AdaStop: adaptive statistical testing for sound comparisons of Deep RL agents

Abstract

Recently, the scientific community has questioned the statistical reproducibility of many empirical results, especially in the field of machine learning. To solve this reproducibility crisis, we propose a theoretically sound methodology to compare the overall performance of multiple algorithms with stochastic returns. We exemplify our methodology in Deep Reinforcement Learning (Deep RL). Indeed, the performance of one execution of a Deep RL algorithm is random. Therefore, several independent executions are needed to accurately evaluate the overall performance. When comparing several RL algorithms, a major question is how many executions must be made and how can we ensure that the results of such a comparison are theoretically sound. Researchers in Deep RL often use less than 5 independent executions to compare algorithms: we claim that this is not enough in general. Moreover, when comparing several algorithms at once, the error of each comparison may accumulate and must be taken into account with a multiple tests procedure to preserve low error guarantees. We introduce AdaStop, a new statistical test based on multiple group sequential tests. When comparing algorithms, AdaStop adapts the number of executions to stop as early as possible while ensuring that we have enough information to distinguish algorithms that perform better than the others in a statistical significant way. We prove theoretically and empirically that AdaStop has a low probability of making a (family-wise) error. Finally, we illustrate the effectiveness of AdaStop in multiple Deep RL use-cases, including toy examples and challenging Mujoco environments. AdaStop is the first statistical test fitted to this sort of comparisons: AdaStop is both a significant contribution to statistics, and a major contribution to computational studies performed in reinforcement learning and in other domains. To summarize our contribution, we introduce AdaStop, a formally grounded statistical tool to let anyone answer the practical question: “Is my algorithm the new state-of-the-art?”

1 Introduction

In many fields of computer science, it is customary to perform an experimental investigation to compare the practical performance of two or more algorithms. When the behavior of an algorithm is non deterministic (for instance because the algorithm is non deterministic, or because the data it is fed upon is random), the performance of the algorithm is a random variable. Then, the way to do this comparison is not clear. Usually, one executes the algorithm (or rather an implementation of the algorithm: this point will soon be clarified) several times in order to obtain an average performance and its variability. How much “several” is depends on the authors, and some contingencies: thanks to the law of large numbers, one may think that the larger the better, the more accurate the estimates of the average and the variability. This may be a satisfactory answer, but when a single execution lasts days or even weeks or months (such as LLM training), performing a large enough number of executions is impossible. To illustrate this point, let us consider the field of deep reinforcement learning (Deep RL), that is reinforcement learning algorithms that use a neural network to represent what they learn. We surveyed all deep RL papers published in the proceedings of the International Conference on Machine Learning in 2022 (see Fig. 1a). In the vast majority of these papers, only a few executions have been performed: among the 18 papers using the Mujoco tasks, only 3 papers performed more than 10 runs, and 11 papers used only from 3 to 5 runs. The fact is that if we can be confident that 3 executions is not enough, how many executions would be enough in a statistical significant way? On the other way around, are 80 executions not too much to draw a statistically significant conclusion considering that each execution consumes resources (energy, CPU time, memory usage, etc) and contributes to pollute our planet and to the climate change. There is no answer to this question today. Moreover, we want the conclusion of the comparison to be reproducible: if someone redoes the comparison of the same algorithms on the same task, the conclusions should be the same: this concept is known as *statistical reproducibility* (Agarwal et al., 2021; Colas et al., 2019; Goodman et al., 2016). This paper provides such an answer: to reach this goal, we introduce AdaStop, a new statistical test to provide a statistically sound answer to this question, and we demonstrate its usage in practice. This paper is accompanied by a software that implements the test and is very easy to use. In short, in this paper, we provide a methodological approach and its actual implementation to compare the performance of algorithms having random performance in a statistically significant way, while trying to minimize the computational effort to reach such a conclusion. Such a methodological question arises in various fields of computational AI like machine learning, and optimization. We will use the field of reinforcement learning to illustrate this paper, but its application to other fields of computational AI is straightforward.

This paper is organized as follows: in an informal way, we detail the requirements that have to fulfil a statistical test to fit our expectations in Section 2. Section 3 introduces the main ingredients of our test, AdaStop. Section 4 in the formal analysis of the properties of AdaStop. Proofs are established in appendices C to F. We illustrate the use of AdaStop in Section 5 before concluding. Appendix A lists all the notation used in this paper. Appendix B provides a minimal exposition of the main concepts of hypothesis testing for readers who are not trained in statistics.

To reproduce the experiments of this paper, the python code is freely available on GitHub at `https://anonymous.4open.science/r/Adaptive_stopping_MC_RL-3434`. In addition, we provide a library and command-line tool that can be used independently: the AdaStop Python package is available at `https://anonymous.4open.science/r/adastop-EC1F`.

2 Informal presentation of AdaStop

2.1 Definitions

First, we define the meaning of a few terms as we will use them in the rest of this paper.

As raised above, we do not compare algorithms but a certain implementation of an algorithm with a certain set of values for its hyperparameters. In D. Knuth's spirit (Knuth, 1968), we use the term **algorithm** in its usual meaning in computer science as the description of the basic operations required to transform a certain input into a certain output. By basic operations we mean the use of variables and simple operations (arithmetical, logical, etc), along with assignments to variables, sequences of instructions, tests and loops. Such an algorithm typically has some hyperparameters that control its behavior (a threshold, the dimension of the input domain, how a certain variable decays in time, a neural network, etc). In this regard, we may say that as stated in (Schulman et al., 2017), PPO is an algorithm. However, one should be cautious that many aspects of PPO may be defined in various ways, and that the notion of "the PPO algorithm" is not as clearly defined as e.g. "the quicksort algorithm". The same may be said for all other "Deep RL algorithms" of which there exists many variations.

Let us make clear the distinction between parameters and hyperparameters: parameters are learned from data, while hyperparameters are set a priori. As such, the weights of a neural network are parameters, while the architecture of the network is a hyperparameter (unless this architecture is also learned during the training of the agent, which is far from being a common practice in DRL).

In experiments, we never compare algorithms: we compare a certain implementation of an algorithm, along a certain set of values for its hyperparameters, to other implementations. We use the term **agent** to refer to a certain implementation of an algorithm along the value of its hyperparameters. Deep RL agents use a seed to initialize their pseudo-random number generator. Though this seed may be considered a hyperparameter, the seed is not specified by this definition of the word "agent": it should be considered as an input to the agent, provided at each run. This definition coincides with the one in (Patterson et al., 2023).

The result of one execution of an agent provides a **score**: this is a numerical value. To compare the performance of two agents, we compare their scores. We are free to define what the score is in a given experiment: this definition has to be aligned with the property of the agents we want to compare. For instance, in Deep RL, the common practice is as follows: an agent that implements a certain algorithm is trained using a certain set of values for its hyperparameters. After training, the trained policy is run a given number of episodes to evaluate its performance. The score may be the mean of these evaluations, but the user is free to define the score according to her own interest.

2.2 Ingredients for an appropriate statistical test

Our goal is to provide a statistical test to decide whether one agent performs better than others. Let us list the requirements and difficulties we have to face in the design of such a statistical test.

First, many tests in statistics consider random variables that are Gaussian. However, one quickly figures out that the observed performance of agents is usually not Gaussian. Fig. 1b illustrates this point: we represent the distribution of the performance of 4 agents implementing 4 different RL algorithms (PPO, SAC, DDPG, TRPO): the performance is usually multi-modal, and it is not even a mixture of a few Gaussian distributions as may seem. This leads us to a nonparametric test. Second, we would like to perform the minimal number of runs, which leads us to the use of a sequential adaptive test that tells us if we need to run the agents one or a few more times, or if we can take a decision in a statistically significant way with the already collected data. Third, we want to be able to compare more than 2 agents which leads us to multiple testing. Fourth, we want the conclusions of the test to be statistically reproducible, that is, if someone reproduces the execution of the agents and applies the test in the same way as someone else, the conclusion is the same. Fifth, we want to keep the number of runs within reasonable limits: for that purpose, we set a maximum number of scores to collect: if no decision can be made using this budget, the test can not decide whether an agent is better than the others. The first 3 requirements call for a nonparametric, sequential, multiple test. A candidate statistical test that may verify all these properties can be found in group sequential permutation test (see the textbook (Jennison & Turnbull, 1999) on general group sequential tests). We use these ingredients to construct AdaStop.

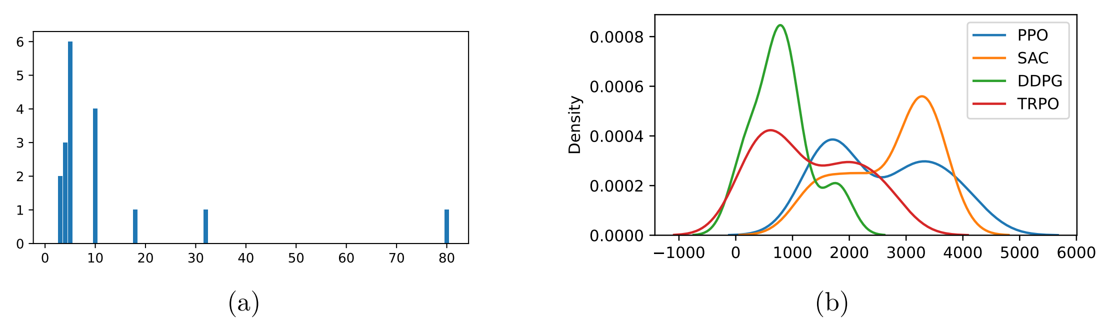


Figure 1: Motivations for AdaStop. (a): a census of the number of scores (N_{score}) used in RL papers using Mujoco environments published in the proceedings of ICML 2022. (b): estimations of the score distributions of several Deep Rl agents on Hopper, a Mujoco environment, with $N_{score} = 30$. We see from the census that most experiments used 5 or less scores ($N_{score} \leq 5$) to draw conclusions. Those conclusions are most likely statistically wrong as they are equivalent to drawing 5 samples from distributions similar to the right plot to draw conclusions about the empirical means.

AdaStop, our proposed statistical test, meets all these expectations. Before diving into the technical details, let us briefly explain how AdaStop is used in practice. Fig. 2 illustrates an execution of AdaStop on a small example. Let us suppose that we want to compare the performance of 2 agents, a green agent and a blue agent. Fig. 2 illustrates the sequential nature of the test from top to bottom. Initially, each agent is executed $N = 5$ times which yields 5 scores for each agent: these 10 initial scores are shown on the top-leftmost part of Fig. 2 labelled $n = 5$, along with their barplots. After the test statistics are computed,

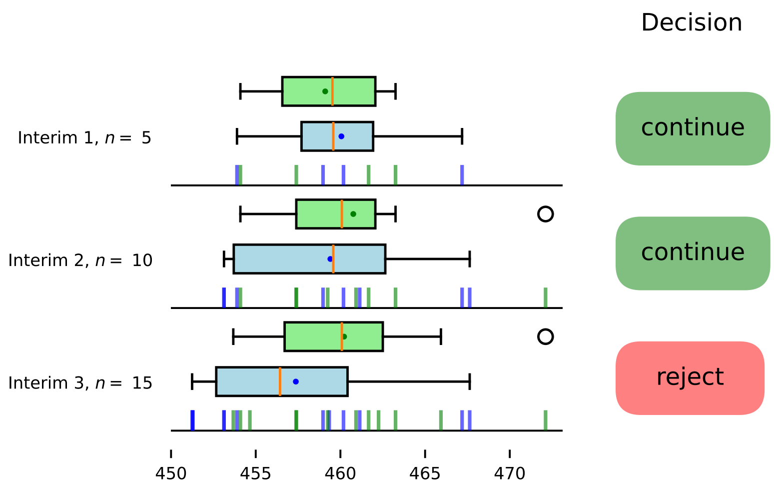


Figure 2: Schematic of Group Sequential Tests with AdaStop.

AdaStop decides that this information is not enough to conclude, and more scores are needed. This set of actions (collection of 5 scores for each agent, computation of the statistics, and decision) is known as an **interim**. As no conclusion can be drawn, a second interim is performed: both agents are run 5 more times, yielding 10 more scores. In the middle of Fig. 2, these additional scores are combined with the first 5 ones of each agent: these 20 scores are represented, as well as the barplot for each agent. The test statistics are computed using these 20 scores. Again, the difference is not big enough to make a decision given available information, so we make a third interim: each agent is executed 5 more times. At the bottom of Figure 2, these 30 scores are represented as well as the barplot for each agent. The test statistics are computed again on these 15 scores per agent. Now, AdaStop decides to reject the null hypothesis, that is that the two set of scores of the two agents are indeed different, and terminates: the compared agents do not perform similarly, and the green agent performs better than the blue one.

2.3 Current approaches for RL agents comparison

In the RL community, different approaches currently exist to compare agents. In (Colas et al., 2018; 2019), the authors show how to use hypothesis testing to test the equality between agents. Compared to our work, their approach is non-adaptive and only compares two agents. In Patterson et al. (2023), the authors explore a similar workflow with added steps specialized to deep RL (hyperparameter optimisation, choice of the testing environment...). Another line of works can be found in (Agarwal et al., 2021) in which the authors compare many agents using confidence intervals.

In this section, we summarize some of the problems we see with the current approaches used to compare two or more RL agents in research articles.

How many scores should we use? The number of scores used in practice in RL is quite arbitrary and often quite small (see Figure 1a). An arbitrary choice of the number of scores does not allow to make a statistically significant comparison of the agents as we have no guarantee whatsoever that we got the information to distinguish the agents. An intuition arises from the law of large numbers. As the performance of an agent is represented by the true mean of its scores, the more scores we have, the more precise the estimation of its performance.

Theoretically sound comparison of multiple agents.

According to statistical theory, in order to compare more than 2 agents, we need more samples from each agent than when we compare only two agents. The basic idea is that there is a higher chance to make an error when we perform multiple comparisons than when we compare only two agents, hence we need more data to have a lower probability of error at each comparison. This informal argument is formalized in the theory of multiple testing. However, the theory of multiple testing has almost never been used to compare RL agents (with the notable exception of (Patterson et al., 2023, Section 4.5)). In this paper, we remedy this with AdaStop giving a theoretically sound workflow to compare 2 or more agents.

Theoretically sound study when comparing agents on a set of tasks.

Atari environments (Bellemare et al., 2013) are famous benchmarks in Deep RL. Due to time constraints, when using these environments, it is customary to use very few scores for one given game (typically 3 scores) and compare the agents on many different games. The comparisons are then aggregated: agent A_1 is better than agent A_2 on more than 20 games out of the 26 games considered. In terms of rigorous statistics, this kind of aggregation is complex to analyse properly because reward distributions are not the same in all games. A_2 may be better than A_1 only on some easy games: does this mean that A_1 is better than A_2? Up to our knowledge, there is not any proper statistical guarantee for this kind of comparison.

Advances have been made in (Agarwal et al., 2021) to interpret and visualize the results of RL agents in Atari environment. In particular the authors advise plotting confidence intervals and using the interquartile mean instead of the mean as aggregation functions. Correctly aggregating the comparisons on several games in Atari is still an *open problem*, and it is *beyond the scope of this article*. In this article, we suppose that we compare the agents on a single task, and we leave the comparison on a set of different tasks for future work. A discussion on these methods and the challenges of aggregating the results from several Atari environments can be found in the Appendix G.

2.4 Some methodologies for comparison of RL agents

Figure 1 of Patterson et al. (2023) defines a workflow for the meaningful comparison of two RL agents given an environment and a performance measure. In addition to the usual considerations regarding which statistics to compare (Colas et al., 2018; Agarwal et al., 2021), (Patterson et al., 2023) also include hyperparameter choice in the workflow. For that, they recommend to use 3 scores per algorithm per set of hyperparameters to identify a good choice of hyperparameters for a given algorithm. When this is done, a fixed number of scores (set using expert knowledge on the environment) are computed for each of these fully-specified agents. These scores are then used for statistical comparison. AdaStop fits at the end of this workflow.

In Fig 3, we showcase the use of AdaStop to compare SAC (Haarnoja et al., 2018) to other Deep RL algorithms on HalfCheetah and Hopper Mujoco tasks. One can imagine a scenario in which SAC inventors follow (Patterson et al., 2023) methodology. After finding the best hyperparameters for TRPO, PPO and DDPG (Schulman et al., 2015; 2017; Lillicrap et al., 2015) agents are compared with AdaStop using a minimal number of scores to get significant statistics. Patterson et al. (2023) recommends 15 scores per agent per environment for a maze environment but this number of scores should vary in other environments, and it is not clear how many scores should be used for HalfCheetah and Hopper, that is why we need to use AdaStop.

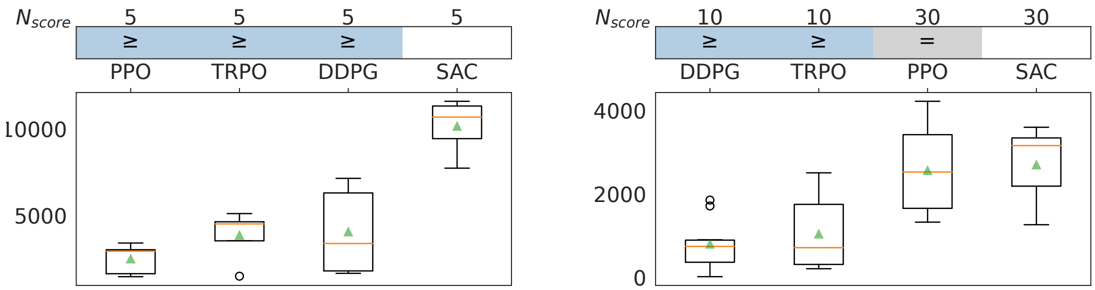
Using AdaStop, we choose the number of scores in an adaptive manner and allows us to say with measured confidence that we used enough scores to conclude that the SAC agent is better than other agents on HalfCheetah, and better than DDPG and TRPO agents on Hopper. A more in-depth study of the agents' performance on Mujoco environments is given in Section 5.3.

3 Hypothesis testing for agents comparisons

In this section, we describe the background material on the statistical tests that we use to construct AdaStop. First we provide a short review of the statistical evaluation methods found in the literature, and then we describe our methodology and some results associated with AdaStop.

3.1 Literature overview of evaluation methods

We present here some relevant references connected to statistical evaluation methodology.



(a) Comparisons on HalfCheetah.

(b) Comparisons on Hopper.

Figure 3: Using AdaStop to benchmark SAC in practice. We set the maximum number of runs B of each agent to 30. The upper row tables represent the conclusions when comparing SAC to the algorithms in the column using N_{score} scores. For example, on HalfCheetah, AdaStop concludes that SAC was better than PPO using 5 scores for SAC and 5 scores for PPO. On Hopper, 10 scores are enough to conclude that SAC outperforms DDPG and TRPO, and AdaStop concludes that SAC is as good as PPO on Hopper using the maximum budget of B scores for SAC and B scores for PPO.

Nonparametric (and non-sequential) hypothesis testing. One of our main challenges in this article is to deal with the nonparametric nature of the data at hand. In the literature there has been a lot of work on nonparametric testing, Lehmann et al. (2005) providing a comprehensive overview. Traditionally, focus has been on asymptotic results due to challenges in deriving optimality with for a nonparametric model. For most nonparametric tests, only rather weak (exact) theoretical results can be given, mostly on type I error (Romano, 1989; Shapiro & Hubert, 1979). Recent work on sequential nonparametric tests (Shin et al., 2021; Howard et al., 2021) show strong non-asymptotic results using concentration inequalities. However, these results often involve non-optimal constants, failing to explain non-asymptotic efficiency. In contrast, we use permutation tests due to their empirical (Ludbrook & Dudley, 1998) and theoretical (Kim et al., 2022) efficiency for small sample sizes.

Sequential tests. A closely related method for adaptive hypothesis testing consists in sequential tests. Two commonly used sequential tests are the Sequential Probability Ratio test (Wald, 1945) and the Generalized Likelihood Ratio test (Kaufmann & Koolen, 2021). In sequential testing, the scores are compared one after the other in a completely online manner. This is not adapted to our situation because in RL practice, one often trains several agents in parallel, obtaining a batch of scores at once. This motivates the use of group sequential tests (Jennison & Turnbull, 1999).

Parametric group sequential tests. Unlike traditional hypothesis testing, where data is analysed in its entirety once it has been fully collected, Group Sequential Test (GST) evaluates data sequentially as it is collected, and the tests are done only at *interim* time points, each time with a new block of N scores (see (Jennison & Turnbull, 1999; Gordon Lan & DeMets, 1983; Pocock, 1977; Pampallona & Tsiatis, 1994) for references on GST). GST are often used in clinical trials to minimize the amount of data needed to conclude and this makes them well adapted for our purpose. The decision to continue sampling or conclude (with a controlled probability of error) depends on pre-defined stopping criteria used to define the tests. GST often makes strong assumptions on the data, in particular it is often assumed that the data is i.i.d. and drawn from a Gaussian distribution (Jennison & Turnbull, 1999), this contrast with our approach which will be more nonparametric.

Bandits (Best arm identification or ranking). Our objective is close to the one of bandit algorithms (Lattimore & Szepesvári, 2020): we *minimize the stopping time* (as in the fixed-confidence setting) of the test, and we have a *fixed maximum budget* (as in a fixed-budget setting). In our test, we allow a type I error with probability $\alpha \in (0, 1)$, which is similar to the fixed confidence setting while still having a fixed budget. Compared to the fixed budget setting, we allow a larger error rate, which results in a test that is more sample efficient than bandit algorithms.

3.2 Background material on hypothesis testing

This section describes the basic building blocks used to construct AdaStop: group sequential testing, permutation tests, and step-down method for multiple hypothesis testing. We explain these items separately, and then we combine them to create AdaStop in Section 4. We also provide a small recap on hypotheses testing in the Appendix B for readers unfamiliar with hypothesis testing.

In order to **perform the minimal number of runs**, we propose to use group sequential testing (GST) with a nonparametric approach using permutation tests. Our approach is similar to (Mehta et al., 1994) but for multiple hypothesis testing. Compared with usual GST, we keep the i.i.d. assumption, but we do not assume that the data are drawn from a specific family of parametric distribution. In (Mehta et al., 1994), the authors use rank tests with group-sequential testing. Contrary to our work, (Mehta et al., 1994) does not provide theoretical guarantees and considers only the case of 2 agents.

3.2.1 Permutation tests

Permutation tests are **nonparametric tests** that are exact for testing the equality of distributions. This means that the type I error of the test (i.e. the probability to make a mistake and reject the equality of two agents when their scores are statistically the same) is controlled by the parameter of the test α, and that this is true for any fixed sample size N. Permutation tests are also well-known to work well in practice on **very small sample sizes** and are used extensively in biology. They were originally introduced by Pitman (1937) and Fisher (1936), see (Lehmann et al., 2005, Chapter 17) for a textbook introduction. More recently Chung & Romano (2013) have studied asymptotic properties of this class of tests, while Romano & Wolf (2003) have focused on stepdown methods for multiple hypothesis testing.

Let us recall the basic formulation of a two-sample permutation test. Let $X_1, \dots, X_N$ be i.i.d. sampled from a law P and $Y_1, \dots, Y_N$ i.i.d sampled from a law Q, we want to test $P = Q$ against $P \neq Q$. Let $Z_i = X_i$ if $i \leq N$ and $Z_i = Y_i$ if $i > N$, $Z_1, \dots, Z_{2N}$. The test proceeds as follows: we reject $P = Q$ if $T(\mathrm{id}) = \left| \frac{1}{N} \sum_{i=1}^{N} (Z_i - Z_{N+i}) \right|$ is larger than a proportion $(1 - \alpha)$ of the values $T(\sigma) = \left| \frac{1}{N} \sum_{i=1}^{N} (Z_{\sigma(i)} - Z_{\sigma(N+i)}) \right|$ where σ enumerates all possible permutations of $\{1, \dots, 2N\}$ where id is the identity permutation such that $\mathrm{id}(i) = i$ for all i. Formally, we define the $(1 - \alpha)$-quantile as

$$B_N = \inf \left\{ b > 0 : \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \mathbb{1}\{T(\sigma) \geq b\} \leq \alpha \right\}$$

and we reject $P = Q$ when $T(\mathrm{id}) \geq B_N$. The idea is that if $P \neq Q$, then $T(\mathrm{id})$ should be large, and due to compensations, most $T(\sigma)$ should be smaller than $T(\mathrm{id})$. Conversely, if $P = Q$, the difference of mean $T(\sigma)$ will be closer to zero. It is then sufficient to compute $T(\mathrm{id})$ and B_N in order to compute the decision of the test. Please note that this is a fairly usual simplification in the nonparametric tests literature to test the equality in distribution instead of the equality of the mean, because equality between distributions is easier to deal with. On the other hand, it can be shown that permutation tests are nonetheless a good approximation of doing a comparison on the means (See Appendix E).

3.2.2 GST comparison of two agents

In this section, we compare two agents A_1 and A_2 through a group sequential test that can be seen as a particular case of AdaStop for two agents (see Section 4). The testing procedure adopted in this simplified case is presented in Algorithm 1, and we leave case with more than 2 agents to compare and the full version of AdaStop, including multiple hypothesis testing, for Section 4 (see in particular Algorithm 3). Algorithm 1 uses a permutation tests, where, at each interim, the boundary deciding the rejection is derived from the permutation distribution of the statistics observed across all previously obtained data. In what follows, we use N to denote the number of scores per interim and k to denote the current interim number.

We denote by $\mathbf{S}_{2N}$ the set of permutations of $\{1,\dots,2N\}$, $\sigma \in \mathbf{S}_{2N}$ one permutation from the previous set and $\sigma(n)$ the n-th element of σ for $n \in \{1,\dots 2N\}$. In GST setting, we perform a permutation test at each interim k, thus $\sigma_k \in \mathbf{S}_{2N}$ denotes the permutation at interim k. For $\sigma_1, \sigma_2, \dots, \sigma_k \in \mathbf{S}_{2N}$, we denote $\sigma_{1:k} = \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_k$ the concatenation of the permutation σ_1 done in interim 1 with σ_2 done on interim 2,..., and σ_k on interim k. Then, $e_{\sigma_i(n),i}$ denotes the score corresponding to the n-th element of the permuted sample at interim i, permuted by σ_i (in the notations of Section 3.2.1 this corresponds to $Z_{\sigma(i)}$ but with added notations to include the interim number). We denote:

$$T_{N,k}(\sigma_{1:k}) = \left| \sum_{i=1}^{k} \left(\sum_{n=1}^{N} e_{\sigma_i(n),i} - \sum_{n=N+1}^{2N} e_{\sigma_i(n),i} \right) \right|, \tag{1}$$

and the decision boundary:

$$B_{N,k} \in \inf \left\{ b > 0 : \frac{1}{((2N)!)^k} \sum_{\sigma_{1:k} \in \widehat{\mathcal{S}}_k} \mathbb{1}\{T_{N,k}(\sigma_{1:k}) \geq b\} \leq \frac{\alpha}{K} \right\}, \tag{2}$$

where K is the total number of interims and $\widehat{\mathcal{S}}_k$ is the set of permutations $\sigma_{1:k} \in (\mathbf{S}_{2N})^k$ such that the test would not have rejected before, e.g.

$$\widehat{\mathcal{S}}_k = \{\sigma_{1:k} \in (\mathbf{S}_{2N})^k : \quad \forall m < k, \quad T_{N,m}(\sigma_{1:m}) \leq B_{N,m}\}. \tag{3}$$

Algorithm 1: Adaptive stopping to compare two agents.

Parameters: Agents A_1, A_2, environment $\mathcal{E}$, number of blocks $K \in \mathbb{N}^*$, size of a block N, level of the test $\alpha \in (0,1)$.

for $k = 1, \dots, K$ **do**
  for $l = 1, 2$ **do**
    Train agent A_l on environment $\mathcal{E}$.
    Collect scores $e_{1,k}(A_l), \dots, e_{N,k}(A_l)$ by running N times the trained agent A_l.
  end
  Compute the boundary $B_{N,k}$ using Equation (2).
  if $T_{N,k}(\mathrm{id}) \geq B_{N,k}$ **then**
    Reject the equality of the agent scores, exit the loop.
  else
    if $k = K$ **then**
      return accept
    end
  end
end
if *the test was never rejected* **then**
  return accept
else
  return reject
end

3.2.3 Multiple hypothesis testing

In order to compare more than two agents we need to perform multiple comparisons which calls for a **multiple simultaneous statistical tests** (Lehmann et al., 2005, Chapter 9). The idea is that the probability to mistakenly reject a null hypothesis (type I error) generally applies only to each test considered individually. On the other hand, in order to conclude on all the tests at once, it is desirable to have an error controlled over the whole family of simultaneous tests. For this purpose, we use the family-wise error rate (Tukey, 1953) which is defined as the probability of making at least one type I error.

Definition 1 (Family-Wise Error (Tukey, 1953)). *Given a set of hypothesis H_j for $j \in \{1, \dots, J\}$, its alternative H'_j, and $\boldsymbol{I} \subset \{1, \dots, J\}$ the set of the true hypotheses among them, then*

$$\mathrm{FWE} = \mathbb{P}_{H_j, j \in \boldsymbol{I}}(\exists j \in \boldsymbol{I} : \quad \textit{reject } H_j) \,.$$

where the notation $\mathbb{P}_{H_j, j \in \boldsymbol{I}}$ denotes the probability distribution for which all hypotheses $j \in \boldsymbol{I}$ hold true[1]*. We say that an algorithm has a weak FWE control at a joint level $\alpha \in (0,1)$ if the FWE is smaller than α when all the hypotheses are true, that is $\boldsymbol{I} = \{1, \dots, J\}$ but not necessarily otherwise. We say it has strong FWE control if FWE is smaller than α for any non-empty set of true hypotheses $\boldsymbol{I} \neq \emptyset$ (while the notation $\boldsymbol{I}^c$ refers to false hypotheses).*

If we want to test the equality of L distributions $P_1, P_2, \dots, P_L$, the straightforward way is to do a pairwise comparison. This creates $J = \frac{L(L-1)}{2}$ hypotheses. We let $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_J\}$ be the set of all possible comparisons between the distributions, where $\mathbf{c}_j = (l_1, l_2) \in \mathbf{C}$ denotes a comparison between distributions P_{l_1} and P_{l_2} for $l_1, l_2 \in \{1, 2, \dots, L\}$. Therefore, for $\mathbf{c}_j = (l_1, l_2)$, $H_{\mathbf{c}_j}$ denotes a hypothesis stating that P_{l_1} and P_{l_2} are equal and its alternative $H'_{\mathbf{c}_j}$ is that P_{l_1} and P_{l_2} are different.

There are several procedures that can be used to control the FWE. The most famous one is Bonferroni's procedure (Bonferroni, 1936) recalled in the Appendix (Section B). As Bonferroni's procedure can be very conservative in general, we prefer a *step-down* method (Romano & Wolf, 2003) that performs better in practice because it implicitly estimates the dependence structure of the test statistic. The step-down method that we use is described in details in Section 3.2.4.

3.2.4 Step-down procedure

Romano & Wolf (2003) proposed the step-down procedure to solve a multiple hypothesis testing problem. It is defined as follows (for a *non-group-sequential-test*): for a permutation $\sigma \in \mathbf{S}_{2N}$ and for $(e_n(j))_{1 \le n \le 2N}$ the random variables being compared in hypothesis j, the permuted test statistics of hypothesis j is defined by:

$$T_N^{(j)}(\sigma) = \left| \sum_{n=1}^{N} e_{\sigma(n)}(j) - \sum_{n=N+1}^{2N} e_{\sigma(n)}(j) \right| . \tag{4}$$

This test statistics is extended to any subset of hypothesis $\mathbf{C} \subset \{1, \dots, J\}$ with the following formula:

$$\overline{T}_N^{(\mathbf{C})}(\sigma) = \max_{j \in \mathbf{C}} T_N^{(j)}(\sigma). \tag{5}$$

To specify the test, one compares $\overline{T}_N^{(\mathbf{C})}(\mathrm{id})$ to some threshold value $B_N^{(\mathbf{C})}$, that is: we accept all hypotheses in $\mathbf{C}$ such that $\overline{T}_N^{(\mathbf{C})}(\mathrm{id}) \le B_N^{(\mathbf{C})}$. The threshold of the test $B_N^{(\mathbf{C})}$ is defined as the quantile of order $1-\alpha$ of the permutation law of $\overline{T}_N^{(\mathbf{C})}(\sigma)$:

$$B_N^{(\mathbf{C})} = \inf \left\{ b > 0 : \left(\frac{1}{(2N)!} \sum_{\sigma \in \mathbf{S}_{2N}} \mathbb{1}\{\overline{T}_N^{(\mathbf{C})}(\sigma) \ge b\} \right) \le \alpha \right\} . \tag{6}$$

In other words, $B_N^{(\mathbf{C})}$ is the real number such that an α proportion of the values of $\overline{T}_N^{(\mathbf{C})}(\sigma)$ exceeds it, when σ enumerates all the permutations of $\{1, \dots, 2N\}$. The permutation test is summarized in Algorithm 2.

Algorithm 2 is initialized with $\mathbf{C} = \mathbf{C}_0$ containing all the comparisons we want to test. Then, it enters a loop where the test decides to reject or not the most extreme hypothesis in $\mathbf{C}$, *i.e.* $H_{j_{\max}}$ where $j_{\max} = \arg\max_{j \in \mathbf{C}} T_N^{(j)}(\mathrm{id})$. Here, $\mathbf{C}$ is the current set of not yet rejected, nor accepted hypotheses. If the test statistic $T_N^{(j)}(\mathrm{id})$ for the most extreme hypothesis in $\mathbf{C}$ (i.e. $\overline{T}_N^{\mathbf{C}}(\mathrm{id})$) does not exceed the given threshold $B_n^{(\mathbf{C})}$, then all hypotheses in $\mathbf{C}$ are accepted, and the loop is exited. Otherwise, the most extreme hypothesis

[1] See also Appendix B for further explanations on this concept.

Algorithm 2: Multiple testing by step-down permutation test.

Parameters: $\alpha \in (0,1)$
Input: $e_n(j)$ for $1 \le n \le 2N$ and $j \in \mathbf{C}_0 = \{\mathbf{c}_1, \dots, \mathbf{c}_J\}$.

```
Initialize C ← C_0.
while C ≠ ∅ do
    Compute T_N^(C)(σ) for every j and every σ using Equation (5).
    Compute B_n^(C) using Equation (6).
    if T̄_N^(C)(id) ≤ B_N^(C) then
        Accept all the hypotheses H_j, j ∈ C and exit the loop.
    else
        Reject H_{c_{j_max}} where c_{j_max} = arg max_{c_j ∈ C} T_N^(j)(id).
        Define C = C \ {c_{j_max}}
    end
end
```

is discarded from the set $\mathbf{C}$ and another iteration is performed until either all remaining hypothesis are accepted, or the set of remaining hypotheses is empty.

The maximum of the statistics in Equation (5) for $\sigma = \mathrm{id}$ allows to test intersections of hypotheses, while the threshold $B_n^{(\mathbf{C})}$, under the null hypotheses of equality of distribution, allows for strong control on the FWE (i.e. FWE $\le \alpha$). This last result follows from (Romano & Wolf, 2003, Corollary 3) and is a particular case of the theoretical result in Theorem 1 for AdaStop. In fact, this procedure is not specific to permutation tests, and it can be used for other tests provided some properties on the thresholds $B_n^{(\mathbf{C})}$.

4 AdaStop: adaptive stopping for nonparametric group-sequential multiple tests

In this section, we present the construction and the theoretical properties of AdaStop (see Algorithm 3) to compare the scores of multiple agents in an *adaptive* rather than fixed way. We consider $L \ge 2$ agents $A_1, \dots, A_L$. As above, we let $\mathbf{C}_0 = \{\mathbf{c}_1, \dots, \mathbf{c}_J\} \subseteq \{1, \dots, L\}^2$ be the set of all the comparisons to make between the agents, while $\mathbf{C}$ denotes a subset of these comparisons. $\mathbf{I}$ denotes the set of indices of the true hypotheses among $\{1, \dots, J\}$.

Algorithm 3 specifies the AdaStop test. It depends on the test statistic $\overline{T}_{N,k}^{(\mathbf{C})}(\sigma_{1:k})$, Equation (8) below, and the boundary thresholds $B_{N,k}^{\mathbf{C}}$, in Equation (9). We discuss a few implementation details in the rest of this section.

Definition of the test statistic. We denote $e_{1,i}(j), \dots, e_{2N,i}(j)$ the $2N$ scores used at interim i, they are obtained through the comparison $\mathbf{c}_j = (l_1, l_2)$ of two agents A_{l_1} and A_{l_2}. We also consider permutations of these scores to define the test statistics $T_{N,k}^{(j)}$ below. For a comparison j, we consider a permutation $\sigma_i \in \mathbf{S}_{2N}$ at interim i that reshuffles the order of the scores sending $n \in \{1, \dots, 2N\}$ to $\sigma_i(n) \in \{1, \dots, 2N\}$. Note that, if $n \in \{1, \dots, N\}$ and $\sigma_i(n) \in \{N+1, \dots, 2N\}$, we are permuting a score of the first agent with a score of the second agent in the comparison and vice versa. It can also happen that we instead permute scores of the same agents. The difference between the two cases is important for the definition of the following permutation statistic:

$$T_{N,k}^{(j)}(\sigma_{1:k}) = \left| \sum_{i=1}^{k} \left(\sum_{n=1}^{N} e_{\sigma_i(n),i}(j) - \sum_{n=N+1}^{2N} e_{\sigma_i(n),i}(j) \right) \right|. \tag{7}$$

In other words, $T_{N,k}^{(j)}(\sigma_{1:k})$ is the absolute value of the sum of differences of all scores until interim k after consecutive permutations of the concatenation of the two agents scores by $\sigma_1, \dots, \sigma_k \in \mathbf{S}_{2N}$. Let $\mathbf{C} \subseteq \mathbf{C}_0$

be a subset of the set of considered hypothesis and let us denote:

$$\overline{T}_{N,k}^{(\mathbf{C})}(\sigma_{1:k}) = \max_{j\in\mathbf{C}} T_{N,k}^{(j)}(\sigma_{1:k})\,, \tag{8}$$

$\overline{T}_{N,k}^{(\mathbf{C})}(\sigma_{1:k})$ is the test statistic used in AdaStop. The construction of the test is inspired by the permutation tests of Equation (4) used to test intersection of hypotheses as from Equation (5) in the step-down method presented in Section 3.2. Still, it also incorporates group sequential tests from Section 3.2.2 and its test statistic introduced in Equation (1).

Choice of permutations. Instead of using all the permutations as it we did for now, one may use a random subset among all permutations $\mathcal{S}_k \subset \{\sigma_{1:k}, \quad \forall i \le k, \sigma_i \in \mathbf{S}_{2N}\}$ to speed-up computations. The theoretical guarantees persist as long as the choice of the permutations is made independent on the data. Using a small number of permutations will decrease the total power of the test, but with a sufficiently large number of random permutations (typically for the values of N and K considered, 10^4 permutations are sufficient) the loss in power is acceptable. Please note that we need to include the identity in addition to the random permutations to keep the type I error guarantee (Phipson & Smyth, 2010).

$T_{N,k}$ does not change when the permutation do not exchange any index from $\{1,\dots,N\}$ with an index of $\{N+1,\dots,2N\}$. In essence, choosing a permutation is equivalent to choosing the signs in $\sum_{n=1}^{N} e_{\sigma_i(n),i}(j) - \sum_{n=N+1}^{2N} e_{\sigma_i(n),i}(j)$. And because we take the absolute value, we obtain that there are $\frac{1}{2}\binom{2N}{N}$ possible permutations in the first interim that give unique values to $T_{N,1}$ (up to ties when the score distributions are discrete). Then, by enumerating all the permutations for the other interims, there are $\frac{1}{2}\binom{2N}{N}^k$ possible permutations giving unique values to $T_{N,k}$.

In practice, we use a parameter $B \in \mathbb{N}$ and the number of permutations used at interim k will be $|\mathcal{S}_k| = m_k = \min\left(B, \frac{1}{2}\binom{2N}{N}^k\right)$, i.e. whenever possible, we use all the permutations and if this is too much, we use permutations drawn at random.

Definition of the boundaries. With these permutations, we define the boundary thresholds $B_{N,k}^{(\mathbf{C})}$ by:

$$B_{N,k}^{(\mathbf{C})} = \inf\left\{ b > 0 : \frac{1}{m_k} \sum_{\sigma\in\widehat{\mathcal{S}}_k} \mathbb{1}\{\overline{T}_{N,k}^{(\mathbf{C})}(\sigma_{1:k}) \ge b\} \le q_k \right\}. \tag{9}$$

where $\sum_{j=1}^{k} q_j \le \frac{k\alpha}{K}$ and where $\widehat{\mathcal{S}}_k$ is the subset of $\mathcal{S}_k$ such that the statistic associated to the permutation would not have rejected before. Formally, $\widehat{\mathcal{S}}_k$ is the following set of permutations:

$$\widehat{\mathcal{S}}_k = \left\{ \sigma_{1:k} : \forall m < k, \quad \overline{T}_{N,m}^{(\mathbf{C})}(\sigma_{1:m}) \le B_{N,m}^{(\mathbf{C})} \right\}.$$

Note that q_1 is not equal to α/K. Due to discreetness (we use an empirical quantile over a finite number of values), q_1 is chosen equal to $\lfloor \frac{\alpha}{2K}\binom{2N}{N} \rfloor / (\frac{1}{2}\binom{2N}{N})$, and similarly q_2 is chosen to be as large as possible while having $q_1 + q_2$ smaller than $2\alpha/K$, and so on for q_i for $3 \le i \le k$ such that $\sum_{i=1}^{K} q_i \le \alpha$ with $\sum_{i=1}^{K} q_i$ as close as possible to α.

4.1 Theoretical guarantees of AdaStop

One of the basic properties of two-sample permutation tests is that when the null hypothesis is true, then all permutations are as likely to give a certain value and permuting the sample should not change the test statistic too much. Following our choice of $\overline{B}_{N,k}$ as a quantile of the law given the data, the algorithm has a probability to wrongly reject the hypothesis bounded by α. This informal statement is made precise in the following theorem.

Theorem 1 (Controlled family-wise error)**.** *Suppose that* $\alpha \in (0,1)$*, and consider the multiple testing problem* $H_j : P_{l_1} = P_{l_2}$ *against* $H_j' : P_{l_1} \ne P_{l_2}$ *for all the couples* $\mathbf{c}_j = (l_1, l_2) \in \{\mathbf{c}_1, \dots, \mathbf{c}_J\}$*. Then, the test resulting*

Algorithm 3: AdaStop (main algorithm).

Parameters: Agents $A_1, A_2, \ldots, A_L$, environment $\mathcal{E}$, comparison pairs $(c_i)_{i \leq L}$ where c_i is a couple of agents that we want to compare. Integers $K, N \in \mathbb{N}^*$, test parameter α.

Set $\mathbf{C} = \{1, \ldots, J\}$ the set of indices for the comparisons to perform.
for $k = 1, \ldots, K$ **do**
  for $l = 1 ... L$ **do**
    Train N times agent A_l on environment $\mathcal{E}$.
    Collect the N scores of agent A_l.
  end
  while *True* **do**
    Compute the boundaries $B_{N,k}^{(\mathbf{C})}$ using Equation (9).
    if $T_{N,k}^{(\boldsymbol{C})}(\mathrm{id}) > B_{N,k}^{(\boldsymbol{C})}$ **then**
      Reject $H_{j_{\max}}$ where $j_{\max} = \arg\max \left(\overline{T}_{N,k}^{(j)}(\mathrm{id}), \quad j \in \mathbf{C} \right)$.
      Update $\mathbf{C} = \mathbf{C} \setminus \{j_{\max}\}$
    else
      exit the while loop.
    end
  end
  if $\boldsymbol{C} = \emptyset$ **then** Exit the loop and return the decision of the test.
  if $k = K$ **then** Exit the loop and accept all hypotheses remaining in $\mathbf{C}$.
end

from Algorithm 3 has a strong control on the Family-wise error for the multiple test, i.e. if we suppose that all the hypotheses $H_i, i \in \mathbf{I}$ are true and the others are false, then

$$\mathbb{P}\left(\exists j \in \mathbf{I}: \quad \text{reject } H_j\right) \leq \alpha.$$

The proof of Theorem 1 is given in the Appendix (Section C).

Hypotheses of the test: in Theorem 1 we show that Algorithm 3 tests the equality of the distributions $P_{l_1} = P_{l_2}$ versus $P_{l_1} \neq P_{l_2}$, whereas in practice we would prefer to compare the means of the distribution $\mu_{l_1} = \mu_{l_2}$ versus $\mu_{l_1} \neq \mu_{l_2}$. Doing the test on distributions is something that is often seen in nonparametric tests (Lehmann et al., 2005) and it is justified because without strong concentration assumptions on the distributions, we lack control on the mean of the distributions. On the other hand, we show in the Appendix E that for N large the test comparing the means $\mu_{l_1} = \mu_{l_2}$ versus $\mu_{l_1} \neq \mu_{l_2}$ has the right guarantees (FWE smaller than α), this shows that even though we test the distributions, we also have an approximate test on the means. More precisely, we show the following in the Appendix E for the comparison of the means of two distributions.

Theorem 2. *Suppose that $\alpha \in (0, 1)$, and consider the two-sample testing problem $H_0 : \mathbb{E}_P[X] = \mathbb{E}_Q[X]$ against $H_0' : \mathbb{E}_P[X] \neq \mathbb{E}_Q[X]$. Then, the test resulting from Algorithm 3 has an asymptotic level of α*

$$\lim_{N \to \infty} \mathbb{P}_{H_0}\left(\textit{reject } H_0\right) = \alpha.$$

Power of the test: in Theorem 1 there is no information on the power of the test. We show that, for any N, K, the test is of level α. Having information on the power would allow us to give a rule for the choice of N and K but power analysis in nonparametric setting is in general hard, and it is outside the scope of this article. Instead, we compute empirically the power of our test and show that we perform well empirically compared to non-adaptive approaches. See Section 5.2 for the empirical power study on a Mujoco environment.

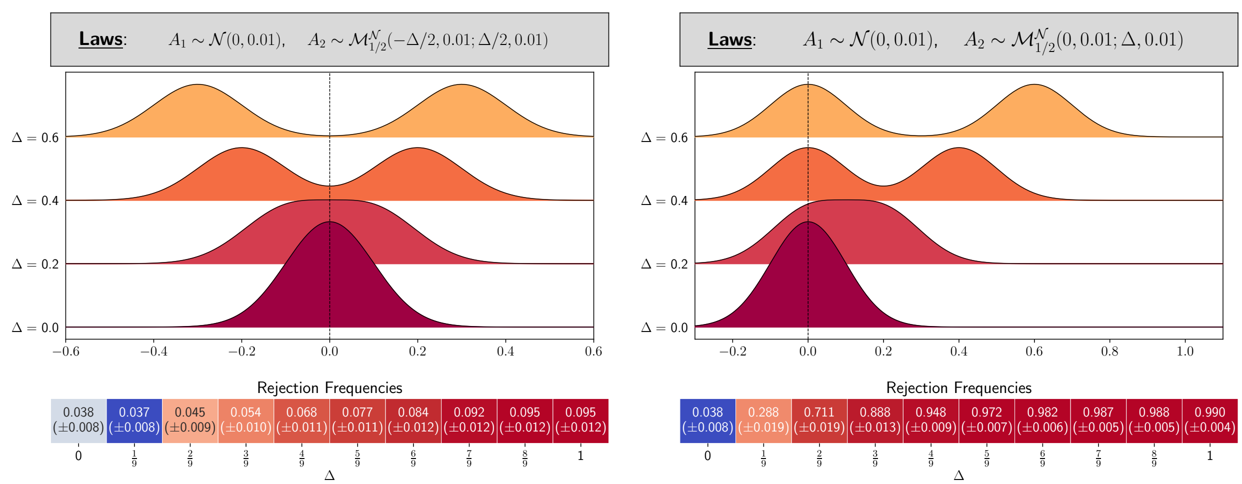
5 Experimental study

In this section, we first illustrate the statistical properties of AdaStop on toy examples in which the scores of the agents are sampled from known distributions. Then, we compare empirically AdaStop to non-adaptive approach. Finally, we exemplify the use of AdaStop on a real case to compare several deep-RL agents. We believe this is a key section demonstrating the strength of our approach from a practitioner perspective.

5.1 Toy examples

To start with, let us follow the execution of our algorithm on toy examples. In what follows, let us denote (i) $\mathcal{N}(\mu, \sigma^2)$ the normal distribution with mean μ and standard deviation σ, (ii) $t(\mu, \nu)$ the t-Student distribution with mean μ and degree of freedom ν, (iii) $\mathcal{M}^{\mathcal{N}}_{\frac{1}{2}}(\mu_1, \sigma_1^2; \mu_2, \sigma_2^2)$ for the mixture of 2 Gaussian distributions $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$, and (iv) $\mathcal{M}^{t}_{\frac{1}{2}}(\mu_1, \nu_1; \mu_2, \nu_2)$ for the mixture of 2 t-Student distributions $t(\mu_1, \nu_1)$ and $t(\mu_2, \nu_2)$.

We compare two agents A_1 and A_2 for which we know the distributions of their scores. We consider two settings in Fig. 4. Δ is the distance between two modes of the mixtures ($\Delta = |\mu_1 - \mu_2|$). For both cases, we run AdaStop with $K = 5$, $N = 5$ and $\alpha = 0.05$. We also limit the maximum number of permutations to $B = 10^4$. At the bottom of Fig. 4, we plot the rejection rate of the null hypothesis that the compared distributions are the same. By varying Δ from 0 to 1, we observe the evolution of the power of tests, i.e. the probability of rejecting the null hypothesis when it is indeed false. Figure 4a shows that the power of the test stays around 0.05 level for all Δ (it is at most 0.1 for the most extreme case). Indeed, even though the distributions in the comparison are different, their means remain the same. If the null hypothesis states that the means are the same, then AdaStop will return the correct answer with type I error not larger than 0.095 (see Figure 4a) for $\alpha = 0.05$. This is an illustration of the fact that in addition to performing a test on the distributions, AdaStop approximates the test on the means as shown theoretically in the asymptotic result in Appendix Section E and as discussed at the end of Section 4.1. In contrast, Figure 4b demonstrates the increasing trend, reaching the level close to 1 after $\Delta = 0.6$, which corresponds to the case where the two modes are separated by 3 standard deviations from both sides. To obtain an estimation of the error, we have executed each comparison $M = 5 \cdot 10^3$ times, and we plot confidence intervals corresponding to $3\sigma/\sqrt{M}$ (more than 99% of confidence) where σ is a standard deviation of the test decision. In addition to Cases 1 and 2, we also provide a third experiment with a comparison of 10 agents in Appendix Section H.1.



(a) Case 1

(b) Case 2

Figure 4: Toy examples 1 and 2 with an illustration of Gaussian mixtures and rejection frequency of null hypothesis according to Δ

5.2 Comparison with non-adaptive approach

N\K	2	3	4	5	6
1	0.0 (2.0)	0.0 (3.0)	0.277 (4.0)	0.465 (5.0)	0.56 (6.0)
2	0.005 (4.0)	0.33 (6.0)	0.531 (6.96)	0.602 (8.345)	0.704 (9.198)
3	0.213 (5.984)	0.506 (8.085)	0.627 (10.212)	0.689 (11.02)	0.785 (11.52)
4	0.371 (7.616)	0.611 (9.648)	0.744 (11.7)	0.82 (12.08)	0.845 (13.89)
5	0.465 (9.044)	0.691 (11.031)	0.78 (13.28)	0.853 (14.27)	0.884 (14.532)
6	0.534 (10.4)	0.73 (12.306)	0.837 (14.124)	0.89 (14.94)	0.911 (15.978)
7	0.599 (11.358)	0.779 (13.404)	0.879 (14.916)	0.92 (15.495)	0.939 (16.404)
8	0.635 (12.322)	0.818 (13.95)	0.885 (15.824)	0.942 (16.03)	0.961 (17.268)

Table 1: Average empirical statistical power and, in parentheses, effective number of scores used by AdaStop as a function of the total number of scores ($N \times K$) when comparing SAC and TD3 agents on Mujoco HalfCheetah task. The number of permutations B is set to 10^4 and α is set to 0.05. AdaStop is run 10^3 times for each (N, K) pair. The shades of blue are proportional to the power, a value in $[0, 1]$ (we use the same color scheme as in (Colas et al., 2018)).

Colas et al. (2019) share the same objective as ours. However, they use non-adaptive tests unlike AdaStop. We follow their experimental protocol and compare AdaStop and non-adaptive approaches empirically in terms of statistical power as a function of the sample size (number of scores). In particular, we use the data they provide for a SAC agent and for a TD3 agent evaluated on HalfCheetah (see Fig. 9 in the Appendix). Similarly to (Colas et al., 2019, Table 15), we compute the empirical statistical power of AdaStop as a function of the number of scores of the RL algorithms (Table 1). To compute the empirical statistical power for a given number of scores, we make the hypothesis that the distributions of SAC and TD3 agents scores are different, and we count how many times AdaStop decides that one agent is better than the other (number of true positives). As the test is adaptive, we also report the effective number of scores that are necessary to make a decision with 0.95 confidence level. For each number of scores, we have run AdaStop 10^3 times. For example, when comparing the scores of SAC and TD3 on HalfCheetah using AdaStop with $N = 4$ and $K = 5$, the maximum number of scores that could be used is $N \times K = 20$ without early stopping. However, we observe in Table 1 that when $N = 4$ and $K = 5$, AdaStop can make a decision with a power of 0.82 using only 12 scores. In (Colas et al., 2019, Table 15), the minimum number of scores required to obtain a statistical power of 0.8 when comparing SAC and TD3 agents is 15 when using a t-test, a Welch test, or a bootstrapping test. With this example, we first show that being an adaptive test, AdaStop may save computations when authors want to perform a "reasonable enough" amount of runs of their agents. We also show that as long as researchers are making the scores of their agents available, AdaStop can use them to provide a statistically original conclusion, and as such, AdaStop may be used to assess the initial conclusions, hopefully strengthening them with a statistically significant argument.

5.3 AdaStop for Deep Reinforcement Learning

In this section, we use AdaStop to compare four commonly-used Deep RL algorithms on the MuJoCo[2] (Todorov et al., 2012) benchmark for high-dimensional continuous control, as implemented in Gymnasium[3]. More specifically, we train agents on the Ant-v3, HalfCheetah-v3, Hopper-v3, Humanoid-v3, and Walker-v3 environments using PPO from rlberry (Domingues et al., 2021), SAC from Stable-Baselines3 (Raffin et al., 2021), DDPG from CleanRL (Huang et al., 2022), and TRPO from MushroomRL (D'Eramo et al., 2021). PPO, SAC, DDPG, and TRPO are all deep reinforcement learning algorithms used for high-dimensional continuous control tasks. We chose these algorithms because they are commonly used and represent a diverse set of approaches from different RL libraries. We use different RL libraries in order to demonstrate the flexibility of AdaStop, as well as to provide examples on how to integrate these popular libraries with AdaStop.

[2]We use MuJoCo version 2.1, as required by https://github.com/openai/mujoco-py
[3]https://github.com/Farama-Foundation/Gymnasium

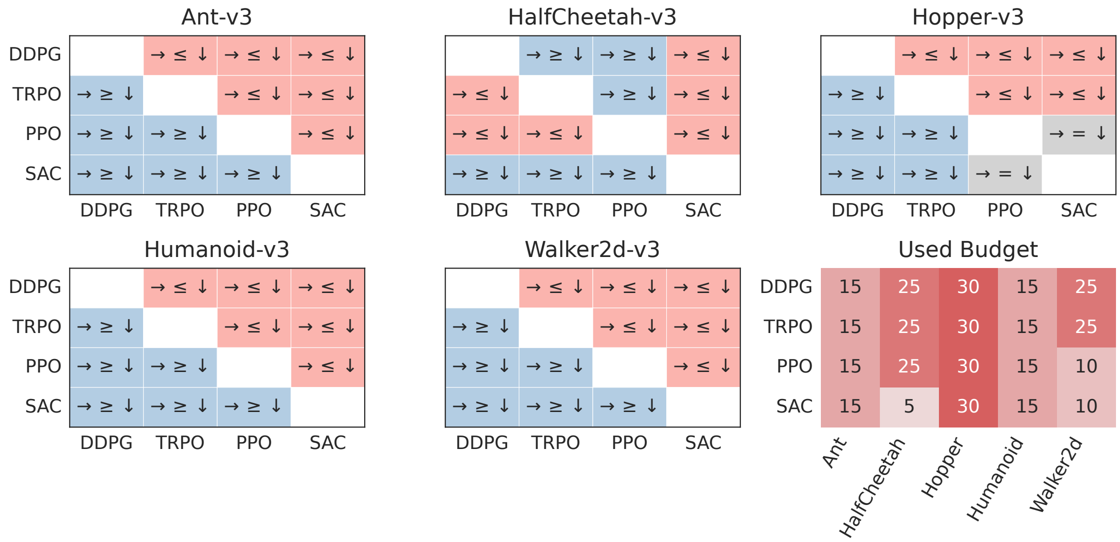


Figure 5: AdaStop decision tables for each MuJoCo environment, and the budget used to make these decisions (bottom right). See Appendix H.3 for further details.

On-policy algorithms, such as PPO and TRPO, update their policies based on the current data they collect during training, while off-policy algorithms, such as SAC and DDPG, can learn from any data, regardless of how it was collected. This difference may make off-policy algorithms more sample-efficient but less stable than on-policy algorithms. Furthermore, SAC typically outperforms DDPG in continuous control robotics tasks due to its ability to handle stochastic policies, while DDPG restricts itself to deterministic policies (Haarnoja et al., 2018). Finally, PPO is generally considered performing better than TRPO in terms of cumulative reward (Engstrom et al., 2020).

For each algorithm, we fix the hyperparameters to those used by the library authors in their benchmarks for one of the MuJoCo environments. Appendix H.3 lists the values that were used and we further discuss the experimental setup.

We compare the four agents in each environment using AdaStop with $N = 5$ and $K = 6$. Fig. 5 shows the AdaStop decision tables for each environment, as well as the number of scores per agent and environment. As expected, SAC ranks first in each environment. In contrast, DDPG is ranked last in four out of five environments; this may be due to the restriction to deterministic policies which hurts exploration in high-dimensional continuous control environments such as the MuJoCo benchmarks. Furthermore, we observe that the expected ordering between PPO and TRPO is generally respected, with TRPO outperforming PPO in only one environment. Finally, we note that PPO performs particularly well in some environments obtaining scores that are comparable to those of SAC, while also being the worst-performing algorithm on HalfCheetah-v3. Overall, the AdaStop rankings in these experiments are not unexpected.

Moreover, our experiments demonstrate that AdaStop can make decisions with fewer scores, thus reducing the computational cost of comparing Deep RL agents. For instance, as expected, SAC outperformed other agents on the environment HalfCheetah-v3, and AdaStop required only five scores to make all decisions involving SAC. Additionally, we observed that the decisions requiring the entire budget of $NK = 30$ scores were the ones in which AdaStop determined that the agents were equivalent in terms of their scores. With the early-accept heuristic detailed in the Appendix (Section F), this process can be sped-up and for instance in the Walker2d-v3 environment, early accept allows us to take all the decisions after only 10 scores.

6 Conclusion and future works

In this paper, we introduce AdaStop which is a sequential group test aiming at ranking agents based on their practical performance. AdaStop may be applied in various fields where algorithms are random, such as machine learning, or optimization. Our goal is to provide statistical grounding to define the number of times a set of agents should be run to be able to confidently rank them, up to some confidence level α. This is the first such test, and we think this is a major contribution to computational studies in reinforcement learning and other domains. From a statistical point of view, we have been able to demonstrate the soundness of AdaStop as a statistical test. Using AdaStop is simple. We provide open source software to use it. Experiments demonstrate how AdaStop may be used in practice, even in a retrospective manner using logged data: this allows one to diagnose prior studies in a statistically significant way, hopefully confirming their conclusions, possibly showing that the same conclusions could have been obtained with less computations.

Currently, AdaStop considers a set of agents facing one single task. Our next step will be to extend the test to experimental settings where a set of agents are compared on a collection of tasks, such as the set of Atari games or the set of Mujoco tasks in reinforcement learning. Properly dealing with such experimental settings requires a careful statistical analysis. Moreover, additional theoretical guarantees for the early accept and to control the power of AdaStop would also greatly improve the interpretability of the conclusions of AdaStop.

As this is illustrated in the experimental section, AdaStop can be run on already collected scores: we do not need to run anew the agents as long as scores are available. This remark calls for an effort of the community to make their scores publicly available so that it is easy for anyone to compare one's new agent with others already proposed.

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A Index of notations

- N_{score}: the total number of scores used for one agent.
- N: the number of scores per interim.
- K: the total number of interims.
- k: the current interim number.
- L: the number of agents to compare.
- $A_1, A_2, \ldots, A_L$: the agents to compare.
- $\mathbb{E}[X]$: expectation of the random variable X.
- $\mathbf{S}_n$: set of all the permutations of $\{1, \ldots, n\}$.
- σ: generic notation for a permutation, $\sigma \in \mathbf{S}_n$ for some $n \in \mathbb{N}^*$.
- id: an identity permutation, *i.e.* $\mathrm{id}(k) = k$ for all $1 \le k \le n$ for $\mathrm{id} \in \mathbf{S}_n$.
- B: the number of random permutations used to approximate the test statistic (see discussion *choice of permutations* in Section 4).
- $\sigma_{1:k}$: shorthand for the concatenation of permutations $\sigma_1, \ldots, \sigma_k$, *i.e.* $\sigma_{1:k}(e_{n,i}) = \sigma_i(e_{n,i})$ for all $1 \le i \le k$.
- H_j: denotes hypothesis j in a multiple test, H'_j denotes the alternative of hypothesis H_j.
- $e_i(j)$ or $e_{i,k}(j)$: score corresponding to run number i when doing the test for comparison $\mathbf{c}_j$. Index k denotes interim k when the scores are used sequentially (e.g. in AdaStop).
- $T_N(\sigma)$ and $T_{N,k}^{(j)}(\sigma)$: test statistics. See Equation (5) and Equation (7).
- $\mathbf{c}_j$: denotes a comparison. This is a couple (l_1, l_2) in $\{1, \ldots, L\}^2$.
- j: shorthand for denoting comparison c_j.
- $\mathbf{C}_0$: set of all the comparisons done in AdaStop.
- $\mathbf{C}$: current set of undecided comparisons in AdaStop, a subset of $\mathbf{C}_0$.
- $\mathbf{C}_k$: state of $\mathbf{C}$ at interim k in AdaStop.
- $B_{N,k}^{(\mathbf{C})}$: boundary for test statistics $T_{N,k}^{(\mathbf{C})}$, *s.t.* if $T_{N,k}^{(\mathbf{C})}(\mathrm{id}) > B_{N,k}^{(\mathbf{C})}$, then reject the set of hypotheses associated to $\mathbf{C}$.
- $\mathbf{I}$: set of true hypotheses and $\mathbf{I}^c$ its complement.
- FWE: family-wise error, see Definition 1.
- $\mathcal{N}(\mu, \sigma^2)$: law of a Gaussian probability distribution with mean μ and variance σ^2.
- $t(\mu, \nu)$: law of a translated Student probability distribution with center of symmetry μ and ν degrees of freedom.
- $\mathcal{M}_{\frac{1}{2}}^{\mathcal{N}}(\mu_1, \sigma_1^2; \mu_2, \sigma_2^2)$: mixture of the two normal probability distributions $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$.
- $\mathcal{M}_{\frac{1}{2}}^{t}(\mu_1, \nu_1; \mu_2, \nu_2)$: mixture of the two Student probability distributions $t(\mu_1, \nu_1)$ and $t(\mu_2, \nu_2)$.
- $\mathbb{P}_{H_j, j \in \mathbf{I}}$: probability distribution when $H_j, j \in \mathbf{I}$ are true and $H_j, j \notin \mathbf{I}$ are false.

B Recap on hypothesis testing

To be fully understood, this paper requires the knowledge of some notions of statistics. In the hope of widening the audience of this paper, we provide a short recap of essential notions of statistics related to hypothesis testing.

B.1 Type I and type II error

In its most simple form, a statistical test is aimed at deciding, whether a given collection of data $X_1, \dots, X_N$ adheres to some hypothesis H_0 (called the null hypothesis), or if it is a better fit for an alternative hypothesis H_1. Typically, the null hypothesis states that the mean of the distribution from which the X_i's are sampled is equal to some μ_0: $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$ where μ is the mean of the distribution of $X_1, \dots, X_N$. Because μ is unknown, it has to be estimated using the data, and often that is done using the empirical mean $\widehat{\mu} = \frac{1}{N}\sum_{i=1}^N X_i$. $\widehat{\mu}$ is random and some deviation from μ is to be expected. The theory of hypothesis tests is concerned in finding a threshold c such that if $|\widehat{\mu} - \mu_0| > c$ then we say that H_0 is false because the deviation is greater than what was expected by the theory.

A slightly more complex problem is to consider two samples $X_1, \dots, X_N$ and $Y_1, \dots, Y_N$ and do a two-sample test deciding whether the mean of the distribution of the X_i's is equal to the mean of the distribution of the Y_i's.

In both cases, the result of a test is either accept H_0 or reject H_0. This answer is not a ground truth: there is some probability that we make an error. However, this probability of error is often controlled and can be decomposed in type I error and type II error (often denoted α and β respectively, see Table 2). Please

	H_0 is true	H_0 is false
We accept H_0	No error	Type II error β
We reject H_0	Type I error α	No error

Table 2: Type I and type II error.

note that the problem is not symmetric: failing to reject the null hypothesis does not mean that the null hypothesis is true. It can be that there is not enough data to reject H_0. It has been shown that a test cannot simultaneously minimize α and β so instead, it is customary to minimize β with a fixed α (typically α is set to 0.05). The probability to reject H_0 when it is false is $1 - \beta$ and is usually called the *power* of a test.

B.2 Multiple tests and FWE

When doing simultaneously L statistical tests for $L > 1$, $L \in \mathbb{N}$, one must be careful that the error of each test accumulates and if one is not cautious, the overall error may become non-negligible. As a consequence, multiple strategies have been developed to deal with multiple testing problem.

To deal with the multiple testing problem, the first step is to define what is an error. There are several definitions of error in multiple testing among which is the False Discovery Rate which measures the expected proportion of false rejections. Another possible measure of error is the Family-Wise Error (this is the error we use in this article) and which is defined as the probability to make at least one false rejection:

$$\mathrm{FWE} = \mathbb{P}_{H_j, j \in \mathbf{I}}\left(\exists j \in \mathbf{I} : \quad \text{reject } H_j\right),$$

where $\mathbb{P}_{H_j, j \in \mathbf{I}}$ is used to denote the probability when $\mathbf{I} \subset \{1, \dots, L\}$ is the set of indices of the hypotheses that are actually true (and $\mathbf{I}^c$ the set of hypotheses that are actually false). To construct a procedure with FWE smaller than α, the simplest method is perhaps Bonferroni's correction (Bonferroni, 1936) in which one would use one statistical test for each of the J couple of hypotheses to be tested. And then, one would tune each hypothesis test to have a type I error α/J where $J = L(L-1)/2$ is the number of tests that have to be done. The union bound then implies that the FWE is bounded by α:

$$\text{FWE } = \mathbb{P}_{H_j, j \in \mathbf{I}}\Big(\bigcup_{j \in \mathbf{I}} \{\text{reject } H_j\}\Big) \leq \sum_{i \in \mathbf{I}} \mathbb{P}_{H_j, j \in \mathbf{I}} (\text{reject } H_j) \leq |\mathbf{I}| \frac{\alpha}{J} \leq \alpha.$$

which is the probability of rejecting the hypothesis given that it is actually true. Bonferroni correction has the advantage of being very simple to implement, but it is often very conservative and the final FWE would be most often a lot smaller than α. An alternative method that performs well in practice is the step-down method that we use in this article and is presented in Section 3.2.4.

C Proof of Theorem 1

The proof of Theorem 1 is based on an extension of the proof of the control of FWE in the non-sequential case and the proof of the step-down method (Romano & Wolf, 2003). The interested reader may refer to Lemma 1, in the Appendix C.1, where we reproduce the proof of the bound on FWE for simple permutation tests as it is a good introduction to permutation tests. The proof proceeds as follows: first, we prove weak control on the FWE by decomposing the error as the sum of the errors on each interim and using the properties of permutation tests to show that the error done at each interim is controlled by α/K. Then, using the step-down method construction, we show that the strong control of the FWE is a consequence of the weak control because of monotony properties on the boundary values of a permutation test.

C.1 Simplified proof for $L = 2$ agents, and $K = 1$

The proof of the theorem for this result is a bit technical. We begin by showing the result in a very simplified case with $L = 2$ agents, and $K = 1$.

Lemma 1. *Let $X_1, \dots, X_N$ be i.i.d from a distribution P and $Y_1, \dots, Y_N$ be i.i.d. from a distribution Q. Denote $Z_1^{2N} = X_1, \dots, X_N, Y_1, \dots, Y_N$ be the concatenation of X_1^N and Y_1^N. Let $\alpha \in (0,1)$ and define B_N such that*

$$B_N = \inf \left\{ b > 0 : \quad \frac{1}{(2N)!} \sum_{\sigma \in \boldsymbol{S}_{2N}} \mathbb{1}\left\{ \frac{1}{N} \sum_{i=1}^{N} (Z_{\sigma(i)} - Z_{\sigma(N+i)}) > b \right\} \leq \alpha \right\}.$$

Then, if $P = Q$, we have

$$\mathbb{P}\left(\frac{1}{N} \sum_{i=1}^{N} (X_i - Y_i) > B_N \right) \leq \alpha$$

Proof. Denote $T(\sigma) = \frac{1}{N}\sum_{i=1}^N (Z_{\sigma(i)} - Z_{\sigma(n+i)})$. Since $P = Q$, for any $\sigma, \sigma' \in \mathbf{S}_{2N}$ we have $T(\sigma) \overset{d}{=} T(\sigma')$. Then, because B_N does not depend on the permutation σ (but it depends on the values of Z_1^{2N}), we have, for any $\sigma \in \mathbf{S}_{2N}$

$$\mathbb{P}\left(T(\text{id}) > B_N\right) = \mathbb{P}\left(T(\sigma) > B_N\right)$$

Now, take the sum over all the permutations,

$$\begin{aligned}
\mathbb{P}\left(T(\text{id}) > B_N\right) &= \frac{1}{(2N)!} \sum_{\sigma \in \mathbf{S}_{2N}} \mathbb{E}\left[\mathbb{1}\{T(\sigma) > B_N\}\right] \\
&= \mathbb{E}\left[\frac{1}{(2N)!} \sum_{\sigma \in \mathbf{S}_{2N}} \mathbb{1}\{T(\sigma) > B_N\} \right] \leq \alpha
\end{aligned}$$

which proves the result. ☐

Next, we prove weak control in the general case.

C.2 Proof of Theorem 1

In this section, we use the shorthand $\mathbb{P}$ instead of $\mathbb{P}_{H_j, j\in \mathbf{I}}$ and omit $H_j, j \in \mathbf{I}$ because $\mathbf{I}$ will always be the set of true hypotheses and the meaning should be clear from the context.

Weak control on FWE: First, we prove weak control on the FWE. This means that we suppose that $\mathbf{I} = \{1, \dots, J\}$: all the hypotheses are true, and we control the probability to make at least one rejection. We have:

$$\mathrm{FWE} = \mathbb{P}\left(\exists j \in \mathbf{I} : \quad H_j \text{ is rejected}\right).$$

We decompose the FWE on the set of interims, using the fact that a rejection happens if and only if we reject a true hypothesis for the first time at interim k for some $k = 1, \dots, K$. These events are mutually exclusive therefore we have that their probabilities sum up and we can rewrite the FWE as:

$$\mathrm{FWE} = \sum_{k=1}^{K} \mathbb{P}\left(\overline{T}_{N,k}^{(\mathbf{I})}(\mathrm{id}) > B_{N,k}^{(\mathbf{I})}, \mathrm{NR}_k(\mathrm{id})\right), \tag{10}$$

where $\mathrm{NR}_k(\mathrm{id})$ is the event on which we did *Not Reject* (NR) before interim k, and can be defined directly for a general concatenation of permutations $\sigma_{1:k}$ as $\mathrm{NR}_k(\sigma_{1:k}) = \{\forall m < k, \overline{T}_{N,m}^{(\mathbf{I})}(\sigma_{1:k}) \leq B_{N,m}^{(\mathbf{I})}\}$. We introduced the definition for $\sigma_{1:k}$ and not only for id since this will be useful later on, for example in Equation (11).

Then, similarly as in the proof of Lemma 1, we want to use the invariance by permutation to make the link with the definition of $B_{N,k}^{(\mathbf{I})}$. For this purpose, we introduce the following lemma, that we prove in Appendix D.

Lemma 2. *We have that for $k \leq K$, for any $\sigma_{1:k}$ concatenation of k permutations:*

$$(\overline{T}_{N,l}^{(\mathbf{I})}(\mathrm{id}), B_{N,l}^{(\mathbf{I})})_{l \leq k} \overset{d}{=} (\overline{T}_{N,l}^{(\mathbf{I})}(\sigma_{1:l}), B_{N,l}^{(\mathbf{I})})_{l \leq k}.$$

Using Lemma 2, we have for any interim k and concatenation of permutations $\sigma_{1:k}$

$$\mathbb{P}\left(\overline{T}_{N,k}^{(\mathbf{I})}(\mathrm{id}) > B_{N,k}^{(\mathbf{I})}, \mathrm{NR}_k(\mathrm{id})\right) = \mathbb{P}\left(\overline{T}_{N,k}^{(\mathbf{I})}(\sigma_{1:k}) > B_{N,k}^{(\mathbf{I})}, \mathrm{NR}_k(\sigma_{1:k})\right) \tag{11}$$

Hence, putting this into Equation (10), we have

$$\begin{aligned}
\mathrm{FWE} &\leq \sum_{k=1}^{K} \frac{1}{m_k} \sum_{\sigma_{1:k} \in \mathcal{S}_k} \mathbb{P}\left(\overline{T}_{N,k}^{(\mathbf{I})}(\sigma_{1:k}) > B_{N,k}^{(\mathbf{I})}, \mathrm{NR}_k(\sigma_{1:k})\right) \\
&= \sum_{k=1}^{K} \mathbb{E}\left[\frac{1}{m_k} \sum_{\sigma_{1:k} \in \mathcal{S}_k} \mathbb{1}\left\{\overline{T}_{N,k}^{(\mathbf{I})}(\sigma_{1:k}) > B_{N,k}^{(\mathbf{I})}, \mathrm{NR}_k(\sigma_{1:k})\right\}\right]
\end{aligned}$$

Then, use that $\sigma_{1:k} \in \widehat{\mathcal{S}}_k$ if and only if $\sigma_{1:k} \in \mathcal{S}_k$ and $\mathrm{NR}_k(\sigma_{1:k})$ is true. Hence,

$$\mathrm{FWE} \leq \sum_{k=1}^{K} \mathbb{E}\left[\frac{1}{m_k} \sum_{\sigma_{1:k} \in \widehat{\mathcal{S}}_k} \mathbb{1}\left\{\overline{T}_{N,k}^{(\mathbf{I})}(\sigma_{1:k}) > B_{N,k}^{(\mathbf{I})}\right\}\right] \leq \sum_{k=1}^{K} q_k \leq \alpha$$

where we used the definition of $B_{N,k}^{(\mathbf{I})}$ to make the link with α

Strong control of FWE: To prove strong control, it is sufficient to show the following Lemma (see Appendix D for a proof), which is an adaptation of the proof of step-down multiple-test strong control of FWE from (Romano & Wolf, 2003).

Lemma 3. *Suppose that* $\mathbf{I} \subset \{1, \dots, J\}$ *is the set of true hypotheses. We have*

$$\mathrm{FWE} = \mathbb{P}\left(\exists j \in \mathbf{I} : \ H_j \ \textit{is rejected}\right) \le \mathbb{P}\left(\exists k \le K : \quad \overline{T}^{(\mathbf{I})}_{N,k}(\mathrm{id}) > B^{(\mathbf{I})}_{N,k}\right).$$

Lemma 3 shows that to control the FWE, it is sufficient to control the probability to reject on $\mathbf{I}$ given by $\mathbb{P}\left(\exists k \le K : \quad \overline{T}^{(\mathbf{I})}_{N,k}(\sigma_{1:k}) > B^{(\mathbf{I})}_{N,k}\right)$ and this quantity, in turns, is exactly the FWE of the restricted problem of testing $(H_j)_{j\in\mathbf{I}}$ against $(H'_j)_{j\in\mathbf{I}}$. In other words, Lemma 3 says that to prove strong FWE control for our algorithm, it is sufficient to prove weak FWE control, and we already did that in the first part of the proof.

D Proof of Lemmas

D.1 Proof of Lemma 2

In this section, for an easier understanding, we change the notation for the score $e^{(j)}_{n,k}(\sigma)$ and denote by $e_{n,k}(A_l)$ the n^{th} score of agent A_l at interim k. In other words, we change the notation of the comparison of agent A_{l_1} versus agent A_{l_2}: in the main text a comparison was denoted by $\mathbf{c}_j \in \{1, \dots, J\}^2$ but here we make explicit the agent from which the score has been computed resulting in the following equalities: $e_{n,k}(A_{l_1}) = e^{(j)}_{n,k}(\mathrm{id})$ for $n \le N$ and $e_{n,k}(A_{l_2}) = e^{(j)}_{N+n,k}(\mathrm{id})$.

We denote the comparisons by $(\mathbf{c}_i)_{i\in\mathbf{I}}$. The set of comparisons can be represented as a graph in which each node represents one of the agents to compare, there exists an edge from j_1 to j_2 denoted (j_1, j_2) if $(j_1, j_2) \in (\mathbf{c}_i)_{i\in\mathbf{I}}$ is one of the comparisons that corresponds to a true hypothesis. This graph is not necessarily connected, we denote $C(i)$ the connected component to which node l (e.g. agent l) belongs, i.e. for any $l_1, l_2 \in C(l)$ there exists a path going from l_1 to l_2. Please note that $C(l)$ cannot be equal to the singleton $\{l\}$, because this would mean that all the comparisons with l are in fact false hypotheses, and then l would not belong to a couple in $\mathbf{I}$.

Then, it follows from the construction of permutation test that jointly on $k \le K$ and $\mathbf{c}_j = (l_1, l_2) \in C(l)$, we have $T^{(j)}_{N,k}(\mathrm{id}) \stackrel{d}{=} T^{(j)}_{N,1}(\sigma_{1:k})$ for any $\sigma_1, \dots, \sigma_k \in \mathbf{S}_{2N}$.

Let us illustrate that on an example. Suppose that $N = 2$ and $J = 3$ so that the comparison are (A_1, A_2), (A_1, A_3), (A_2, A_3). Consider the permutation

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Because all the scores are i.i.d., we have the joint equality in distribution

$$\begin{pmatrix} |e_{1,1}(A_1) + e_{2,1}(A_1) - e_{1,1}(A_2) - e_{2,1}(A_2)| \\ |e_{1,1}(A_3) + e_{2,1}(A_3) - e_{1,1}(A_2) - e_{2,1}(A_2)| \\ |e_{1,1}(A_1) + e_{2,1}(A_1) - e_{1,1}(A_3) - e_{2,1}(A_3)| \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} |e_{2,1}(A_1) + e_{1,1}(A_2) - e_{1,1}(A_1) - e_{2,1}(A_2)| \\ |e_{2,1}(A_3) + e_{1,1}(A_2) - e_{1,1}(A_3) - e_{2,1}(A_2)| \\ |e_{2,1}(A_1) + e_{1,1}(A_3) - e_{1,1}(A_1) - e_{2,1}(A_3)| \end{pmatrix}$$

and hence,

$$(T^{(j)}_{N,1}(\mathrm{id}))_{1\le j\le 3} \stackrel{d}{=} (T^{(j)}_{N,1}(\sigma_1))_{1\le j\le 3}.$$

For $k = 2$, we have for $\sigma_2 = \sigma_1$,

$$
\begin{pmatrix}
|e_{1,1}(A_1) + e_{2,1}(A_1) - e_{1,1}(A_2) - e_{2,1}(A_2)| \\
|e_{1,1}(A_3) + e_{2,1}(A_3) - e_{1,1}(A_2) - e_{2,1}(A_2)| \\
|e_{1,1}(A_1) + e_{2,1}(A_1) - e_{1,1}(A_3) - e_{2,1}(A_3)| \\
|e_{1,1}(A_1) + e_{2,1}(A_1) - e_{1,1}(A_2) - e_{2,1}(A_2) + e_{1,2}(A_1) + e_{2,2}(A_1) - e_{1,2}(A_2) - e_{2,2}(A_2)| \\
|e_{1,1}(A_3) + e_{2,1}(A_3) - e_{1,1}(A_2) - e_{2,1}(A_2) + e_{1,2}(A_3) + e_{2,2}(A_3) - e_{1,2}(A_2) - e_{2,2}(A_2)| \\
|e_{1,1}(A_1) + e_{2,1}(A_1) - e_{1,1}(A_3) - e_{2,1}(A_3) + e_{1,2}(A_1) + e_{2,2}(A_1) - e_{1,2}(A_3) - e_{2,2}(A_3)|
\end{pmatrix}
$$

$$
\stackrel{d}{=} \begin{pmatrix}
|e_{1,1}(A_1) + e_{2,1}(A_2) - e_{1,1}(A_1) - e_{2,1}(A_2)| \\
|e_{1,1}(A_3) + e_{2,1}(A_2) - e_{1,1}(A_3) - e_{2,1}(A_2)| \\
|e_{1,1}(A_1) + e_{2,1}(A_3) - e_{1,1}(A_1) - e_{2,1}(A_3)| \\
|e_{1,1}(A_1) + e_{2,1}(A_2) - e_{1,1}(A_1) - e_{2,1}(A_2) + e_{1,2}(A_1) + e_{2,2}(A_2) - e_{1,2}(A_1) - e_{2,2}(A_2)| \\
|e_{1,1}(A_3) + e_{2,1}(A_2) - e_{1,1}(A_3) - e_{2,1}(A_2) + e_{1,2}(A_3) + e_{2,2}(A_2) - e_{1,2}(A_3) - e_{2,2}(A_2)| \\
|e_{1,1}(A_1) + e_{2,1}(A_3) - e_{1,1}(A_1) - e_{2,1}(A_3) + e_{1,2}(A_1) + e_{2,2}(A_3) - e_{1,2}(A_1) - e_{2,2}(A_3)|
\end{pmatrix}
$$

and then, we get jointly

$$(T^{(j)}_{N,k}(\mathrm{id}))_{1\le j\le 3, k\le 2} \stackrel{d}{=} (T^{(j)}_{N,k}(\sigma_1 \cdot \sigma_2))_{1\le j\le 3, k\le 2}.$$

This reasoning can be generalized to general N, J and K:

$$(T^{(j)}_{N,k}(\mathrm{id}))_{k\le K, \mathbf{c}_j \in C(l)^2} \stackrel{d}{=} (T^{(j)}_{N,k}(\sigma_{1:k}))_{k\le K, \mathbf{c}_j \in C(l)^2}.$$

Then, use that by construction, the different connected component $C(l)$ are independent of one another and hence,

$$(T^{(j)}_{N,k}(\mathrm{id}))_{k\le K, j\in \mathbf{I}} \stackrel{d}{=} (T^{(j)}_{N,k}(\sigma_{1:k}))_{k\le K, j\in \mathbf{I}}.$$

The result follows from taking the maximum on all the comparisons and because the boundaries do not depend on the permutation.

D.2 Proof of Lemma 3

Denote by $\mathbf{C}_k$ the (random) value of $\mathbf{C}$ at the beginning of interim k. We have,

$$
\begin{aligned}
\mathrm{FWE} &= \mathbb{P}\left(\exists j \in \mathbf{I} : \quad H_j \text{ is rejected}\right) \\
&= \mathbb{P}\left(\exists k \le K : \quad \overline{T}^{(\mathbf{C}_k)}_{N,k}(\mathrm{id}) > B^{(\mathbf{C}_k)}_{N,k}, \operatorname*{arg\,max}_{j, \mathbf{c}_j \in \mathbf{C}_k} \overline{T}^{(j)}_{N,k}(\mathrm{id}) \in \mathbf{I}\right). \qquad (12)
\end{aligned}
$$

Then, let k_0 correspond to the very first rejection (if any) in the algorithm. Having that the argmax is attained in $\mathbf{I}$,

$$\overline{T}^{(\mathbf{C}_{k_0})}_{N,k_0}(\mathrm{id}) = \max\{T^{(j)}_{N,k_0}(\mathrm{id}), \mathbf{c}_j \in \mathbf{C}_{k_0}\} = \max\{T^{(j)}_{N,k_0}(\mathrm{id}), j \in \mathbf{I}\} = \overline{T}^{(\mathbf{I})}_{N,k_0}(\mathrm{id})$$

Moreover, having that the comparisons indiced by $\mathbf{I}$ are included into $\mathbf{C}_{k_0}$, we have $B^{(\mathbf{C}_{k_0})}_{N,k_0} \ge B^{(\mathbf{I})}_{N,k_0}$. Injecting these two relations in Equation (12), we obtain

$$
\begin{aligned}
\mathrm{FWE} &\le \mathbb{P}\left(\exists k \le K : \quad \overline{T}^{(\mathbf{I})}_{N,k}(\mathrm{id}) > B^{(\mathbf{I})}_{N,k}, \operatorname*{arg\,max}_{j \in \mathbf{C}_k} \overline{T}^{(j)}_{N,k}(\mathrm{id}) \in \mathbf{I}\right) \\
&\le \mathbb{P}\left(\exists k \le K : \quad \overline{T}^{(\mathbf{I})}_{N,k}(\mathrm{id}) > B^{(\mathbf{I})}_{N,k}\right).
\end{aligned}
$$

This proves the desired result.

E Asymptotic results for two agents

E.1 Convergence of boundaries and comparing the means

Because there are only two agents and no early stopping, we simplify the notations and denote

$$t_{N,i}(\sigma_i) = \sum_{n=1}^{N} e_{\sigma_i(n),i}(2) - \sum_{n=N+1}^{2N} e_{\sigma_i(n),i}(1)$$

and

$$\begin{aligned} T_{N,k}(\sigma_{1:k}) &= \left| \sum_{i=1}^{k} \left(\sum_{n=1}^{N} e_{\sigma_i(n),i}(2) - \sum_{n=N+1}^{2N} e_{\sigma_i(n),i}(1) \right) \right| \\ &= \left| \sum_{i=1}^{k} t_{N,i}(\sigma_i) \right| \end{aligned}$$

and

$$B_{N,k} = \inf \left\{ b > 0 : \frac{1}{((2N)!)^k} \sum_{\sigma_1,\dots,\sigma_k \in \mathbf{S}_{2N}^k} \mathbb{1}\{T_{N,k}(\sigma_{1:k}) \geq b\} \leq q_k \right\}$$

When there is only one interim ($K = 1$), we have the following convergence of the randomization law of $T_{N,1}(\sigma)$.

Proposition 1 (Theorem 17.3.1 in (Lehmann et al., 2005))**.** *Suppose $e_{1,1}(1), \dots, e_{N,1}(1)$ are i.i.d from P and $e_{1,1}(2), \dots, e_{N,1}(2)$ are i.i.d from Q and both P and Q has finite variance. Then, we have*

$$\sup_t \left| \frac{1}{(2N)!} \sum_{\sigma \in \boldsymbol{S}_{2N}} \mathbb{1}\left\{ \frac{1}{\sqrt{N}} T_{N,1}(\sigma) \leq t \right\} - \Phi\left(t/\tau(P,Q)\right) \right| \xrightarrow[N\to\infty]{P} 0$$

where Φ is the standard normal c.d.f. and $\tau(P,Q)^2 = \sigma_P^2 + \sigma_Q^2 + \frac{(\mu_P - \mu_Q)^2}{2}$.

Using the non-sequential result from proposition 1, we can show the following theorem that controls the asymptotic law of the sequential test.

Theorem 3. *We have that for any $1 \leq k \leq K$, $\frac{1}{\sqrt{N}} B_{N,k} \xrightarrow[N\to\infty]{} b_k$ where the real numbers b_k are defined as follows. Let $W_1, \dots, W_K$ be i.i.d random variable with law $\mathcal{N}(0,1)$, then b_1 is the solution of the following equation:*

$$\mathbb{P}\left(|W_1| \geq \frac{b_1}{\tau(P,Q)} \right) = \frac{\alpha}{K},$$

and for any $1 < k \leq K$, b_k is the solution of

$$\mathbb{P}\left(\left| \frac{1}{k} \sum_{j=1}^{k} W_j \right| > \frac{b_l}{\tau(P,Q)}, \quad \forall j < k, \left| \frac{1}{j} \sum_{i=1}^{j} W_i \right| \leq \frac{b_j}{\tau(P,Q)} \right) = \frac{\alpha}{K}.$$

Please note that the test we do corresponds to testing

$$\mathbb{1}\{\exists k \leq K : \frac{1}{\sqrt{N}} T_{N,k}(\text{id}) > \frac{1}{\sqrt{N}} B_{N,k}\}$$

and from Theorem 3 and central-limit theorem $\frac{1}{\sqrt{N}} T_{N,k}(\text{id})$ converges to $\sum_{j=1}^{k} W_j \sqrt{\sigma_P^2 + \sigma_Q^2}$ and $B_{N,k}/\sqrt{N}$ converges to b_k, hence the test is asymptotically equivalent to

$$\mathbb{1}\left\{ \exists k \leq K : \sum_{j=1}^{k} W_j \sqrt{\sigma_P^2 + \sigma_Q^2} > b_k \right\}.$$

Then, in the case in which $\mu_P = \mu_Q$, we have $\tau(P,Q) = \sqrt{\sigma_P^2 + \sigma_Q^2}$ and

$$\begin{aligned}
\mathrm{FWE} &= \mathbb{P}\left(\exists k \leq K : \sum_{j=1}^{k} W_j \sqrt{\sigma_P^2 + \sigma_Q^2} > b_k\right) \\
&= \sum_{k=1}^{K} \mathbb{P}\left(\left|\frac{1}{k}\sum_{j=1}^{k} W_j\right| > \frac{b_l}{\tau(P,Q)}, \quad \forall j < k, \left|\frac{1}{j}\sum_{i=1}^{j} W_i\right| \leq \frac{b_j}{\tau(P,Q)}\right) \\
&= \sum_{k=1}^{K} \frac{\alpha}{K} = \alpha.
\end{aligned}$$

Hence, for the test $H_0: \quad \mu_P = \mu_Q$ versus $H_1: \quad \mu_P \neq \mu_Q$, our test is asymptotically of level α.

E.2 Proof of Theorem 3

For $x \in \mathbb{R}$, we denote:

$$R_{N,k}(x) = \frac{1}{(2N)!} \sum_{\sigma_k \in \mathbf{S}_{2N}} \mathbb{1}\{t_{N,k}(\sigma_k) \leq x\}.$$

$R_{N,k}$ is the c.d.f of the randomization law of $t_{N,k}(\sigma_k)$, and by Proposition 1, it converges uniformly to a Gaussian c.d.f when N goes to infinity.

Convergence of $B_{N,1}$

$$\frac{1}{\sqrt{N}} B_{N,1} = \frac{1}{\sqrt{N}} \min\left\{b > 0 : \frac{1}{(2N)!} \sum_{\sigma_1 \in \mathbf{S}_{2N}} \mathbb{1}\{|T_{N,1}(\sigma_1)| > b\} \leq \frac{\alpha}{K}\right\}$$

This implies

$$\begin{aligned}
\frac{1}{(2N)!} \sum_{\sigma_1 \in \mathbf{S}_{2N}} \mathbb{1}\{|T_{N,1}(\sigma_1)| \leq B_{N,1}\} &= \widehat{R}_{N,1}\left(\frac{1}{\sqrt{N}} B_{N,1}\right) - \widehat{R}_{N,1}\left(-\frac{1}{\sqrt{N}} B_{N,1}\right) \\
&\geq 1 - \frac{\alpha}{K}
\end{aligned}$$

and for any $b < B_{N,1}$, we have

$$\frac{1}{(2N)!} \sum_{\sigma_1 \in \mathbf{S}_{2N}} \mathbb{1}\{|T_{N,1}(\sigma_1)| \leq b\} = \widehat{R}_{N,1}\left(\frac{b}{\sqrt{N}}\right) - \widehat{R}_{N,1}\left(-\frac{b}{\sqrt{N}}\right) < 1 - \frac{\alpha}{K}$$

Then,

$$\begin{aligned}
&\Phi\left(\frac{B_{N,1}}{\tau(P,Q)\sqrt{N}}\right) - \Phi\left(-\frac{B_{N,1}}{\tau(P,Q)\sqrt{N}}\right) \\
\geq &\widehat{R}_{N,1}\left(\frac{B_{N,1}}{\sqrt{N}}\right) - \widehat{R}_{N,1}\left(-\frac{B_{N,1}}{\sqrt{N}}\right) - \left|\Phi\left(\frac{B_{N,1}}{\tau(P,Q)\sqrt{N}}\right) - \widehat{R}_{N,1}\left(\frac{B_{N,1}}{\sqrt{N}}\right)\right| \\
&- \left|\Phi\left(-\frac{B_{N,1}}{\tau(P,Q)\sqrt{N}}\right) - \widehat{R}_{N,1}\left(-\frac{B_{N,1}}{\sqrt{N}}\right)\right| \\
\geq &1 - \frac{\alpha}{K} - 2\sup_t \left|\Phi\left(\frac{t}{\tau(P,Q)}\right) - \widehat{R}_{N,1}(t)\right|
\end{aligned}$$

Hence, by taking N to infinity, we have from Proposition 1,

$$\liminf_{N\to\infty} \Phi\left(\frac{B_{N,1}}{\tau(P,Q)\sqrt{N}}\right) - \Phi\left(-\frac{B_{N,1}}{\tau(P,Q)\sqrt{N}}\right) \geq 1 - \frac{\alpha}{K}.$$

and for any $\varepsilon > 0$, because of the definition of $B_{N,1}$ as a supremum, we have

$$\limsup_{N\to\infty} \Phi\left(\frac{B_{N,1}+\varepsilon}{\tau(P,Q)\sqrt{N}}\right) - \Phi\left(-\frac{B_{N,1}+\varepsilon}{\tau(P,Q)\sqrt{N}}\right) < 1 - \frac{\alpha}{K}.$$

By continuity of Φ, this implies that $\frac{1}{\sqrt{N}}B_{N,1}$ converges almost surely and its limit is such that

$$\Phi\left(\frac{\lim_{N\to\infty} B_{N,1}/\sqrt{N}}{\tau(P,Q)}\right) - \Phi\left(-\frac{\lim_{N\to\infty} B_{N,1}/\sqrt{N}}{\tau(P,Q)}\right) = 1 - \frac{\alpha}{K}.$$

Or said differently, let $W \sim \mathcal{N}(0,1)$, then we have the almost sure convergence $\lim_{N\to\infty} \frac{1}{\sqrt{N}}B_{N,1} = b_1$ where b_1 is the real number defined by

$$\mathbb{P}\left(|W| \geq \frac{b_1}{\tau(P,Q)}\right) = \frac{\alpha}{K}.$$

Convergence of $B_{N,k}$ for $k > 1$. We proceed by induction. Suppose that $\frac{1}{\sqrt{N}}B_{N,k-1}$ converges to some $b_{k-1} > 0$ and that for any $d_1, \ldots, d_{k-1}$, the randomization probability

$$\sup_{d_1,\ldots,d_{k-1}} \left| \frac{1}{((2N)!)^{k-1}} \sum_{\sigma_1,\ldots,\sigma_{k-1}\in\mathbf{S}_{2N}} \mathbb{1}\left\{\forall j \leq k-1, \sum_{i=1}^{j} t_{N,i}\left(\sigma_i\right) \leq d_j\sqrt{N}\right\} - \mathbb{P}\left(\forall j \leq k-1, \sum_{i=1}^{j} W_i \leq \frac{d_j}{\tau(P,Q)}\right)\right| \xrightarrow[a.s.]{N\to\infty} 0, \quad (13)$$

where $W_1, \ldots, W_{k-1}$ are i.i.d $\mathcal{N}(0,1)$ random variables. In other words, the randomization law converges uniformly to the joint law described above with the sum of Gaussian random variables.

Then, by uniform convergence and by convergence of the $B_{N,j}$, we have

$$\frac{1}{((2N)!)^{k-1}} \sum_{\sigma_1,\ldots,\sigma_{k-1}\in\mathbf{S}_{2N}} \mathbb{1}\left\{T_{N,j}\left(\sigma_{1:j}\right) > B_{N,k-1}, \quad \forall j < k-1,\, T_{N,j}\left(\sigma_{1:j}\right) \leq B_{N,j}\right\}$$

converges to

$$\mathbb{P}\left(\left|\sum_{i=1}^{l} W_i\right| > \frac{b_l}{\tau(P,Q)}, \quad \forall j < l,\, \left|\sum_{i=1}^{j} W_i\right| \leq \frac{b_j}{\tau(P,Q)}\right). \quad (14)$$

which is equal to $\frac{\alpha}{K}$ by construction of $B_{N,j}$ for $j < k$,

We have

$$B_{N,k} = \min\left\{b > 0 : \frac{1}{((2N)!^k)} \sum_{\sigma_1,\ldots,\sigma_k\in\mathbf{S}_{2N}} \mathbb{1}\left\{\begin{matrix} \left|\sum_{j=0}^{k} t_{N,j}(\sigma_j)\right| \geq b, \\ \forall j<k, \left|\sum_{i=0}^{j} t_{N,i}(\sigma_i)\right| \leq B_{N,j}\end{matrix}\right\} + \sum_{i=1}^{k-1} q_i \leq \frac{k\alpha}{K}\right\}.$$

By the induction hypothesis, we have $q_i \xrightarrow[n\to\infty]{} \alpha/K$ for any $i < k$.

Let $W_1, \ldots, W_k$ be i.i.d $\mathcal{N}(0,1)$ random variables. We show the following lemma that prove part of the step k of the induction hypothesis, and proved in Section E.3.

Lemma 4. *Suppose Equation* (14) *is true. Then,*

$$\sup_{d_1,\ldots,d_k} \left| \frac{1}{((2N)!)^k} \sum_{\sigma_1,\ldots,\sigma_k\in\boldsymbol{S}_{2N}} \mathbb{1}\left\{\forall j \leq k, \sum_{i=1}^{j} t_{N,i}\left(\sigma_i\right) \leq d_j\sqrt{N}\right\} - \mathbb{P}\left(\forall j \leq k, \sum_{i=1}^{j} W_i \leq \frac{d_j}{\tau(P,Q)}\right)\right| \xrightarrow[a.s.]{N\to\infty} 0,$$

Then, what remains is to prove the convergence of $B_{N,k}$. Denote

$$\Psi_k(d_k) = \mathbb{P}\left(\left|\sum_{i=1}^k W_i\right| > \frac{d_k}{\tau(P,Q)}, \forall j \leq k-1, \left|\sum_{i=1}^j W_i\right| \leq \frac{b_j}{\tau(P,Q)}\right),$$

we have, from Lemma 4, that

$$\left|\Psi_k\left(\frac{B_{N,k}}{\sqrt{N}}\right) - \frac{1}{((2N)!)^k} \sum_{\sigma_1,\dots,\sigma_k \in \mathbf{S}_{2N}} \mathbb{1}\left\{T_{N,k}\left(\sigma_{1:k}\right) > B_{N,k}, \quad \forall j < k,\, T_{N,j}\left(\sigma_{1:j}\right) \leq B_{N,j}\right\}\right|$$

converges to 0 as N goes to infinity. Hence,

$$\limsup_{N\to\infty} \Psi_k\left(\frac{B_{N,k}}{\sqrt{N}}\right) \leq \alpha/K.$$

Then, similarly to the case $k=1$, we also have for any $\varepsilon > 0$,

$$\liminf_{N\to\infty} \Psi_k\left(\frac{B_{N,k}-\varepsilon}{\sqrt{N}}\right) \geq \alpha/K$$

and by continuity of Ψ_k (which is a consequence of the continuity of the joint c.d.f. of Gaussian random variables) we conclude that $B_{N,k}/\sqrt{N}$ converges almost surely to b_k.

E.3 Proof of Lemma 4

In this proof, we denote by $E_{\sigma_{1:k}}(x)$ the expectation of the randomization law defined for some function $f : \mathbf{S}_{2N}^k \to \mathbb{R}$ by

$$E_{\sigma_{1:k}}[f(\sigma_{1:k})] = \frac{1}{((2N)!)^k} \sum_{\sigma_1,\dots,\sigma_k \in \mathbf{S}_{2N}} f(\sigma_{1:k}).$$

Remark that this is still random and should be differentiated from the usual expectation $\mathbb{E}$.

First, let us first handle the convergence of step k. We have,

$$\begin{aligned}
&\frac{1}{(2N)!} \sum_{\sigma_k \in \mathbf{S}_{2N}} \mathbb{1}\left\{\sum_{j=1}^k t_{N,j}\left(\sigma_j\right) \leq d_k\sqrt{N}\right\} \\
&\quad = \frac{1}{(2N)!} \sum_{\sigma_k \in \mathbf{S}_{2N}} \mathbb{1}\left\{\frac{1}{\sqrt{N}} t_{N,k}\left(\sigma_k\right) \leq d_k - \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} t_{N,j}\left(\sigma_j\right)\right\} \\
&\quad = \widehat{R}_{n,k}\left(d_k - \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} t_{N,j}\left(\sigma_j\right)\right)
\end{aligned}$$

We have, because the convergence in Proposition 1 is uniform,

$$\begin{aligned}
&\left|\widehat{R}_{n,k}\left(d_k - \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} t_{N,j}\left(\sigma_j\right)\right) - \Phi\left(\frac{1}{\tau(P,Q)}\left(d_k - \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} t_{N,j}\left(\sigma_j\right)\right)\right)\right| \\
&\leq \sup_t \left|\widehat{R}_n(t) - \Phi\left(\frac{t}{\tau(P,Q)}\right)\right| \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

Then, using this convergence we have that

$$E_{\sigma_{1:k}}\left[\mathbb{1}\left\{\forall j < k,\, \sum_{i=1}^j t_{N,i}\left(\sigma_i\right) \leq d_j\sqrt{N}\right\}\right]$$

converges uniformly on $d_1,\dots,d_k$ when N goes to infinity to

$$E_{\sigma_{1:k}}\left[\mathbb{1}\left\{\forall j<k-1,\ \sum_{i=1}^{j} t_{N,i}\left(\sigma_i\right)\le d_j\sqrt{N}\right\}\mathbb{P}\left(W_k\le \frac{1}{\tau(P,Q)}\left(d_k-\frac{1}{\sqrt{N}}\sum_{i=1}^{k-1}t_{N,i}\left(\sigma_i\right)\right)\right)\right]$$
$$=\mathbb{E}\left[E_{\sigma_{1:k-1}}\left[\mathbb{1}\left\{\substack{\forall j<k-2,\ \sum_{i=1}^{j}t_{N,i}(\sigma_i)\le d_j\sqrt{N},\\ \frac{1}{\sqrt{N}}\sum_{i=1}^{k-1}t_{N,j}(\sigma_i)\le \min(d_k-\tau(P,Q)W_k,d_{k-1})}\right\}\right]\right]$$

Then, using the induction hypothesis, this converges to Equation (13),

$$\mathbb{E}\left[\mathbb{1}\left\{\substack{\forall j<k-2,\ \sum_{i=1}^{j}W_i\le\frac{d_j}{\tau(P,Q)},\\ \frac{1}{\sqrt{N}}\sum_{i=1}^{k-1}W_i\le\frac{1}{\tau(P,Q)}\min(d_k-W_k,d_{k-1})}\right\}\right]=\mathbb{E}\left[\mathbb{1}\left\{\forall j<k,\ \sum_{i=1}^{j}W_i\le\frac{d_j}{\tau(P,Q)}\right\}\right]$$
$$=\mathbb{P}\left(\forall j\le k,\ \sum_{i=1}^{j}W_i\le\frac{d_j}{\tau(P,Q)}\right).$$

F On early accept in AdaStop

Let $\mathbf{C}\subset\{\mathbf{c}_1,\dots,\mathbf{c}_J\}$ be a subset of the set of comparisons that we want to do, denote

$$\overline{T}_{N,k}^{(\mathbf{C})}(\sigma_1^k)=\max\left(T_{N,k}^{(j)}(\sigma_1^k),\quad \mathbf{c}_j\in\mathbf{C}\right)\quad\text{and}\quad \underline{T}_{N,k}^{(\mathbf{C})}(\sigma_1^k)=\min\left(T_{N,k}^{(j)}(\sigma_1^k),\quad \mathbf{c}_j\in\mathbf{C}\right)$$

$$\overline{B}_{N,k}^{(\mathbf{C})}=\inf\left\{b>0:\frac{1}{m_k}\sum_{\sigma\in\widehat{\mathcal{S}}_k}\mathbb{1}\{\overline{T}_{N,k}^{(\mathbf{C})}(\sigma_1^k)\ge b\}\le\overline{q}_k\right\}\tag{15}$$

and

$$\underline{B}_{N,k}^{(\mathbf{C})}=\sup\left\{b>0:\frac{1}{m_k}\sum_{\sigma\in\widehat{\mathcal{S}}_k}\mathbb{1}\{\underline{T}_{N,k}^{(\mathbf{C})}(\sigma_1^k)\le b\}\le\underline{q}_k\right\}.\tag{16}$$

where $\sum_{j=1}^{k}\underline{q}_j\le\frac{k\alpha}{K}$ and $\sum_{j=1}^{k}\overline{q}_j\le\frac{k\beta}{K}$ and where $\widehat{\mathcal{S}}_k$ is the subset of $\mathcal{S}_k$ such that it would not have accepted or rejected before: for each $\sigma_1^k\in\widehat{\mathcal{S}}_k$, we have the following property

$$\forall m<k,\quad \overline{T}_{N,m}^{(\mathbf{C})}(\sigma_1^k)\le\overline{B}_{N,m}^{(\mathbf{C})}\text{ and }\underline{T}_{N,m}^{(\mathbf{C})}(\sigma_1^k)\ge\underline{B}_{N,m}^{(\mathbf{C})}.$$

In AdaStop, modify the decision step (line 10 to 15 in Algorithm 3) to The resulting algorithm have a small

Algorithm 4: Early accept.

if $\overline{T}_{N,k}^{(C)}(\mathrm{id})>\overline{B}_{N,k}^{(C)}$ **then**
 Reject $H_{j_{\max}}$ where $\mathbf{c}_{j_{\max}}=\arg\max\left(T_{N,k}^{(j)}(\mathrm{id}),\quad \mathbf{c}_j\in\mathbf{C}\right)$.
 Update $\mathbf{C}=\mathbf{C}\setminus\{\mathbf{c}_{j_{\max}}\}$
else if $\underline{T}_{n,k}^{(C)}(\mathrm{id})<\underline{B}_{N,k}^{(C)}$ **then**
 Accept $H_{j_{\min}}$ where $\mathbf{c}_{j_{\min}}=\arg\min\left(T_{N,k}^{(j)}(\mathrm{id}),\quad \mathbf{c}_j\in\mathbf{C}\right)$.
 Update $\mathbf{C}=\mathbf{C}\setminus\{\mathbf{c}_{j_{\min}}\}$

probability to accept a decision early, and as a consequence it may be unnecessary to compute some of the agent in the subsequent steps.

As an illustration of the performance of early accept, if one was to execute AdaStop with early parameter $\beta=0.01$ for the Walked2D-v3 experiment from Section 5.3, the experiment would stop at interim 2 and 10 scores would have been made for each agent. By comparison, in Section 5.3 we showed that without early accept, AdaStop uses 30 scores for DDPG and TRPO. In this example, early stopping saves a lot of computations, and results in a significant speed-up without affecting the final decisions.

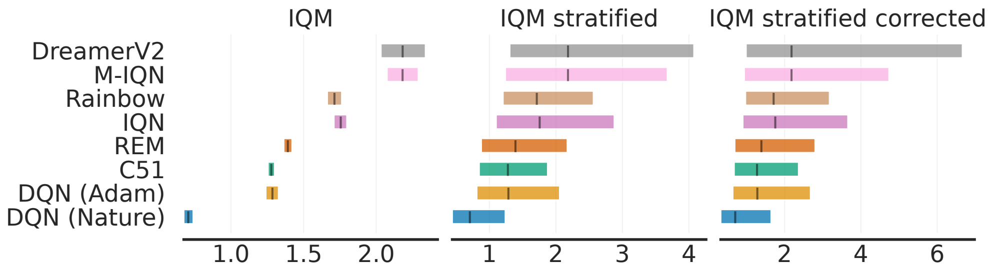

Figure 6: 95% confidence intervals on the IQM of the "Human Normalized Score" of a set of algorithms with various methods using the results from (Agarwal et al., 2021).

G Agent comparison on Atari environments

In this section, we discuss the approach of Agarwal et al. (2021) on the problem of comparing RL agents on Atari environments. The methodology from Agarwal et al. (2021) prescribes to give confidence intervals around some measure of location (typically mean or interquartile mean, IQM) of the agents' score. We demonstrate this approach in the left and middle sub-figures of Figure 6. However, it is not clear how to draw a conclusion from such graphs (see (Cumming & Finch, 2005) for a discussion on doing inference with confidence intervals). Moreover, there is no definite criterion on how to construct the confidence interval. For example, this approach gives rise to three very different confidence interval plots in Figure 6: the plots on the left and in the middle are the ones presented in (Agarwal et al., 2021) and the plot on the right is a modified version of the plot in the middle, where we added the error of performing multiple comparisons.

In the first plot of Figure 6, it is easy to draw conclusions but on the other hand the theoretical interpretation is not clear because the inter-game randomness is not taken into account. The second and third plots in Figure 6 are a naive approach to the problem that assumes that all the games are very different from one another, and as such the confidence intervals are too large to draw conclusions. We think that there should be an approach in the middle that considers a cluster of similar games (all the easy games, all the maze-like games, all the difficult games, etc.) and treat these games as having all the same law. This approach would produce smaller (and more interpretable) confidence intervals compared to the naive approach, providing a middle-ground between the first plot and second plot.

H Implementation details and additional plots

H.1 Complementary experiment for Section 5.1

In this third example, we suppose we have 10 agents which score distributions are listed in Fig. 7, where the first column indicates the labels of the agents as they are used in Fig. 8.

Similarly to Cases 1 and 2 (see Section 5.1), we execute AdaStop with $K = 5$, $N = 5$, $\alpha = 0.05$. As indicated in Section 4, we do not enumerate all the permutations for our permutation test as this would be too exphensive and instead we use 10^4 randomly selected permutations to compute our test statistic at each interim.

In contrast to Cases 1 and 2, in Case 3 we use early accept (with $\beta = 0.01$) to avoid situations when all agents are compared with the maximum number of scores, *i.e.* NK scores, which may occur when each agent has a similar distribution to at least one another agent in comparison.

We show the performance of AdaStop for multiple agents comparison in Fig. 8, which corresponds to the output of one execution of AdaStop. The table (left) summarizes the decisions of the algorithm for every pair of comparisons, and violin plots (right) reflect empirically measured distributions in the comparison. From this figure, we can see that almost all agents are grouped in clusters of distributions with equal means, except for *MG3 that is assigned to two different groups at the same time. Interestingly, except for *MG3,

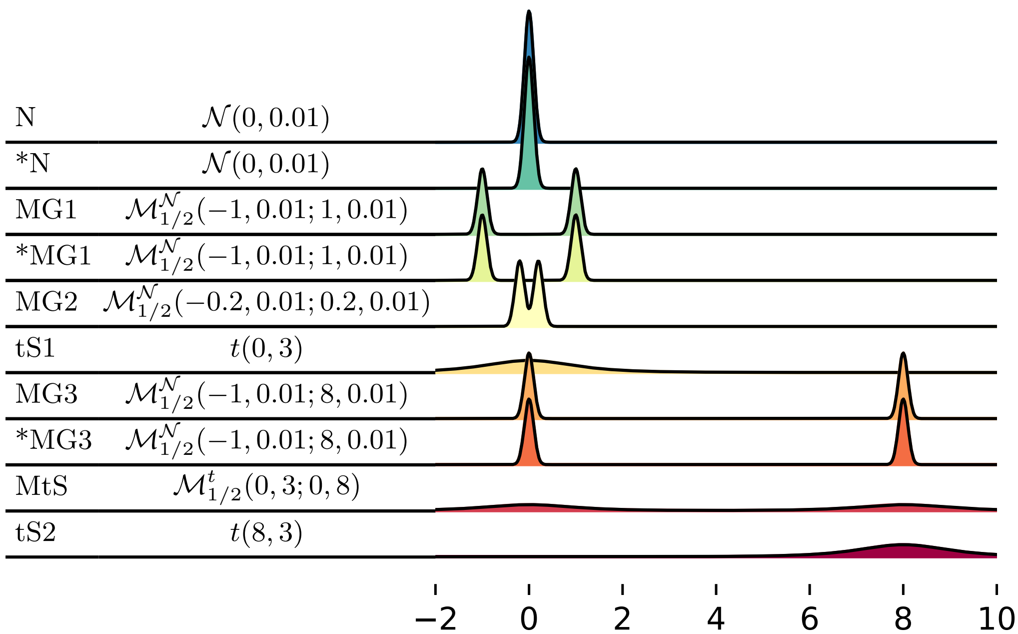

Figure 7: Toy example 3, with an illustration of the involved distributions.

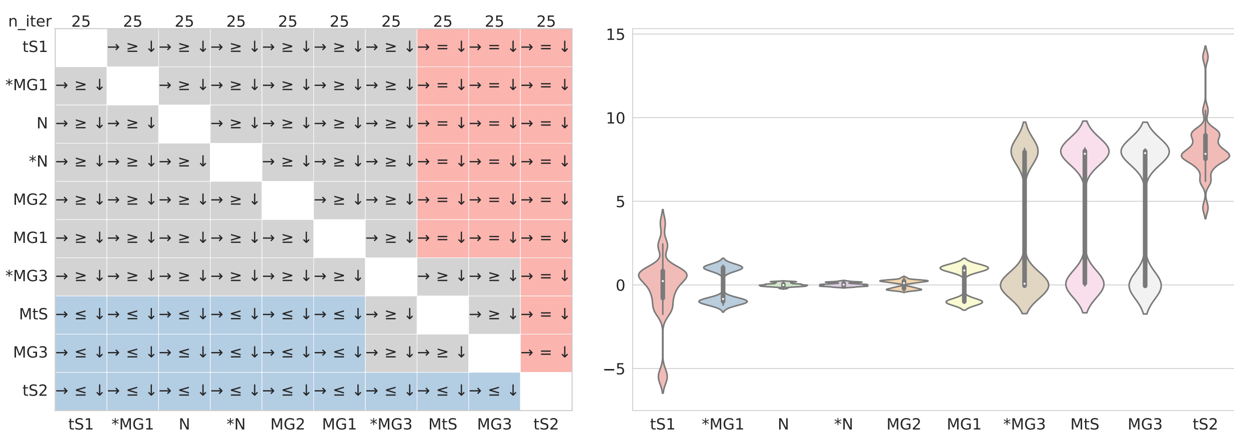

Figure 8: Case 3. AdaStop decision table (left) and measured empirical distributions (right).

these clusters are correctly formed. Moreover, similarly to the two previous cases, we have executed AdaStop $M = 5\ 000$ times to measure FWE of the test. The empirical measurements are 0.0178 of rejection rate of at least one correct hypothesis when comparing distributions and 0.0472 of rejection rate when comparing means: both are below 0.05. This example illustrates the fact that AdaStop can be efficiently used to compare the score of several agents simultaneously.

H.2 Additional plot for Section 5.2

H.3 MuJoCo Experiments

In this section, we go into details about the experimental setup of the MuJoCo experiments, as well as present additional plots.

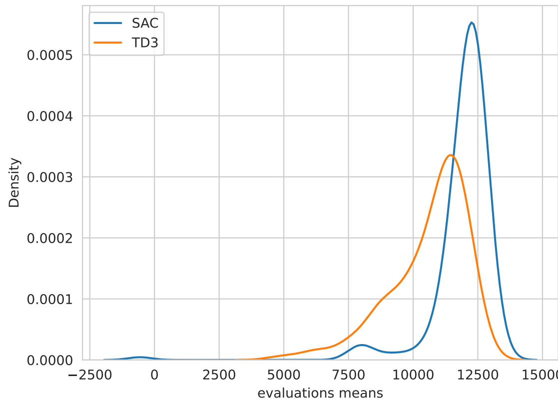

Figure 9: Scores distributions for a SAC and a TD3 agents on HalfCheetah obtained with 192 independent scores, each made of 2 million steps.

	DDPG	TRPO	PPO	SAC
γ	0.99	0.99	0.99	0.99
Learning Rate	1×10^{-3}	1×10^{-3}	3×10^{-4}	3×10^{-4}
Batch Size	128	64	64	256
Buffer Size	10^6	1024	2048	10^6
Value Loss	MSE	MSE	AVEC (Flet-Berliac et al., 2021)	MSE
Use gSDE	No	No	No	Yes
Entropy Coef.	-	0	0	`auto`
GAE λ	-	0.95	0.95	-
Advantage Norm.	-	Yes	Yes	-
Target Smoothing	0.005	-	-	0.005
Learning Starts	10^4	-	-	10^4
Policy Frequency	32	-	-	-
Exploration Noise	0.1	-	-	-
Noise Clip	0.5	-	-	-
Max KL	-	10^{-2}	-	-
Line Search Steps	-	10	-	-
CG Steps	-	100	-	-
CG Damping	-	10^{-2}	-	-
CG Tolerance	-	10^{-10}	-	-
LR Schedule	-	-	Linear to 0	-
Clip ϵ	-	-	0.2	-
PPO Epochs	-	-	10	-
Value Coef.	-	-	0.5	-
Train Freq.	-	-	-	1 step
Gradient Steps	-	-	-	1

Table 3: Hyperparameters used for the MuJoCo experiments.

Hyperparameters. Table 3 details the hyperparameters used with each Deep RL on the MuJoCo benchmark. For all agents, we use a budget of one million time steps for HalfCheetah-v3, Hopper-v3, and Walker2d-v3, and a budget of two million time steps for Ant-v3 and Humanoid-v3. We use a maximum horizon of one thousand steps for all environments.

Scores. Agents' scores are constructed by stopping the training procedure on predetermined time steps and averaging the results of 50 evaluation episodes.

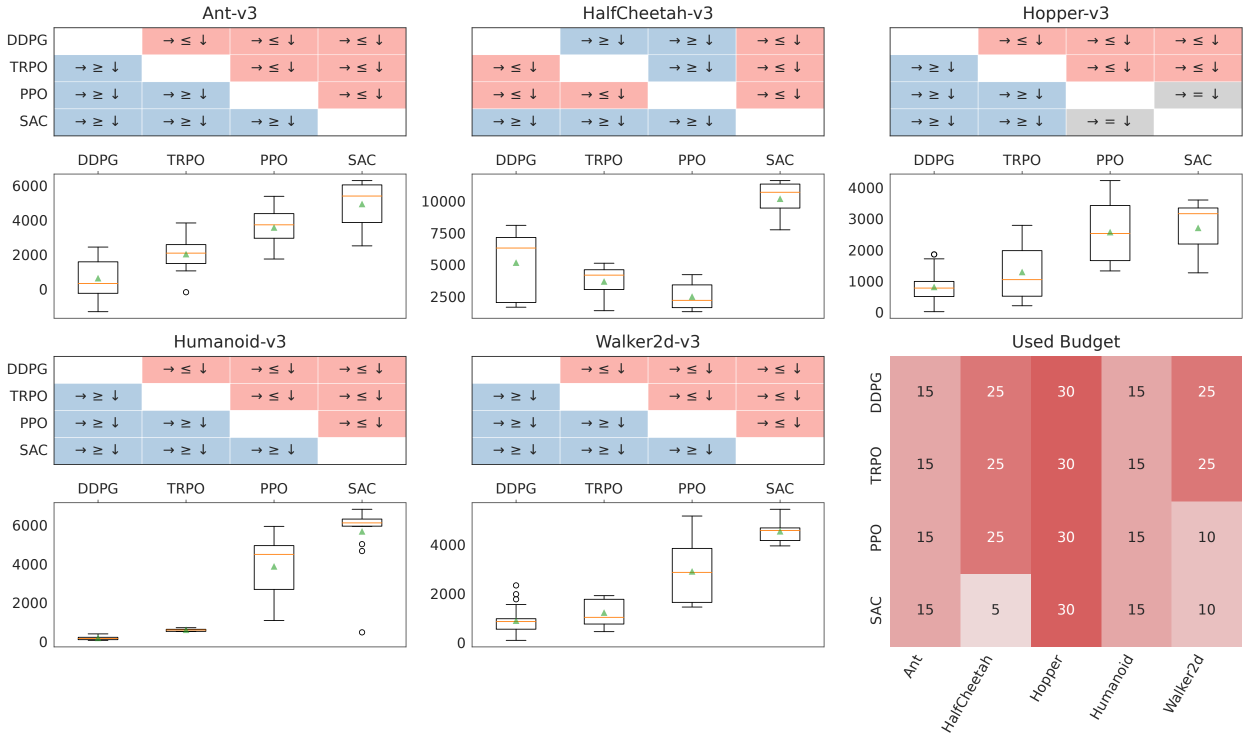

Figure 10: AdaStop decision tables (top) and score distributions (bottom) for each MuJoCo environment, and the budget used to make these decisions (bottom right). The medians are represented as the green triangles and the means as the horizontal orange lines.

Learning Curves. Fig. 11 presents sample efficiency curves for all algorithms in each environment. The shaded areas represent 95% bootstrapped confidence intervals, computed using `rliable` (Agarwal et al., 2021). Note that each curve may be an aggregation of a different number of scores, which can be found in the bottom right of Fig. 10.

Additional Comparison Plots. Fig. 10 expands upon the comparisons given in the main text (in Fig. 5) by also plotting the score distributions of each agent using boxplots.

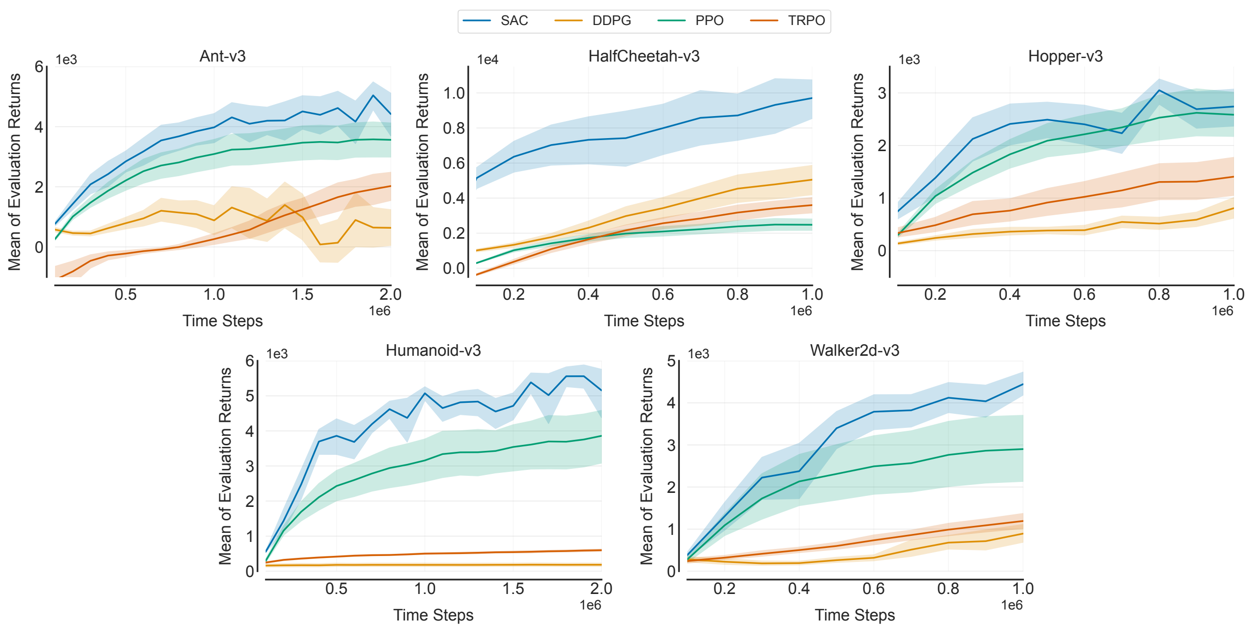

Figure 11: Mean of score Returns with 95% stratified bootstrap CIs. Note that curves in the same figure may use a different number of scores, depending on when AdaStop made the decisions.