varepsilon}\left(\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{2\sqrt{2}+1} - g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+D_0}\right).
\end{aligned}
$$

Case II.2: $D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2 > \varepsilon^2$. This case is quite similar to Case I.2, and we have

$$\begin{aligned}
&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{2\sqrt{D_1\|\nabla f(\boldsymbol{w}_{k,0})\|^2+\varepsilon^2}+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\geq&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{(2\sqrt{2}+1)\sqrt{D_1}\|\nabla f(\boldsymbol{w}_{k,0})\|} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\geq&\frac{\eta_k\|\nabla f(\boldsymbol{w}_{k,0})\|}{(2\sqrt{2}+1)\sqrt{D_1}} - \sqrt{d}\sqrt{\frac{2n}{\beta_2^n}}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\|\nabla f(\boldsymbol{w}_{k,0})\|\\
=&\eta_k\left(\frac{1}{(2\sqrt{2}+1)\sqrt{D_1}} - \sqrt{d}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}}\right)\|\nabla f(\boldsymbol{w}_{k,0})\|\\
\geq&\eta_k\frac{1}{2(2\sqrt{2}+1)\sqrt{D_1}}\|\nabla f(\boldsymbol{w}_{k,0})\|.
\end{aligned}$$

Therefore, we have either (1). there exists a iteration $k\in[T]$, such that

$$\|\nabla f(\boldsymbol{w}_{k,0})\| \leq 2\sqrt{d}(2\sqrt{2}+1)\sqrt{D_0}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}},$$

or (2). for all $k\in[1,T]$,

$$\begin{aligned}
&\sum_{l\in[d]}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\geq&\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon}\right\}.
\end{aligned}$$

As a conclusion of Case I and Case II, we have that either there exists a iteration $k\in[T]$, such that

$$\|\nabla f(\boldsymbol{w}_{k,0})\| \leq 2\sqrt{d}(2\sqrt{2}+1)\sqrt{D_0}g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\sqrt{\frac{2n}{\beta_2^n}},$$

or for all iteration $k\in[1,T]$, we have that

$$\begin{aligned}
&\sum_{l\in[d]}\frac{\eta_k\partial_l f(\boldsymbol{w}_{k,0})^2}{2\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon} - \sum_{l\in[d]}\eta_k g(\beta_2)\left(n-1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\partial_l f(\boldsymbol{w}_{k,0})|}{\sqrt{\frac{\beta_2^n}{2n}}\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|+\varepsilon}\left(\max_{i\in[n]}|\partial_l f_i(\boldsymbol{w}_{k,0})|\right)\\
\geq&\eta_k\frac{1}{2(2\sqrt{2}+1)}\min\left\{\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|}{\sqrt{D_1}},\frac{\|\nabla f(\boldsymbol{w}_{k,0})\|^2}{\varepsilon+\sqrt{D_0}}\right\}.
\end{aligned}$$

The proof is completed. □

D Experiment Details

This section collects experiments and their corresponding settings, and is arranged as follows: to begin with, we show that Adam works well under the different reshuffling order; we then provide the experiment settings of Figure 1.

D.1 Adam works well under different reshuffling order

We run Adam on ResNet 110 for CIFAR 10 across different random seeds and plot the 10-run mean and variance in Figure 4. One can observe that the performance of Adam is robust with respect to random seed, and support Theorem 1 in terms of trajectory-wise convergence. The experiment is based on this repo, where we adopt the default hyperparameters settings.

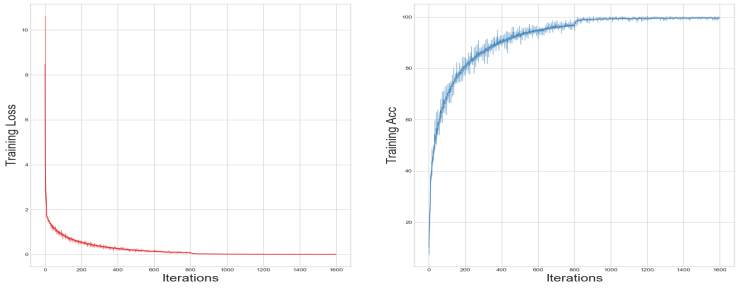


Figure 4: Performance of Adam with different shuffling orders. We respectively plot the training loss and the training accuracy of Adam together with their variances over 10 runs with different random shuffling order. The result indicate the performance of Adam is robust w.r.t. the shuffling order.

D.2 Local smoothness vs. gradient norm

In this section, we provide the models and hyperparameter settings of Figures 1. We will also illustrate how we evaluate the local smoothness.

Models and hyper-parameter settings in Figures 1. In Figure 1, we use exactly the same setting as Vaswani et al. (2017) on WMT 2014 dataset, based on this repo.

How we evaluate the local smoothness. We use the same method as Zhang et al. (2019a). Specifically, with a finite-difference step α, we calculate the smoothness at $\boldsymbol{w}_k$ as

$$\text{local smoothness} = \max_{\gamma \in \{\alpha, 2\alpha, \cdots, 1\}} \frac{\|\nabla f(\boldsymbol{w}_k + \gamma(\boldsymbol{w}_{k+1} - \boldsymbol{w}_k)) - \nabla f(\boldsymbol{w}_k)\|}{\gamma \|\boldsymbol{w}_{k+1} - \boldsymbol{w}_k\|}.$$