Adaptivity Complexity for Causal Graph Discovery

Abstract

Causal discovery from interventional data is an important problem, where the task is to design an interventional strategy that learns the hidden ground truth causal graph $G(V, E)$ on $|V| = n$ nodes while minimizing the number of performed interventions. Most prior interventional strategies broadly fall into two categories: non-adaptive and adaptive. Non-adaptive strategies decide on a single fixed set of interventions to be performed while adaptive strategies can decide on which nodes to intervene on sequentially based on past interventions. While adaptive algorithms may use exponentially fewer interventions than their non-adaptive counterparts, there are practical concerns that constrain the amount of adaptivity allowed. Motivated by this trade-off, we study the problem of r-adaptivity, where the algorithm designer recovers the causal graph under a total of r sequential rounds whilst trying to minimize the total number of interventions. For this problem, we provide a r-adaptive algorithm that achieves $\mathcal{O}(\min\{r, \log n\} \cdot n^{1/\min\{r,\log n\}})$ approximation with respect to the verification number, a well-known lower bound for adaptive algorithms. Furthermore, for every r, we show that our approximation is tight. Our definition of r-adaptivity interpolates nicely between the non-adaptive ($r = 1$) and fully adaptive ($r = n$) settings where our approximation simplifies to $\mathcal{O}(n)$ and $\mathcal{O}(\log n)$ respectively, matching the best-known approximation guarantees for both extremes. Our results also extend naturally to the bounded size interventions.

1 INTRODUCTION

Learning causal relationships from data is a fundamental problem that has found applications across various scientific disciplines, including biology [King et al., 2004, Cho et al., 2016, Tian, 2016, Sverchkov and Craven, 2017, Rotmensch et al., 2017, Pingault et al., 2018, de Campos et al., 2019], epidemiology, philosophy [Reichenbach, 1956, Woodward, 2005, Eberhardt and Scheines, 2007], and econometrics [Hoover, 1990, Rubin and Waterman, 2006]. Directed acyclic graphs (DAGs) are a popular choice to model causal relationships and it is well known that using observational data, the causal structure can only be learned up to its Markov equivalence class (MEC) and additional assumptions or interventional data is required for the recovery task. Here, we focus our attention on causal discovery using interventions. As interventions are often costly, our objective is to minimize interventions during the recovery process.

There is a rich literature on causal discovery from interventional data, and causal discovery algorithms can be broadly classified into two categories: adaptive [Shanmugam et al., 2015, Greenewald et al., 2019, Squires et al., 2020, Choo et al., 2022, Choo and Shiragur, 2023] versus non-adaptive [Eberhardt et al., 2005, 2006, Eberhardt, 2010, Hu et al., 2014]. Given an essential graph, *non-adaptive* algorithms have to decide beforehand a collection of interventions such that *any* plausible causal graph can be recovered while *adaptive* algorithms can decide on interventions sequentially while using information gleaned from past interventions.

Adaptive algorithms are powerful and the interventional cost of an optimal adaptive algorithm can even be exponentially better than any non-adaptive algorithms in some cases[1]. However, the sequential nature of adaptive algorithms fundamentally hinders parallelization [Dean et al., 2008, Balkanski and Singer, 2018] and may even be practically infeasible, e.g. hard constraints like timed deadlines

[1] On tree causal graphs, $\Omega(n)$ non-adaptive interventions are needed in the worst case while $\mathcal{O}(\log n)$ adaptive ones suffice.

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may restrict how many rounds of adaptivity one can afford to do whilst minimizing the number of interventions.

The fundamental concept of adaptivity has been studied across a wide spectrum of problems in computer science including parellel algorithms [Valiant, 1975, Cole, 1988, Braverman et al., 2016], communication complexity [Papadimitriou and Sipser, 1982, Duris et al., 1984, Nisan and Widgerson, 1991, Alon et al., 2015], multi-armed bandits [Agarwal et al., 2017], sparse recovery problems [Malioutov et al., 2008, Indyk et al., 2011, Haupt et al., 2009], and many more [Scarlett, 2018, Canonne, Clément L and Gur, Tom, 2018, Raskhodnikova and Smith, 2006]. Typically, one can obtain a lower objective cost if more rounds of adaptivity is allowed. This tradeoff between cost and adaptivity is well-studied and researchers have tried to characterize the complexity of adaptivity for various problems. For instance, the adaptivity gap/complexity has been studied for knapsack [Dean et al., 2008], submodular optimization [Balkanski and Singer, 2018], and stochastic probing [Gupta et al., 2016, 2017].

In this work, we study the tradeoff between cost and adaptivity for causal graph discovery as described below.

Problem setup Given r adaptivity rounds and a partially oriented observational essential graph of an underlying causal graph $G^* = (V, E)$, adaptively design[2] intervention sets $\mathcal{I}_1, \ldots, \mathcal{I}_r \subseteq 2^V$ such that all arc orientations in G^* will be recovered after intervening on them while minimizing the total number of interventions performed.

Formally, we require $\mathcal{E}_{\mathcal{I}_1 \cup \ldots \cup \mathcal{I}_r}(G^*) = G^*$ while minimizing $\cup_{i=1}^{r} |\mathcal{I}_i|$, where $\mathcal{E}_{\mathcal{I}}(G^*)$ denotes the partially oriented interventional essential graph after performing interventions $\mathcal{I}$. Furthermore, each intervention $I \in \mathcal{I}_i$ in any intervention set $\mathcal{I}_i$ has size $|I| \leq k$. Atomic interventions when $k = 1$ and bounded size interventions when $k > 1$.

1.1 CONTRIBUTIONS

Under standard assumptions of causal sufficiency, faithfulness and infinite sample regime, we provide a r-adaptive search algorithm that recovers the ground truth causal graph by performing at most $\mathcal{O}(\min\{r, \log n\} \cdot n^{1/\min\{r, \log n\}} \cdot \nu_1(G^*))$ atomic interventions, where $\nu_1(G^*)$ is the atomic verification number of G^* [Squires et al., 2020, Choo et al., 2022, Choo and Shiragur, 2023][3], a natural lower bound for the adaptive algorithms. See Definition 6 for definition of $\nu_1(G^*)$ and $\nu_k(G^*)$, the bounded size analog of $\nu_1(G^*)$.

Theorem 1 (Atomic upper bound)**.** *Let $\mathcal{E}(G^*)$ be the observational essential graph of an underlying causal DAG G^* on n nodes. There is a polynomial time r-adaptive algorithm that uses $\mathcal{O}(\min\{r, \log n\} \cdot n^{1/\min\{r, \log n\}} \cdot \nu_1(G^*))$ atomic interventions to recover G^* from $\mathcal{E}(G^*)$.*

When $r = 1$, the upper bound becomes $\mathcal{O}(n)$, which is worst case optimal since $\Omega(n)$ non-adaptive interventions are necessary when the given essential graph is a path on n vertices. Meanwhile, when $r = \log n$, the upper bound becomes $\mathcal{O}(\log n \cdot \nu_1(G^*))$, matching the upper bound guarantees of Choo et al. [2022]. Again, this is worst case optimal when the given essential graph is a path on n vertices.

In fact, for any r, our approximation factor is tight. We formally show this in the following lower bound result.

Theorem 2 (Atomic worst case)**.** *In the worst case, any r-adaptive algorithm needs to use at least $\Omega(\min\{r, \log n\} \cdot n^{1/\min\{r, \log n\}} \cdot \nu_1(G^*))$ atomic interventions against an adaptive adversary.*

We also extend this to bounded size interventions where each intervention involves at most k vertices, for some predetermined upper bound $k \geq 1$.

Theorem 3 (Bounded upper bound)**.** *Let $\mathcal{E}(G^*)$ be the observational essential graph of an underlying causal DAG G^* on n nodes. There is a polynomial time r-adaptive algorithm that uses $\mathcal{O}(\min\{r, \log n\} \cdot n^{1/\min\{r, \log n\}} \cdot \log k \cdot \nu_k(G^*))$ bounded sized interventions to recover G^* from $\mathcal{E}(G^*)$, where each intervention involves at most $k > 1$ vertices.*

While the approximation ratio is worse, note that we are comparing against $\nu_k(G^*)$. Since using bounded size interventions typically translates to using a smaller number of interventions performed as k increases, $\nu_k(G^*)$ could roughly be k times smaller than $\nu_1(G^*)$.

Remark Since non-adaptive results and the upper bound guarantees of Choo et al. [2022] already matches our upper bounds for regime of $r \geq \log n$, it suffices for us to design and analyze algorithms for regime of $1 < r < \log n$.

1.2 OUTLINE

After giving preliminaries and related work in Section 2, we slowly build up intuition towards our main results. We first solve the problem using atomic interventions when the input graph is a path or a tree (Section 3) before solving the problem in full generality on any graph input, and also generalizing to the setting of bounded size interventions (Section 4). Then, we conclude with some interesting future work directions in Section 5. For a cleaner exposition, some details and full proofs are deferred to the appendix.

[2] The decision of $\mathcal{I}_i$ may depend on outcomes after recovering arc orientations from intervening on $\mathcal{I}_1, \ldots, \mathcal{I}_{i-1}$.

[3] Given a MEC of an unknown ground truth causal graph G^* and a graph G from the MEC, the goal of the verification problem is determining whether G is G^*. By plugging in G with G^* in the verification problem, we see that the optimal solution to the verification is a natural lower bound for the search problem.

2 PRELIMINARIES

We write $\{1, \ldots, n\}$ as $[n]$ and use standard asymptotic notations such as $\mathcal{O}(\cdot)$ and $\Omega(\cdot)$.

2.1 GRAPH NOTATIONS

Let $G = (V, E)$ be a graph on $|V| = n$ vertices. We use $V(G)$, $E(G)$ and $A(G) \subseteq E(G)$ to denote its vertices, edges, and oriented arcs respectively. The graph G is said to be directed or fully oriented if $A(G) = E(G)$, and partially oriented otherwise. For any two vertices $u, v \in V$, we write $u \sim v$ if these vertices are connected in the graph and $u \not\sim v$ otherwise. To specify the arc directions, we use $u \to v$ or $u \leftarrow v$. We use $G[V']$ to denote the vertex-induced subgraph for any subset $V' \subseteq V$ of vertices.

A *clique* is a graph where $u \sim v$ for any pair of vertices $u, v \in V$. A *maximal clique* is an vertex-induced subgraph of a graph that is a clique and ceases to be one if we add any other vertex to the subgraph. For an undirected graph G, $\omega(G)$ refers to the size of its maximum clique.

The *skeleton* $skel(G)$ of a (partially oriented) graph G is the underlying graph where all edges are made undirected. A *v-structure* refers to three distinct vertices $u, v, w \in V$ such that $u \to v \leftarrow w$ and $u \not\sim w$. A simple cycle is a sequence of $k \geq 3$ vertices where $v_1 \sim v_2 \sim \ldots \sim v_k \sim v_1$. The cycle is partially directed if at least one of the edges is directed and all directed arcs are in the same direction along the cycle. A partially directed graph is a *chain graph* if it contains no partially directed cycle. In the undirected graph $G[E \setminus A]$ obtained by removing all arcs from a chain graph G, each connected component in $G[E \setminus A]$ is called a *chain component*. We use $CC(G)$ to denote the set of chain components, where each $H \in CC(G)$ is a subgraph of G and $V = \dot{\cup}_{H \in CC(G)} V(H)$.

2.2 CHORDAL GRAPHS

An undirected graph is chordal if every cycle of length at least 4 has an edge connecting two non-adjacent vertices of the cycle. There are many known characterizations and properties of chordal graphs; see Blair and Peyton [1993] for an introduction. One of the most common characterization is the following: A graph G is chordal if and only if perfect elimination ordering (PEO)[4] exists [Fulkerson and Gross, 1965]. Furthermore, a PEO can be computed in linear time via lexicographic BFS [Rose et al., 1976] and can used to prove the following lemma which implies that a chordal graph on n nodes has at most n maximal cliques.

Lemma 4 (Fulkerson and Gross [1965]; Lemma 6 in Blair and Peyton [1993])**.** *The set of maximal cliques of a graph G is precisely the sets $\{v_i\} \cup (\{v_{i+1}, \ldots, v_n\} \cap N(v_i))$ for which $\{v_i\} \cup (\{v_{i+1}, \ldots, v_n\} \cap N(v_i))$ is* not *in $\{v_j\} \cup (\{v_{j+1}, \ldots, v_n\} \cap N(v_j))$ for any vertex v_j with $j < i$.*

It is known that chordal graphs have a clique tree representation. One way to construct a clique tree T_G from a chordal graph G in polynomial time is via the "maximum-weight spanning tree property": Let the vertex set of T_G be all maximal cliques of G, assign edge weight as the size of intersection between every pair of maximal cliques, and then compute the *maximum* weight spanning tree.

The following result is one of the many useful properties of clique trees which we exploit in our algorithm later.

Lemma 5 (Lemma 5 of Blair and Peyton [1993])**.** *Let $T_G = (K, S)$ be the clique tree of a chordal graph $G = (V, E)$. For any two adjacent maximal cliques K_i and K_j in T_G, let T_i and T_j be the subtrees obtained by removing the edge $\{K_i, K_j\}$ from T_G. Then, vertices v_i and v_j are disconnected in $G[V \setminus (V(K_i) \cap V(K_j))]$.*

2.3 CAUSAL GRAPH BASICS

Directed acyclic graphs (DAGs), a special case of chain graphs where *all* edges are directed, are commonly used as graphical causal models [Pearl, 2009] where vertices represents random variables and the joint probability density f factorizes according to the Markov property: $f(v_1, \ldots, v_n) = \prod_{i=1}^{n} f(v_i \mid pa(v_i))$, where $pa(v_i)$ is the values taken by v_i's parents in the DAG.

For any DAG G, we denote its *Markov equivalence class* (MEC) by $[G]$ and *essential graph* by $\mathcal{E}(G)$. Two graphs are Markov equivalent if and only if they have the same skeleton and v-structures [Verma and Pearl, 1990, Andersson et al., 1997]. DAGs in the same MEC have the same skeleton; Essential graph is a partially directed graph such that an arc $u \to v$ is directed if $u \to v$ in *every* DAG in MEC $[G]$, and an edge $u \sim v$ is undirected if there exists two DAGs $G_1, G_2 \in [G]$ such that $u \to v$ in G_1 and $v \to u$ in G_2.

An *intervention* $S \subseteq V$ is an experiment where all variables $s \in S$ are forcefully set to some value, independent of the underlying causal structure. An intervention is *atomic* if $|S| = 1$ and *bounded* if $|S| \leq k$ for some $k > 0$; observational data is a special case where $S = \emptyset$. The effect of interventions is formally captured by Pearl's do-calculus [Pearl, 2009]. We call any $\mathcal{I} \subseteq 2^V$ a *intervention set*: an intervention set is a set of interventions where each intervention corresponds to a subset of variables. An *ideal intervention* on $S \subseteq V$ in G induces an interventional graph G_S where all incoming arcs to vertices $v \in S$ are removed [Eberhardt et al., 2012]. It is known that intervening on S allows us to infer the edge orientation of any edge cut by S and $V \setminus S$ [Eberhardt, 2007, Hyttinen et al., 2013, Hu et al., 2014, Shanmugam et al., 2015, Kocaoglu et al., 2017].

[4] An ordering σ is a PEO if for $1 \leq i \leq n$, the node-induced subgraph $G[\{v_1, \ldots, v_{i-1}\} \cap N(v_i)]$ on v_i's neighbors is a clique.

For ideal interventions, an $\mathcal{I}$-essential graph $\mathcal{E}_{\mathcal{I}}(G)$ of G is the essential graph representing the Markov equivalence class of graphs whose interventional graphs for each intervention is Markov equivalent to G_S for any intervention $S \in \mathcal{I}$. There are several known properties about $\mathcal{I}$-essential graph properties (e.g. see [Hauser and Bühlmann, 2012, 2014]). For instance, every $\mathcal{I}$-essential graph is a chain graph with chordal chain components; this includes the case of $S = \emptyset$. Also, orientations in one chain component do not affect orientations in other components. Thus, to fully orient any essential graph $\mathcal{E}(G^*)$, it is necessary and sufficient to orient every chain component in $\mathcal{E}(G^*)$.

A *verifying set* $\mathcal{I}$ for a DAG $G \in [G^*]$ is an intervention set that fully orients G from $\mathcal{E}(G^*)$, possibly with repeated applications of Meek rules (see Appendix A). In other words, for any graph $G = (V, E)$ and any verifying set $\mathcal{I}$ of G, we have $\mathcal{E}_{\mathcal{I}}(G)[V'] = G[V']$ for *any* subset of vertices $V' \subseteq V$. Furthermore, if $\mathcal{I}$ is a verifying set for G, then $\mathcal{I} \cup S$ is also a verifying set for G for any additional intervention $S \subseteq V$. While DAGs may have multiple verifying sets in general, we are often interested in finding one with minimum size.

Definition 6 (Verifying set and verifying number)**.** An intervention set $\mathcal{I}$ is called a verifying set for a DAG G^* if $\mathcal{E}_{\mathcal{I}}(G^*) = G^*$. $\mathcal{I}$ is a *minimum size verifying set* if $\mathcal{E}_{\mathcal{I}'}(G^*) \neq G^*$ for any $|\mathcal{I}'| < |\mathcal{I}|$. The *verification number* $\nu_k(G^*)$ denotes the size of the minimum size verifying set of G^* when each intervention has size at most $k \geq 1$.

Recently, Choo et al. [2022] proved two fundamental results with respect to verification number. Theorem 7 characterizes verification number of any given causal DAG while Theorem 8 gives an adaptive search algorithm that is competitive to the verification number of the underlying causal DAG.

Theorem 7 (Choo et al. [2022])**.** *Fix an essential graph $\mathcal{E}(G^*)$ and $G \in [G^*]$. An atomic intervention set $\mathcal{I}$ is a minimal sized verifying set for G if and only if $\mathcal{I}$ is a minimum vertex cover of covered edges $\mathcal{C}(G)$ of G. A minimal sized atomic verifying set can be computed in polynomial time since the edge-induced subgraph on $\mathcal{C}(G)$ is a forest.*

Theorem 8 (Choo et al. [2022])**.** *Fix an unknown underlying DAG G^*. Given an essential graph $\mathcal{E}(G^*)$ and intervention set bound $k \geq 1$, there is a deterministic polynomial time algorithm that computes an intervention set $\mathcal{I}$ adaptively such that $\mathcal{E}_{\mathcal{I}}(G^*) = G^*$, and $|\mathcal{I}|$ has size*
1. $\mathcal{O}(\log(n) \cdot \nu_1(G^*))$ when $k = 1$
2. $\mathcal{O}(\log(n) \cdot \log(k) \cdot \nu_k(G^*))$ when $k > 1$.

To obtain a competitive bound with respect to $\nu_1(G^*)$, Choo et al. [2022] proved a stronger (but non-computable) lower bound Theorem 8 on $\nu_1(G^*)$. We will rely on this lemma to show a competitive bound for our algorithm later.

Lemma 9. *For any causal DAG G^*,*

$$\nu_1(G^*) \geq \max_{\mathcal{I} \subseteq V} \sum_{H \in CC(\mathcal{E}_{\mathcal{I}}(G^*))} \left\lfloor \frac{\omega(H)}{2} \right\rfloor$$

For any intervention set $\mathcal{I} \subseteq 2^V$, we write $R(G, \mathcal{I}) \subseteq E$ to mean the set of oriented arcs in the $\mathcal{I}$-essential graph of a DAG G, and define $G^{\mathcal{I}} = G[E \setminus R(G, \mathcal{I})]$ as the *fully directed* subgraph DAG induced by the *unoriented arcs* in G, where $G^{\emptyset}$ is the graph obtained after removing all the oriented arcs in the observational essential graph due to v-structures. The next result explains why it suffices to study causal graph discovery via interventions on causal graphs without v-structures: since $R(G, \mathcal{I}) = R(G^{\emptyset}, \mathcal{I}) \dot{\cup} R(G, \emptyset)$, any oriented arcs in the observational graph can be removed *before performing any interventions* as the optimality of the solution is unaffected.

Theorem 10 (Choo and Shiragur [2023])**.** *For any DAG $G = (V, E)$ and intervention sets $\mathcal{A}, \mathcal{B} \subseteq 2^V$,*

$$\begin{aligned} &R(G, \mathcal{A} \cup \mathcal{B}) \\ &\quad = R(G^{\mathcal{A}}, \mathcal{B}) \dot{\cup} R(G^{\mathcal{B}}, \mathcal{A}) \dot{\cup} (R(G, \mathcal{A}) \cap R(G, \mathcal{B})) \end{aligned}$$

We will also borrow the following definition of *relevant nodes* from Choo and Shiragur [2023].

Definition 11 (Relevant nodes)**.** Fix a DAG $G^* = (V, E)$ and arbitrary subset $V' \subseteq V$. For any intervention set $\mathcal{I} \subseteq V$ and resulting interventional essential graph $\mathcal{E}_{\mathcal{I}}(G^*)$, we define the *relevant nodes* $\rho(\mathcal{I}, V') \subseteq V'$ as the set of nodes within V' that is adjacent to some unoriented arc within the node-induced subgraph $\mathcal{E}_{\mathcal{I}}(G^*)[V']$.

2.4 RELATED WORK

There are two broad classes of causal graph discovery algorithsm. Given the observational essential graph, non-adaptive algorithms need to recover any underlying causal graph using a single fixed set of interventions while adaptive algorithms are allowed to adapt their interventional decisions based on the outcomes of earlier interventions.

Non-adaptive search Separating systems are the central mathematical objects for non-adaptive intervention design. Roughly speaking, a separating system on a set of elements is a collection of subsets such that for every pair of elements from the set, there exists at least one subset which contains exactly one element from the pair. Instead of all pairs of elements, let us consider the (typically smaller) G-separating system for a given undirected graph G.

Definition 12 (G-separating system; Definition 3 of Kocaoglu et al. [2017])**.** Given an undirected graph $G = (V, E)$, a set of subsets $\mathcal{I} \subseteq 2^V$ is a G-separating system if for every edge $\{u, v\} \in E$, there exists $I \in \mathcal{I}$ such that either ($u \in I_i$ and $v \notin I_i$) or ($u \notin I_i$ and $v \in I_i$).

To simplify notation, we write G-separating system to refer to $\mathrm{skel}(G)$-separation system or $\mathrm{skel}(\mathcal{E}(G))$-separation system, for any causal DAG G. It is known that the optimal non-adaptive intervention set to learn a moral DAG G^* is a G^*-separating system [Kocaoglu et al., 2017].

Theorem 13 (Theorem 1 of Kocaoglu et al. [2017])**.** *For any undirected graph G, an intervention set $\mathcal{I}$ learns* every *possible causal graph D with $\mathrm{skel}(D) = G$ if and only if $\mathcal{I}$ is a G-separating system.*

Adaptive search There exists essential graphs[5] where non-adaptive interventions require exponentially more interventions than if one could use adaptive interventions.

Adaptive search has been studied for special graph classes by Shanmugam et al. [2015], Greenewald et al. [2019], Squires et al. [2020]. More recently, Choo et al. [2022] showed that $\mathcal{O}(\log n \cdot \nu_1(G^*))$ atomic inteventions suffices to fully recover for any general causal graph G^* on n nodes while Choo and Shiragur [2023] showed that $\mathcal{O}(\log |\rho(\emptyset, V(H))| \cdot \nu_1(G^*))$ atomic inteventions suffices to fully recover edge directions within a subgraph H of G^*. Both papers also gave results in terms of bounded size interventions where their guarantees incur an additional $\mathcal{O}(\log k)$ factor when comparing to $\nu_k(G^*)$ by invoking Lemma 14 suitably.

Lemma 14 (Lemma 1 of Shanmugam et al. [2015])**.** *Let (n, k, a) be parameters where $k \leq n/2$. There is a polynomial time labeling scheme that produces distinct ℓ length labels for all elements in $[n]$ using letters from the integer alphabet $\{0\} \cup [a]$ where $\ell = \lceil \log_a n \rceil$. In every label index, any integer letter is used at most $\lceil n/a \rceil$ times. This labelling scheme is a separating system: for any $i, j \in [n]$, there exists some digit $d \in [\ell]$ where the labels of i and j differ.*

3 WARMUP: PATHS AND TREES

When the essential graph $\mathcal{E}(G^*)$ is a path or tree, there are n possible DAGs, each associated with setting a node as a root and orienting all edges away from it. It is known that $\nu(G^*) = 1$ because the covered edges for each DAG are precisely the edges incident to the hidden root (e.g. see Greenewald et al. [2019], Choo et al. [2022]).

In this section, we investigate how to optimally solve the r-adaptivity on these special classes of graphs.

3.1 PATHS

When the adaptivity parameter $r = 1$, any 1-adaptive algorithm behaves *exactly* like a non-adaptive algorithm. Thus, it is necessary and sufficient to intervene on a G^*-seperating system, and such a system on a path has size $\Theta(n)$.

Meanwhile, when the adaptive parameter $r = 2$, we already see an interesting trade-off occurring: *how much (and how) should we intervene now versus later?* Our next lemma tells us how to balance the number of interventions done in the first and second adaptive rounds in a worst case fashion.

Lemma 15. *Suppose $\mathcal{E}(G^*)$ is a path on n vertices. There is a 2-adaptive algorithm that uses at most $\mathcal{O}(\sqrt{n})$ interventions in total. In the worst case, $\Omega(\sqrt{n})$ is necessary for any 2-adaptive algorithm.*

Proof. Without loss of generality, up to the inclusion of floors and ceilings, let us assume that n is a square number.

Suppose the vertices on the path are labelled $v_1, v_2, \ldots, v_n$ where $v_i \sim v_{i+1}$ for $1 \leq i \leq n-1$.

In the first round, we intervene on $\sqrt{n}$ evenly spaced vertices: $v_{\sqrt{n}}, v_{2\sqrt{n}}, v_{3\sqrt{n}}, \ldots, v_n$. After applying Meek's rule R1, at most one segment of $v_{i\cdot\sqrt{n}}, v_{i\cdot\sqrt{n}+1}, v_{i\cdot\sqrt{n}+2}, \ldots, v_{(i+1)\cdot\sqrt{n}}$ will remain unoriented. That is, the total number of relevant vertices is at most $\sqrt{n}$. In the second round, we intervene on all these relevant vertices, incurring a total of $\mathcal{O}(\sqrt{n})$ interventions.

For the worst case lower bound, consider any arbitrary 2-adaptive algorithm A. If A makes strictly less than $\sqrt{n}$ interventions in the first round, then there exists a consecutive sequence of $\sqrt{n}$ vertices $v_i, v_{i+1}, \ldots, v_{i+\sqrt{n}}$ which is not intervened by A. If the root node of the path was v_i, then this entire segment $v_i \sim v_{i+1} \sim \ldots \sim v_{i+\sqrt{n}}$ remains unoriented after the first round of interventions. Thus, in the second round, A needs to at least intervene on a separating system of this segment, which has a size at least $\Omega(\sqrt{n})$. Therefore, in the worst case, any 2-adaptive algorithm needs to perform at least $\Omega(\sqrt{n})$ number of interventions to orient $\mathcal{E}(G^*)$ when it is a path on n vertices. □

The key technical algorithmic idea in Lemma 15 is the strategy of "balanced partitioning", which balances the worst case size of the largest possible unoriented component after a round of intervention. Interestingly, we can generalize this "balanced partitioning" strategy to larger values of adaptivity parameter r as follows: Suppose we perform L interventions per round in the first $r-1$ rounds and then intervene on the G-separating system on the remaining relevant vertices in the final round, where L is an integer that depends on r which we define later. In each of the $r-1$ rounds, we will choose the L vertices judiciously by a "balanced partitioning". If we further insist that the final G-separating system has size at most L, then we get a recurrence relation $\frac{n}{(L+1)^{r-1}} \leq L$ while incurring a total of $\approx r \cdot L$ interventions. We formally prove this in the next lemma.

[5] If the essential graph is an undirected path on n vertices, then a G-sparating system has size $\Omega(n)$ while adaptive search only requires $\mathcal{O}(\log n)$ atomic interventions by "binary search".

Lemma 16. *Suppose $\mathcal{E}(G^*)$ is a path on n vertices. There is a r-adaptive algorithm that uses at most $\mathcal{O}(r \cdot n^{1/r})$ interventions in total.*

Proof. Observe that no matter how we intervene on a path, the remaining relevant vertices will form a subpath.

Let $L = \lceil n^{1/r} \rceil$. In the first $r-1$ rounds, we will partition the remaining unoriented subpath into equal length segments. For instance, if the subpath length is currently ℓ, then we partition it into segments of length $\lceil \ell/(L+1) \rceil$ or $\lfloor \ell/(L+1) \rfloor$. Thus, the length of the subpath with relevant vertices in the next round will be at most $\lceil \ell/(L+1) \rceil$.

After $r-1$ rounds, the length of the remaining unoriented subpath is at most (via repeated applications of Lemma 17):

$$\left\lceil \left\lceil \left\lceil \frac{n}{L+1} \right\rceil / (L+1) \right\rceil \dots \right\rceil \leq \dots \leq \left\lceil \frac{n}{(L+1)^{r-1}} \right\rceil$$

Since $L = \lceil n^{1/r} \rceil$, we see that $\left\lceil \frac{n}{(L+1)^{r-1}} \right\rceil \leq L$. Therefore, by intervening on all the remaining relevant vertices in the final r-th round, the total number of interventions performed will be $\mathcal{O}(r \cdot L) \subseteq \mathcal{O}(r \cdot n^{1/r})$. □

Lemma 17 (See appendix for proof)**.** *For positive integer n, and arbitrary real numbers m, x, we have $\left\lceil \frac{\lceil \frac{x}{m} \rceil}{n} \right\rceil = \lceil \frac{x}{mn} \rceil$.*

3.2 TREES

While the high-level strategy employed in Lemma 16 also works for trees, how we perform the "balanced partitioning" is slightly different since the vertices are no longer arranged in a line. Instead, we rely on Algorithm 1 to partition a tree on n nodes into subtrees using L node removals such that each subtree has size at most $\lceil n/(L+1) \rceil$.

The correctness of Algorithm 1 is proven in Lemma 18 and the resulting r-adaptive search result is given in Lemma 19.

Lemma 18. *Given a tree $G = (V, E)$ with $|V| = n$ and an integer $L \leq n$, Algorithm 1 runs in polynomial time and returns a subset of vertices $A \subseteq V$ such that $|A| \leq L$ and subtrees in $G[V \setminus A]$ have size at most $\lceil n/(L+1) \rceil$.*

Proof. We first prove the correctness then the running time.

Correctness Consider an arbitrary iteration of the while loop. We will argue two things:

1. The size of G decreases by at least $\lceil \frac{n}{L+1} \rceil$.
2. The newly pruned subtree(s) have size at most $\lceil \frac{n}{L+1} \rceil$.

Since $n - L \cdot \lceil \frac{n}{L+1} \rceil \leq \lceil \frac{n}{L+1} \rceil$, the while loop terminates after at most L iterations. Thus, $|A| \leq L$ as we only add one vertex per iteration to A.

Algorithm 1 Balanced partitioning on trees.

Input: A tree $G = (V, E)$ with $|V| = n$, an integer L.
Output: A subset of vertices $A \subseteq V$ such that $|A| \leq L$ and subtrees in $G[V \setminus A]$ have size at most $\lceil n/(L+1) \rceil$.

1: Initialize $A \leftarrow \emptyset$
2: **while** tree $G = (V, E)$ has size $|V| > \lceil \frac{n}{L+1} \rceil$ **do**
3: Root G arbitrarily.
4: Compute size of subtrees T_u at each node $u \in V$
5: **if** there is a subtree T_u of size $1 + \lceil \frac{n}{L+1} \rceil$ **then**
6: Add u to A; Update $G \leftarrow G[V \setminus V(T_u)]$.
7: **else**
8: Find a subtree T_u of size $|V(T_u)| > 1 + \lceil \frac{n}{L+1} \rceil$, s.t. $|V(T_w)| \leq \lceil \frac{n}{L+1} \rceil$ for *all* children w of u.
9: Add u to A; Update $G \leftarrow G[V \setminus V(T_u)]$.
10: **end if**
11: **end while**
12: **return** A

if-case: Since $|V(T_u)| = 1 + \lceil \frac{n}{L+1} \rceil$, removing $V(T_u)$ from V decreases the size of G by at least $\lceil \frac{n}{L+1} \rceil$. Furthermore, the subtree pruned by u's removal has size exactly $\lceil \frac{n}{L+1} \rceil$.

else-case: This case happens when *no* subtrees has size exactly $1 + \lceil \frac{n}{L+1} \rceil$. So, if a subtree has size strictly larger than $\lceil \frac{n}{L+1} \rceil$, then it must have size at least $2 + \lceil \frac{n}{L+1} \rceil$.

We first show how to find such a subtree T_u in Line 8:

1. Initialize u as the root node.
2. If all children w of u have subtree sizes $|V(T_w)| \leq \lceil \frac{n}{L+1} \rceil$, we have found T_u and can stop.
3. Otherwise, update u to any child w with subtree size $|V(T_w)| > \lceil \frac{n}{L+1} \rceil$, and go to step 2. Note that $|V(T_w)| \geq 2 + \lceil \frac{n}{L+1} \rceil$ from the above discussion.

Since $|V(T_u)| > 1 + \lceil \frac{n}{L+1} \rceil$, removing $V(T_u)$ from V decreases the size of G by at least $\lceil \frac{n}{L+1} \rceil$. Furthermore, each subtree pruned by u's removal has size at most $\lceil \frac{n}{L+1} \rceil$ since all subtrees of children w of u has size at most $\lceil \frac{n}{L+1} \rceil$.

Running time Rooting the tree and computing sizes of each rooted subtree can be done in polynomial time via depth-first search and dynamic programming. Finding the node u in within the while loop can also be done in polynomial time. Since the while loop executes at most $L \leq n$ times, the algorithm runs in polynomial time. □

Lemma 19. *Suppose $\mathcal{E}(G^*)$ is a tree on n vertices. There is a r-adaptive algorithm that uses at most $\mathcal{O}(r \cdot n^{1/r})$ interventions in total.*

Proof. Since the underlying causal DAG is a tree, whenever $\mathcal{E}(G^*)$ is not fully oriented, the chain components are singletons and at most one tree (of size at least two).

Let $L = \lceil n^{1/r} \rceil$. Consider the following algorithm:

1. Run Algorithm 1 on the tree chain component of $\mathcal{E}(G^*)$ to obtain a subset of vertices A of size $|A| \leq L$.
2. Intervene on all vertices in A.
3. If the essential graph is still not fully oriented and we have used less than $r-1$ rounds, go back to step 1.
4. In the final round, we intervene on all vertices[6] in the tree chain component (if it exists).

In the first $r-1$ adaptive rounds, the new tree chain component must have size at most one of the pruned subtrees, i.e. size at most $1/(L+1)$ factor smaller than before. After $r-1$ rounds, the largest possible size of the tree chain component is at most (via repeated applications of Lemma 17):

$$\left\lceil \left\lceil \left\lceil \frac{n}{L+1} \right\rceil / (L+1) \right\rceil \dots \right\rceil \leq \dots \leq \left\lceil \frac{n}{(L+1)^{r-1}} \right\rceil$$

Since $L = \lceil n^{1/r} \rceil$, we see that $\left\lceil \frac{n}{(L+1)^{r-1}} \right\rceil \leq L$. Therefore, by intervening on all the remaining relevant vertices in the final r-th round, the total number of interventions performed will be $\mathcal{O}(r \cdot L) \subseteq \mathcal{O}(r \cdot n^{1/r})$. □

4 FULL GENERALITY

Building upon the intuition developed in Section 3, we now show how to obtain Theorem 1. We also present a matching worst case lower bound (Theorem 2), and extend our results to the bounded size intervention settings (Theorem 3).

Let us now describe the algorithmic idea behind our atomic intervention result. By Theorem 10, we can remove all oriented edges from earlier rounds of interventions and focus on each undirected chain component. Let L be a suitably chosen value that depends on r, which we define later. Since these chain components are chordal graphs, we can efficiently compute their clique tree representation for each chain component G. Invoking Algorithm 1 to compute a "balanced partitioning" on the clique tree T_G, we obtain L maximal cliques from each chain component G. Intervening on all these cliques incurs a cost of $\mathcal{O}(L \cdot \nu_1(G^*))$ interventions, as the sum of maximal cliques in each chain component is a lower bound for the verification number (see Lemma 9). Performing this operation for $r-1$ rounds leaves us with chain components, each with at most L maximal cliques. In the final r-th round, we intervene on all the vertices in these unoriented connected components which additionally incurs a cost of at most $\mathcal{O}(L \cdot \nu_1(G^*))$. Therefore, the total cost in all iterations is at most $\mathcal{O}(r \cdot L \cdot \nu_1(G^*))$. Setting $L \approx n^{1/r}$ above gives the desired guarantees.

[6] We could have further optimized by intervening on a G-separating system of the remaining tree component but this does not affect the asymptotics in the worst case. For instance, if the underlying causal DAG is a single path, then the final round will still be a path and thus the G-separating system size is still roughly at least half the remaining nodes.

To formally prove our result, we first begin with following lemma which tells us that intervening on all vertices within a maximal clique of a chordal graph breaks up the chain component according to its clique tree representation.

Lemma 20. *Let $T_G = (K, S)$ be the clique tree of a chordal graph $G = (V, E)$. For any maximal clique K_i in T_G, if two maximal cliques K_1 and K_2 are disjoint in $T_G[K \setminus \{K_i\}]$, then $v_1 \in V(K_1)$ and $v_2 \in V(K_2)$ are not in the same connected component of $G[V \setminus V(K_i)]$.*

Proof. Since K_1 and K_2 are disjoint in $T_G[K \setminus \{K_i\}]$, the removed K_i must lie on the unique path between K_1 and K_2 in the tree T_G. As $V(K_i) = \cup_{\{K_i, K_j\} \in S}(V(K_i) \cap V(K_j))$, Lemma 5 tells us that the removal of K_i will makes v_1 and v_2 disconnected in $G[V \setminus V(K_i)]$. □

We are now ready to prove Theorem 1 using Algorithm 2.

Algorithm 2 Adaptivity-sensitive search.

Input: Essential graph $\mathcal{E}(G^*)$, adaptivity param. $r \geq 1$.
Output: A sequence of intervention sets $\mathcal{I}_1, \dots, \mathcal{I}_r$ such that $\mathcal{E}_{\mathcal{I}_1, \dots, \mathcal{I}_r}(G^*) = G^*$.

```
Initialize L = ⌈n^{1/r}⌉.
for i = 1, ..., r − 1 do
    Initialize I_i ← ∅
    for chain comp. H ∈ CC(E_{I_1,...,I_{i−1}}(G*)) do
        if H is a clique then Set V' ← V(H).
        else Compute clique tree T_H of H. Compute
            L-balanced partitioning S of T_H via
            Algorithm 1. Let V' ← ∪_{K_j ∈ S} V(K_j).
        Add V' to I_i.
    end for
    Intervene on all vertices in I_i.
end for
Define I_r as all remaining relevant vertices and intervene on vertices in I_r.
return I_1, ..., I_r
```

Theorem 1 (Atomic upper bound)**.** *Let $\mathcal{E}(G^*)$ be the observational essential graph of an underlying causal DAG G^* on n nodes. There is a polynomial time r-adaptive algorithm that uses $\mathcal{O}(\min\{r, \log n\} \cdot n^{1/\min\{r, \log n\}} \cdot \nu_1(G^*))$ atomic interventions to recover G^* from $\mathcal{E}(G^*)$.*

Proof. Consider Algorithm 2. Since we always intervene on all relevant vertices outside of the for loop, we are guaranteed to fully recover G^*.

Number of interventions

By Lemma 18, we know that invoking Algorithm 1 for any clique tree T_H on maximal cliques K_H of H returns a set $A_H \subseteq K_H$ of at most $|A_H| \leq L$ clique nodes such that subtrees in $T_H[K_H \setminus A_H]$ have size at most $\lceil |T_H|/(L+1) \rceil$.

By Lemma 9,

$$\left|\bigcup_{H\in CC(\mathcal{E}_{\mathcal{I}_1,\ldots,\mathcal{I}_{i-1}}(G^*))} A_H\right| = \sum_{H\in CC(\mathcal{E}_{\mathcal{I}_1,\ldots,\mathcal{I}_{i-1}}(G^*))} |A_H| \in \mathcal{O}(L\cdot\nu_1(G^*))$$

So, within the for loop, we incur at most $\mathcal{O}(r\cdot L\cdot\nu_1(G^*))$ interventions.

By Lemma 20, we know that each iteration reduces the maximum number of maximal cliques in any chain component size by a factor of $L+1$. After $r-1$ rounds, the largest clique tree in any chain component has at most

$$\frac{n}{(L+1)^{r-1}} = \frac{n}{(\lceil n^{1/r}\rceil+1)^{r-1}} \le \lceil n^{1/r}\rceil = L$$

maximal cliques. So, if we intervene on all remaining relevant vertices in the final r-th round, this incurs at most $\mathcal{O}(L\cdot\nu_1(G^*))$ interventions via Lemma 9.

Therefore, in total, we use $\mathcal{O}(r\cdot L\cdot\nu_1(G^*)) \subseteq \mathcal{O}(r\cdot n^{1/r}\cdot\nu_1(G^*))$ atomic interventions.

Running time

The overall run time is polynomial because each of the following steps runs in polynomial time: Intervening and propagating Meek rules; Computing chain components (there are at most n chain components); Computing the clique tree; Computing the L-balanced partitioning. □

Our lower bound Theorem 2 generalizes the idea behind the lower bound proof in Lemma 15 (for the special case of $r=2$): on a path essential graph with $\nu_1(G^*)=1$, the adaptive adversary repeatedly hides the source node in the largest possible unoriented segment based on the current round of interventions. See Appendix C for the full proof.

Theorem 2 (Atomic worst case)**.** *In the worst case, any r-adaptive algorithm needs to use at least $\Omega(\min\{r,\log n\}\cdot n^{1/\min\{r,\log n\}}\cdot\nu_1(G^*))$ atomic interventions against an adaptive adversary.*

With bounded size interventions, each intervention is now allowed to involve $k\ge 1$ vertices, for some pre-determined upper bound k. Algorithmically, we tweak Lines 7 and 11 in Algorithm 2 to use the labelling lemma of Lemma 14, which incurs additional $\mathcal{O}(\log k)$ multiplicative factor when comparing with $\nu_k(G^*)$. See Appendix C for the full proof.

Theorem 3 (Bounded upper bound)**.** *Let $\mathcal{E}(G^*)$ be the observational essential graph of an underlying causal DAG G^* on n nodes. There is a polynomial time r-adaptive algorithm that uses $\mathcal{O}(\min\{r,\log n\}\cdot n^{1/\min\{r,\log n\}}\cdot\log k\cdot\nu_k(G^*))$ bounded sized interventions to recover G^* from $\mathcal{E}(G^*)$, where each intervention involves at most $k>1$ vertices.*

5 CONCLUSION

In our work, we define r-adaptivity that interpolates between non-adaptivity (for $r=1$) and full adaptivity (for $r=n$). We provide a r-adaptive algorithm that achieves $\mathcal{O}(\min\{r,\log n\}\cdot n^{1/\min\{r,\log n\}})$ approximation with respect to the verification number.

The verification number is a benchmark for fully adaptive approaches which might turn out to be a weak lower bound for r-adaptive algorithms, especially for smaller values of r. Therefore, it is an interesting open direction to define a new benchmark that depends on the value of r and lower bounds the minimum number of interventions needed for any r-adaptive algorithms to fully orient the causal graph. Ideally, this new benchmark should interpolate between the size of G-separating systems and verification numbers for $r=1$ and $r=n$ respectively. Furthermore, it would be important to provide an corresponding r-adaptive search algorithm that achieves a good approximation with respect to this new benchmark, possibly better than Algorithm 2.

Here, we studied the complexity of adaptivity for causal discovery under the standard assumptions of causal sufficiency, faithfulness, and infinite sample regime. When these assumptions are violated, wrong causal conclusions may be drawn and possibly lead to unintended downstream consequences. Hence, it is of great interest to remove/weaken these assumptions while maintaining theoretical guarantees.

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