Understanding the Effect of Stochasticity in Policy Optimization

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Abstract

The convergence of policy gradient and its variants heavily relies on true gradient update [1–5], which is impractical. We study the stochasticity effect in policy optimization for the on-policy case to bridge the gap, and make the following four contributions. *First*, we show that the preferability of optimization methods depends critically on whether stochastic versus exact gradients are used. In particular, unlike the true gradient setting, geometric information *cannot* be easily exploited in the stochastic case for accelerating policy optimization without detrimental consequences or impractical assumption usually used in [6, 7]. *Second*, to explain these findings we introduce the concept of *committal rate* for stochastic policy optimization, and show that this can serve as a criterion for determining almost sure convergence to global optimality. *Third*, we show that in the absence of external oracle information, which allows an algorithm to determine the difference between optimal and sub-optimal actions given only on-policy samples, there is an inherent trade-off between exploiting geometry to accelerate convergence versus achieving optimality almost surely. That is, an uninformed algorithm either converges to a globally optimal policy with probability 1 but at a rate no better than $O(1/t)$, or it achieves faster than $O(1/t)$ convergence but then must fail to converge to the globally optimal policy with some positive probability. *Finally*, we use the committal rate theory to explain why practical policy optimization methods are sensitive to random initialization, then develop an ensemble method that can be guaranteed to achieve near-optimal solutions with high probability.

1 Introduction

Policy optimization is a central problem in reinforcement learning (RL) that provides a foundation for both policy-based and actor-critic RL methods. Until recently it had generally been assumed that methods based on following the policy gradient (PG) [8] could not be guaranteed to converge to globally optimal solutions, given that the policy value function is not concave. However, this assumption has been contradicted by recent findings that policy gradient methods can indeed be proved to converge to global optima, at least in the tabular setting. In particular, the standard softmax PG method with a constant learning rate has been shown to converge to a globally optimal policy at a $\Theta(1/t)$ rate for finite MDPs [2], albeit with challenging problem and initialization dependent constants [9, 10]. Several techniques have been developed to further improve standard PG and achieve better rates and constants. For example, adding *entropy regularization* has been shown to produce faster $O(e^{-c \cdot t})$ convergence to the optimal regularized policy [2, 4, 7]. By exploiting natural geometries based on Bregman divergences, *Natural PG (NPG) or mirror descent (MD)* have been shown to achieve better constants than standard PG [1, 4] and faster $O(e^{-c \cdot t})$ rates ($c > 0$), with [4, 7] and without regularization [5]. Alternative policy parameterizations, such as the escort parameterization, have been shown to improve the constants achieved by softmax and yield faster

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plateau escaping [9]. More recently, a *Geometry-aware Normalized PG (GNPG)* approach has been proposed to exploit the non-uniformity of the value function, achieving even faster $O(e^{-c \cdot t})$ rates with improved constants [3].

A key observation is that each of these four techniques—*(i)* entropy regularization, *(ii)* NPG (or MD), *(iii)* alternative policy parameterization, and *(iv)* GNPG—accelerate the convergence of standard softmax PG by better exploiting the geometry of the optimization landscape. In particular, entropy regularization makes the regularized objective behave more like a quadratic [2, 4, 7], which significantly improves the near-linear character of the softmax policy value. Natural PG (or MD) perform non-Euclidean updates in the parameter space, which is quite different from the Euclidean geometry characterizing standard softmax PG updates [1, 4, 5]. The escort policy parameterization induces an alternative policy-parameter relation [9]. GNPG exploits the non-uniform smoothness in the optimization landscape via a simple gradient normalization operation [3].

However, these advantages have only been established for the true gradient setting. A natural question therefore is whether geometry can also be exploited to accelerate convergence to global optimality in *stochastic* gradient settings. In this paper, we show that in a certain fundamental sense, the answer is *no*. That is, there exists a fundamental trade-off between leveraging geometry to accelerate convergence and overcoming the variance introduced by stochastic gradients (possibly infinite); in particular, no uninformed algorithm can improve the $O(1/t)$ convergence rate without incurring a positive probability of failure (i.e. diverging or converging to a sub-optimal stationary point).

The conditions used in vanilla stochastic gradient convergence analysis, *i.e.*, unbiased and variance-bounded gradient estimator [11], has been exploited to attempt to explain such a trade-off in policy gradients [6, 7]. However, the bounded variance requires the sample policy bounded away from zero everywhere, which is impractical. Meanwhile, the a variant of NPG can convergence even with unbounded variance [12]. These gaps raise the question that if not the bounded variance, then what is the key factor to ensure the convergence of stochastic policy optimization algorithms? Motivated by this question, we introduce the concept *committal rate* to characterize the behavior policies, which significantly affect whether convergence to a correct solution can be guaranteed in stochastci on-policy setting. In particular, we make the following contributions.

- *First*, we illustrate the anomaly that the preferability of policy optimization algorithms (softmax PG vs. NPG and GNPG) changes dramatically depending on whether true versus on-policy stochastic gradients are considered, and reveal the impracticality and unnecessity of bounded variance requirement in Section 2;
- *Second*, we introduce the concept of the *committal rate* in Section 3 to characterize how quickly a sampled actions' probability approaches 1, which provide us tools for analyzing the stochasticity effect in convergences;
- *Third*, we use the committal rate to study general stochastic policy optimization behaviors rigorously and reveal the inherent geometry-convergence trade-off in Section 4;
- *Finally*, we explain the sensitivity to random initialization in practical policy optimization algorithms. From these results, we then develop an ensemble method that can achieve fast convergence to global optima with high probability in Section 5.

2 Understanding Algorithm Preferability in On-line Policy Optimization

To illustrate the key aspects of policy optimization methods and their comparative preferability, it suffices to consider deterministic, single-state, finite-action Markov decision processes (MDPs). The main results below extend to general finite MDPs, but for clarity of exposition we move details for general cases to the appendix.

A deterministic, single-state, finite-action MDP can be simply be specified by an action space is $[K] \coloneqq \{1, 2, \dots K\}$ and a K-dimensional reward vector $r \in (0,1)^K$. The problem is to maximize the expected reward of a parametric policy π_θ,

$$\max_{\theta:[K]\to\mathbb{R}} \mathop{\mathbb{E}}_{a\sim\pi_\theta}[r(a)]. \tag{1}$$

where π_θ is parameterized by θ using the standard softmax transform,

$$\pi_\theta(a) = \exp\{\theta(a)\} / \sum_{a'\in[K]} \exp\{\theta(a')\} \text{ for all } a \in [K]. \tag{2}$$

Without loss of generality, we assume there exists a unique optimal action $a^* = \arg\max_{a\in[K]} r(a)$, hence there exists a unique optimal deterministic policy π^* such that ${\pi^*}^\top r = \sup_\theta \pi_\theta^\top r = r(a^*)$.

2.1 Exact Gradient Setting

It is known that Eq. (1) is a non-concave maximization over the parameters θ [2]. Nevertheless, it has recently become better understood how policy gradient (PG) methods still converge to global optima for Eq. (1) when exact gradients are used. To illustrate the main considerations, we focus on the following three representative algorithms that have recently been proved to achieve convergence to globally optimal solutions but at different rates: softmax policy gradient (PG), natural PG (NPG), and geometry-aware normalized PG (GNPG), while similar conclusion can be drawn for other variants, such as entropy regularization [2, 4, 7].

2.1.1 Standard Softmax PG

The standard softmax PG method is specified by the following update.

Update 1 (Softmax PG, true gradient)**.** $\theta_{t+1} \leftarrow \theta_t + \eta \cdot \frac{d\pi_{\theta_t}^\top r}{d\theta_t}$, *where* $\frac{d\pi_\theta^\top r}{d\theta} = \left(diag(\pi_\theta) - \pi_\theta \pi_\theta^\top\right) r$, *and thus* $\frac{d\pi_\theta^\top r}{d\theta(a)} = \pi_\theta(a) \cdot (r(a) - \pi_\theta^\top r)$ *for all* $a \in [K]$.

As shown in Mei et al. [2], the convergence of this update to a globally optimal policy, given exact gradients, can be established by considering the following non-uniform Łojasiewicz (NŁ) inequality,

Lemma 1 (NŁ, [2])**.** $\left\| \frac{d\pi_\theta^\top r}{d\theta} \right\|_2 \geq \pi_\theta(a^*) \cdot (\pi^* - \pi_\theta)^\top r$.

By considering smoothness of the optimization landscape, Mei et al. [2] shows that the progress in each iteration of PG can be lower bounded by the squared norm of the gradient, $\left\| \frac{d\pi_{\theta_t}^\top r}{d\theta_t} \right\|_2^2$, which leads to a $O(1/t)$ rate.

Proposition 1 (PG upper bound [2])**.** *Using Update 1 with* $\eta = 2/5$, *we have* $(\pi^* - \pi_{\theta_t})^\top r \leq 5/(c^2 \cdot t)$ *for all* $t \geq 1$, *such that* $c = \inf_{t\geq 1} \pi_{\theta_t}(a^*) > 0$ *is a constant that depends on* r *and* θ_1, *but it does not depend on the time* t. *In particular, if* $\pi_{\theta_1}(a) = 1/K \ \forall a$ *then* $c \geq 1/K$.

Proposition 2 (PG lower bound [2])**.** *For sufficiently large* $t \geq 1$, *Update 1 with* $\eta \in (0, 1]$ *exhibits* $(\pi^* - \pi_{\theta_t})^\top r \geq \Delta^2/(6 \cdot t)$, *where* $\Delta = r(a^*) - \max_{a\neq a^*} r(a) > 0$ *is the reward gap of* r.

Remark 1. *The constant dependence of PG follows a* $\Omega(1/c)$ *lower bound for one-state MDPs [9], while* c *can be exponentially small in terms of the number of states for general finite MDPs [10].*

To summarize, using $\eta \in O(1)$, softmax PG achieves convergence to a global optima, but with a $\Theta(1/t)$ rate that exhibits poor constant dependence.

2.1.2 Natural PG (NPG)

An alternative method, natural PG (NPG) [13], provides the prototype for many practical policy optimization methods, such as TRPO and PPO [14, 15]. NPG is based on the following update.

Update 2 (Natural PG (NPG), true gradient)**.** $\theta_{t+1} \leftarrow \theta_t + \eta \cdot r$.

For softmax policies, it turns out that Update 2 is identical to mirror descent (MD) with a Kullback Leibler (KL) divergence. Therefore a standard MD analysis shows that Update 2 achieves convergence to a global optimum at a rate of $O(1/t)$ [1]. Very recently, work concurrent to this submission [5] has shown that Update 2 actually enjoys a much faster $O(e^{-c\cdot t})$ rate. In fact, here too we can establish the same $O(e^{-c\cdot t})$ rate, but using a simpler argument based on the following variant of the NŁ inequality for natural gradients. These results are new to this paper. **Due to space limitation, we postpone all the proofs to the appendix.**

Lemma 2 (Natural NŁ inequality, continuous)**.** $\left\langle \frac{d\pi_\theta^\top r}{d\theta}, r \right\rangle \geq \pi_\theta(a^*) \cdot \Delta \cdot (\pi^* - \pi_\theta)^\top r$.

Lemma 3 (Natural NŁ, discrete)**.** *Let* $\pi'(a) := \frac{\pi(a)\cdot e^{\eta\cdot r(a)}}{\sum_{a'} \pi(a')\cdot e^{\eta\cdot r(a')}}$, $\forall\, a \in [K]$, *where* $\eta > 0$. *Then,*

$$(\pi' - \pi)^\top r \geq \left[1 - \frac{1}{\pi(a^*)\cdot(e^{\eta\cdot\Delta} - 1) + 1}\right] \cdot (\pi^* - \pi)^\top r. \tag{3}$$

In particular, by using a non-Euclidean update and analysis, the progress of each iteration of NPG can lower bounded by the larger bound $\left\langle \frac{d\pi_{\theta_t}^\top r}{d\theta_t}, r \right\rangle$ instead of the weaker bound $\left\| \frac{d\pi_{\theta_t}^\top r}{d\theta_t} \right\|_2^2$ established for standard PG. Based on this inequality, one can easily establish a much faster $O(e^{-c\cdot t})$ convergence to a globally optimal solution for NPG, making it far preferable to PG if true gradients are available.

Theorem 1. *Using Update 2 with any $\eta > 0$, we have, for all $t \geq 1$,*

$$\left(\pi^* - \pi_{\theta_t}\right)^\top r \leq \left(\pi^* - \pi_{\theta_1}\right)^\top r \cdot e^{-c\cdot(t-1)}, \tag{4}$$

where $c := \log\left(\pi_{\theta_1}(a^*) \cdot \left(e^{\eta\cdot\Delta} - 1\right) + 1\right) > 0$ for any $\eta > 0$, and $\Delta = r(a^*) - \max_{a \neq a^*} r(a) > 0$.

2.1.3 Geometry-aware Normalized PG (GNPG)

The Geometry-aware Normalized PG (GNPG) update is investigated in [3] to accelerate the convergence of PG by exploiting local smoothness properties of the optimization landscape.

Update 3 (Geometry-aware Normalized PG (GNPG), true gradient)**.** $\theta_{t+1} = \theta_t + \eta \cdot \frac{d\pi_{\theta_t}^\top r}{d\theta_t} \Big/ \left\| \frac{d\pi_{\theta_t}^\top r}{d\theta_t} \right\|_2$.

The analysis in [3] focuses on exploiting non-uniform smoothness (NS) rather than improving the NŁ inequality as for NPG above.

Lemma 4 (NS, [3])**.** *The spectral radius of Hessian matrix $\frac{d^2\pi_\theta^\top r}{d\theta^2}$ is upper bounded by $3 \cdot \left\| \frac{d\pi_\theta^\top r}{d\theta} \right\|_2$.*

Given this NS property, [3] then show how the progress in each iteration of GNPG can be lower bounded by the larger quantity $\left\| \frac{d\pi_{\theta_t}^\top r}{d\theta_t} \right\|_2$ instead of the weaker $\left\| \frac{d\pi_{\theta_t}^\top r}{d\theta_t} \right\|_2^2$ bound for standard PG. Then, using the same NŁ inequality as for PG, Mei et al. [3] shows that GNPG also converges to a globally optimal solution at rate $O(e^{-c\cdot t})$. Again, one naturally concludes that GNPG is preferable to PG if exact gradients are used.

Proposition 3 ([3])**.** *Using Update 3 with $\eta = 1/6$, we have, for all $t \geq 1$,*

$$\left(\pi^* - \pi_{\theta_t}\right)^\top r \leq \left(\pi^* - \pi_{\theta_1}\right)^\top r \cdot e^{-\frac{c\cdot(t-1)}{12}}, \tag{5}$$

where $c = \inf_{t\geq 1} \pi_{\theta_t}(a^*) > 0$ does not depend on t. If $\pi_{\theta_1}(a) = 1/K$, $\forall a$, then $c \geq 1/K$.

2.2 On-policy Stochastic Gradient Setting: Anomalies

Although the above results show that exploiting geometric information can allow linear convergence to an optimal solution given true gradients—obviously $O(e^{-c\cdot t})$ represents an exponential speedup over the $\Omega(1/t)$ lower bound established for standard PG—it is critical to understand whether such advantages can also be obtained in the more natural stochastic gradient setting. Given the previous results, it would seem natural to prefer accelerated algorithms over PG in practice, and there is some evidence that such thinking has become mainstream based on the popularity of TRPO and PPO over PG. However, by more closely examining the behavior of these algorithms when true gradients are replaced by on-policy stochastic estimates, serious shortcomings begin to emerge, as empirical observed in Chung et al. [12], and it is far from obvious that similar advantages from the true gradient case might be recoverable in the more practical stochastic scenario.

We begin by examining the behavior of the previous algorithms in the context of on-policy stochastic gradients. To enable this analysis, first note that each of the above PG methods, Updates 1 to 3, can be adapted to the stochastic setting by using on-policy importance sampling (IS) to provide an unbiased estimate of the true reward. We do not make assumptions like each action is sufficiently explored, since π_{θ_t} is the behaviour policy as well as the policy to be optimized. It is possible that π_{θ_t} approaches near deterministic policy, ruling out positive results based on such assumptions [6].

2.2.1 Softmax PG

Definition 1 (On-policy IS)**.** *At iteration t, sample one action $a_t \sim \pi_{\theta_t}$. The IS reward estimator $\hat{r}_t$ is constructed as $\hat{r}_t(a) = \frac{\mathbb{I}\{a_t = a\}}{\pi_{\theta_t}(a)} \cdot r(a)$ for all $a \in [K]$.*

Remark 2. *We consider sampling one action in each iteration for convenience, and the results hold for sampling a constant $B > 0$ mini-batch of actions in each iteration.*

Update 4 (Softmax PG, stochastic gradient)**.** $\theta_{t+1} \leftarrow \theta_t + \eta \cdot \frac{d\pi_{\theta_t}^\top \hat{r}_t}{d\theta_t}$, *where* $\frac{d\pi_{\theta_t}^\top \hat{r}_t}{d\theta_t(a)} = \pi_{\theta_t}(a) \cdot (\hat{r}_t(a) - \pi_{\theta_t}^\top \hat{r}_t)$ *for all* $a \in [K]$.

Using IS estimation, the softmax PG is unbiased and its variance is upper bounded by constant.

Lemma 5. *Let* $\hat{r}$ *be the IS estimator using on-policy sampling* $a \sim \pi_\theta$*. Denote* $H(\pi_\theta) := diag(\pi_\theta) - \pi_\theta \pi_\theta^\top = \left(\frac{d\pi_\theta}{d\theta}\right)^\top$ *as the Jacobian of* $\theta \mapsto \mathrm{softmax}(\theta)$*. We have,* $\mathbb{E}_{a\sim\pi_\theta}\left[H(\pi_\theta)\hat{r}\right] = H(\pi_\theta) r$*. Also,* $\mathbb{E}_{a\sim\pi_\theta} \left\|H(\pi_\theta)\hat{r}\right\|_2^2 = 2 \cdot r^\top \left(diag(\pi_\theta) - diag(\pi_\theta \odot \pi_\theta)\right) r \leq K/2$.

These observations imply that stochastic softmax PG converges to a global optimum almost surely.

Theorem 2. *Using Update 4,* $\left(\pi^* - \pi_{\theta_t}\right)^\top r \to 0$ *as* $t \to \infty$ *with probability* 1.

2.2.2 NPG

Similarly, we can use on-policy IS estimation to adapt NPG to the stochastic setting.

Update 5 (NPG, stochastic gradient)**.** $\theta_{t+1} \leftarrow \theta_t + \eta \cdot \hat{r}_t$.

Although the NPG is unbiased, its variance can be possibly unbounded in the on-policy setting, as established by the following direct calculation.

Lemma 6. *For NPG, we have,* $\mathbb{E}_{a\sim\pi_\theta}\left[\hat{r}\right] = r$*, and* $\mathbb{E}_{a\sim\pi_\theta} \left\|\hat{r}\right\|_2^2 = \sum_{a\in[K]} \frac{r(a)^2}{\pi_\theta(a)}$.

Note that such an unbounded variance does not satisfy the convergence conditions for stochastic gradient [11]. In fact, we provide a more direct result showing that stochastic NPG has a positive probability of converging to a sub-optimal deterministic policy.

Theorem 3. *Using Update 5, there is a positive probability that* $\pi_{\theta_t}(a) \to 1,\ \forall a \in [K]$ *as* $t \to \infty$.

This finding extends some of the recent results for the two-action case [12] to general K-action cases.

2.2.3 GNPG

Finally, we consider the stochastic version of GNPG.

Update 6 (GNPG, stochastic gradient)**.** $\theta_{t+1} = \theta_t + \eta \cdot \frac{d\pi_{\theta_t}^\top \hat{r}_t}{d\theta_t} \Big/ \left\|\frac{d\pi_{\theta_t}^\top \hat{r}_t}{d\theta_t}\right\|_2$.

Unfortunately, this estimator involves a ratio of random variables, and the bias of stochastic GNPG can possibly be unbounded. As for NPG we can show that stochastic GNPG fails with positive probability in the stochastic case.

Theorem 4. *Using Update 6, there is a positive probability that* $\pi_{\theta_t}(a) \to 1,\ \forall a \in [K]$ *as* $t \to \infty$.

2.3 Motivating the On-policy Stochastic Setting

We summarize the preferability of alternative optimization strategies in the exact versus on-policy stochastic gradient settings in Table 1: there appears to be a major reversal in going from one scenario to the other.

	Softmax PG	NPG	GNPG
True gradient	converges $\Theta(1/t)$	converges $O(e^{-c\cdot t})$	converges $O(e^{-c\cdot t})$
Stochastic on-policy	converges w.p. 1	fails w.p. > 0	fails w.p. > 0

Table 1: Convergence properties of softmax PG, NPG and GNPG in the alternative settings.

Of course it is possible to study the convergence of algorithms when the gradient estimates are assumed to be unbiased and have bounded variance as in the analysis of vanilla SGD [11], and in some other work for policy gradients [6, 7, 16, 17]. However, **first**, such conditions are only *sufficient* conditions which are difficult to be satisfied, because a bounded variance assumption requires that the probabilities induced by a behaviour policy are bounded away from 0 everywhere [12], which is impractical for large state and action spaces and impossible when they are infinite. **Second**, PPO [15] and TRPO [14] use on-policy sampling *without* any explicit correction for fulfill the exploratory

behaviour requirement, while still solving practical problems [18]. **Third**, the on-policy setting under practical requirement is technically more challenging. In off-policy settings, the sampling procedure is independent of the parameter update [12], which makes the analysis much more straightforward with some extra assumptions on the estimate of value functions or gradients [19], while these become coupled in the on-policy setting and more subtle analysis is required. **Finally**, Chung et al. [12] investigate the baselines and show that variance reduction techniques are not able to overcome unbounded variance, while NPG can still achieve global convergence almost surely with judicious choice of baseline even though its variance remains *unbounded* (see Update 7 for details). This means bounded variance is not necessary for convergence, and some other factors rather than variance account for the convergence behaviours of stochastic policy optimization algorithms.

This leave us an important question to be answered to bridge the gap between theory and practice,

What are the key factors determining the convergence of stochastic policy optimization?

We propose the *committal rate* to characterize the behaviors of algorithms to answer this question.

3 Committal Rate of Stochastic Policy Optimization Algorithms

Although the baseline study [12] only focuses on two- and three-action bandits primarily, it develops a useful intuition that stochastic policy optimization in practical setting consists of separate "sampling" and "updating" steps that become coupled in the on-policy setting. Building from this observation, and seeking to explain the outcomes in Section 2, we formalize the following "committal rate" function of an policy optimization algorithm.

Definition 2 (Committal Rate)**.** *Consider a policy optimization algorithm $\mathcal{A}$ that maintains the parameters θ_t. Suppose $\pi_{\theta_1}(a) \in \Theta(1)$ for all $a \in [K]$. Let t denote the number of times action a is sampled after initialization. The committal rate of the algorithm $\mathcal{A}$ on action a is then defined as*

$$\kappa(\mathcal{A}, a) = \sup\left\{\alpha \geq 0 : \lim_{t\to\infty} t^{\alpha} \cdot [1 - \pi_{\theta_t}(a)] < \infty\right\}. \tag{6}$$

This definition accounts for the sampling process in characterizing the speed at which a policy approaches repeatedly sampling an action, using $1/t^{\alpha}$ to characterize how fast $\pi_{\theta_t}(a)$ is approaching 1 after sampling action a for t times. A larger value of $\kappa(\mathcal{A}, a)$ indicates an algorithm that quickly produces committal behaviour. For example, if $\pi_{\theta_t}(a) = 1 - 1/(t \cdot \log(t))$, then $\kappa(\mathcal{A}, a) = 1$. Similarly, if $\pi_{\theta_t}(a) = 1 - 1/e^t$, then $\kappa(\mathcal{A}, a) = \infty$, which means $\pi_{\theta_t}(a)$ approaches 1 extremely quickly. On the other hand, if $\pi_{\theta_t}(a) \in O(1)$, then $\kappa(\mathcal{A}, a) = 0$, implying that π_{θ_t} never becomes committal, since $\pi_{\theta_t}(a)$ never approaches 1.

Using this definition we can show that a slow committal rate for a stochastic policy optimization algorithm with respect to sub-optimal actions is a necessary condition for obtaining almost sure convergence to a globally optimal policy.

Theorem 5. *Consider a stochastic policy optimization algorithm $\mathcal{A}$ that maintains the parameters θ_t for all $t \geq 1$. If $\kappa(\mathcal{A}, a) > 1$ for a sub-optimal action $a \in [K]$, then,*

$$\prod_{t=1}^{\infty} \pi_{\theta_t}(a) > 0, \tag{7}$$

which implies that there is a positive probability the sub-optimal action $a \in [K]$ is sampled forever using π_{θ_t}, hence π_{θ_t} converges to sub-optimal deterministic policy with positive probability.

That is, Theorem 5 shows that $\kappa(\mathcal{A}, a) \leq 1$ for all sub-optimal actions $a \in [K]$ is a necessary condition for ensuring convergence to a global optimum with probability 1.

Using this result, we can now interrogate the committal rates of the above algorithms.

Theorem 6. *For the stochastic updates NPG and GNPG from Updates 5 and 6 we obtain $\kappa(NPG, a) = \infty$ and $\kappa(GNPG, a) = \infty$ for all $a \in [K]$ respectively.*

Theorem 6 explains why stochastic NPG and GNPG have a non-zero failure probability in the on-policy stochastic setting: they do not obey a necessary condition for almost sure global convergence. Intuitively, these algorithms can fail by prematurely allocating too much probability to a sub-optimal

action: each sampling of an action $a \in [K]$ increments its parameter by $\Theta(1)$, so if an action is sampled t times successively, the sum of all the other actions' probability decreases at rate $O(e^{-c \cdot t})$, which implies $\kappa(\mathcal{A}, a) = \infty$. According to Theorem 5, this means there is a positive probability that a single sub-optimal action can receive a long enough sampling run to ensure the other actions will never again be sampled.

By contrast, we can compare these outcomes to the committal rate of the softmax PG algorithm.

Theorem 7. *Softmax PG obtains $\kappa(PG, a) = 1$ for all $a \in [K]$, and π_{θ_t} converges to a globally optimal deterministic policy with probability* 1.

Theorems 5 and 7 provide (partial) explanations of the observations in Section 2: stochastic NPG and GNPG can fail while PG almost surely converges to a global optimum, but their committal rates lie on different sides of the necessary condition. Since $\kappa(\text{PG}, a) = 1$ for softmax PG, it follows that $\prod_{t=1}^{\infty} \pi_{\theta_t}(a) = 0$, hence it is not possible to sample sub-optimal actions forever, and the optimal action a^* always has a sufficient chance to be sampled, which ensures learning.

Next, we consider NPG using special baselines [12], which enjoys almost sure global convergence but has unbounded variance, and thus cannot be explained using variance.

Update 7 (NPG with oracle baseline)**.** $\theta_{t+1} \leftarrow \theta_t + \eta \cdot \left(\hat{r}_t - \hat{b}_t\right)$, *where* $\hat{b}_t(a) = \left(\frac{\mathbb{I}\{a_t = a\}}{\pi_{\theta_t}(a)} - 1\right) \cdot b$ *for all* $a \in [K]$, *and* $b \in (r(a^*) - \Delta, r(a^*))$.

Theorem 8. *Using Update 7,* $(\pi^* - \pi_{\theta_t})^\top r \to 0$ *as* $t \to \infty$ *with probability* 1.

For Update 7, the variance is still unbounded [12], and thus the convergence is **not** due to variance. However, the necessary condition in Theorem 5 is satisfied. We have, $\pi_{\theta_{t+1}}(a_t) < \pi_{\theta_t}(a_t)$, if $a_t \neq a^*$, i.e., whenever a sub-optimal action is selected its probability decreases, while the optimal action's probability always increases after any update. Therefore, we have $\kappa(\mathcal{A}, a^*) = \infty$ and $\kappa(\mathcal{A}, a) = 0$ for all $a \neq a^*$. This example means committal rate works for cases beyond the bounded variance condition used widely in optimization and reinforcement learning community.

4 The Geometry-Convergence Trade-off in Stochastic Policy Optimization

Theorem 7 raises the question of whether $\kappa(\mathcal{A}, a) = 1$ for all sub-optimal actions $a \in [K]$ is sufficient to ensure an algorithm $\mathcal{A}$ converges to an optimal policy almost surely. Unfortunately, this is not the case, and the complete picture of global optimality in stochastic policy optimization is more complex and requires detailed study of different iteration behaviors.

4.1 Iteration Behaviours

Remark 3. *The condition that $\kappa(\mathcal{A}, a) \leq 1$ for all sub-optimal actions $a \in [K]$ is **not** sufficient for ensuring almost sure convergence to global optimality. In addition to "convergence to a sub-optimal policy with positive probability" and "convergence to a globally optimal policy with probability* 1*" there exist other possible optimization behaviours, such as "not converging to any policy".*

In particular, consider the following update behaviors.

Staying. For the stationary update $\mathcal{A} : \theta_{t+1} \leftarrow \theta_t$ we obtain $\kappa(\mathcal{A}, a) = 0 \leq 1$ for all $a \in [K]$, yet $\pi_{\theta_t} = \pi_{\theta_1}$ does not converge to the optimal policy nor any sub-optimal deterministic policy.

Wandering. [NPG with a large baseline] Consider $\mathcal{A} : \theta_{t+1} \leftarrow \theta_t + \eta \cdot \left(\hat{r}_t - \hat{b}_t\right)$ with $\hat{b}_t(a) = \left(\frac{\mathbb{I}\{a_t = a\}}{\pi_{\theta_t}(a)} - 1\right) \cdot b$ for all $a \in [K]$. If $b > r(a^*)$, then we have $\pi_{\theta_{t+1}}(a_t) < \pi_{\theta_t}(a_t)$, i.e., a selected action's probability will decrease after updating, hence $\kappa(\mathcal{A}, a) = 0$ for all $a \in [K]$. However, $\pi_{\theta_t}(a) \not\to 1$ as $t \to \infty$ for all $a \in [K]$, therefore π_{θ_t} will wander within the simplex forever.

The above examples show that not converging to a sub-optimal policy does not necessarily imply converging to an optimal policy almost surely, and a stronger condition is needed to eliminate unreasonable behaviors like $\theta_{t+1} \leftarrow \theta_t$. We leave it as an open question to identify necessary and sufficient conditions for almost sure convergence to a global optimum.

4.2 Geometry-Convergence Trade-off

In Section 2 we saw that NPG and GNGP can use true gradients to significantly accelerate PG by better exploiting geometry. However, in the stochastic setting, any estimated geometry might be inaccurate, and intuitively, accelerated methods risk leveraging inaccurate information too aggressively. On the one hand, if progress is sufficiently fast (i.e., with a large committal rate), then an algorithm might never recover from aggressive yet inaccurate updates (Theorem 5). On the other hand, large progress is necessary for fast convergence. The tension between these observations suggest that there might be an inherent trade-off between exploiting geometry and avoiding premature convergence in stochastic policy optimization. We formalize this intuition with the following results.

Theorem 9. *If an algorithm $\mathcal{A}$ converges to a globally optimal policy at a rate $O(1/t^{\alpha})$ with positive probability, for $\alpha > 0$, then $\kappa(\mathcal{A}, a^*) \geq \alpha$.*

This theorem implies that a large committal rate for the optimal action is necessary for achieving fast convergence to the globally optimal policy, since the sub-optimality dominates how close the optimal action's probability is to 1, i.e., $(\pi^* - \pi_{\theta_t})^\top r \geq (1 - \pi_{\theta_t}(a^*)) \cdot \Delta$. Therefore $(\pi^* - \pi_{\theta_t})^\top r \in O(1/t^{\alpha})$ implies $1 - \pi_{\theta_t}(a^*) \in O(1/t^{\alpha})$. Combining this result with Theorem 5 formally establishes the following trade-off, which indicates that if there is no external mechanism that allows an algorithm to determine the difference between optimal and sub-optimal actions using only on-policy samples, then there must be an inherent trade-off between exploiting geometry to accelerate convergence versus achieving global optimality almost surely.

Theorem 10 (Geometry-Convergence trade-off)**.** *For any algorithm $\mathcal{A}$, if $\kappa(\mathcal{A}, a^*) = \kappa(\mathcal{A}, a)$ for at least one sub-optimal action $a \in [K]$, then $\mathcal{A}$ can only exhibit at most one of the following two behaviors:* ***(i)*** *$\mathcal{A}$ converges to a globally optimal policy almost surely;* ***(ii)*** *$\mathcal{A}$ converges at a rate faster than $O(1/t)$ with positive probability, but then must also converge to a non-optimal policy with positive probability.*

In other words, if $\mathcal{A}$ has a chance to converge to a global optimum, then either $\mathcal{A}$ converges to the globally optimal policy with probability 1 but at a rate no better than $\Omega(1/t)$, or it achieves a faster than $O(1/t)$ convergence rate but fails to converge to the globally optimal policy with some positive probability. This trade-off between the geometry and convergence is faced by any stochastic policy optimization algorithm that is not informed by external oracle information that allows it to distinguish optimal and sub-optimal actions based on on-policy samples.

Remark 4. *Theorem 10 means an algorithm can achieve at most one of the mentioned two results. It is possible that an algorithm achieves neither (e.g., staying or wandering).*

4.3 Exploiting External Information

Property I	$\kappa(\mathcal{A}, a) > 1$	$\kappa(\mathcal{A}, a) \leq 1$, for all sub-optimal action $a \in [K]$		
Algorithm	NPG GNPG	Softmax PG SAMBA	Staying Wandering (NPG + large baseline)	NPG + oracle baseline
Property II	$\kappa(\mathcal{A}, a^*) = \kappa(\mathcal{A}, a)$, for at least one sub-optimal action $a \in [K]$			$\kappa(\mathcal{A}, a^*) \neq \kappa(\mathcal{A}, a)$

Figure 1: Different algorithmic behaviours subdivided by two properties of committal rate. SAMBA [20] does not use parametric policies and is discussed in the appendix.

In Theorem 10, the condition of $\kappa(\mathcal{A}, a^*) = \kappa(\mathcal{A}, a)$ for at least one sub-optimal action $a \in [K]$ is necessary for the trade-off to hold. If this condition can somehow be bypassed, for example, by providing problem specific information, then it is possible to simultaneously achieve faster rates and almost sure convergence to a global optimum. For example, consider the NPG with oracle baseline of Update 7. As mentioned before, we have $\kappa(\mathcal{A}, a^*) = \infty$ and $\kappa(\mathcal{A}, a) = 0$ for all $a \neq a^*$, breaking the mentioned condition, which allows $\mathcal{A}$ to enjoy almost sure global convergence as well as a $O(e^{-c \cdot t})$ rate. Of course, such a fortuitous outcome required a very specific baseline that is aware of both the optimal reward and the reward gap. Without introducing external mechanisms that inform an on-policy algorithm it appears that such information cannot be recovered sufficiently quickly from sample data alone [21]. Nevertheless, it remains an open question to prove that this is not possible, or whether some other strategy might allow an on-policy stochastic policy optimization algorithm to avoid the condition of Theorem 10 and achieve both fast rates and almost sure global convergence.

Figure 1 summarizes all the iteration behaviours we studied in this paper, organized by two properties of committal rate: *(i)* possible failure if $\kappa(\mathcal{A}, a) > 1$ for at least one sub-optimal action a; and *(ii)* an inherent geometry-convergence trade-off if $\kappa(\mathcal{A}, a^*) = \kappa(\mathcal{A}, a)$ for at least one sub-optimal action a. It remains open to study where other algorithms suit themselves in this diagram.

5 Initialization Sensitivity and Ensemble Methods

In this section, we will keep exploiting the newly introduced concept, committal rate, to further reveal the mystery observations in practice about the sensitivity of the initialization in policy optimization [22]. With the understanding of this unavoidable phenomenon, we introduce ensemble method and quantitatively characterize the successful rate in terms of number of trials.

5.1 Initialization Sensitivity

It has been observed empirically that RL algorithms are sensitive to initialization in practice: the same algorithm can produce remarkably different performance given different random seeds [22]. Some existing work has attempted to explain initialization sensitivity due to the softmax transform [9], but such results only hold for true gradients and apply to standard PG methods.

Using the committal rate theory developed above, we can provide a new explanation and additional understanding of the initialization sensitivity of practical policy optimization algorithms. Most well-performing policy optimization algorithms in practice, such as TRPO and PPO [14, 15], are based on NPG, which exploits geometry to accelerate PG in true gradient settings. However, according to Theorem 10, such fast convergence must incur a positive probability of failing to reach a global optimum, even in bandit settings. Therefore, the need to attempt multiple random seeds to achieve success is an unavoidable consequence of using these algorithms according to this theory.

5.2 Ensemble Methods

The committal rate theory also explains why ensemble methods [23–25], *i.e.*, running a policy optimization algorithm in multiple parallel threads and picking the best performing one, can provably work well. This is because a fast algorithm for the true gradient setting can have a positive probability of success or failure across different initializations while always converging quickly. In which case, multiple independent runs can then be used to reduce the failure probability to any desired positive value, while retaining efficiency (if full parallelism can be maintained).

Theorem 11. *With probability $1 - \delta$, the best single run among $O(\log(1/\delta))$ independent runs of NPG (GNPG) converges to a globally optimal policy at an $O(e^{-c \cdot t})$ rate.*

It is known that softmax PG can get stuck on long plateaus for even true gradient settings [9, 10], which means almost sure global convergence does not necessarily imply good practical performance. Therefore, it is a reasonable choice to perform well with the compromise of small failure probability. Here we consider simply best selection, and it remains open to study whether and how practical training tricks, such as proximal update [15, 26], momentum [27], and regularization [28] stabilize training by increasing the success probability under stochastic settings.

6 Conclusion and Future Work

This paper introduces the committal rate theory, which not only explains why faster policy optimization algorithms in the true gradient setting become dominated by slower counterparts in the on-policy stochastic setting, but also reveals an inherent geometry-convergence trade-off in stochastic policy optimization. The theory also explains empirical observations of sensitivity to random initialization for practical policy optimization algorithms as well as the effectiveness of ensemble methods.

One interesting future direction is to study necessary and sufficient conditions for almost sure global convergence, which could be much weaker than the bounded variance assumption. Another important direction is to investigate whether other techniques might be integrated into on-policy stochastic optimization to break the condition of Theorem 10 and bypass the geometry-convergence trade-off, to achieve both almost sure global convergence and a faster than $O(1/t)$ rate.

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1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
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   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]
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