Quasi-Newton Methods for Saddle Point Problems

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Abstract

This paper studies quasi-Newton methods for strongly-convex-strongly-concave saddle point problems. We propose random Broyden family updates, which have explicit local superlinear convergence rate of $\mathcal{O}((1-1/(d\varkappa^2))^{k(k-1)/2})$, where d is the dimension of the problem, $\varkappa$ is the condition number and k is the number of iterations. The design and analysis of proposed algorithm are based on estimating the square of indefinite Hessian matrix, which is different from classical quasi-Newton methods in convex optimization. We also present two specific Broyden family algorithms with BFGS-type and SR1-type updates, which enjoy the faster local convergence rate of $\mathcal{O}((1-1/d)^{k(k-1)/2})$. Our numerical experiments show proposed algorithms outperform classical first-order methods.

1 Introduction

In this paper, we focus on the following smooth saddle point problem

$$\min_{\mathbf{x}\in\mathbb{R}^{d_x}} \max_{\mathbf{y}\in\mathbb{R}^{d_y}} f(\mathbf{x},\mathbf{y}), \tag{1}$$

where f is strongly-convex in $\mathbf{x}$ and strongly-concave in $\mathbf{y}$. We target to find the saddle point $(\mathbf{x}^*,\mathbf{y}^*)$ which holds that

$$f(\mathbf{x}^*,\mathbf{y}) \le f(\mathbf{x}^*,\mathbf{y}^*) \le f(\mathbf{x},\mathbf{y}^*)$$

for all $\mathbf{x}\in\mathbb{R}^{d_x}$ and $\mathbf{y}\in\mathbb{R}^{d_y}$. This formulation is widely used in game theory [2, 44], AUC maximization [18, 48], fairness-aware machine learning [49], robust optimization [3, 13, 15, 41], empirical risk minimization [51] and reinforcement learning [12].

There are a great number of first-order optimization algorithms for solving problem (1), including extragradient method [22, 43], optimistic gradient descent ascent [9, 33], proximal point method [36] and dual extrapolation [28]. These algorithms iterate with first-order oracle and achieve linear convergence. Lin et al. [25], Wang and Li [45] used Catalyst acceleration to reduce the complexity for unbalanced saddle point problem, nearly matching the lower bound of first-order algorithms [30, 50] under some specific assumptions. Compared with first-order methods, second-order methods usually enjoy superior convergence in numerical optimization. Huang et al. [20] extended cubic regularized Newton (CRN) method [27, 28] to solve saddle point problem (1), which has quadratic local convergence. However, each iteration of CRN requires accessing the exact Hessian matrix and solving the corresponding linear systems. These steps arise $\mathcal{O}\left(d_x^3+d_y^3\right)$ time complexity, which is too expensive for high dimensional problems.

Quasi-Newton methods [4–6, 10, 42] are popular ways to avoid accessing exact second-order information applied in standard Newton methods. They approximate the Hessian matrix based on the Broyden family updating formulas [4], which significantly reduces the computational cost.

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Table 1: We summarize the convergence behaviors of proposed algorithms for solving saddle point problem in the view of gradient norm $\lambda_{k+k_0} \stackrel{\text{def}}{=} \|\nabla f(\mathbf{z}_{k+k_0})\|$ after $(k+k_0)$ iterations, where $d \stackrel{\text{def}}{=} d_x + d_y$ is dimensions of the problem and $\varkappa$ is the condition number. The results come from 3.18 and the upper bound holds with high probability at least $1-\delta$.

Algorithms	Upper Bound of λ_{k+k_0}	k_0
Broyden (Alg. 5)	$\left(1-\frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{4\varkappa^2}\right)^{k_0}$	$\mathcal{O}\left(d\varkappa^2\ln\left(\frac{d\varkappa}{\delta}\right)\right)$
BFGS/SR1 (Alg. 6 and 7)	$\left(1-\frac{1}{d+1}\right)^{k(k-1)/2}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{4\varkappa^2}\right)^{k_0}$	$\mathcal{O}\left((d+\varkappa^2)\ln\left(\frac{d\varkappa}{\delta}\right)\right)$

These algorithms are well studied for convex optimization. The famous quasi-Newton methods including Davidon-Fletcher-Powell (DFP) method [10, 14], Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [5, 6, 42] and symmetric rank 1 (SR1) method [4, 10] enjoy local superlinear convergence [7, 11, 34] when the objective function is strongly-convex. Recently, Rodomanov and Nesterov [37, 38, 39] proposed greedy and random variants of quasi-Newton methods, which first achieves non-asymptotic superlinear convergence. Later, Lin et al. [24] established a better convergence rate which is condition-number-free. Jin and Mokhtari [21], Ye et al. [47] showed the non-asymptotic superlinear convergence rate also holds for classical DFP, BFGS and SR1 methods.

In this paper, we study quasi-Newton methods for saddle point problem (1). Note that when the Hessian matrix of the objective function is indefinite, the existing Broyden family update formulas and their convergence analysis cannot be applied directly. To overcome this issue, we propose a variant framework of random quasi-Newton methods for saddle point problems, which approximates the square of the Hessian matrix during the iteration. Our theoretical analysis characterizes the convergence rate by the gradient norm, rather than the weighted norm of gradient used in convex optimization [21, 24, 37–39, 47]. We summarize the theoretical results for proposed algorithms in Table 1. The local convergence behaviors for all of the algorithms have two periods. The first period has k_0 iterations with a linear convergence rate $\mathcal{O}((1-1/\varkappa^2)^{k_0})$. The second one enjoys superlinear convergence, that is,

- For general Broyden family methods, we have $\mathcal{O}\big(\big(1-1/(d\varkappa^2+1)\big)^{k(k-1)/2}\big)$.
- For BFGS method and SR1 method, we have $\mathcal{O}\big(\big(1-1/(d+1)\big)^{k(k-1)/2}\big)$, which is condition-number-free.

Paper Organization In Section 2, we state the notation and preliminaries of this paper. In Section 3, we first propose random quasi-Newton methods for quadratic saddle point problem which enjoys local superlinear convergence. Then we extend it to solve general strongly-convex-strongly-concave saddle point problems. In Section 5, we provide numerical experiments to validate our algorithms on popular machine learning models. All proofs, experiment details and extensions are deferred to appendix.

2 Notation and Preliminaries

We use $\|\cdot\|$ to present spectral norm and Euclidean norm of matrix and vector respectively. We denote the standard basis for $\mathbb{R}^d$ by $\{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ and let $\mathbf{I}$ be the identity matrix. The trace of a square matrix is denoted by $\text{tr}(\cdot)$. Given two positive definite matrices $\mathbf{G}$ and $\mathbf{H}$, we define their inner product as $\langle\mathbf{G},\mathbf{H}\rangle \stackrel{\text{def}}{=} \text{tr}(\mathbf{GH})$.

We introduce the quantity $\sigma_{\mathbf{H}}(\mathbf{G}) \stackrel{\text{def}}{=} \langle\mathbf{H}^{-1},\mathbf{G}-\mathbf{H}\rangle = \langle\mathbf{H}^{-1},\mathbf{G}\rangle - d$, which is used to measure how well does matrix $\mathbf{G}$ approximate matrix $\mathbf{H}$. If we further suppose $\mathbf{G}\succeq\mathbf{H}$, then according to Rodomanov and Nesterov [37] it holds that $\mathbf{G}-\mathbf{H}\preceq\langle\mathbf{H}^{-1},\mathbf{G}-\mathbf{H}\rangle\mathbf{H} = \sigma_{\mathbf{H}}(\mathbf{G})\mathbf{H}$.

Using the notation of problem (1), we let $\mathbf{z} = [\mathbf{x}; \mathbf{y}] \in \mathbb{R}^d$ where $d \stackrel{\text{def}}{=} d_x + d_y$ and denote the gradient and Hessian matrix of f at $(\mathbf{x}, \mathbf{y})$ as $\mathbf{g}(\mathbf{z}) \in \mathbb{R}^d$ and $\hat{\mathbf{H}}(\mathbf{z}) \in \mathbb{R}^{d \times d}$.

We suppose the saddle point problem (1) satisfies the following assumptions.

Assumption 2.1. The objective function $f(\mathbf{x}, \mathbf{y})$ is twice differentiable and has L-Lipschitz continuous gradient and L_2-Lipschitz continuous Hessian , i.e., there exists constants $L > 0$ and $L_2 > 0$ such that for any $\mathbf{z} = [\mathbf{x}; \mathbf{y}], \mathbf{z}' = [\mathbf{x}'; \mathbf{y}'] \in \mathbb{R}^d$, we have $\|\mathbf{g}(\mathbf{z}) - \mathbf{g}(\mathbf{z}')\| \leq L\|\mathbf{z} - \mathbf{z}'\|$ and $\|\hat{\mathbf{H}}(\mathbf{z}) - \hat{\mathbf{H}}(\mathbf{z}')\| \leq L_2\|\mathbf{z} - \mathbf{z}'\|$.

Assumption 2.2. The objective function $f(\mathbf{x}, \mathbf{y})$ is twice differentiable, μ-strongly-convex in $\mathbf{x}$ and μ-strongly-concave in $\mathbf{y}$, i.e., there exists constant $\mu > 0$ such that $\nabla^2_{\mathbf{xx}} f(\mathbf{x}, \mathbf{y}) \succeq \mu \mathbf{I}$ and $\nabla^2_{\mathbf{yy}} f(\mathbf{x}, \mathbf{y}) \preceq -\mu \mathbf{I}$ for any $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$.

The L-Lipschitz continuous of gradient means the spectral norm of Hessian matrix $\hat{\mathbf{H}}(\mathbf{z})$ can be upper bounded, that is $\|\hat{\mathbf{H}}(\mathbf{z})\| \leq L$. Additionally, the condition number of the objective function is defined as $\varkappa \stackrel{\text{def}}{=} L/\mu$.

3 Quasi-Newton Methods for Saddle Point Problems

In this section, we focus on designing quasi-Newton methods for saddle point problem and showing their superlinear local convergence rate. The update rule of standard Newton's method can be written as $\mathbf{z}_+ = \mathbf{z} - \left(\hat{\mathbf{H}}(\mathbf{z})\right)^{-1}\mathbf{g}(\mathbf{z})$ for solving problem (1). It has quadratic local convergence, but takes $\mathcal{O}(d^3)$ time complexity per iteration. For convex minimization, quasi-Newton methods including BFGS/SR1 [4–6, 10, 42] and their variants [24, 37, 38, 47] focus on approximating the Hessian which can reduce the computational cost to $\mathcal{O}\left(d^2\right)$ for each round. However, all these algorithms and related convergence analysis are based on the assumption that the Hessian matrix is positive definite, which is not suitable for our saddle point problems since $\hat{\mathbf{H}}(\mathbf{z})$ is indefinite.

We introduce the auxiliary matrix $\mathbf{H}(\mathbf{z})$ be the square of Hessian $\mathbf{H}(\mathbf{z}) \stackrel{\text{def}}{=} \left(\hat{\mathbf{H}}(\mathbf{z})\right)^2$. The following lemma shows that $\mathbf{H}(\mathbf{z})$ is positive definite.

Lemma 3.1. *Under Assumption 2.1 and 2.2, we have $\mu^2\mathbf{I} \preceq \mathbf{H}(\mathbf{z}) \preceq L^2\mathbf{I}$ for all $\mathbf{z} \in \mathbb{R}^d$.*

Hence, we can reformulate the update of Newton's method by

$$\begin{aligned} \mathbf{z}_+ =& \mathbf{z} - [(\hat{\mathbf{H}}(\mathbf{z}))^2]^{-1}\hat{\mathbf{H}}(\mathbf{z})\mathbf{g}(\mathbf{z}) \\ =& \mathbf{z} - \mathbf{H}(\mathbf{z})^{-1}\hat{\mathbf{H}}(\mathbf{z})\mathbf{g}(\mathbf{z}). \end{aligned} \tag{2}$$

Then it is natural to characterize the second-order information by estimating the auxiliary matrix $\mathbf{H}(\mathbf{z})$, rather than the indefinite Hessian $\hat{\mathbf{H}}(\mathbf{z})$. If one can obtain a symmetric positive definite matrix $\mathbf{G} \in \mathbb{R}^{d \times d}$ as an estimator for $\mathbf{H}(\mathbf{z})$, the update rule of (2) can be approximated by

$$\mathbf{z}_+ = \mathbf{z} - \mathbf{G}^{-1}\hat{\mathbf{H}}(\mathbf{z})\mathbf{g}(\mathbf{z}). \tag{3}$$

The remainder of this section introduce strategies to construct $\mathbf{G}$, resulting the quasi-Newton methods for saddle point problem with local superlinear convergence. We should point out the implementation of iteration (3) is unnecessary to construct Hessian matrix $\hat{\mathbf{H}}(\mathbf{z})$ explicitly, since we are only interested in the Hessian-vector product $\hat{\mathbf{H}}(\mathbf{z})\mathbf{g}(\mathbf{z})$ which can be computed efficiently [31, 40].

3.1 The Broyden Family Updates

We first review some basic results for quasi-Newton methods in convex optimization. We introduce the Broyden family [29, Section 6.3] of quasi-Newton updates for approximating an positive definite matrix $\mathbf{H} \in \mathbb{R}^{d \times d}$ by using the information of current estimator $\mathbf{G} \in \mathbb{R}^{d \times d}$.

Definition 3.2. Suppose two positive definite matrices $\mathbf{H} \in \mathbb{R}^{d\times d}$ and $\mathbf{G} \in \mathbb{R}^{d\times d}$ satisfy $\mathbf{H} \preceq \mathbf{G}$. For any $\mathbf{u} \in \mathbb{R}^d$, if $\mathbf{G}\mathbf{u} = \mathbf{H}\mathbf{u}$, we define $\mathrm{Broyd}_\tau(\mathbf{G}, \mathbf{H}, \mathbf{u}) \stackrel{\text{def}}{=} \mathbf{H}$. Otherwise, we define

$$\begin{aligned}\mathrm{Broyd}_\tau(\mathbf{G}, \mathbf{H}, \mathbf{u}) \stackrel{\text{def}}{=} &(1-\tau)\left[\mathbf{G} - \frac{(\mathbf{G}-\mathbf{H})\mathbf{u}\mathbf{u}^\top(\mathbf{G}-\mathbf{H})}{\mathbf{u}^\top(\mathbf{G}-\mathbf{H})\mathbf{u}}\right] \\ &+ \tau\left[\mathbf{G} - \frac{\mathbf{H}\mathbf{u}\mathbf{u}^\top\mathbf{G} + \mathbf{G}\mathbf{u}\mathbf{u}^\top\mathbf{H}}{\mathbf{u}^\top\mathbf{H}\mathbf{u}} + \left(\frac{\mathbf{u}^\top\mathbf{G}\mathbf{u}}{\mathbf{u}^\top\mathbf{H}\mathbf{u}} + 1\right)\frac{\mathbf{H}\mathbf{u}\mathbf{u}^\top\mathbf{H}}{\mathbf{u}^\top\mathbf{H}\mathbf{u}}\right].\end{aligned} \tag{4}$$

The different choices of parameter τ for above formula contain several popular quasi-Newton updates:

- For $\tau = \mathbf{u}^\top\mathbf{H}\mathbf{u}/\left(\mathbf{u}^\top\mathbf{G}\mathbf{u}\right) \in [0,1]$, it corresponds to the BFGS update

$$\mathrm{BFGS}(\mathbf{G}, \mathbf{H}, \mathbf{u}) \stackrel{\text{def}}{=} \mathbf{G} - \frac{\mathbf{G}\mathbf{u}\mathbf{u}^\top\mathbf{G}}{\mathbf{u}^\top\mathbf{G}\mathbf{u}} + \frac{\mathbf{H}\mathbf{u}\mathbf{u}^\top\mathbf{H}}{\mathbf{u}^\top\mathbf{H}\mathbf{u}}. \tag{5}$$

- For $\tau = 0$, it corresponds to the SR1 update

$$\mathrm{SR1}(\mathbf{G}, \mathbf{H}, \mathbf{u}) \stackrel{\text{def}}{=} \mathbf{G} - \frac{(\mathbf{G}-\mathbf{H})\mathbf{u}\mathbf{u}^\top(\mathbf{G}-\mathbf{H})}{\mathbf{u}^\top(\mathbf{G}-\mathbf{H})\mathbf{u}}. \tag{6}$$

Now we introduce the update rule [24, 37] by choosing $\mathbf{u}$ as

$$\mathbf{u} \sim (\mathbf{0}, \mathbf{I}) \qquad \text{or} \qquad \mathbf{u} \sim \mathrm{Unif}\left(\mathcal{S}^{d-1}\right). \tag{7}$$

The following lemma shows applying the update rule (4) with (7) leads to a new estimator with tighter error bound in the measure of $\sigma_{\mathbf{H}}(\cdot)$.

Lemma 3.3 (Modified from Lin et al. [24, Theorem 3.1])**.** *Suppose two positive definite matrices $\mu^2\mathbf{I} \preceq \mathbf{H} \preceq L^2\mathbf{I}$ and $\mathbf{G} \in \mathbb{R}^{d\times d}$ satisfy that $\mathbf{H} \preceq \mathbf{G}$. Let $\mathbf{G}_+ = \mathrm{Broyd}_\tau\left(\mathbf{G}, \mathbf{H}, \mathbf{u}\right)$), where $\mathbf{u}$ is chosen random method as* (7)*, then for any $\tau \in [0,1]$, we have $\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_+)\right] \leq \left(1 - 1/(d\varkappa^2)\right)\sigma_{\mathbf{H}}(\mathbf{G})$*

For specific Broyden family updates BFGS and SR1 shown in (5) and (6), the update can achieve a better convergence result. Concretely, for BFGS method, we first find $\mathbf{L}$ such that $\mathbf{G}^{-1} = \mathbf{L}^\top\mathbf{L}$, where $\mathbf{L}$ is an upper triangular matrix. This step can be implemented with $\mathcal{O}(d^2)$ complexity [24, Proposition 1]. We present the subroutine for factorizing $\mathbf{G}^{-1}$ in Algorithm 1 and give its detailed implementation in the appendix. And we use the direction $\mathbf{L}^\top\mathbf{u}$ instead of $\mathbf{u}$ for the BFGS update. Applying the BFGS update rule (5) with formula (7), we obtain a condition-number-free result as follows.

Lemma 3.4 (Modified from Lin et al. [24, Theorem 4.2])**.** *Suppose two positive definite matrices $\mathbf{H} \in \mathbb{R}^{d\times d}$ and $\mathbf{G} \in \mathbb{R}^{d\times d}$ satisfy $\mathbf{H} \preceq \mathbf{G}$. Let $\mathbf{G}_+ = \mathrm{BFGS}\left(\mathbf{G}, \mathbf{H}, \mathbf{L}^\top\mathbf{u}\right)$, where $\mathbf{u}$ is chosen by the random method in* (7) *and $\mathbf{L}$ is an upper triangular matrix such that $\mathbf{G}^{-1} = \mathbf{L}^\top\mathbf{L}$. Then we have $\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_+)\right] \leq (1 - 1/d)\,\sigma_{\mathbf{H}}(\mathbf{G})$.*

Remark 3.5. Note that the step of conducting $\mathbf{G}^{-1} = \mathbf{L}^\top\mathbf{L}$ requires QR decomposition of rank-1 change matrix which requires $\mathcal{O}(26d^2)$ flops [16, Section 12.5.1]. We do not recommend using this BFGS update strategy in practice when n is large.

The convergence of the SR1 update can be characterized by the measure $\tau_{\mathbf{H}}(\mathbf{G}) \stackrel{\text{def}}{=} \mathrm{tr}(\mathbf{G} - \mathbf{H})$. Applying the SR1 update rule (6) with formula (7), the convergence also holds a condition-number-free result.

Lemma 3.6 (Modified from Lin et al. [24, Theorem 4.1])**.** *Suppose two positive definite matrices $\mathbf{H} \in \mathbb{R}^{d\times d}$ and $\mathbf{G} \in \mathbb{R}^{d\times d}$ satisfy $\mathbf{H} \preceq \mathbf{G}$. Let $\mathbf{G}_+ = \mathrm{SR1}\left(\mathbf{G}, \mathbf{H}, \mathbf{u}\right)$, where $\mathbf{u}$ is chosen by the random method in* (7)*. Then we have $\mathbb{E}\left[\tau_{\mathbf{H}}\left(\mathbf{G}_+\right)\right] \leq (1 - 1/d)\,\tau_{\mathbf{H}}(\mathbf{G})$.*

3.2 Algorithms for Quadratic Saddle Point Problems

We now solve simple quadratic saddle point problem that $f(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{z}^\top\mathbf{A}\mathbf{z} - \mathbf{b}^\top\mathbf{z}$ in (1) where $\mathbf{z} = [\mathbf{x}; \mathbf{y}]$, $\mathbf{b} \in \mathbb{R}^d$, $\mathbf{A} \in \mathbb{R}^{d\times d}$ is symmetric and $d = d_x + d_y$. We suppose $\mathbf{A}$ could be partitioned as $\mathbf{A} = \begin{bmatrix}\mathbf{A}_{\mathbf{xx}} & \mathbf{A}_{\mathbf{xy}} \\ \mathbf{A}_{\mathbf{yx}} & \mathbf{A}_{\mathbf{yy}}\end{bmatrix}$ where the sub-matrices $\mathbf{A}_{\mathbf{xx}} \in \mathbb{R}^{d_x\times d_x}$, $\mathbf{A}_{\mathbf{xy}} \in \mathbb{R}^{d_x\times d_y}$, $\mathbf{A}_{\mathbf{yx}} \in \mathbb{R}^{d_y\times d_x}$

Algorithm 1 Fast-Chol$(\mathbf{H}, \mathbf{L}, \mathbf{u})$

Input: positive definite matrix $\mathbf{H} \in \mathbb{R}^{d\times d}$, upper triangular matrix $\mathbf{L} \in \mathbb{R}^{d}$, direction $\mathbf{u} \in \mathbb{R}^{d}$

Using QR-decomposition to obtain $[\mathbf{Q}, \mathbf{R}] = \mathrm{QR}\left(\mathbf{L}\left(\mathbf{I} - \mathbf{H}\mathbf{u}\mathbf{u}^\top/(\mathbf{u}^\top\mathbf{H}\mathbf{u})\right)\right)$

Calculate $\mathbf{v} = \mathbf{u}/\sqrt{\mathbf{u}^\top\mathbf{H}\mathbf{u}}$ and $[\mathbf{Q}', \mathbf{R}'] = \mathrm{QR}(\begin{bmatrix}\mathbf{v} & \mathbf{R}^\top\end{bmatrix}^\top)$

Output: $\hat{\mathbf{L}} = \mathbf{R}'$

Algorithm 2 Random-Broyden-Quadratic

Input: $\mathbf{z}_0 \in \mathbb{R}^d$, $\mathbf{G}_0 = L^2\mathbf{I}$ and $\tau_k \in [0, 1]$

for $k = 0, 1, \ldots$

  $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}\mathbf{g}(\mathbf{z}_k)$

  Randomly choose $\mathbf{u}_k$ from (7) and update $\mathbf{G}_{k+1} = \mathrm{Broyd}_{\tau_k}\left(\mathbf{G}_k, \mathbf{H}, \mathbf{u}_k\right)$

end for

and $\mathbf{A}_{\mathbf{yy}} \in \mathbb{R}^{d_y\times d_y}$ satisfy $\mathbf{A}_{\mathbf{xx}} \succeq \mu\mathbf{I}$, $\mathbf{A}_{\mathbf{yy}} \preceq -\mu\mathbf{I}$ and $\|\mathbf{A}\| \le L$. Using notations introduced in Section 2, we have $\mathbf{z} = [\mathbf{x}; \mathbf{y}]$, $\mathbf{g}(\mathbf{z}) = \mathbf{A}\mathbf{z} - \mathbf{b}$, $\hat{\mathbf{H}} \stackrel{\mathrm{def}}{=} \hat{\mathbf{H}}(\mathbf{z}) = \mathbf{A}$ and $\mathbf{H} \stackrel{\mathrm{def}}{=} \mathbf{H}(\mathbf{z}) = \mathbf{A}^2$.

We present the detailed procedure of random quasi-Newton methods for quadratic saddle point problem by using the Broyden family update, BFGS update and SR1 update in Algorithm 2, 3 and 4 respectively.

We define λ_k as the gradient norm at $\mathbf{z}_k$ for our convergence analysis, that is $\lambda_k \stackrel{\mathrm{def}}{=} \|\mathbf{g}(\mathbf{z}_k)\|$. The definition of λ_k in this paper is different from the measure used in convex optimization [24, 37][2], but it also holds the similar property as follows.

Lemma 3.7. *Assume we have $\eta_k \ge 1$ and $\mathbf{G}_k \in \mathbb{R}^{d\times d}$ such that $\mathbf{H} \preceq \mathbf{G}_k \preceq \eta_k\mathbf{H}$ for Algorithm 2-4, then we have $\lambda_{k+1} \le (1 - 1/\eta_k)\,\lambda_k$.*

The next theorem states the assumptions of Lemma 3.7 always holds with $\eta_k = \varkappa^2 \ge 1$, which means λ_k converges to 0 linearly.

Theorem 3.8. *For all $k \ge 0$, Algorithm 2, 3, 4 hold that $\lambda_k \le \left(1 - \frac{1}{\varkappa^2}\right)^k \lambda_0$ and $\mathbf{H} \preceq \mathbf{G}_k \le \varkappa$.*

Lemma 3.7 also implies superlinear convergence can be obtained if there exists η_k which converges to 1. Applying Lemma 3.3, 3.4 and 3.6, we can show it holds for proposed algorithms.

Theorem 3.9. *Solving quadratic saddle point problem by the proposed quasi-Newton Algorithms, for all $k \ge 0$, we have:*

1. *For the Broyden family method (Algorithm 2), we have $\mathbb{E}\left[\lambda_{k+1}/\lambda_k\right] \le \left(1 - 1/(d\varkappa^2)\right)^k d\varkappa^2$.*
2. *For the BFGS method (Algorithm 3), we have $\mathbb{E}\left[\lambda_{k+1}/\lambda_k\right] \le \left(1 - 1/d\right)^k d\varkappa^2$.*
3. *For the SR1 method (Algorithm 4), we have $\mathbb{E}\left[\lambda_{k+1}/\lambda_k\right] \le \left(1 - k/d\right) d\varkappa^4$.*

Combining the results of Theorem 3.8 and 3.9, we achieve the two-stages convergence behavior, that is, the algorithm has the global linear convergence and local superlinear convergence. We leave the formal description in appendix.

3.3 Algorithms for General Saddle Point Problems

In this section, we consider the general saddle point problem where $f(\mathbf{x}, \mathbf{y})$ in (1) satisfies Assumption 2.1 and 2.2. We propose quasi-Newton methods for solving the problem with local superlinear convergence and $\mathcal{O}(d^2)$ time complexity for each iteration.

[2] In later section, we will see the measure $\lambda_k \stackrel{\mathrm{def}}{=} \|\mathbf{g}_k\|$ is suitable to convergence analysis of quasi-Newton methods for saddle point problems.

Algorithm 3 Random-BFGS-Quadratic

Input: $\mathbf{z}_0 \in \mathbb{R}^d$, $\mathbf{G}_0 = L^2\mathbf{I}$, $\mathbf{L}_0 = L^{-1}\mathbf{I}$
for $k = 0, 1 \ldots$
  $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}\mathbf{g}(\mathbf{z}_k)$
  Randomly choose $\tilde{\mathbf{u}}_k$ from (7)
  $\mathbf{u}_k = \mathbf{L}_k^\top \tilde{\mathbf{u}}_k$
  $\mathbf{G}_{k+1} = \mathrm{BFGS}\left(\mathbf{G}_k, \mathbf{H}, \mathbf{u}_k\right)$
  $\mathbf{L}_{k+1} = \text{Fast-Chol}(\mathbf{H}, \mathbf{L}_k, \mathbf{u}_k)$
end for

Algorithm 4 Random-SR1-Quadratic

Input: $\mathbf{z}_0 \in \mathbb{R}^d$, $\mathbf{G}_0 = L^2\mathbf{I}$
$K \leq d+1$
for $k = 0, 1 \ldots, K$
  $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}\mathbf{g}(\mathbf{z}_k)$
  Randomly choose $\mathbf{u}_k$ from (7)
  $\mathbf{G}_{k+1} = \mathrm{SR1}\big(\mathbf{G}_k, \mathbf{H}, \mathbf{u}_k\big)$
end for

3.3.1 Algorithms

The key idea of designing quasi-Newton methods for saddle point problems is approximating the auxiliary matrix $\mathbf{H}(\mathbf{z}) \stackrel{\text{def}}{=} \big(\hat{\mathbf{H}}(\mathbf{z})\big)^2$ to characterize the second-order information. Since the Hessian of f is Lipschitz continuous and bounded by Assumption 2.1 and 2.2, which means the auxiliary matrix operator $\mathbf{H}(\mathbf{z})$ is also Lipschitz continuous.

Lemma 3.10. *Under Assumption 2.1 and 2.2, we have $\mathbf{H}(\mathbf{z})$ is $2LL_2$-Lipschitz continuous.*

Combining Lemma 3.1 and 3.10, we achieve the following properties of $\mathbf{H}(\mathbf{z})$, which analogize the strongly self-concordance in convex optimization [37].

Lemma 3.11. *Under Assumption 2.1 and 2.2, for all $\mathbf{z}, \mathbf{z}', \mathbf{w} \in \mathbb{R}^d$, the auxiliary matrix operator $\mathbf{H}(\cdot)$ satisfies $\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}') \preceq M\|\mathbf{z} - \mathbf{z}'\|\mathbf{H}(\mathbf{w})$, where $M = 2\varkappa^2 L_2/L$.*

Corollary 3.12. *Let $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^d$ and $r = \|\mathbf{z}' - \mathbf{z}\|$. Suppose the objective function f satisfies Assumption 2.1 and 2.2, then for all $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^d$ and $M = 2\varkappa^2 L_2/L$, the auxiliary matrix operator $\mathbf{H}(\cdot)$ holds that*

$$\frac{\mathbf{H}(\mathbf{z})}{(1+Mr)} \preceq \mathbf{H}(\mathbf{z}') \preceq (1+Mr)\mathbf{H}(\mathbf{z})$$

Different from the quadratic case, the auxiliary matrix $\mathbf{H}(\mathbf{z})$ is not fixed for general saddle point problem. Based on the smoothness of $\mathbf{H}(\mathbf{z})$, we apply Corollary 3.12 to generalize Lemma 3.13 as follows.

Lemma 3.13. *Let $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{G} \in \mathbb{R}^{d\times d}$ be a positive definite matrix such that $\mathbf{H}(\mathbf{z}) \preceq \mathbf{G} \preceq \eta\mathbf{H}(\mathbf{z})$ for some $\eta \geq 1$. In addition, define $\mathbf{z}_+ \in \mathbb{R}^d$ and $r = \|\mathbf{z}_+ - \mathbf{z}\|$, then for all $\mathbf{u} \in \mathbb{R}^d$, $\tau \in [0,1]$ and $M = 2\varkappa^2 L_2/L$, we have*

$$\mathbf{H}(\mathbf{z}_+) \preceq \mathrm{Broyd}_\tau(\tilde{\mathbf{G}}, \mathbf{H}(\mathbf{z}_+), \mathbf{u}) \preceq (1+Mr)^2\eta\mathbf{H}(\mathbf{z}_+),$$

where $\tilde{\mathbf{G}} \stackrel{\text{def}}{=} (1+Mr)\mathbf{G} \succeq \mathbf{H}(\mathbf{z}_+)$.

Lemma 3.13 implies it is reasonable to have the algorithms using $\mathbf{G}_{k+1} = \mathrm{Broyd}_{\tau_k}(\tilde{\mathbf{G}}_k, \mathbf{H}_{k+1}, \mathbf{u}_k)$ with $\tilde{\mathbf{G}}_k = (1+Mr_k)\mathbf{G}_k$ and $r_k = \|\mathbf{z}_{k+1} - \mathbf{z}_k\|$. Similarly, we can also achieve $\mathbf{G}_{k+1}$ by such $\tilde{\mathbf{G}}_k$ for specific BFGS and SR1 update. Combining this with iteration (3), we propose several quasi-Newton methods for general strongly-convex-strongly-concave saddle point problems. The details are shown in Algorithm 5, 6 and 7 for Broyden family, BFGS and SR1 updates respectively.

3.3.2 Convergence Analysis

Let us consider the convergence guarantee for algorithms proposed in Section 3.3.1. We introduce the following notations to simplify the presentation. We let $\{\mathbf{z}_k\}$ be the sequence generated from Algorithm 5, 6 or 7 and denote

$$\mathbf{g}_k \stackrel{\text{def}}{=} \mathbf{g}(\mathbf{z}_k), \quad r_k \stackrel{\text{def}}{=} \|\mathbf{z}_{k+1} - \mathbf{z}_k\|, \quad \hat{\mathbf{H}}_k \stackrel{\text{def}}{=} \hat{\mathbf{H}}(\mathbf{z}_k) \quad \text{and} \quad \mathbf{H}_k \stackrel{\text{def}}{=} \big(\hat{\mathbf{H}}(\mathbf{z}_k)\big)^2.$$

We still use gradient norm $\lambda_k \stackrel{\text{def}}{=} \|\nabla f(\mathbf{z}_k)\|$ for analysis and establish the relationship between λ_k and λ_{k+1}, which is shown in Lemma 3.14.

Algorithm 5 Random-Broyden-General

Input: $\mathbf{z}_0 \in \mathbb{R}^d$, $\mathbf{G}_0 \succeq \mathbf{H}$, $\tau_k \in [0,1]$ and $M \geq 0$.
for $k = 0, 1 \ldots$
  $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}_k \mathbf{g}_k$
  Compute $r_k = \|\mathbf{z}_{k+1} - \mathbf{z}_k\|$ and $\tilde{\mathbf{G}}_k = (1 + M r_k)\mathbf{G}_k$
  Randomly choose $\mathbf{u}_k$ from (7) and update $\mathbf{G}_{k+1} = \mathrm{Broyd}_{\tau_k}(\tilde{\mathbf{G}}_k, \mathbf{H}_{k+1}, \mathbf{u}_k)$.
end for

Algorithm 6 Random-BFGS-General

Input: $\mathbf{z}_0 \in \mathbb{R}^d$, $\mathbf{G}_0 \succeq \mathbf{H}$, $M \geq 0$.
$\mathbf{L}_0 = \mathbf{G}_0^{-1/2}$
for $k = 0, 1 \ldots$
  $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}_k \mathbf{g}_k$
  $r_k = \|\mathbf{z}_{k+1} - \mathbf{z}_k\|$
  $\tilde{\mathbf{G}}_k = (1 + M r_k)\mathbf{G}_k$
  $\tilde{\mathbf{L}}_k = \mathbf{L}_k / \sqrt{1 + M r_k}$
  Randomly choose $\mathbf{u}_k$ from (7)
  $\mathbf{G}_{k+1} = \mathrm{BFGS}(\tilde{\mathbf{G}}_k, \mathbf{H}_{k+1}, \tilde{\mathbf{L}}_k \mathbf{u}_k)$
  $\mathbf{L}_{k+1} = \text{Fast-Chol}(\mathbf{H}_{k+1}, \tilde{\mathbf{L}}_k, \tilde{\mathbf{L}}_k \mathbf{u}_k)$
end for

Algorithm 7 Random-SR1-General

Input: $\mathbf{z}_0 \in \mathbb{R}^d$, $\mathbf{G}_0 \succeq \mathbf{H}$ and $M \geq 0$
for $k = 0, 1 \ldots$
  $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}_k \mathbf{g}_k$
  $r_k = \|\mathbf{z}_{k+1} - \mathbf{z}_k\|$
  $\tilde{\mathbf{G}}_k = (1 + M r_k)\mathbf{G}_k$
  Randomly choose $\mathbf{u}_k$ from (7)
  $\mathbf{G}_{k+1} = \mathrm{SR1}(\tilde{\mathbf{G}}_k, \mathbf{H}_{k+1}, \mathbf{u}_k)$
end for

Lemma 3.14. *Using Algorithm 5, 6 and 7, suppose we have $\mathbf{H}_k \preceq \mathbf{G}_k \preceq \eta_k \mathbf{H}_k$, for some $\eta_k \geq 1$ and let $\beta = L_2/(2\mu^2)$, then we have*

$$\lambda_{k+1} \leq \left(1 - \frac{1}{\eta_k}\right)\lambda_k + \beta\lambda_k^2 \quad \textit{and} \quad r_k \leq \frac{\lambda_k}{\mu}.$$

Rodomanov and Nesterov [37, Lemma 4.3] derive a result similar to Lemma 3.14 for minimizing the strongly-convex function $\hat{f}(\cdot)$ on the different measure $\lambda_{\hat{f}}(\cdot) \stackrel{\text{def}}{=} \langle \nabla \hat{f}(\cdot), \nabla^2 \hat{f}(\cdot)^{-1} \nabla \hat{f}(\cdot)\rangle$.[3] Note that our algorithms are based on the iteration rule $\mathbf{z}_{k+1} = \mathbf{z}_k - \mathbf{G}_k^{-1}\hat{\mathbf{H}}_k \mathbf{g}_k$. Compared with quasi-Newton methods for convex optimization, there exists an additional term $\hat{\mathbf{H}}_k$ between $\mathbf{G}_k^{-1}$ and $\mathbf{g}_k$, which leads to the fact that the convergence analysis based on $\lambda_{\hat{f}}(\mathbf{z}_k)$ is difficult. Fortunately, we find using gradient norm λ_k directly makes the analysis achievable.

For further analysis, we also denote $\sigma_k \stackrel{\text{def}}{=} \sigma_{\mathbf{H}_k}(\mathbf{G}_k) = \langle \mathbf{H}_k^{-1}, \mathbf{G}_k \rangle - d$.

Then we establish the linear convergence for the first period of iterations, which can be viewed as the extension of Theorem 3.8. Note that the following result holds for Algorithm 5, 6 and 7; and it does not depend on the choice of $\mathbf{u}_k$.

Theorem 3.15. *Using Algorithm 5, 6 and 7 by $\mathbf{G}_0 = L^2\mathbf{I}$ and $M = 2\varkappa^2 L_2/L$, suppose the initial point $\mathbf{z}_0$ is sufficiently close to $\mathbf{z}^*$ such that $M\lambda_0/\mu \leq \ln b/(4b\varkappa^2)$ with $1 < b < 5$, then we have $\lambda_k \leq \left(1 - 1/2b\varkappa^2\right)^k \lambda_0$ and $\mathbf{H}_k \preceq \mathbf{G}_k \preceq \exp\left(2\sum_{i=0}^{k-1}\rho_i\right)\varkappa^2\mathbf{H}_k \preceq b\varkappa^2\mathbf{H}_k$ for all $k \geq 0$, where $\rho_k \stackrel{\text{def}}{=} M\lambda_k/\mu$.*

We now analyze how σ_k changes after one iteration to show the local superlinear convergence for the Broyden family method (Algorithm 5) and the BFGS method (Algorithm 6). Recall that σ_k is defined to measure how well does matrix $\mathbf{G}_k$ approximate $\mathbf{H}_k$.

Lemma 3.16. *Solving the general strongly-convex-strongly-concave saddle point problem* (1) *under Assumption 2.1 and 2.2 by our quasi-Newton algorithms and supposing the sequences $\mathbf{G}_0, \cdots, \mathbf{G}_k$ generated by the Algorithm 5 and 6 are given, then we have the following results for σ_k for all k:*

[3] The original notations of Rodomanov and Nesterov [37] is minimizing the strongly-convex function $f(\cdot)$, to avoid ambiguity, we use notations $\hat{f}(\cdot)$ and $\lambda_{\hat{f}}$ to describe their work in this paper.

1. *The random Broyden family method (Algorithm 5) holds that*

$$\mathbb{E}\left[\sigma_{k+1}\right] \leq \left(1 - 1/(d\varkappa^2)\right)\left(1 + Mr_k\right)^2\left(\sigma_k + 2dMr_k/(1 + Mr_k)\right).$$

2. *The random BFGS method (Algorithm 6) holds that*

$$\mathbb{E}\left[\sigma_{k+1}\right] \leq \left(1 - 1/d\right)\left(1 + Mr_k\right)^2\left(\sigma_k + 2dMr_k/(1 + Mr_k)\right).$$

The analysis for the SR1 method is based on constructing η_k such that $\eta_k = \mathrm{tr}(\mathbf{G}_k - \mathbf{H}_k)/\mathrm{tr}(\mathbf{H}_k)$ and the technical details are showed in appendix. Based on Lemma 3.16, one can show that our algorithms enjoy the local superlinear convergence for the general saddle point problems.

Theorem 3.17. *Solving general saddle point problem* (1) *under Assumption 2.1 and 2.2 by proposed random quasi-Newton methods (Algorithm 5, 6 and 7) by* $\mathbf{G}_0 = L^2\mathbf{I}$ *and* $M = 2\varkappa^2 L_2/L$*, we have the following results for all* $k > 0$*:*

1. *For the random Broyden family method (Algorithm 5), if* λ_0 *satisfies that* $\frac{M\lambda_0}{\mu} \leq \frac{\ln 2}{8(1+2d)\varkappa^2}$*, we have* $\mathbb{E}\left[\lambda_{k+1}/\lambda_k\right] \leq \left(1 - 1/(d\varkappa^2)\right)^k 2d\varkappa^2$.
2. *For the random BFGS method (Algorithm 6), if* λ_0 *satisfies that* $\frac{M\lambda_0}{\mu} \leq \frac{\ln 2}{8(1+d)\varkappa^2}$*, we have* $\mathbb{E}\left[\lambda_{k+1}/\lambda_k\right] \leq \left(1 - 1/d\right)^k 2d\varkappa^2$.
3. *For the random SR1 method (Algorithm 7), if* λ_0 *satisfies that* $\frac{M\lambda_0}{\mu} \leq \frac{\ln 2}{8(1+2d\varkappa^2)\varkappa^2}$*, we have* $\mathbb{E}\left[\lambda_{k+1}/\lambda_k\right] \leq \left(1 - 1/d\right)^k 2d\varkappa^4$.

Finally, combining the results of Theorem 3.15 and 3.17, we can prove the algorithms achieve the two-stages convergence behaviors as follows.

Corollary 3.18. *Solving the general saddle point problem* (1) *under Assumption 2.1 and 2.2 by our random quasi-Newton methods (Algorithm 5, 6 and 7) with* $\mathbf{G}_0 = L^2\mathbf{I}$ *and* $M = 2\varkappa^2 L_2/L$*, if the initial point is sufficiently close to the saddle point such that* $M\lambda_0/\mu \leq \ln 2/(8\varkappa^2)$*, then with probability* $1 - \delta$ *for any* $\delta \in (0, 1)$*, we have the following results:*

1. *The random Broyden family method (Algorithm 5) holds that*

$$\lambda_{k_0+k} \leq \left(1 - \frac{1}{d\varkappa^2 + 1}\right)^{\frac{k(k-1)}{2}} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{4\varkappa^2}\right)^{k_0} \lambda_0$$

   for all $k_0 = \mathcal{O}\left(d\varkappa^2 \ln(d\varkappa/\delta)\right)$ *and* $k \geq 0$.

2. *The random BFGS/SR1 method (Algorithm 6/7) holds that*

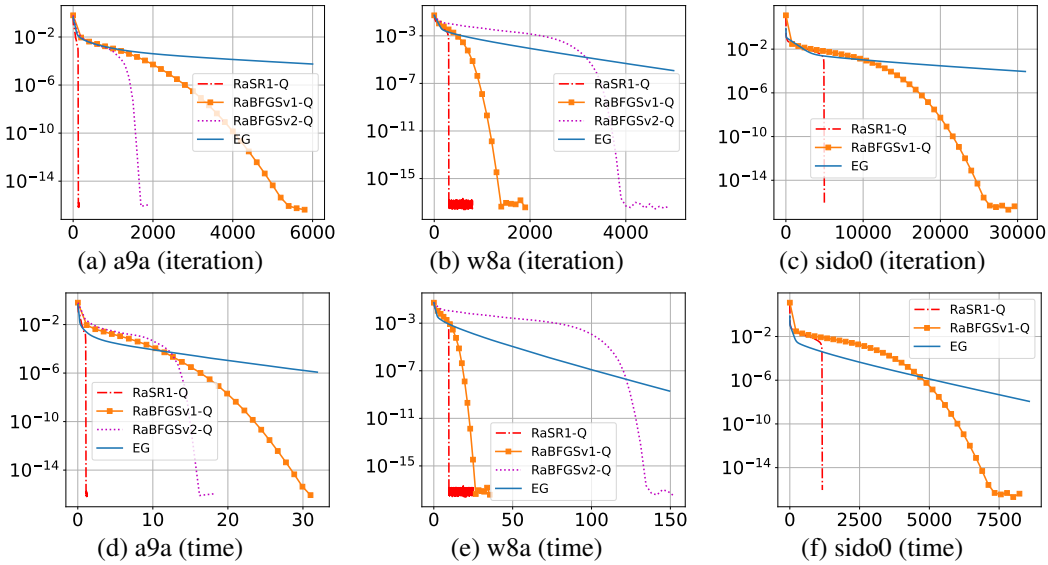
$$\lambda_{k_0+k} \leq \left(1 - \frac{1}{d + 1}\right)^{\frac{k(k-1)}{2}} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{4\varkappa^2}\right)^{k_0} \lambda_0$$

   for all $k_0 = \mathcal{O}\left(\max\left\{d, \varkappa^2\right\} \ln(d\varkappa/\delta)\right)$ *and* $k \geq 0$.

4 Discussion

The relationship between our quasi-Newton methods (Algorithm 5, 6 and 7) and existing first-order method for minimax problem is similar to the one for minimization problem. For our BFGS and SR1 methods, the dependency on $\varkappa^2$ only appears on the first period of linear convergence, which matches the convergence rate of gradient descent ascent. As an analogy, minimizing strongly-convex function by quasi-Newton methods [24, 37] has dependency on $\varkappa$ in the first period of linear convergence, which matches the convergence rate of gradient descent.

Our quasi-Newton methods also can be extended for solving non-linear equations, and the two-stage convergence behavior still hold. We provide the detailed discussion and the comparison with related work [23, 46] in Appendix F.



(a) a9a (iteration) (b) w8a (iteration) (c) sido0 (iteration)

(d) a9a (time) (e) w8a (time) (f) sido0 (time)

Figure 1: We demonstrate iteration numbers vs. $\|\mathbf{g}(\mathbf{z})\|_2$ and CPU time (second) vs. $\|\mathbf{g}(\mathbf{z})\|_2$ for AUC model on datasets "a9a" ($d = 126$, $n = 32561$), "w8a" ($d = 303$, $n = 45546$) and "sido" ($d = 4935$, $n = 12678$).

5 Numerical Experiments

In this section, we conduct the experiments on machine learning applications of AUC Maximization and adversarial debiasing to verify our theory. We refer to Algorithm 2 and 5 with parameter $\tau_k = \mathbf{u}^\top \mathbf{H}_{k+1}/(\mathbf{u}^\top \mathbf{G}_k \mathbf{u})$ as RaBFGSv1-Q and RaBFGSv1-G; refer to Algorithm 3, 4, 6 and 7 as RaBFGSv2-Q, RaSR1-Q, RaBFGSv2-G and RaSR1-G respectively. We compare these proposed algorithms with classical first-order method extragradient (EG) [22, 43].

5.1 AUC Maximization

AUC maximization [18, 48] aims to find the classifier $\mathbf{w} \in \mathbb{R}^m$ on the training set $\{\mathbf{a}_i, b_i\}_{i=1}^n$ where $\mathbf{a}_i \in \mathbb{R}^m$ and $b_i \in \{+1, -1\}$. We denote n^+ be the number of positive instances and $p = n^+/n$. The minimax formulation for AUC maximization can be written as

$$\min_{\mathbf{x}\in\mathbb{R}^{m+2}} \max_{y\in\mathbb{R}} f(\mathbf{x}, y) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^n f_i(\mathbf{x}, \mathbf{y}; \mathbf{a}_i, b_i, \lambda),$$

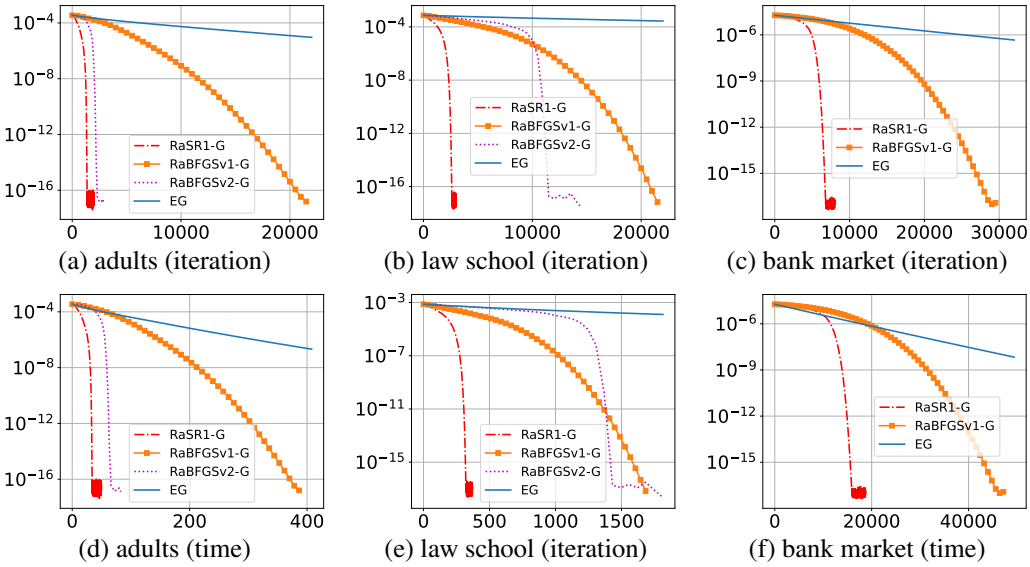
where $\mathbf{x} = [\mathbf{w}; u; v] \in \mathbb{R}^{m+2}$, $\lambda > 0$ is the regularization parameter and f_i is defined as

$$\begin{aligned} f_i(\mathbf{x}, \mathbf{y}; \mathbf{a}_i, b_i, \lambda) =& \frac{\lambda}{2}\|\mathbf{x}\|_2^2 - p(1-p)y^2 + p((\mathbf{w}^T\mathbf{a}_i - v)^2 + 2(1+y)\mathbf{w}^T\mathbf{a}_i)\mathbb{I}_{b_i=-1} \\ &+ (1-p)((\mathbf{w}^T\mathbf{a}_i - u)^2 - 2(1+y)\mathbf{w}^T\mathbf{a}_i)\mathbb{I}_{b_i=1}. \end{aligned}$$

The objective function of AUC maximization is quadratic, hence we conduct the algorithms in Section 3.2 (Algorithm 2, 3 and 4) for this model. We set $\lambda = 100/n$ and evaluate all algorithms on three real-world datasets "a9a", "w8a" and "sido0". The dimension of the problem is $d = m + 3$. The results of iteration numbers against $\|\mathbf{g}(\mathbf{z})\|_2$ and CPU time against $\|\mathbf{g}(\mathbf{z})\|_2$ are presented in Figure 1. These results show that our algorithms perform better than the EG method.

5.2 Adversarial Debiasing

Adversarial learning [26, 49] can be used on fairness-aware machine learning issues. Give the training set $\{\mathbf{a}_i, b_i, c_i\}_{i=1}^n$, where $\mathbf{a}_i \in \mathbb{R}^d$ contains all input variables, $b_i \in \mathbb{R}$ is the output and $c_i \in \mathbb{R}$ is the input variable which we want to protect and make it unbiased. Our experiments are based on the



(a) adults (iteration) (b) law school (iteration) (c) bank market (iteration)

(d) adults (time) (e) law school (iteration) (f) bank market (time)

Figure 2: We demonstrate iteration numbers vs. $\|\mathbf{g}(\mathbf{z})\|_2$ and CPU time (second) vs. $\|\mathbf{g}(\mathbf{z})\|_2$ for adversarial debiasing model on datasets "adults" ($d = 123$, $n = 32561$), "law school" ($d = 380$, $n = 20427$) and "bank market" ($d = 3880$, $n = 45211$).

fairness-aware binary classification dataset "adult", "bank market" and "law school"[35], leading to $b_i, c_i \in \{+1, -1\}$. The model is formulated by the minimax problem

$$\min_{\mathbf{x}\in\mathbb{R}^m} \max_{y\in\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n}(l_1(\mathbf{a}_i, b_i, \mathbf{x}) - \beta l_2(\mathbf{a}_i^\top \mathbf{x}, c_i, y)) + \lambda\|\mathbf{x}\|^2 - \gamma y^2,$$

where l_1, l_2 are the logit functions: $\mathrm{logit}(\mathbf{a}, b, \mathbf{c}) = \log(1 + \exp(-b\mathbf{a}^\top \mathbf{c}))$. We set the parameters β, λ and γ as $0.5, 10^{-4}$ and 10^{-4} respectively. The dimension of the problem is $d = m + 1$. Since the objective function is non-quadratic, we conduct the proposed algorithms in Section 3.3 (Algorithm 5, 6 and 7) here. We use extragradient as warm up to achieve the local condition for proposed algorithms. The results of iteration numbers against $\|\mathbf{g}(\mathbf{z})\|_2$ and CPU time against $\|\mathbf{g}(\mathbf{z})\|_2$ are presented in Figure 2, which indicate that our algorithms significantly outperform the baseline algorithm.

6 Conclusion

In this work, we propose quasi-Newton methods for solving strongly-convex-strongly-concave saddle point problems. We characterize the second-order information by approximating the square of Hessian matrix, which avoids the issue of dealing with the indefinite Hessian directly. We present the explicit local superlinear convergence rates for Broyden's family update and faster convergence rates for two specific methods: SR1 and BFGS updates. However, our algorithms still require to compute the exact gradient and store the approximate Hessian by $\mathcal{O}(d^2)$ space complexity. In future work, it is interesting to design the stochastic algorithms to further reduce the computational cost of the iteration. It is also possible to design the limited-memory quasi-Newton methods for more scalable saddle point problems.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See Section 6.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Section 2.
   (b) Did you include complete proofs of all theoretical results? [Yes] See Appendix A-D.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] We present them in the supplemental material.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] We present them in Section 5 and in Appendix E.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See Appendix E.
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] See Section 5 and Appendix E.
   (b) Did you mention the license of the assets? [Yes] See Section 5 and Appendix E, the data is public available.
   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] See Appendix E.
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

In Section A, we present the details for Algorithm 1 and Proof for Section 3.1. In Section B, we give the two useful lemmas of symmetric positive definite matrices and the proof of Lemma 3.1. In Section C, we give the detailed proofs for the results and the two-stage convergence results for quadratic case in Section 3.2. In Section D, we give the detailed proofs for the results in Section 3.3. We provide some details for our experiments in Section E. Finally, in Section F, we extend our methods for solving non-linear equations.

A Efficient Implementation for Algorithm 1 and Proof of Section 3.1

For the self-completeness of this paper, we present Proposition 1 of Lin et al. [24] to show Algorithm 1 can be implemented with $\mathcal{O}\left(d^2\right)$ flops.

Lemma A.1 (Lin et al. [24, Proposition 1])**.** *In this Lemma, we show how to construct upper triangle matrix $\hat{\mathbf{L}}$ from $\mathbf{L}$, $\mathbf{H}$ and the direction $\mathbf{u}$ with $\mathcal{O}(d^2)$ flops. From the inverse BFGS update rule of $\mathbf{A} = \mathbf{G}^{-1}$, we have*

$$\begin{aligned}\mathbf{A}_+ &= \left(\mathrm{BFGS}\left(\mathbf{G}, \mathbf{H}, \mathbf{u}\right)\right)^{-1} \\ &= \left(\mathbf{I} - \frac{\mathbf{u}\mathbf{u}^\top \mathbf{H}}{\mathbf{u}^\top \mathbf{H}\mathbf{u}}\right) \mathbf{A} \left(\mathbf{I} - \frac{\mathbf{H}\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top \mathbf{H}\mathbf{u}}\right) + \frac{\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top \mathbf{H}\mathbf{u}}.\end{aligned} \tag{8}$$

Suppose we already have $\mathbf{A} = \mathbf{L}^\top \mathbf{L}$ where $\mathbf{L}$ is an upper triangular matrix, now we construct $\hat{\mathbf{L}}$ such that $\mathbf{A}_+ = \hat{\mathbf{L}}^\top \hat{\mathbf{L}}$ with $\mathcal{O}(d^2)$ flops.

1. *First we can obtain QR decomposition of*
$$\mathbf{L}\left(\mathbf{I} - \frac{\mathbf{H}\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top \mathbf{H}\mathbf{u}}\right) = \mathbf{L} - \frac{\mathbf{L}(\mathbf{H}\mathbf{u})}{\mathbf{u}^\top \mathbf{H}\mathbf{u}}\mathbf{u}^\top$$
with $\mathcal{O}(d^2)$ flops since it is a rank-one changes of $\mathbf{L}$.

2. *Second, we have*
$$\mathbf{L}\left(\mathbf{I} - \frac{\mathbf{H}\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top \mathbf{H}\mathbf{u}}\right) = \mathbf{Q}\mathbf{R},$$
with an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{d\times d}$ and an upper triangular matrix $\mathbf{R} \in \mathbb{R}^{d\times d}$. Denote $\mathbf{v} = \mathbf{u}/\sqrt{\mathbf{u}^\top \mathbf{H}\mathbf{u}}$, then we have
$$\mathbf{A}_+ = \mathbf{R}^\top \mathbf{Q}^\top \mathbf{Q}\mathbf{R} + \frac{\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top \mathbf{H}\mathbf{u}} = \mathbf{R}^\top \mathbf{R} + \mathbf{v}\mathbf{v}^\top = \begin{bmatrix}\mathbf{v} & \mathbf{R}^\top\end{bmatrix}\begin{bmatrix}\mathbf{v}^\top \\ \mathbf{R}\end{bmatrix}.$$
we still can obtain QR decomposition of $\begin{bmatrix}\mathbf{v}^\top \\ \mathbf{R}\end{bmatrix}$ with only $\mathcal{O}(d^2)$ flops, leading to $\begin{bmatrix}\mathbf{v}^\top \\ \mathbf{R}\end{bmatrix} = \mathbf{Q}'\mathbf{R}'$, with an column orthogonal matrix $\mathbf{Q}' \in \mathbb{R}^{(d+1)\times d}$ and an upper triangular matrix $\mathbf{R}' \in \mathbb{R}^{d\times d}$, and
$$\mathbf{A}_+ = \mathbf{R}'^\top \mathbf{R}'.$$
Thus $\mathbf{R}'$ satisfies our requirements.

A.1 Proofs of Lemma 3.3, 3.4 and 3.6

Proof. The results of Lemma 3.3, 3.4 and 3.6 can be directly derived from Theorem 3.1, 4.2 and 4.1 of Lin et al. [24].

Take $k = 1$ in Theorem 3.1, 4.1 and 4.2 of Lin et al. [24] we have

$$\begin{aligned}\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_1)\right] &\leq \left(1 - \frac{1}{d\kappa^2}\right)\sigma_{\mathbf{H}}(\mathbf{G}_0) \\ \mathbb{E}\left[\tau_{\mathbf{H}}(\mathbf{G}_1)\right] &\leq \left(1 - \frac{1}{d}\right)\tau_{\mathbf{H}}(\mathbf{G}_0) \\ \mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_1)\right] &\leq \left(1 - \frac{1}{d}\right)\sigma_{\mathbf{H}}(\mathbf{G}_0).\end{aligned}$$

Replacing $\mathbf{G}_1$ and $\mathbf{G}_0$ with $\mathbf{G}_+$ and $\mathbf{G}$ in the above inequalities, we obtain the results of Lemma 3.3, 3.6 and 3.4. □

B Useful Lemmas for Convergence Analysis and Proof for Lemma 3.1

This section provides two useful lemmas of symmetric positive definite matrices for our further analysis.

Lemma B.1 (Rodomanov and Nesterov [37, Lemma 2.1 and Lemma 2.2])**.** *Suppose two positive definite matrices* $\mathbf{H}, \mathbf{G} \in \mathbb{R}^{d\times d}$ *satisfy* $\mathbf{H} \preceq \mathbf{G} \preceq \eta\mathbf{H}$ *for some* $\eta \geq 1$*, then for any* $\mathbf{u} \in \mathbb{R}^d$ *and* $\tau_1 < \tau_2$*, we have*

$$\mathrm{Broyd}_{\tau_1}(\mathbf{G}, \mathbf{H}, \mathbf{u}) \preceq \mathrm{Broyd}_{\tau_2}(\mathbf{G}, \mathbf{H}, \mathbf{u}).$$

Additionally, for any $\tau \in [0, 1]$*, we have*

$$\mathbf{H} \preceq \mathrm{Broyd}_{\tau}(\mathbf{G}, \mathbf{H}, \mathbf{u}) \preceq \eta\mathbf{H}. \tag{9}$$

Lemma B.2. *Suppose* $\mathbf{H}, \mathbf{G} \in \mathbb{R}^{d\times d}$ *are symmetric positive definite matrices,* $\hat{\mathbf{H}} \in \mathbb{R}^{d\times d}$ *are symmetric non-degenerate matrix where* $\mathbf{H} = (\hat{\mathbf{H}})^2$ *and*

$$\mathbf{H} \preceq \mathbf{G} \preceq \eta\mathbf{H},$$

where $\eta > 1$*. Then we have*

$$\|\mathbf{I} - \hat{\mathbf{H}}\mathbf{G}^{-1}\hat{\mathbf{H}}\| \leq 1 - \frac{1}{\eta}. \tag{10}$$

Proof. We have the following inequality for $\mathbf{G}^{-1}$ and $\mathbf{H}^{-1}$

$$\frac{1}{\eta}\mathbf{H}^{-1} \preceq \mathbf{G}^{-1} \preceq \mathbf{H}^{-1},$$

which means

$$\mathbf{0} \preceq \mathbf{H}^{-1} - \mathbf{G}^{-1} \preceq \left(1 - \frac{1}{\eta}\right)\mathbf{H}^{-1}.$$

Thus we have

$$\mathbf{0} \preceq \hat{\mathbf{H}}(\mathbf{H}^{-1} - \mathbf{G}^{-1})\hat{\mathbf{H}} \preceq \left(1 - \frac{1}{\eta}\right)\hat{\mathbf{H}}\mathbf{H}^{-1}\hat{\mathbf{H}} \preceq \left(1 - \frac{1}{\eta}\right)\mathbf{I}.$$

So we have

$$\|\mathbf{I} - \hat{\mathbf{H}}\mathbf{G}^{-1}\hat{\mathbf{H}}\| = \|\hat{\mathbf{H}}(\mathbf{H}^{-1} - \mathbf{G}^{-1})\hat{\mathbf{H}}\| \leq \left(1 - \frac{1}{\eta}\right).$$

□

B.1 Proof of Lemma 3.1

Proof. We partition $\hat{\mathbf{H}}(\mathbf{z}) \in \mathbb{R}^{d\times d}$ as

$$\hat{\mathbf{H}}(\mathbf{z}) = \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{xx}}(\mathbf{z}) & \hat{\mathbf{H}}_{\mathbf{xy}}(\mathbf{z}) \\ \hat{\mathbf{H}}_{\mathbf{xy}}(\mathbf{z})^\top & \hat{\mathbf{H}}_{\mathbf{yy}}(\mathbf{z}), \end{bmatrix} \in \mathbb{R}^{d\times d}$$

where the sub-matrices $\hat{\mathbf{H}}_{\mathbf{xx}}(\mathbf{z}) \in \mathbb{R}^{d_x\times d_x}$, $\hat{\mathbf{H}}_{\mathbf{xy}}(\mathbf{z}) \in \mathbb{R}^{d_x\times d_y}$, $\hat{\mathbf{H}}_{\mathbf{yx}}(\mathbf{z}) \in \mathbb{R}^{d_y\times d_x}$ and $\hat{\mathbf{H}}_{\mathbf{yy}}(\mathbf{z}) \in \mathbb{R}^{d_y\times d_y}$ satisfy $\hat{\mathbf{H}}_{\mathbf{xx}}(\mathbf{z}) \succeq \mu\mathbf{I}$, $\hat{\mathbf{H}}_{\mathbf{yy}}(\mathbf{z}) \preceq -\mu\mathbf{I}$ and $\|\hat{\mathbf{H}}(\mathbf{z})\| \leq L$ for all $\mathbf{z} \in \mathbb{R}^n$ by Assumption 2.1 and 2.2. We denote

$$\mathbf{J} = \begin{bmatrix} \mathbf{I}_{d_x} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{d_y} \end{bmatrix} \in \mathbb{R}^{d\times d},$$

where $\mathbf{I}_{d_x}$ and $\mathbf{I}_{d_y}$ are $d_x \times d_x$ identity matrix and $d_y \times d_y$ identity matrix respectively. We can verified that $\mathbf{J}^\top\mathbf{J} = \mathbf{I}$ and

$$\mathbf{J}\hat{\mathbf{H}}(\mathbf{z}) = \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{xx}}(\mathbf{z}) & \hat{\mathbf{H}}_{\mathbf{xy}}(\mathbf{z}) \\ -\hat{\mathbf{H}}_{\mathbf{xy}}(\mathbf{z})^\top & -\hat{\mathbf{H}}_{\mathbf{yy}}(\mathbf{z}) \end{bmatrix}.$$

Thus we have

$$\mathbf{H}(\mathbf{z}) = \hat{\mathbf{H}}(\mathbf{z})^\top\hat{\mathbf{H}}(\mathbf{z}) = \hat{\mathbf{H}}^\top(\mathbf{z})(\mathbf{J}^\top\mathbf{J})\hat{\mathbf{H}}(\mathbf{z}) = (\mathbf{J}\hat{\mathbf{H}})^\top(\mathbf{J}\hat{\mathbf{H}}).$$

Following Lemma 4.5 of Abernethy et al. [1], we have

$$\mathbf{H}(\mathbf{z}) \succeq \mu^2 \mathbf{I} \succ \mathbf{0}.$$

Since $\left\|\hat{\mathbf{H}}\right\| \leq L$ and $\mathbf{H} \succ \mathbf{0}$, we have

$$\|\mathbf{H}(\mathbf{z})\| = \left\|\hat{\mathbf{H}}(\mathbf{z})^2\right\| \leq \left\|\hat{\mathbf{H}}(\mathbf{z})\right\|^2 \leq L^2$$

and

$$\mathbf{H}(\mathbf{z}) \preceq L^2 \mathbf{I}.$$

□

C The Proof Details for Section 3.2

This section give the detailed proofs for the results and the two-stage convergence results for quadratic case in Section 3.2.

C.1 The Proof of Lemma 3.7

Proof. We have $\mathbf{g}_k = \mathbf{A}\mathbf{z}_k - b$ and $\mathbf{g}_{k+1} = \mathbf{A}\mathbf{z}_{k+1} - b$ where $\mathbf{A} = \hat{\mathbf{H}}$, which means

$$\mathbf{g}_{k+1} - \mathbf{g}_k = \mathbf{A}(\mathbf{z}_{k+1} - \mathbf{z}_k) = -\hat{\mathbf{H}}\mathbf{G}_k^{-1}\hat{\mathbf{H}}\mathbf{g}_k.$$

So we have

$$\lambda_{k+1} = \|(\mathbf{I} - \hat{\mathbf{H}}\mathbf{G}_k^{-1}\hat{\mathbf{H}})\mathbf{g}_k\| \leq \|\mathbf{I} - \hat{\mathbf{H}}\mathbf{G}_k^{-1}\hat{\mathbf{H}}\|\lambda_k.$$

Since $\mathbf{H} \preceq \mathbf{G}_k \preceq \eta_k \mathbf{H}$ and $\mathbf{H} = (\hat{\mathbf{H}})^2$, According to Lemma B.2, we have $\|\mathbf{I} - \hat{\mathbf{H}}\mathbf{G}_k^{-1}\hat{\mathbf{H}}\| \leq \left(1 - \frac{1}{\eta_k}\right)$, which means

$$\lambda_{k+1} \leq \left(1 - \frac{1}{\eta_k}\right)\lambda_k.$$

□

C.2 The Proof of Theorem 3.8

Proof. Lemma 3.1 means $\mu^2 \mathbf{I} \preceq \mathbf{H} \preceq L^2 \mathbf{I}$. Combining with $\mathbf{G}_0 = L^2 \mathbf{I}$, we have $\mathbf{H} \preceq \mathbf{G}_0 \preceq \varkappa^2 \mathbf{H}$. According to Lemma B.1, we achieve

$$\mathbf{H} \preceq \mathbf{G}_k \preceq \varkappa^2 \mathbf{H}.$$

Applying Lemma 3.7, we obtain

$$\lambda_{k+1} \leq \left(1 - \frac{1}{\varkappa^2}\right)\lambda_k \tag{11}$$

for all $k \geq 0$ which leads to

$$\lambda_k \leq \left(1 - \frac{1}{\varkappa^2}\right)^k \lambda_0. \tag{12}$$

□

C.3 The Proof of Theorem 3.9

Proof. The proof is modified from Lin et al. [24, Theorem 4.4]. Denote $\eta_k = \|\mathbf{H}^{-1/2}\mathbf{G}_k\mathbf{H}^{-1/2}\| \geq 1$, then we have

$$\mathbf{H} \preceq \mathbf{G}_k \preceq \eta_k \mathbf{H}$$

and

$$1 - \frac{1}{\eta_k} \leq \eta_k - 1 \overset{*}{\leq} \sigma_{\mathbf{H}}(\mathbf{G}_k) \leq \frac{\mathrm{tr}(\mathbf{G}_k - \mathbf{H})}{\mu^2} = \frac{\tau_{\mathbf{H}}(\mathbf{G}_k)}{\mu^2}. \tag{13}$$

Inequality $*$ comes from the fact that

$$\eta_k - 1 \leq \|\mathbf{H}^{-1/2}(\mathbf{G}_k - \mathbf{H})\mathbf{H}^{-1/2}\| \leq \operatorname{tr}\left(\mathbf{H}^{-1/2}(\mathbf{G}_k - \mathbf{H})\mathbf{H}^{-1/2}\right) = \sigma_k.$$

BFGS method: According to Lemma 3.4, we have:

$$\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_{k+1})\right] \leq \left(1 - \frac{1}{d}\right)\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_k)\right].$$

which implies

$$\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_k)\right] \leq \left(1 - \frac{1}{d}\right)^k \sigma_{\mathbf{H}}(\mathbf{G}_0).$$

Thus we have

$$\mathbb{E}\left[(\eta_k - 1)\right] \overset{(13)}{\leq} \mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_k)\right] \leq \left(1 - \frac{1}{d}\right)^k \sigma_{\mathbf{H}}(\mathbf{G}_0). \tag{14}$$

The upper bound of $\sigma_{\mathbf{H}}(\mathbf{G}_0)$ means

$$\sigma_{\mathbf{H}}(\mathbf{G}_0) = \langle \mathbf{H}^{-1}, \mathbf{G}_0\rangle - d \leq \langle \mathbf{H}^{-1}, \varkappa^2\mathbf{H}\rangle - d = d\left(\varkappa^2 - 1\right) \leq d\varkappa^2. \tag{15}$$

Combining with Lemma 3.7, we have

$$\begin{aligned}\mathbb{E}\left[\frac{\lambda_{k+1}}{\lambda_k}\right] \leq & \mathbb{E}\left[1 - \frac{1}{\eta_k}\right] \leq \mathbb{E}\left[\eta_k - 1\right] \\ \overset{(14)}{\leq} & \left(1 - \frac{1}{d}\right)^k \sigma_{\mathbf{H}}(\mathbf{G}_0) \\ \overset{(15)}{\leq} & \left(1 - \frac{1}{d}\right)^k d\varkappa^2.\end{aligned}$$

Broyden Family Method: The proof for the Broyden family method is similar to the one of BFGS. From Lemma 3.3, we have

$$\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_{k+1})\right] \leq \left(1 - \frac{1}{d\varkappa^2}\right)\mathbb{E}\,\sigma_{\mathbf{H}}(\mathbf{G}_k).$$

which means

$$\mathbb{E}\left[\sigma_{\mathbf{H}}(\mathbf{G}_k)\right] \leq \left(1 - \frac{1}{d\varkappa^2}\right)^k \sigma_{\mathbf{H}}(\mathbf{G}_0).$$

The upper bound bound of $\sigma_{\mathbf{H}}(\mathbf{G}_0)$ is the same with the BFGS which implies

$$\mathbb{E}\left[\frac{\lambda_{k+1}}{\lambda_k}\right] \leq \mathbb{E}\left[\eta_k - 1\right] \leq \left(1 - \frac{1}{d\varkappa^2}\right)^k \sigma_{\mathbf{H}}(\mathbf{G}_0) \overset{(15)}{\leq} \left(1 - \frac{1}{d\varkappa^2}\right)^k d\varkappa^2.$$

SR1 Method: Using Theorem 4.1 of Lin et al. [24], we have

$$\mathbb{E}\left[\tau_{\mathbf{H}}(\mathbf{G}_k)\right] \leq \left(1 - \frac{k}{d}\right)\tau_{\mathbf{H}}(\mathbf{G}_0).$$

The upper bound of $\tau_{\mathbf{H}}(\mathbf{G}_k)$ means

$$\tau_{\mathbf{H}}(\mathbf{G}_0) = \operatorname{tr}(\mathbf{G}_0 - \mathbf{H}) \leq (\varkappa^2 - 1)\operatorname{tr}(\mathbf{H}) \leq d\varkappa^2 L^2.$$

Combining with Lemma 3.7, we have

$$\mathbb{E}\left[\frac{\lambda_{k+1}}{\lambda_k}\right] \leq \mathbb{E}\left[\eta_k - 1\right] \overset{(13)}{\leq} \mathbb{E}\left[\frac{\tau_{\mathbf{H}}(\mathbf{G}_k)}{\mu^2}\right] \leq \left(1 - \frac{k}{d}\right)\frac{d\varkappa^2 L^2}{\mu^2} = \left(1 - \frac{k}{d}\right) d\varkappa^4.$$

□

Combining the results of Theorem 3.8 and 3.9, we achieve the two-stages convergence behavior, that is, the algorithm has global linear convergence and local superlinear convergence. The formal description is summarized as follows.

Corollary C.1. *Solving simple quadratic (SQ) saddle point problem by proposed random quasi-Newton algorithms, then with probability $1-\delta$ for any $\delta \in (0,1)$, we have the following results:*

1. *Using the random Broyden family method (Algorithm 2), we have*

$$\lambda_{k_0+k} \leq \left(1-\frac{1}{d\varkappa^2+1}\right)^{\frac{k(k-1)}{2}} \left(\frac{1}{2}\right)^k \left(1-\frac{1}{\varkappa^2}\right)^{k_0} \lambda_0$$

for all $k>0$ and $k_0 = \mathcal{O}\left(d\varkappa^2 \ln(d\varkappa/\delta)\right)$.

2. *Using the random BFGS method (Algorithm 3), we have*

$$\lambda_{k_0+k} \leq \left(1-\frac{1}{d+1}\right)^{\frac{k(k-1)}{2}} \left(\frac{1}{2}\right)^k \left(1-\frac{1}{\varkappa^2}\right)^{k_0} \lambda_0 \tag{16}$$

for all $k>0$ and $k_0 = \mathcal{O}\left(d\ln(d\varkappa/\delta)\right)$.

3. *Using the random SR1 method (Algorithm 4), we have*

$$\lambda_{k+k_0} \leq \frac{(d-k_0-k+1)!}{(d-k_0+1)!} \left(\frac{1}{2(d-k_0)}\right)^k \left(1-\frac{1}{\varkappa^2}\right)^{k_0} \lambda_0$$

for all $d-k_0+1 \geq k > 0$ and $k_0 = \left\lceil \left(1-\delta/(4d^2(d+1)\varkappa^4)\right) d \right\rceil$.

Note that for random SR1 method, λ_k decrease to 0 with probability $1-\delta$ when $k+k_0 = d+1$.

Proof. The convergence behaviors of the random algorithms also have two stages, the first one is global linear convergence and the second one is local superlinear convergence. We consider the random variable $X_k = \lambda_{k+1}/\lambda_k$ for all $k \geq 0$ in the following derivation.

Broyden Family Method: Note that given $X_k \geq 0$, using Markov's inequality and Theorem 3.8, we have

$$\mathbb{P}\left(X_k \geq \frac{d\varkappa^2}{\epsilon}\left(1-\frac{1}{d\varkappa^2}\right)^k\right) \leq \frac{\mathbb{E}\left[\, X_k\right]}{\frac{d\varkappa^2}{\epsilon}\left(1-\frac{1}{d\varkappa^2}\right)^k} \leq \epsilon \tag{17}$$

for any $\epsilon > 0$. Choosing $\epsilon_k = \delta(1-q)q^k$ for some positive $q<1$, then we have

$$\begin{aligned}\mathbb{P}\left(X_k \geq \frac{d\varkappa^2}{\epsilon_k}\left(1-\frac{1}{d\varkappa^2}\right)^k, \exists k \in \mathbb{N}\right) &\leq \sum_{k=0}^{\infty} \mathbb{P}\left(X_k \geq \frac{d\varkappa^2}{\epsilon_k}\left(1-\frac{1}{d\varkappa^2}\right)^k\right) \\ &\overset{(17)}{\leq} \sum_{k=0}^{\infty} \epsilon_k = \sum_{k=0}^{\infty} \delta(1-q)q^k = \delta.\end{aligned}$$

Therefore, we obtain

$$X_k \leq \left(\frac{1-\frac{1}{d\varkappa^2}}{q}\right)^k \cdot \frac{d\varkappa^2}{(1-q)\delta}$$

for all $k \in \mathbb{N}$ with probability $1-\delta$.

If we set $q = 1-1/(d^2\varkappa^4)$, then it holds that

$$X_k \leq \frac{d^3\varkappa^6}{\delta}\left(1+\frac{1}{d\varkappa^2}\right)^{-k} = \frac{d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^k.$$

for all $k \in \mathbb{N}$ probability $1-\delta$.

Furthermore, it holds that

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^k \tag{18}$$

for all $k \in \mathbb{N}$ with probability $1-\delta$.

Telescoping from k to 0 in Eq. (18), we get

$$\begin{aligned}\lambda_k =& \lambda_0 \cdot \prod_{i=1}^{k} \frac{\lambda_i}{\lambda_{i-1}} \leq \lambda_0 \cdot \left(\frac{d^3\varkappa^6}{\delta}\right)^k \prod_{i=1}^{k} \left(1 - \frac{1}{d\varkappa^2+1}\right)^{i-1} \\ =& \left(\frac{d^3\varkappa^6}{\delta}\right)^k \left(1 - \frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2} \lambda_0.\end{aligned}$$

In the view of (18), we denote $k_0 \geq 0$ as the number of the first iteration satisfying

$$\frac{d^3\varkappa^6}{\delta}\left(1 - \frac{1}{d\varkappa^2+1}\right)^{k_0} \leq \frac{1}{2}.$$

Clearly, we have $k_0 \leq (d\varkappa^2+1)\ln(2d^3\varkappa^6/\delta)$. Thus for all $k \geq 0$, we have

$$\lambda_{k_0+k+1} \overset{(18)}{\leq} \frac{d^3\varkappa^6}{\delta}\left(1 - \frac{1}{d\varkappa^2+1}\right)^{k_0+k} \lambda_{k_0+k} \leq \frac{1}{2}\left(1 - \frac{1}{d\varkappa^2+1}\right)^{k} \lambda_{k_0+k}.$$

Therefore, it holds that

$$\lambda_{k_0+k} \leq \left(1 - \frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2} \left(\frac{1}{2}\right)^k \lambda_{k_0},$$

and by Theorem 3.8 we have

$$\lambda_{k_0} \leq \left(1 - \frac{1}{\varkappa^2}\right)^{k_0} \lambda_0.$$

Finally, choose $k_0 = \mathcal{O}\left(d\varkappa^2 \ln(d\varkappa/\delta)\right)$, we obtain

$$\lambda_{k_0+k} \leq \left(1 - \frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2} \cdot \left(\frac{1}{2}\right)^k \cdot \left(1 - \frac{1}{\varkappa^2}\right)^{k_0} \lambda_0.$$

BFGS Method: Similar to the analysis for the random Broyden family method, we obtain with probability $1-\delta$,

$$X_k \leq \left(\frac{1-\frac{1}{d}}{q}\right)^k \cdot \frac{d\varkappa^2}{(1-q)\delta}, \qquad \text{for all } k \in \mathbb{N}.$$

If we set $q = 1 - 1/d^2$, we could obtain with probability $1-\delta$,

$$\lambda_{k+1} \leq \frac{d^3\varkappa^2}{\delta}\left(1 - \frac{1}{d+1}\right)^k \lambda_k, \qquad \text{for all } k \in \mathbb{N}. \tag{19}$$

We require the point $\mathbf{z}_{k_0}$ satisfies

$$\frac{2d^3\varkappa^2}{\delta}\left(1 - \frac{1}{d+1}\right)^{k_0} \leq \frac{1}{2},$$

which can be guaranteed by setting $k_0 = \mathcal{O}\left(d\ln(d\varkappa/\delta)\right)$.

The remainder of the proof can follow the analysis in the random Broyden methods. We only need to replace all the term of $\left(1 - 1/(d\varkappa^2+1)\right)$ to $(1 - 1/(d+1))$. The reason is that (19) provides a faster convergence result for the random BFGS update, rather than (18) for random Broyden method.

SR1 Method: Similar to the analysis for random Broyden family method, we obtain with probability $1-\delta$,

$$X_k \leq \left(\frac{1-\frac{k}{d}}{q}\right) \cdot \frac{d\varkappa^4}{(1-q)\delta} \qquad \text{for all } k \in \mathbb{N}.$$

If we set $q = 1 - 1/d^2$, we could obtain with probability $1-\delta$,

$$\lambda_{k+1} \leq \frac{d^2(d+1)\varkappa^4}{\delta}\left(1 - \frac{k}{d}\right)\lambda_k \qquad \text{for all } 0 \leq k \leq d. \tag{20}$$

Recall that we denote k_0 the first iteration such that

$$\left(1-\frac{k_0}{d}\right)\frac{d^2(d+1)\varkappa^4}{\delta}\leq\frac{1}{2}.$$

Clearly, we can set

$$k_0=\left\lceil\left(1-\frac{\delta}{2d^2(d+1)\varkappa^4}\right)d\right\rceil.$$

Then it holds that

$$\begin{aligned}\lambda_{k_0+k+1} &\overset{(20)}{\leq} \frac{d^2(d+1)\varkappa^4}{\delta}\left(1-\frac{k+k_0}{d}\right)\\ &= \left(\frac{d-k-k_0}{d-k_0}\right)\left(1-\frac{k_0}{d}\right)\frac{d^2(d+1)\varkappa^4}{\delta}\\ &\leq \frac{1}{2}\left(\frac{d-k-k_0}{d-k_0}\right)\lambda_{k_0+k}.\end{aligned}$$

Thus for $k_0=\left\lceil\left(1-\delta/(2d^2(d+1)\varkappa^4)\right)d\right\rceil$ and $0<k\leq d-k_0+1$, we have

$$\begin{aligned}\lambda_{k+k_0} &\leq \left(\frac{d-k+1-k_0}{d-k_0}\cdots\frac{d-k_0-1}{d-k_0}\right)\left(\frac{1}{2}\right)^k\lambda_{k_0}\\ &\leq \frac{(d-k_0-k+1)!}{(d-k_0+1)!}\left(\frac{1}{2(d-k_0)}\right)^k\left(1-\frac{1}{\varkappa^2}\right)^{k_0}\lambda_0\end{aligned}$$

□

D The Proof Details of Section 3.3

This section give the detailed proofs for the results in Section 3.3.

D.1 The Proof of Lemma 3.10

Proof. Assumption 2.1 mean operator $\hat{\mathbf{H}}(\cdot)$ is L_2-Lipschitz continuous and $\|\hat{\mathbf{H}}(\mathbf{z})\|\leq L$ for all $\mathbf{z}\in\mathbb{R}^d$ then we have

$$\begin{aligned}\|\mathbf{H}(\mathbf{z})-\mathbf{H}(\mathbf{z}')\| &= \|\hat{\mathbf{H}}(\mathbf{z})\hat{\mathbf{H}}(\mathbf{z})-\hat{\mathbf{H}}(\mathbf{z}')\hat{\mathbf{H}}(\mathbf{z}')\|\\ &\leq \|\hat{\mathbf{H}}(\mathbf{z})(\hat{\mathbf{H}}(\mathbf{z})-\hat{\mathbf{H}}(\mathbf{z}'))\|+\|(\hat{\mathbf{H}}(\mathbf{z})-\hat{\mathbf{H}}(\mathbf{z}'))\hat{\mathbf{H}}(\mathbf{z}')\|\\ &\leq 2\|\hat{\mathbf{H}}\|\cdot\|\hat{\mathbf{H}}(\mathbf{z})-\hat{\mathbf{H}}(\mathbf{z}')\|\\ &\leq 2L_2L\|\mathbf{z}-\mathbf{z}'\|.\end{aligned}$$

□

D.2 The Proof of Lemma 3.11

Proof. Lemma 3.1 shows that

$$\mu^2\mathbf{I}\preceq\mathbf{H}(\mathbf{z})\preceq L^2\mathbf{I} \tag{21}$$

and Lemma 3.10 implies that

$$\|\mathbf{H}(\mathbf{z})-\mathbf{H}(\mathbf{z}')\|\leq 2L_2L\|\mathbf{z}-\mathbf{z}'\|. \tag{22}$$

Combining the above lemmas, we have

$$\begin{aligned}\mathbf{H}(\mathbf{z})-\mathbf{H}(\mathbf{z}') &\overset{(22)}{\preceq} 2L_2L\|\mathbf{z}-\mathbf{z}'\|\mathbf{I}\\ &\overset{(21)}{\preceq} \frac{2L_2L}{\mu^2}\|\mathbf{z}-\mathbf{z}'\|\mathbf{H}(\mathbf{w})\\ &= \frac{2\varkappa^2L_2}{L}\|\mathbf{z}-\mathbf{z}'\|\mathbf{H}(\mathbf{w}).\end{aligned}$$

□

D.3 The Proof of Corollary 3.12

Proof. According to Lemma 3.11, we have

$$\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}') \preceq M\|\mathbf{z} - \mathbf{z}'\|\mathbf{H}(\mathbf{w}) \tag{23}$$

Taking interchanging of $\mathbf{z}'$ and $\mathbf{z}$ and letting $\mathbf{w} = \mathbf{z}$ in (23), we have

$$\mathbf{H}(\mathbf{z}') - \mathbf{H}(\mathbf{z}) \preceq M\|\mathbf{z} - \mathbf{z}'\|\mathbf{H}(\mathbf{z}) = Mr\mathbf{H}(\mathbf{z}).$$

Then taking $\mathbf{w} = \mathbf{z}'$, we have

$$\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}') \preceq M\|\mathbf{z} - \mathbf{z}'\|\mathbf{H}(\mathbf{z}') = Mr\mathbf{H}(\mathbf{z}').$$

Combining above results we have

$$\frac{\mathbf{H}(\mathbf{z})}{1 + Mr} \preceq \mathbf{H}(\mathbf{z}') \preceq (1 + Mr)\mathbf{H}(\mathbf{z}), \tag{24}$$

. □

D.4 The Proof of Lemma 3.13

Proof. By the conditiod, we have

$$\mathbf{H}(\mathbf{z}) \preceq \mathbf{G} \preceq \eta\mathbf{H}(\mathbf{z}), \tag{25}$$

Combining with Corollary 3.12 we have

$$\mathbf{H}(\mathbf{z}_+) \overset{(24)}{\preceq} (1 + Mr)\mathbf{H}(\mathbf{z}) \overset{(25)}{\preceq} (1 + Mr)\mathbf{G} = \tilde{\mathbf{G}} \tag{26}$$

and

$$\tilde{\mathbf{G}} = (1 + Mr)\mathbf{G} \overset{(25)}{\preceq} (1 + Mr)\eta\mathbf{H}(\mathbf{z}) \overset{(24)}{\preceq} (1 + Mr)^2\eta\mathbf{H}(\mathbf{z}_+),$$

which means

$$\mathbf{H}(\mathbf{z}_+) \preceq \tilde{\mathbf{G}} \preceq (1 + Mr)^2\eta\mathbf{H}(\mathbf{z}_+).$$

Then we obtain

$$\mathbf{H}(\mathbf{z}_+) \preceq \mathrm{Broyd}_\tau(\tilde{\mathbf{G}}, \mathbf{H}(\mathbf{z}_+), \mathbf{u}) \preceq (1 + Mr)^2\eta\mathbf{H}(\mathbf{z}_+). \tag{27}$$

□

D.5 The Proof of Lemma 3.14

Proof. Recall that the objective satisfies the Assumption 2.1, that is

$$\|\mathbf{g}(\mathbf{z}_k) - \mathbf{g}(\mathbf{z}_{k+1})\| \leq L\,\|\mathbf{z}_k - \mathbf{z}_{k+1}\| \quad \text{and} \quad \|\hat{\mathbf{H}}(\mathbf{z}_k) - \hat{\mathbf{H}}(\mathbf{z}_{k+1})\| \leq L_2\,\|\mathbf{z}_k - \mathbf{z}_{k+1}\|\,. \tag{28}$$

We rewrite $\nabla f(\mathbf{z}_{k+1}) - \nabla f(\mathbf{z}_k)$ below

$$\begin{aligned}
&\nabla f(\mathbf{z}_{k+1}) - \nabla f(\mathbf{z}_k) = \int_0^1 \nabla^2 f(\mathbf{z}_k + s(\mathbf{z}_{k+1} - \mathbf{z}_k))(\mathbf{z}_{k+1} - \mathbf{z}_k)ds \\
&= \int_0^1 \left[\nabla^2 f(\mathbf{z}_k + s(\mathbf{z}_{k+1} - \mathbf{z}_k)) - \nabla^2 f(\mathbf{z}_k)\right](\mathbf{z}_{k+1} - \mathbf{z}_k)ds + \nabla^2 f(\mathbf{z}_k)(\mathbf{z}_{k+1} - \mathbf{z}_k) \\
&= \int_0^1 \left[\nabla^2 f(\mathbf{z}_k + s(\mathbf{z}_{k+1} - \mathbf{z}_k)) - \nabla^2 f(\mathbf{z}_k)\right](\mathbf{z}_{k+1} - \mathbf{z}_k)ds - \nabla^2 f(\mathbf{z}_k)\mathbf{G}_k^{-1}\nabla^2 f(\mathbf{z}_k)\nabla f(\mathbf{z}_k),
\end{aligned}$$

which means

$$\mathbf{g}_{k+1} = \underbrace{(\mathbf{I} - \hat{\mathbf{H}}_k\mathbf{G}_k^{-1}\hat{\mathbf{H}}_k)\mathbf{g}_k}_{\mathbf{a}_k} + \underbrace{\int_0^1 \left[\hat{\mathbf{H}}(\mathbf{z}_k + s(\mathbf{z}_{k+1} - \mathbf{z}_k)) - \hat{\mathbf{H}}(\mathbf{z}_k)\right](\mathbf{z}_{k+1} - \mathbf{z}_k)ds}_{\mathbf{b}_k}\,.$$

We first bound the term $\|\mathbf{a}_k\|$ by Lemma 3.13

$$\|\mathbf{a}_k\| \leq \|\mathbf{I} - \hat{\mathbf{H}}_k \mathbf{G}_k^{-1} \hat{\mathbf{H}}_k\| \|\mathbf{g}_k\| \leq \left(1 - \frac{1}{\eta_k}\right) \lambda_k. \tag{29}$$

Before we bound $\mathbf{b}_k$, we first try to bound $\hat{\mathbf{H}}_k \mathbf{G}_k^{-2} \hat{\mathbf{H}}_k$

$$\begin{aligned}\hat{\mathbf{H}}_k \mathbf{G}_k^{-2} \hat{\mathbf{H}}_k &= (\hat{\mathbf{H}}_k \mathbf{G}_k^{-1/2}) \mathbf{G}_k^{-1} (\mathbf{G}_k^{-1/2} \hat{\mathbf{H}}_k) \preceq (\hat{\mathbf{H}}_k \mathbf{G}_k^{-1/2}) \frac{1}{\mu^2} \mathbf{I} (\mathbf{G}_k^{-1/2} \hat{\mathbf{H}}_k) = \frac{1}{\mu^2} \hat{\mathbf{H}}_k \mathbf{G}_k^{-1} \hat{\mathbf{H}}_k \\ &\preceq \frac{1}{\mu^2} \hat{\mathbf{H}}_k \mathbf{H}_k^{-1} \hat{\mathbf{H}}_k \preceq \frac{1}{\mu^2} \mathbf{I}.\end{aligned}$$

And $\|\mathbf{b}_k\|$ can be bounded by the L_2-Lipschitz continuity of the objective function

$$\begin{aligned}\|\mathbf{b}_k\| &\leq \int_0^1 \left\| \left(\hat{\mathbf{H}} \left(\mathbf{z}_k + s(\mathbf{z}_{k+1} - \mathbf{z}_k) \right) - \hat{\mathbf{H}}(\mathbf{z}_k) \right) (\mathbf{z}_{k+1} - \mathbf{z}_k) \right\| ds \\ &\overset{(28)}{\leq} \int_0^1 \|L_2 s(\mathbf{z}_{k+1} - \mathbf{z}_k)\| \, \|\mathbf{z}_{k+1} - \mathbf{z}_k\| ds \leq \frac{L_2}{2} \|\mathbf{z}_{k+1} - \mathbf{z}_k\|^2 \\ &= \frac{L_2}{2} \|\mathbf{G}_k^{-1} \hat{\mathbf{H}}_k \mathbf{g}_k\|^2 = \frac{L_2}{2} \langle \mathbf{g}_k, \hat{\mathbf{H}}_k \mathbf{G}_k^{-2} \hat{\mathbf{H}}_k \mathbf{g}_k \rangle \leq \frac{L_2}{2\mu^2} \|\mathbf{g}_k\|^2 = \beta \lambda_k^2.\end{aligned}$$

Combining the above results, we have

$$\lambda_{k+1} \leq \|\mathbf{a}_k\| + \|\mathbf{b}_k\| \leq \left(1 - \frac{1}{\eta_k}\right) \lambda_k + \beta \lambda_k^2 \tag{30}$$

The relation of r_k and λ_k can be directly prove by the update formula

$$r_k = \|\mathbf{z}_{k+1} - \mathbf{z}_k\| = \|\mathbf{G}_k^{-1} \hat{\mathbf{H}}_k \mathbf{g}_k\| = \langle \mathbf{g}_k, \hat{\mathbf{H}}_k \mathbf{G}_k^{-2} \hat{\mathbf{H}}_k \mathbf{g}_k \rangle^{1/2} \leq \frac{1}{\mu} \lambda_k \tag{31}$$

□

D.6 The proof of Theorem 3.15

Proof. We use induction to prove the following statements

$$\mathbf{H}_k \preceq \mathbf{G}_k \preceq \exp\left(2 \sum_{i=0}^{k-1} \rho_i\right) \varkappa^2 \mathbf{H}_k \preceq b \varkappa^2 \mathbf{H}_k, \tag{32}$$

$$\lambda_k \leq \left(1 - \frac{1}{2b\varkappa^2}\right)^k \lambda_0, \tag{33}$$

$$\eta_k \overset{\text{def}}{=} \exp\left(\sum_{i=0}^{k-1} 2\rho_i\right) \varkappa^2 \leq b\varkappa^2 \tag{34}$$

hold for all $k \geq 0$. The initial assumption promise that λ_0 is small enough such that

$$\frac{M}{\mu} \lambda_0 \leq \frac{\ln b}{4b\varkappa^2} \tag{35}$$

For $k = 0$, the initialization $\mathbf{G}_0 = L^2 \mathbf{I}$ leads to $\mathbf{H}_0 \preceq \mathbf{G}_0 \preceq \varkappa^2 \mathbf{H}_0$ and $\eta_0 = \varkappa^2$, which satisfy (32), (33) and (34). Then we prove these results for $k' = k + 1$.

The induction assumption means $\eta_k \leq b\varkappa^2$ and $\mathbf{H}_k \preceq \mathbf{G}_k \leq \eta_k \mathbf{H}_k$. Using Lemma 3.14, we have

$$\begin{aligned}\lambda_{k+1} &\leq \left(1 - \frac{1}{b\varkappa^2}\right) \lambda_k + \beta \lambda_k^2 \overset{(33)}{\leq} \left(1 - \frac{1}{b\varkappa^2} + \beta \lambda_0\right) \lambda_k \\ &\overset{(35)}{\leq} \left(1 - \frac{1}{2b\varkappa^2}\right) \lambda_k \overset{(33)}{\leq} \left(1 - \frac{1}{2b\varkappa^2}\right)^{k+1} \lambda_0.\end{aligned}$$

Recall that we've defined $\rho_i = \frac{M\lambda_i}{\mu}$. Based on the fact $e^x \geq x+1$ and Lemma 3.13, we have

$$\begin{aligned}
\mathbf{H}_{k+1} \preceq \mathbf{G}_{k+1} &\overset{(26)}{\preceq} (1+Mr_k)^2 \eta_k \mathbf{H}_{k+1} \preceq \left(1+\frac{M\lambda_k}{\mu}\right)^2 \eta_k \mathbf{H}_{k+1} \\
&= (1+\rho_k)^2 \eta_k \mathbf{H}_{k+1} \preceq e^{2\rho_k} \eta_k \mathbf{H}_{k+1} \\
&\overset{(32)}{\preceq} \exp\left(2\sum_{i=0}^{k} \rho_i\right) \varkappa^2 \mathbf{H}_{k+1};
\end{aligned}$$

and the term $\sum_{i=0}^{k} \rho_i$ can be bounded by

$$\begin{aligned}
\sum_{i=0}^{k} \rho_i &= \frac{M}{\mu} \sum_{i=0}^{k} \lambda_i \overset{(33)}{\leq} \frac{M}{\mu} \lambda_0 \sum_{i=0}^{k} \left(1 - \frac{1}{2b\varkappa^2}\right)^i \\
&\leq \frac{2bML^2}{\mu^3} \lambda_0 \overset{(35)}{\leq} \frac{\ln b}{2}.
\end{aligned} \tag{36}$$

Hence, for $k+1$, we have

$$\mathbf{G}_{k+1} \preceq \exp\left(2\sum_{i=1}^{k} \rho_i\right) \varkappa^2 \mathbf{H}_{k+1} \preceq b\varkappa^2 \mathbf{H}_{k+1} \quad \text{and} \quad \eta_{k+1} = \exp\left(2\sum_{i=1}^{k} \rho_i\right) \varkappa^2 \leq b\varkappa^2.$$

Thus we have complete the proof for statements (32), (33) and (34) by induction. □

D.7 The Proof of Lemma 3.16

Proof. The proof of two algorithms are similar, the only difference is because two update formulas have different convergence rate. And from the Lemma 3.13, for both BFGS and Broyden family updates, we have

$$\tilde{\mathbf{G}} \overset{\text{def}}{=} (1+Mr)\mathbf{G} \succeq \mathbf{H}(\mathbf{z}). \tag{37}$$

We first give the proof for BFGS method.

BFGS Method: Using Lemma 3.4, we have

$$\mathbb{E}_{\mathbf{u}_k}[\sigma_{k+1}] = \mathbb{E}_{\mathbf{u}_k}[\sigma_{\mathbf{H}_{k+1}}(\mathbf{G}_{k+1})] \leq \left(1 - \frac{1}{d}\right) \sigma_{\mathbf{H}_{k+1}}(\tilde{\mathbf{G}}_k).$$

Then we bound the term $\sigma_{\mathbf{H}_{k+1}}(\tilde{\mathbf{G}}_k)$ by σ_k as follows

$$\begin{aligned}
\sigma_{\mathbf{H}_{k+1}}(\tilde{\mathbf{G}}_k) &= \langle \mathbf{H}_{k+1}^{-1}, \tilde{\mathbf{G}}_k \rangle - d \\
&\overset{(37)}{=} (1+Mr_k) \langle \mathbf{H}_{k+1}^{-1}, \mathbf{G}_k \rangle - d \\
&\overset{(24)}{\leq} (1+Mr_k)^2 \langle \mathbf{H}_k^{-1}, \mathbf{G}_k \rangle - d \\
&= (1+Mr_k)^2 \sigma_k + d((1+Mr_k)^2 - 1) \\
&= (1+Mr_k)^2 \sigma_k + 2dMr_k \left(1 + \frac{Mr_k}{2}\right) \\
&\leq (1+Mr_k)^2 \left(\sigma_k + \frac{2dMr_k}{1+Mr_k}\right).
\end{aligned} \tag{38}$$

Combining above results, we obtain

$$\mathbb{E}_{\mathbf{u}_k}[\sigma_{k+1}] \leq \left(1 - \frac{1}{d}\right)(1+Mr_k)^2 \left(\sigma_k + \frac{2dMr_k}{1+Mr_k}\right) \quad \text{for all } k \geq 0. \tag{39}$$

Broyden Family Method: If we use the Broyden family update

$$\mathbf{G}_{k+1} = \mathrm{Broyd}_{\tau_k}(\tilde{\mathbf{G}}_k, \mathbf{H}_{k+1}, \mathbf{u}_k)$$

instead of BFGS, Lemma 3.3 means

$$\mathbb{E}_{\mathbf{u}_k}[\sigma_{k+1}] \leq \left(1 - \frac{1}{d\varkappa^2}\right) \sigma_{\mathbf{H}_{k+1}}(\tilde{\mathbf{G}}_k).$$

Combining with (38) which holds for both Broyden method and BFGS method, we obtain

$$\mathbb{E}_{\mathbf{u}_k}[\sigma_{k+1}] \leq \left(1 - \frac{1}{d\varkappa^2}\right)(1 + Mr_k)^2 \left(\sigma_k + \frac{2dMr_k}{1 + Mr_k}\right) \quad \text{for all } k \geq 0. \tag{40}$$

□

D.8 The Proof of Theorem 3.17

We give the proof for the BFGS and Broyden Family methods in Section D.8.1 and the proof for the SR1 method in Section D.8.2. Taking $b = 2$, all algorithms have $\frac{M\lambda_0}{\mu} \leq \frac{\ln 2}{8\varkappa^2}$, which implies the properties of λ_k shown in Theorem 3.15.

Recall the definition of σ_k

$$\sigma_k \overset{\text{def}}{=} \sigma_{\mathbf{H}_k}(\mathbf{G}_k) = \langle \mathbf{H}_k^{-1}, \mathbf{G}_k \rangle - d. \tag{41}$$

If $\mathbf{G}_k \succeq \mathbf{H}_k$, then according to Rodomanov and Nesterov [37] it holds that

$$\mathbf{G}_k - \mathbf{H}_k \preceq \langle \mathbf{H}_k^{-1}, \mathbf{G}_k - \mathbf{H}_k \rangle \mathbf{H}_k = \sigma_k \mathbf{H}_k. \tag{42}$$

From Theorem 3.14, we have

$$\lambda_{k+1} \leq \left(1 - \frac{1}{1 + \sigma_k}\right) \lambda_k + \beta \lambda_k^2, \tag{43}$$

and

$$r_k \leq \frac{\lambda_k}{\mu} \tag{44}$$

for each $k \geq 0$.

D.8.1 The Proofs of BFGS and Broyden Methods

BFGS Method: First we give the proof for the BFGS method. From Lemma 3.16, we obtain

$$\mathbb{E}_{\mathbf{u}_k}[\sigma_{k+1}] \leq \left(1 - \frac{1}{d}\right)(1 + Mr_k)^2 \left(\sigma_k + \frac{2dMr_k}{1 + Mr_k}\right). \tag{45}$$

Since we have defined $\rho_k = M\lambda_k/\mu$ and the constant $\beta,\ M$ satisfy $\frac{\beta}{M} < \frac{1}{4L}$, it holds that

$$\rho_k \overset{(43)}{\leq} \frac{\sigma_k}{1 + \sigma_k}\rho_k + \frac{\beta\mu}{2M}\rho_k^2 \leq \sigma_k\rho_k + \frac{\beta\mu}{2M}\rho_k^2 < \sigma_k\rho_k + \frac{\mu}{8L}\rho_k^2 \leq \sigma_k\rho_k + \frac{1}{8}\rho_k^2, \tag{46}$$

and

$$\mathbb{E}[\sigma_{k+1}] \overset{(45),(44)}{\leq} \left(1 - \frac{1}{d}\right) \mathbb{E}\left[(1 + \rho_k)^2 \left(\sigma_k + \frac{2d\rho_k}{1 + \rho_k}\right)\right]. \tag{47}$$

We set

$$\theta_k \overset{\text{def}}{=} \sigma_k + 2d\rho_k$$

and consider Theorem 3.15 with $b = 2$. Then the convergence result of (36) and the initial assumption of $\mathbf{z}_0$ implies that

$$\rho_k \leq \left(1 - \frac{1}{4\varkappa^2}\right)^k \rho_0, \tag{48}$$

and

$$\rho_0 \leq \frac{\ln 2}{8\varkappa^2(1+2d)}. \tag{49}$$

We now use induction to show that

$$\mathbb{E}\left[\theta_k\right] \leq \left(1-\frac{1}{d}\right)^k 2d\varkappa^2. \tag{50}$$

In the case of $k=0$, we have

$$\begin{aligned}\sigma_0 + 2d\rho_0 &\overset{(41)}{=} \langle \mathbf{H}_0^{-1}, \mathbf{G}_0\rangle - d + 2d\rho_0 \leq \langle \mathbf{H}_0^{-1}, \varkappa^2 \mathbf{H}_0\rangle - d + 2d\rho_0 \\ &= d\left(\varkappa^2 - 1\right) + 2d\rho_0 \leq d\varkappa^2.\end{aligned} \tag{51}$$

Thus for $k=0$, (50) is satisfied.

Suppose inequality (50) holds for $0 \leq k' \leq k$. For $k+1$, using the inequality $e^x \geq 1+x$, we have

$$\begin{aligned}\mathbb{E}\left[\sigma_{k+1}\right] &\overset{(47)}{\leq} \left(1-\frac{1}{d}\right)\mathbb{E}\left[(1+\rho_k)^2\left(\sigma_k + \frac{2d\rho_k}{1+\rho_k}\right)\right] \\ &\leq \left(1-\frac{1}{d}\right)\mathbb{E}\left[(1+\rho_k)^2(\sigma_k + 2d\rho_k)\right] \\ &= \left(1-\frac{1}{d}\right)\mathbb{E}\left[(1+\rho_k)^2\theta_k\right] \\ &\leq \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2\rho_k\right)\theta_k\right],\end{aligned} \tag{52}$$

and

$$\rho_{k+1} \leq \rho_k\left(\sigma_k + \frac{1}{8}\rho_k\right) \leq \rho_k\left(\sigma_k + 2d\rho_k\right) \leq \left(1-\frac{1}{d}\right)2\exp\left(2\rho_k\right)\theta_k\rho_k. \tag{53}$$

The last inequality of (53) comes from

$$2\left(1-\frac{1}{d}\right)\exp(2\rho_k) \geq 1,$$

where we use the fact $\rho_k \geq 0$ and assume $n \geq 2$.

Thus we obtain

$$\begin{aligned}\mathbb{E}\left[\sigma_{k+1} + 2d\rho_{k+1}\right] &\overset{(53),(52)}{\leq} \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2\rho_k\right)\theta_k\right] + \left(1-\frac{1}{d}\right)4d\mathbb{E}\left[\exp\left(2\rho_k\right)\theta_k\rho_k\right] \\ &\leq \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2\rho_k\right)\theta_k(1+4d\rho_k)\right] \\ &\leq \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2\rho_k\right)\exp\left(4d\rho_k\right)\theta_k\right] \\ &\leq \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2(1+2d)\rho_k\right)\theta_k\right] \\ &\overset{(48)}{\leq} \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2(1+2d)\left(1-\frac{1}{4\varkappa^2}\right)^k\rho_0\right)\theta_k\right] \\ &= \left(1-\frac{1}{d}\right)\exp\left(2(1+2d)\left(1-\frac{1}{4\varkappa^2}\right)^k\rho_0\right)\mathbb{E}\left[\theta_k\right].\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[\theta_{k+1}\right] \leq \left(1-\frac{1}{d}\right)\exp\left(2(1+2d)\left(1-\frac{1}{4\varkappa^2}\right)^k\rho_0\right)\mathbb{E}\left[\theta_k\right]$$

$$
\begin{aligned}
&\leq \quad \left(1-\frac{1}{d}\right)^{k+1} \exp\left(2(1+2d)\rho_0 \sum_{i=0}^{k}\left(1-\frac{1}{4\varkappa^2}\right)^i\right)\mathbb{E}\left[\theta_0\right] \\
&\leq \quad \left(1-\frac{1}{d}\right)^{k+1} \exp\left(8\varkappa^2(1+2d)\rho_0\right)\mathbb{E}\left[\theta_0\right] \\
&\overset{(49),(51)}{\leq} \quad \left(1-\frac{1}{d}\right)^{k+1} 2d\varkappa^2,
\end{aligned}
$$

which proves (50). Hence, for any $k \geq 0$, we have

$$
\mathbb{E}\left[\sigma_k\right] \leq \mathbb{E}\left[\theta_k\right] \leq \left(1-\frac{1}{d}\right)^k 2d\varkappa^2,
$$

which implies

$$
\mathbb{E}\left[\frac{\lambda_{k+1}}{\lambda_k}\right] = \mathbb{E}\left[\frac{\rho_{k+1}}{\rho_k}\right] \overset{(53)}{\leq} \mathbb{E}\left[\sigma_k + 2d\rho_k\right] \leq \mathbb{E}\left[\theta_k\right] \leq \left(1-\frac{1}{d}\right)^k 2d\varkappa^2. \tag{54}
$$

Broyden Family Method: The proof for the Broyden method is almost the same as the one in the BFGS method. The reason that produce the different convergence result between Broyden and BGGS is that Lemma 3.16 only provides a slower convergence rate

$$
\mathbb{E}[\sigma_{k+1}] \leq \left(1-\frac{1}{d\varkappa^2}\right)(1+Mr_k)^2\left(\sigma_k + \frac{2dMr_k}{1+Mr_k}\right)
$$

for Broyden method, rather than

$$
\mathbb{E}[\sigma_{k+1}] \leq \left(1-\frac{1}{d}\right)(1+Mr_k)^2\left(\sigma_k + \frac{2dMr_k}{1+Mr_k}\right)
$$

for BFGS. Thus we can directly replace the term $\left(1-\frac{1}{d}\right)$ to the term $\left(1-\frac{1}{d\varkappa^2}\right)$ in the proof of BFGS method and obtain the convergence result of Broyden method

$$
\mathbb{E}\left[\sigma_k\right] \leq \left(1-\frac{1}{d\varkappa^2}\right)^k 2d\varkappa^2,
$$

and

$$
\mathbb{E}\left[\frac{\lambda_{k+1}}{\lambda_k}\right] \leq \left(1-\frac{1}{d\varkappa^2}\right)^k 2d\varkappa^2.
$$

D.8.2 The Proof of SR1 Method

Note that we use τ_k to replace σ_k for measuring how well does $\mathbf{G}_k$ approximate $\mathbf{H}_k$. We modify the proof in Lin et al. [24, Lemma 4.6] to obtain our results.

SR1 Method: First, we define the random sequence $\left\{\eta_k\right\}$ as follows

$$
\eta_k \overset{\text{def}}{=} \frac{\operatorname{tr}(\mathbf{G}_k - \mathbf{H}_k)}{\operatorname{tr}(\mathbf{H}_k)}. \tag{55}
$$

Since $\mathbf{G}_{k+1} = \mathrm{SR1}(\widetilde{\mathbf{G}}_\mathrm{k}, \mathbf{H}_{\mathrm{k}+1}, \mathbf{u}_\mathrm{k})$ and according to Lemma 3.6 we have

$$
\begin{aligned}
\mathbb{E}_{\mathbf{u}_k}[\operatorname{tr}(\mathbf{G}_{k+1} - \mathbf{H}_{k+1})] \quad &\leq \quad (1-\frac{1}{d})\operatorname{tr}(\widetilde{\mathbf{G}}_k - \mathbf{H}_{k+1}) \\
&\overset{(24)}{\leq} \quad \left(1-\frac{1}{d}\right)\operatorname{tr}\left((1+Mr_k)\mathbf{G}_k - \frac{1}{1+Mr_k}\mathbf{H}_k\right) \\
&\overset{(55)}{=} \quad \left(1-\frac{1}{d}\right)\left((1+Mr_k)(1+\eta_k) - \frac{1}{1+Mr_k}\right)\operatorname{tr}(\mathbf{H}_k) \\
&\leq \quad \left(1-\frac{1}{d}\right)\left((1+Mr_k)^2(1+\eta_k) - 1\right)\operatorname{tr}(\mathbf{H}_{k+1}).
\end{aligned}
$$

Thus, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbf{u}_k}\left[\eta_{k+1}\right] &\leq \left(1-\frac{1}{d}\right)\left((1+Mr_k)^2(1+\eta_k)-1\right)\\ &\leq \left(1-\frac{1}{d}\right)\left[(1+Mr_k)^2\eta_k+Mr_k(Mr_k+2)\right]\\ &\leq \left(1-\frac{1}{d}\right)(1+Mr_k)^2(\eta_k+2Mr_k).\end{aligned} \tag{56}$$

The last inequality comes from the fact that

$$t(t+2)=t^2+2t\leq 2t+4t^2+2t^3=(1+t^2)2t$$

for all $t>0$.

Since we have $\mu^2\mathbf{I}\preceq\mathbf{H}_k\preceq L^2\mathbf{I}$, then

$$\sigma_k=\mathrm{tr}((\mathbf{G}_k-\mathbf{H}_k)\mathbf{H}_k^{-1})\leq\frac{1}{\mu^2}\mathrm{tr}(\mathbf{G}_k-\mathbf{H}_k)=\frac{\eta_k}{\mu^2}\mathrm{tr}(\mathbf{H}_k)\leq d\eta_k\varkappa^2.$$

It also holds that

$$\lambda_{k+1}\overset{(43)}{\leq}\left(1-\frac{1}{1+\sigma_k}\right)\lambda_k+\beta\lambda_k^2\leq\sigma_k\lambda_k+\beta\lambda_k^2\leq(d\varkappa^2\eta_k)\lambda_k+\beta\lambda_k^2.$$

Recall that $\rho_k=M\lambda_k/\mu$, and we have

$$\begin{aligned}2\rho_{k+1} &\leq (2d\varkappa^2\eta_k)\rho_k+\frac{1}{4}\rho_k^2\\ &\leq 2d\varkappa^2\rho_k(\eta_k+2\rho_k) \qquad (57)\\ &\leq \left(1-\frac{1}{d}\right)4d\varkappa^2\rho_k(\eta_k+2\rho_k)\\ &\leq \left(1-\frac{1}{d}\right)(1+\rho_k)^2 4d\varkappa^2\rho_k(\eta_k+2\rho_k), \qquad (58)\end{aligned}$$

and

$$\mathbb{E}\left[\eta_{k+1}\right]\overset{(56)}{\leq}\left(1-\frac{1}{d}\right)\mathbb{E}\left[(1+\rho_k)^2(\eta_k+2\rho_k)\right]. \tag{59}$$

Combing above results, we obtain

$$\begin{aligned}\mathbb{E}\left[\eta_{k+1}+2\rho_{k+1}\right] &\overset{(59),(58)}{\leq}\left(1-\frac{1}{d}\right)\mathbb{E}\left[(1+\rho_k)^2(1+4d\varkappa^2\rho_k)(\eta_k+2\rho_k)\right]\\ &\leq\left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2\rho_k+4d\varkappa^2\rho_k\right)(\eta_k+2\rho_k)\right].\end{aligned} \tag{60}$$

Let $\theta_k\overset{\mathrm{def}}{=}\eta_k+2\rho_k$, then

$$\mathbb{E}\left[\theta_{k+1}\right]\leq\left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2(1+2d\varkappa^2)\rho_k\right)\theta_k\right].$$

The initial condition means that

$$\rho_0\leq\frac{\ln 2}{8(1+2d\varkappa^2)\varkappa^2}. \tag{61}$$

In the following, we use induction to prove the fact that

$$\mathbb{E}\left[\theta_k\right]\leq\left(1-\frac{1}{d}\right)^k 2\varkappa^2.$$

For $k=0$, the initial condition $\mathbf{G}_0\preceq\varkappa^2\mathbf{H}_0$ means $\eta_0\leq\varkappa^2-1$. Thus we obtain

$$\theta_0 \;=\; 2\rho_0+\eta_0\overset{(61)}{\leq}\frac{\ln 2}{4(1+2d\varkappa^2)\varkappa^2}+(\varkappa^2-1)\leq\varkappa^2. \tag{62}$$

For $k \geq 1$, we have

$$
\begin{aligned}
\mathbb{E}\left[\theta_{k+1}\right] &\overset{(60)}{\leq} \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2(1+2d\varkappa^2)\rho_k\right)\theta_k\right] \\
&\overset{(48)}{\leq} \left(1-\frac{1}{d}\right)\mathbb{E}\left[\exp\left(2(1+2d\varkappa^2)\left(1-\frac{1}{4\varkappa^2}\right)^k\rho_0\right)\theta_k\right] \\
&= \left(1-\frac{1}{d}\right)\exp\left(2(1+2d\varkappa^2)\left(1-\frac{1}{4\varkappa^2}\right)^k\rho_0\right)\mathbb{E}\left[\theta_k\right] \\
&\leq \left(1-\frac{1}{d}\right)^{k+1}\exp\left(2(1+2d\varkappa^2)\rho_0\sum_{i=0}^{k}\left(1-\frac{1}{4\varkappa^2}\right)^i\right)\mathbb{E}\left[\theta_0\right] \\
&\leq \left(1-\frac{1}{d}\right)^{k+1}\exp\left(8\varkappa^2(1+2d\varkappa^2)\rho_0\right)\mathbb{E}\left[\theta_0\right] \\
&\overset{(61),(62)}{\leq} \left(1-\frac{1}{d}\right)^{k+1} 2\varkappa^2,
\end{aligned}
\tag{63}
$$

which implies

$$
\mathbb{E}\left[\eta_k\right] \leq \mathbb{E}\left[\theta_k\right] \leq \left(1-\frac{1}{d}\right)^k 2\varkappa^2.
$$

Finally, we have

$$
\mathbb{E}\left[\frac{\lambda_{k+1}}{\lambda_k}\right] = \mathbb{E}\left[\frac{\rho_{k+1}}{\rho_k}\right] \overset{(57)}{\leq} \mathbb{E}\left[d\varkappa^2\theta_k\right] \overset{(63)}{\leq} \mathbb{E}\left[\left(1-\frac{1}{d}\right)^k 2d\varkappa^4\right]. \tag{64}
$$

D.9 The Proof of Corollary 3.18

Proof. We split the proofs of three algorithms into different subsections.

D.9.1 Random Broyden Family and BFGS Method

We consider the random variable $X_k = \sigma_k$ or $X_k = \lambda_{k+1}/\lambda_k$ for all $k \geq 0$ in the following derivation.

Broyden Family Method The proof is modified from the proof of Theorem 4.7 in Lin et al. [24]. Recall the Section D.8, we obtained the following results for Broyden update

$$
\mathbb{E}\left[X_k\right] \leq \left(1-\frac{1}{d\varkappa^2}\right)^k 2d\varkappa^2. \tag{65}
$$

Note that $\mathbf{X}_k \geq 0$, using Markov's inequality, we have for any $\epsilon > 0$,

$$
\mathbb{P}\left(X_k \geq \frac{2d\varkappa^2}{\epsilon}\left(1-\frac{1}{d\varkappa}\right)^k\right) \leq \frac{\mathbb{E}\left[\,X_k\right]}{\frac{2d\varkappa^2}{\epsilon}\left(1-\frac{1}{d\varkappa^2}\right)^k} \overset{(65)}{\leq} \epsilon. \tag{66}
$$

Choosing $\epsilon_k = \delta(1-q)q^k$ for some positive $q < 1$, then we have

$$
\begin{aligned}
\mathbb{P}\left(X_k \geq \frac{2d\varkappa^2}{\epsilon_k}\left(1-\frac{1}{d\varkappa^2}\right)^k, \exists k \in \mathbb{N}\right) &\leq \sum_{k=0}^{\infty}\mathbb{P}\left(X_k \geq \frac{2d\varkappa^2}{\epsilon_k}\left(1-\frac{1}{d\varkappa^2}\right)^k\right) \\
&\overset{(66)}{\leq} \sum_{k=0}^{\infty}\epsilon_k = \sum_{k=0}^{\infty}\delta(1-q)q^k = \delta.
\end{aligned}
$$

Therefore, we obtain with probability $1-\delta$,

$$
X_k \leq \left(\frac{1-\frac{1}{d\varkappa^2}}{q}\right)^k \cdot \frac{2d\varkappa^2}{(1-q)\delta}, \quad \text{for all } k \in \mathbb{N}.
$$

If we set $q = 1 - \frac{1}{d^2\varkappa^4}$, we could obtain with probability $1-\delta$, for all $k \in \mathbb{N}$,

$$X_k \leq \frac{2d^3\varkappa^6}{\delta}\left(1+\frac{1}{d\varkappa^2}\right)^{-k} = \frac{2d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^{k}.$$

Furthermore, it holds with probability $1-\delta$ that

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{4d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^{k}, \text{ for all } k \in \mathbb{N}, \tag{67}$$

and

$$\sigma_k \leq \frac{4d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^{k}, \text{ for all } k \in \mathbb{N}.$$

Telescoping from k to 0 in Eq. (67), we get

$$\begin{aligned}\lambda_k = \lambda_0 \cdot \prod_{i=1}^{k}\frac{\lambda_i}{\lambda_{i-1}} &\overset{(67)}{\leq} \lambda_0 \cdot \left(\frac{4d^3\varkappa^6}{\delta}\right)^k \prod_{i=1}^{k}\left(1-\frac{1}{d\varkappa^2+1}\right)^{i-1} \\ &= \left(\frac{4d^3\varkappa^6}{\delta}\right)^k \left(1-\frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2}\lambda_0.\end{aligned}$$

Now we combine this result with Theorem 3.15,we give the entire period convergence estimator. Denote by $k_1 \geq 0$ the number of the first iteratiod, for which

$$\left(1-\frac{1}{4\varkappa^2}\right)^{k_1} \leq \frac{1}{2d+1}.$$

Clearly, $k_1 \leq 4\varkappa^2\ln(2d+1)$. Since the initial point $\mathbf{z}_0$ is close to the saddle point: $M\lambda_0/\mu \leq \ln 2/(8\varkappa^2)$, Combining with Theorem 3.15 by choosing $b=2$, we have

$$\frac{M\lambda_{k_1}}{\mu} \leq M\left(1-\frac{1}{4\varkappa^2}\right)^{k_1}\frac{\lambda_0}{\mu} \leq \frac{\ln 2}{8(2d+1)\varkappa^2}, \tag{68}$$

and thus satisfies the initial condition for the Broyden family and the BFGS method in Theorem 3.17. In view of (67) denote $k_2 \geq 0$ the number of the first iteratiod, for which

$$\frac{4d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^{k_2} \leq \frac{1}{2}.$$

Clearly, $k_2 \leq (d\varkappa^2+1)\ln(8d^3\varkappa^6/\delta)$.

Thus for all $k \geq 0$, we have

$$\lambda_{k_1+k_2+k+1} \overset{(67)}{\leq} \frac{4d^3\varkappa^6}{\delta}\left(1-\frac{1}{d\varkappa^2+1}\right)^{k_2+k}\lambda_{k_1+k_2+k} \leq \frac{1}{2}\left(1-\frac{1}{d\varkappa^2+1}\right)^{k}\lambda_{k_1+k_2+k}.$$

Therefore,

$$\lambda_{k_1+k_2+k} \leq \left(1-\frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2}\left(\frac{1}{2}\right)^k\lambda_{k_1+k_2},$$

and by Theorem 3.15 we have

$$\lambda_{k_1+k_2} \leq \left(1-\frac{1}{4\varkappa^2}\right)^{k_1+k_2}\lambda_0.$$

Finally, choose $k_0 = k_1+k_2 = \mathcal{O}\left(d\varkappa^2\ln(d\varkappa/\delta)\right)$, we obtain

$$\lambda_{k_0+k} \leq \left(1-\frac{1}{d\varkappa^2+1}\right)^{k(k-1)/2}\cdot\left(\frac{1}{2}\right)^k\cdot\left(1-\frac{1}{4\varkappa^2}\right)^{k_0}\lambda_0.$$

BFGS Method Similar to the analysis for the random Broyden family method, we obtain with probability $1-\delta$,

$$X_k \leq \left(\frac{1-\frac{1}{d}}{q}\right)^k \cdot \frac{2d\varkappa^2}{(1-q)\delta}$$

for all $k \in \mathbb{N}$.

If we set $q = 1 - 1/d^2$, we could obtain with probability $1-\delta$,

$$\lambda_{k+1} \leq \frac{4d^3\varkappa^2}{\delta}\left(1-\frac{1}{d+1}\right)^k \lambda_k, \text{ for all } k \in \mathbb{N}, \tag{69}$$

and

$$\sigma_{k+1} \leq \frac{4d^3\varkappa^2}{\delta}\left(1-\frac{1}{d+1}\right)^k \sigma_k, \text{ for all } k \in \mathbb{N}.$$

Similar to the above proof, we denote $k_1 \geq 0$ as the number of the first iteration satisfies

$$\left(1-\frac{1}{4\varkappa^2}\right)^{k_1} \leq \frac{1}{2d+1}. \tag{70}$$

And we denote $k_2 \geq 0$ as the number of the first iteration satisfies

$$\frac{4d^3\varkappa^2}{\delta}\left(1-\frac{1}{d+1}\right)^{k_2} \leq \frac{1}{2}. \tag{71}$$

Clearly, $k_1 \leq 4\varkappa^2 \ln(2d+1)$ and $k_2 \leq (d+1)\ln(8d^3\varkappa^2/\delta)$. The remainder of the proof can follow the analysis in random Broyden methdos. We only need to replace all the term of $\left(1-\frac{1}{d\varkappa^2+1}\right)$ to $\left(1-\frac{1}{d+1}\right)$. The reason is that (69) provides a faster convergence result for random BFGS update, rather than (67). Set $k_0 = k_1 + k_2 = \mathcal{O}(\max\{d, \varkappa^2\}\ln(d\varkappa/\delta))$, we obtain

$$\lambda_{k_0+k} \leq \left(1-\frac{1}{d+1}\right)^{k(k-1)/2} \cdot \left(\frac{1}{2}\right)^k \cdot \left(1-\frac{1}{4\varkappa^2}\right)^{k_0} \lambda_0.$$

D.9.2 Random SR1 Method

SR1 Method We denote $X_k = \lambda_{k+1}/\lambda_k$ or $X_k = 2d\varkappa^2\theta_k$ for all $k \geq 0$ in the following derivation.

According to the results of (63), (64) and similar to the analysis for Broyden family method, we obtain with probability $1-\delta$ for SR1 update that

$$X_k \leq \left(\frac{1-\frac{1}{d}}{q}\right)^k \cdot \frac{2d\varkappa^4}{(1-q)\delta}$$

for all $k \in \mathbb{N}$.

If we set $q = 1 - 1/d^2$, we have

$$\lambda_{k+1} \leq \frac{4d^3\varkappa^4}{\delta}\left(1-\frac{1}{d+1}\right)^k \lambda_k$$

for all $k \in \mathbb{N}$. Similar to the above proof, we denote $k_1 \geq 0$ as the number of the first iteration satisfies

$$\left(1-\frac{1}{4\varkappa^2}\right)^{k_1} \leq \frac{1}{(2d\varkappa^2+1)}.$$

Since the initial point $\mathbf{z}_0$ is close to the saddle point: $M\lambda_0/\mu \leq \ln 2/(8\varkappa^2)$, Combining with Theorem 3.15 by choosing $b = 2$, we have

$$\frac{M\lambda_{k_1}}{\mu} \leq M\left(1-\frac{1}{4\varkappa^2}\right)^{k_1}\frac{\lambda_0}{\mu} \leq \frac{\ln 2}{8(2d\varkappa^2+1)\varkappa^2},$$

which satisfies the initial condition for SR1 method in Theorem 3.17. And we denote $k_2 \geq 0$ as the number of the first iteration satisfies

$$\frac{4d^3\varkappa^4}{\delta}\left(1-\frac{1}{d+1}\right)^{k_2} \leq \frac{1}{2}.$$

We have $k_1 \leq 4\varkappa^2 \ln(2d\varkappa^2+1)$ and $k_2 \leq (d+1)\ln(8d^3\varkappa^4/\delta)$.

Similar to above analysis, we obtain

$$\lambda_{k_0+k} \leq \left(1-\frac{1}{d+1}\right)^{k(k-1)/2} \cdot \left(\frac{1}{2}\right)^k \cdot \left(1-\frac{1}{4\varkappa^2}\right)^{k_0} \lambda_0$$

by setting $k_0 = k_1 + k_2 = \mathcal{O}(\max\{d, \varkappa^2\}\ln(d\varkappa/\delta))$.

□

E Experimental Details

We provide some details for our experiments in this section. Our experiments are conducted on a work station with 56 Intel(R) Xeon(R) Gold 6132 CPU @ 2.60GHz and 256GB memory. We use MATLAB 2021a to run the code and the operating system is Ubuntu 20.04.2.

E.1 AUC Maximization

The gradient of the object function at $\mathbf{z} = [\mathbf{x}; \mathbf{y}] = [\mathbf{w}; u; v; y]$ is

$$\mathbf{g}(\mathbf{z}) = \nabla f(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}\nabla_{\mathbf{w}} f_i(\mathbf{z}) \\ \nabla_u f_i(\mathbf{z}) \\ \nabla_v f_i(\mathbf{z}) \\ \nabla_y f_i(\mathbf{z})\end{bmatrix},$$

where

$$\begin{aligned}
\nabla_{\mathbf{w}} f_i(\mathbf{z}) &= \lambda\mathbf{w} + 2(1-p)(\mathbf{w}^\top\mathbf{a}_i - u - 1 - y))\mathbf{a}_i\mathbb{I}_{b_i=1} + 2p(\mathbf{w}_i^\top\mathbf{a}_i - v + 1 + y)\mathbf{a}_i\mathbb{I}_{b_i=-1}, \\
\nabla_u f_i(\mathbf{z}) &= \lambda u - 2(1-p)(\mathbf{w}^\top\mathbf{a}_i - u)\mathbb{I}_{b_i=1}, \\
\nabla_v f_i(\mathbf{z}) &= \lambda v - 2p(\mathbf{w}^\top\mathbf{a}_i - v)\mathbb{I}_{b_i=-1}, \\
\nabla_y f_i(\mathbf{z}) &= -2p(1-p)y + 2p\mathbf{w}^\top\mathbf{a}_i\mathbb{I}_{b_i=-1} - 2(1-p)\mathbf{w}^\top\mathbf{a}_i\mathbb{I}_{b_i=1}.
\end{aligned}$$

The Hessian-vector of the object function is

$$\nabla^2 f(\mathbf{z})\mathbf{h} = \hat{\mathbf{H}}(\mathbf{z})\mathbf{h} = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}(\nabla^2 f_i(\mathbf{z})\mathbf{h})_{\mathbf{w}} \\ (\nabla^2 f_i(\mathbf{z})\mathbf{h})_u \\ (\nabla^2 f_i(\mathbf{z})\mathbf{h})_v \\ (\nabla^2 f_i(\mathbf{z})\mathbf{h})_y\end{bmatrix},$$

where $\mathbf{h} = [\mathbf{h}_{\mathbf{w}}; \mathbf{h}_u; \mathbf{h}_v; \mathbf{h}_y]$ such that

$$\begin{aligned}
(\nabla^2 f_i(\mathbf{z})\mathbf{h})_{\mathbf{w}} &= \lambda\mathbf{h}_{\mathbf{w}} + 2(1-p)(\langle\mathbf{a}_i, \mathbf{h}_{\mathbf{w}}\rangle - \mathbf{h}_u - \mathbf{h}_y)\mathbf{a}_i\mathbb{I}_{b_i=1} \\
&\quad + 2p(\langle\mathbf{a}_i, \mathbf{h}_{\mathbf{w}}\rangle - \mathbf{h}_v + \mathbf{h}_y)\mathbf{a}_i\mathbb{I}_{b_i=-1}, \\
(\nabla^2 f_i(\mathbf{z})\mathbf{h})_u &= -2(1-p)\mathbf{a}_i^\top\mathbf{h}_{\mathbf{w}}\mathbb{I}_{b_i=1} + (\lambda + 2(1-p)\mathbb{I}_{b_i=1})\mathbf{h}_u, \\
(\nabla^2 f_i(\mathbf{z})\mathbf{h})_v &= -2p\mathbf{a}_i^\top\mathbf{h}_{\mathbf{w}}\mathbb{I}_{b_i=-1} + (\lambda + 2p\mathbb{I}_{b_i=-1})\mathbf{h}_v, \\
(\nabla^2 f_i(\mathbf{z})\mathbf{h})_y &= -2p(1-p)y + 2p\mathbf{a}_i^\top\mathbf{h}_{\mathbf{w}}\mathbb{I}_{b_i=-1} - 2(1-p)\mathbf{a}_i^\top\mathbf{h}_{\mathbf{w}}\mathbb{I}_{b_i=1}.
\end{aligned}$$

Note that the Hessian-vector can be achieved in $\mathcal{O}(nd)$ flops and it guarantees $\mathcal{O}(nd+d^2)$ complexity for each iteration.

For baseline method extragradient (Algorithm 8), we tune the stepsize from $\{0.01, 0.05, 0.1, 0.5\}$. For RaBFGSv1-Q (Algorithm 2), we let $\mathbf{G}_0 = 3\mathbf{I}$ for "a9a", "w8a" and $\mathbf{G}_0 = 30\mathbf{I}$ for "sido0". For RaBFGSv2-Q (Algorithm 3), we let $\mathbf{G}_0 = 3\mathbf{I}$ for "a9a" and "w8a". We do not run RaBFGSv2-Q on

Algorithm 8 Extragradient Method

Input: $\mathbf{z}_0 \in \mathbb{R}^n$, and $\eta > 0$.
for $k = 0, 1, \ldots$
  $\mathbf{x}_{k+1/2} = \mathbf{x}_k - \eta \nabla_{\mathbf{x}} f(\mathbf{x}_k, \mathbf{y}_k)$
  $\mathbf{y}_{k+1/2} = \mathbf{y}_k + \eta \nabla_{\mathbf{y}} f(\mathbf{x}_k, \mathbf{y}_k)$
  $\mathbf{x}_{k+1} = \mathbf{x}_k - \eta \nabla_{\mathbf{x}} f(\mathbf{x}_{k+1/2}, \mathbf{x}_{k+1/2})$
  $\mathbf{y}_{k+1} = \mathbf{y}_k + \eta \nabla_{\mathbf{y}} f(\mathbf{x}_{k+1/2}, \mathbf{x}_{k+1/2})$
end for

"sido0" because this algorithm is not efficient for high-dimensional problem as we have mentioned in Remark 3.5. For RaSR1-Q (Algorithm 4), we let $\mathbf{G}_0 = 5\mathbf{I}$ for "a9a", "w8a" and $\mathbf{G}_0 = 30\mathbf{I}$ for "sido0".

The dataset "sido0" comes from Causality Workbench [17] and the other datasets can be downloaded from LIBSVM repository [8].

E.2 Adversarial Debiasing

The minimax formulation can be rewritten as

$$\min_{\mathbf{x} \in \mathbb{R}^m} \max_{y \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{x}, y; \mathbf{a}_i, b_i, c_i, \lambda, \gamma, \beta)$$

where f_i is defined as

$$f_i(\mathbf{x}, y; \mathbf{a}_i, b_i, c_i, \lambda, \gamma, \beta) = \log(1 + \exp(-b_i \mathbf{a}_i^\top \mathbf{x})) - \beta \log(1 + \exp(-c_i \mathbf{a}_i^\top \mathbf{x} y)) + \lambda \|\mathbf{x}\|^2 - \gamma y^2.$$

We define

$$p_i = \frac{1}{1 + \exp(b_i \mathbf{a}_i^\top \mathbf{x})} \qquad \text{and} \qquad q_i = \frac{1}{1 + \exp(c_i y \mathbf{a}_i^\top \mathbf{x})}.$$

Then the gradient of the object function at $\mathbf{z} = [\mathbf{x}; y]$ is

$$\mathbf{g}(\mathbf{z}) = \nabla f(\mathbf{z}) = \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix} \nabla_{\mathbf{x}} f_i(\mathbf{z}) \\ \nabla_y f_i(\mathbf{z}) \end{bmatrix},$$

where

$$\nabla_{\mathbf{x}} f_i(\mathbf{z}) = -p_i b_i \mathbf{a}_i + \beta q_i c_i y \mathbf{a}_i + 2\lambda \mathbf{x} \qquad \text{and} \qquad \nabla_y f_i(\mathbf{z}) = \beta c_i \mathbf{a}_i^\top x q_i - 2\gamma y.$$

The Hessian-vector of the object function is

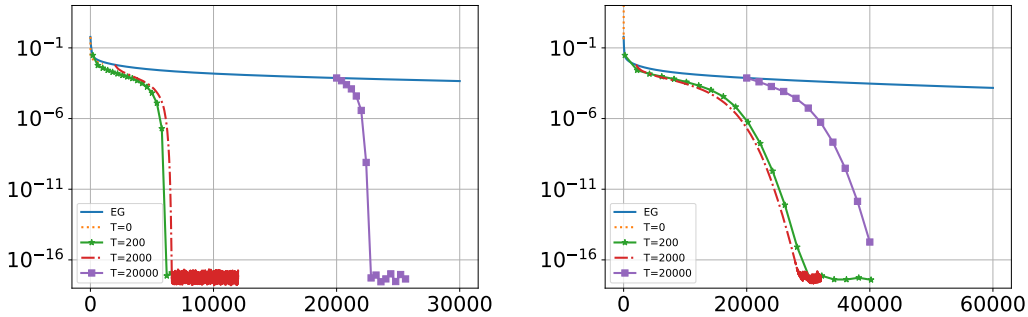
$$\nabla^2 f(\mathbf{z}) \mathbf{h} = \hat{\mathbf{H}}(\mathbf{z}) \mathbf{h} = \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix} (\nabla^2 f_i(\mathbf{z}) \mathbf{h})_{\mathbf{x}} \\ (\nabla^2 f_i(\mathbf{z}) \mathbf{h})_y \end{bmatrix}.$$

where $\mathbf{h} = [\mathbf{h}_{\mathbf{x}}; \mathbf{h}_y]$ such that

$$\begin{aligned} (\nabla^2 f_i(\mathbf{z}) \mathbf{h})_{\mathbf{x}} &= (p_i(1 - p_i) \mathbf{a}_i^\top \mathbf{h}_{\mathbf{x}} - q_i(1 - q_i) \beta y^2 \mathbf{a}_i^\top \mathbf{h}_{\mathbf{x}}) \mathbf{a}_i + 2\lambda \mathbf{h}_{\mathbf{x}} \\ &\quad - (q_i(1 - q_i) \beta y \mathbf{a}_i^\top \mathbf{x} - q_i \beta c) \mathbf{h}_y \mathbf{a}_i, \\ (\nabla^2 f_i(\mathbf{z}) \mathbf{h})_y &= -\beta y q_i (1 - q_i) \mathbf{a}_i^\top \mathbf{x} \mathbf{a}_i^\top \mathbf{h}_{\mathbf{x}} + q_i \beta c \mathbf{a}_i^\top \mathbf{h}_{\mathbf{x}} - q_i(1 - q_i) \beta (\mathbf{a}_i^\top \mathbf{x})^2 \mathbf{h}_y - 2\gamma \mathbf{h}_y. \end{aligned}$$

The Hessian-vector can be achieved in $\mathcal{O}(nd)$ flops and it guarantees $\mathcal{O}(nd + d^2)$ complexity for each iteration.

Data Preparation The experiments are based on the datasets of fairness aware machine learning [35]. Following the reprocessing of Chang and Lin [8], Platt [32], we convert the features of the original datasets into binary for our experiments. Concretely, the continuous features are discretized into quantiles, and each quantile is represented by a binary feature. Also, a categorical feature with C categories is converted to C binary features. More specifically, for the "adults" dataset, we transform the 13 features of it into 122 binary features and choose the feature of "gender" as the protected feature. For the "law school" dataset, we transform the 11 features of it into 379 binary features and choose the feature of "gender" as the protected feature. For the "bank marketing" dataset, we transform 16 features of it into 3879 binary features and choose "marital" as the protected feature.



(a) RaSR1 with different T (b) RaBFGSv2 with different T

Figure 3: We demonstrate iteration numbers vs. $\|\mathbf{g}(\mathbf{z})\|_2$ for adversarial debiasing model on datasets "law school" ($d = 380$, $n = 20427$).

Assumptions Validation We can verify the objective function satisfies our assumptions. Since the convergence results in Section 3.3 are local, we only need to show that f satisfies Assumption 2.1 and 2.2 in a local region around $\mathbf{z}^*$. Hence, we only needs to consider $(\mathbf{x}, y)$ such that $\|\mathbf{x}\| \leq D_1$ and $\|y\| \leq D_2$ for some $D_1 > 0$ and $D_2 > 0$.

Using the notation in Appendix E.2, we have

$$\left(2\lambda - \left\|\frac{\beta y^2}{4n}\sum_{i=1}^n \mathbf{a}_i^\top \mathbf{a}_i\right\|\right)\mathbf{I} \preceq \nabla^2_{\mathbf{xx}} f(\mathbf{z}) \preceq \left(\|\frac{1}{4n}\sum_{i=1}^n \mathbf{a}_i^\top \mathbf{a}_i\| + 2\lambda\right)\mathbf{I},$$

$$2\gamma \leq -\nabla^2_{yy} f(\mathbf{z}) \leq \left(\frac{1}{4n}\sum_{i=1}^n \left(\beta \mathbf{a}_i^\top \mathbf{x}\right)^2 + 2\gamma\right).$$

Additionally, the feature $\{\mathbf{a}_i\}$ is sparse due to the transform, which means the upper bound of $\left\|\sum_{i=1}^n \mathbf{a}_i^\top \mathbf{a}_i\right\|$ is small. We can guarantee $2\lambda - \left\|\frac{\beta D_2^2}{4n}\sum_{i=1}^n \mathbf{a}_i^\top \mathbf{a}_i\right\| > 0$ by choosing λ properly. Theoretically, setting $\mu = \min\left\{2\lambda - \left\|\frac{\beta D_2^2}{4n}\sum_{i=1}^n \mathbf{a}_i^\top \mathbf{a}_i\right\|, 2\gamma\right\}$ can guarantee f satisfies Assumption 2.2.

Experiment Parameters For experiments, we set $\mathbf{G}_0 = 2\mathbf{I}$ for the proposed quasi-Newton methods. For RaBFGSv1-G (Algorithm 5), we set $M = 0$ for "adults", $M = 1$ for "law school" and "bank market". For RaBFGSv2-G (Algorithm 6), we set $M = 1$ for "adults" and "law school". For RaSR1-G (Algorithm 7), we set $M = 1$ for all three datasets.

We tune the stepsize of EG from $\{0.01, 0.05, 0.1, 0.5\}$ and run it with 4000, 20000 and 40000 iterations for the "adults", "law school" and "Bank market" respectively as warm up to obtain $\mathbf{z}_0$ as initial point. Then we evaluate all algorithms (including the baseline algorithm EG) by starting with $\mathbf{z}_0$ and achieve the result shown in Figure 2. Since each algorithm has the identical behavior in the warm up stage, we only present the curves of iterations vs. $\|\mathbf{g}(\mathbf{z})\|_2$ and CPU time vs. $\|\mathbf{g}(\mathbf{z})\|_2$ after warm up stage in Figure 2.

We run additional experiments on RaSR1 and RaBFGSv2 for "law school" with different number rounds of extragradient iteration as warm-up. The number of iterations T is selected from $\{0, 200, 2000, 20000\}$. The results in Figure 3 show that our methods could converge even for small T.

E.3 Two Stages Convergence Behavior

For most of cases, our algorithms enter the local region of superlinear convergence quickly, so that the two-period convergence is not very clear on the figure. Let's have a look at RaBFGSv2-Q in (b) of Figure 1 (the purple curve). We can observe two-period convergence behavior clearly. The first period (linear convergence) roughly corresponds to the first 2000 rounds of iterations on the figure and the second period (superlinear convergence) roughly corresponds to the later iterations. The

convergence behavior of first period for RaBFGSv2-Q looks worse than the first-order method (EG), which is reasonable since the linear convergence rate of RaBFGSv2-Q in the first period depends on κ^2 while the linear convergence rate of EG depends on κ.

F Extension for Solving Nonlinear Equations

In this section, we extend our algorithms for solving general nonlinear equations. Concretely, we consider finding the solution of the system

$$\mathbf{g}(\mathbf{z}) = \mathbf{0}. \tag{72}$$

The remainders of this paper do not require the operator $\mathbf{g}(\cdot)$ related to some minimax problem and it could be any differentiable function from $\mathbb{R}^d$ to $\mathbb{R}^d$. We use $\hat{\mathbf{H}}(\mathbf{z})$ to present the Jacobian of $\mathbf{g}(\cdot)$ at $\mathbf{z} \in \mathbb{R}^d$ and still follow the notation $\mathbf{H}(\mathbf{z}) \stackrel{\text{def}}{=} \left(\hat{\mathbf{H}}(\mathbf{z})\right)^2$. We suppose the nonlinear equation satisfies the following conditions.

Assumption F.1. The function $\mathbf{g} : \mathbb{R}^d \to \mathbb{R}^d$ is differentiable and its Jacobian $\hat{\mathbf{H}} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is L_2-Lipschitz continuous. That is, for all $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^d$, we have

$$\left\|\hat{\mathbf{H}}(\mathbf{z}) - \hat{\mathbf{H}}(\mathbf{z}')\right\| \le L_2\|\mathbf{z} - \mathbf{z}'\|. \tag{73}$$

Assumption F.2. There exists a solution $\mathbf{z}^*$ of equation (72) such that $\hat{\mathbf{H}}(\mathbf{z}^*)$ is non-singular. Additionally, we assume the smallest and largest singular values of $\hat{\mathbf{H}}(\mathbf{z}^*)$ are μ and L respectively.

We still denote the condition number as $\varkappa \stackrel{\text{def}}{=} L/\mu$. Note that the saddle point problem (1) under Assumption 2.1 and 2.2 is a special case of solving nonlinear equation (72) under Assumption F.1 and F.2.

Similar to previous section, the design of the algorithms is based on approximating the auxiliary matrix $\mathbf{H}(\mathbf{z})$. Hence, we start from considering its smoothness.

Lemma F.3. *Under Assumption F.1 and F.2, we have*

$$\|\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}')\| \le 4L_2L\|\mathbf{z} - \mathbf{z}'\|, \tag{74}$$

and

$$\frac{\mu^2}{2}\mathbf{I} \preceq \mathbf{H}(\mathbf{z}) \preceq 4L^2\mathbf{I}, \tag{75}$$

for all $\mathbf{z}, \mathbf{z}' \in \left\{\mathbf{z} : \|\mathbf{z} - \mathbf{z}^*\| \le \frac{L}{8\varkappa^2 L_2}\right\}$.

Proof. We define the local neighbor of the solution $\mathbf{z}^*$ as follows

$$\Omega^* \stackrel{\text{def}}{=} \{\mathbf{z} : \|\mathbf{z} - \mathbf{z}^*\| \le D\}, \quad \text{where } D = \frac{\mu^2}{8L_2L}.$$

Assumption F.2 implies $\sigma_{\min}(\mathbf{H}(\mathbf{z}^*)) = \mu^2$ and $\sigma_{\max}(\mathbf{H}(\mathbf{z}^*)) = L^2$.

Since $\mathbf{H}(\mathbf{z}^*)$ is invertible and symmetric, we have $\mathbf{H}(\mathbf{z}^*) \succ 0$ and we have restricted $\mathbf{z}, \mathbf{z}' \in \Omega^*$, it holds that

$$\left\|\hat{\mathbf{H}}(\mathbf{z})\right\| \le \left\|\hat{\mathbf{H}}(\mathbf{z}) - \hat{\mathbf{H}}(\mathbf{z}^*)\right\| + \left\|\hat{\mathbf{H}}(\mathbf{z}^*)\right\| \le L_2\|\mathbf{z} - \mathbf{z}^*\| + \left\|\hat{\mathbf{H}}(\mathbf{z}^*)\right\| \le L_2D + L, \tag{76}$$

which implies

$$\mathbf{H}(\mathbf{z}) \preceq (L_2D + L)^2\mathbf{I} \tag{77}$$

for all $\mathbf{z} \in \Omega^*$. According to (77), we have

$$\begin{aligned}
\|\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}')\| &= \left\|\hat{\mathbf{H}}(\mathbf{z})^2 - \hat{\mathbf{H}}(\mathbf{z}')^2\right\| \\
&\le \left\|\hat{\mathbf{H}}(\mathbf{z})\left(\hat{\mathbf{H}}(\mathbf{z}) - \hat{\mathbf{H}}(\mathbf{z}')\right)\right\| + \left\|\left(\hat{\mathbf{H}}(\mathbf{z}) - \hat{\mathbf{H}}(\mathbf{z}')\right)\hat{\mathbf{H}}(\mathbf{z}')\right\| \\
&\le L_2\left(\left\|\hat{\mathbf{H}}(\mathbf{z})\right\| + \left\|\hat{\mathbf{H}}(\mathbf{z}')\right\|\right)\|\mathbf{z} - \mathbf{z}'\|
\end{aligned}$$

$$\overset{(76)}{\leq} \quad 2L_2(L_2D+L)\|\mathbf{z}-\mathbf{z}'\|.$$

Combining above result with the Weyl's inequality for singular values [19, Theorem 3.3.16], we have

$$\sigma_{\min}(\mathbf{H}(\mathbf{z}^*)) - 2L_2(L_2D+L)\|\mathbf{z}-\mathbf{z}^*\| \leq \sigma_{\min}(\mathbf{H}(\mathbf{z})).$$

Since the definition of D indicates $2L_2(L_2D+L)D \leq \frac{\mu^2}{2}$ and $L_2D \leq L$, we have

$$\frac{\mu^2}{2}\mathbf{I} \preceq \mathbf{H}(\mathbf{z}) \preceq 4L^2\mathbf{I},$$

and

$$\|\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}')\| \leq 4L_2L\|\mathbf{z}-\mathbf{z}'\|$$

□

Then we have the property similar to strongly self-concordance like Lemma 3.11.

Lemma F.4. *For all* $\mathbf{z}, \mathbf{z}', \mathbf{w} \in \left\{\mathbf{z} : \|\mathbf{z}-\mathbf{z}^*\| \leq \frac{L}{8\varkappa^2 L_2}\right\}$*, we have:*

$$\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}') \preceq M\|\mathbf{z}-\mathbf{z}'\|\mathbf{H}(\mathbf{w}) \tag{78}$$

with $M = 8\varkappa^2 L_2/L$

Proof. According to Lemma F.3, we have

$$\begin{aligned}
\mathbf{H}(\mathbf{z}) - \mathbf{H}(\mathbf{z}') &\overset{(74)}{\preceq} 4L_2L\|\mathbf{z}-\mathbf{z}'\|\mathbf{I} \\
&\overset{(75)}{\preceq} \frac{8L_2L}{\mu^2}\|\mathbf{z}-\mathbf{z}'\|\mathbf{H}(\mathbf{w}) \\
&= \frac{8\varkappa^2 L_2}{L}\|\mathbf{z}-\mathbf{z}'\|\mathbf{H}(\mathbf{w}).
\end{aligned}$$

for all $\mathbf{z}, \mathbf{z}' \in \Omega^*$. □

After above preparation, we can directly apply Algorithm 5, 6 and 7 with $M = 8\varkappa^2 L_2/L$ and $\mathbf{G}_0 = 4L^2\mathbf{I}$ to find the solution of (72). Our convergence analysis is still based on the measure of $\lambda_k \overset{\text{def}}{=} \|\nabla(\mathbf{z}_k)\|$. Different from the setting of saddle point problems, the properties shown in Lemma F.3 and F.4 only hold locally. Hence, we introduce the following lemma to show $\mathbf{z}_k$ generated from the algorithms always lies in the neighbor of solution $\mathbf{z}^*$.

Lemma F.5. *Solving general nonlinear equations* (72) *under Assumption F.1 and F.2 by proposed greedy and random quasi-Newton methods (Algorithm 5, 6 and 7) with* $M = 8\varkappa^2 L_2/L$ *and* $\mathbf{G}_0 = 4L^2\mathbf{I}$*, if the initial point* $\mathbf{z}_0$ *is sufficiently close to the solution* $\mathbf{z}^*$ *such that*

$$\frac{M\lambda_0}{\mu} \leq \frac{\ln 2}{64\sqrt{2}\varkappa^2}, \tag{79}$$

then for all $k \geq 0$*, we have*

$$\lambda_k \leq \left(1 - \frac{1}{32\varkappa^2}\right)^k \lambda_0, \tag{80}$$

which means $\mathbf{z}_k \in \left\{\mathbf{z} : \|\mathbf{z}-\mathbf{z}^*\| \leq \frac{L}{8\varkappa^2 L_2}\right\}$.

Proof. We prove this lemma by induction. For $k = 0$, it is obviously. Suppose the statement holds for all $k' \leq k$. Then for all $k' = 0, \ldots, k$, we have $\mathbf{z}_{k'} \in \Omega^*$ and $\mathbf{z}_{k'}$ holds that (75), (74) and (78). By Theorem 3.15, we guarantee

$$\mathbf{H}_{k'} \leq \mathbf{G}_{k'} \leq 16\varkappa^2\mathbf{H}_{k'}. \tag{81}$$

For $k' = k+1$, according to the proof of Lemma 3.14, we have

$$\mathbf{g}_{k+1} = \underbrace{(\mathbf{I} - \hat{\mathbf{H}}_k\mathbf{G}_k^{-1}\hat{\mathbf{H}}_k)\mathbf{g}_k}_{\mathbf{a}_k} + \underbrace{\int_0^1 \left[\hat{\mathbf{H}}(\mathbf{z}_k + s(\mathbf{z}_{k+1}-\mathbf{z}_k)) - \hat{\mathbf{H}}(\mathbf{z}_k)\right](\mathbf{z}_{k+1}-\mathbf{z}_k)ds}_{\mathbf{b}_k}.$$

Using Lemma B.2 and the result of (81), we have

$$\|\mathbf{I} - \hat{\mathbf{H}}_k \mathbf{G}_k^{-1} \hat{\mathbf{H}}_k\| \leq 1 - \frac{1}{16\varkappa^2}$$

and

$$\|\mathbf{b}_k\| \leq \frac{L_2}{\mu^2} \lambda_k^2.$$

Combing above results, we have

$$\lambda_{k+1} \leq \left(1 - \frac{1}{16\varkappa^2}\right) \lambda_k + \frac{L_2}{\mu^2} \lambda_k^2 \leq \left(1 - \frac{1}{32\varkappa^2}\right) \lambda_k.$$

Thus we always have $\|\mathbf{z}_{k+1} - \mathbf{z}_k\| \leq \frac{1}{\mu} \lambda_{k+1} \leq \lambda_0 \leq D$. By induction, we finish the proof. □

Based on Lemma F.5, we establish the following theorem to show the algorithms also have local superlinear convergence for solving nonlinear equations.

Theorem F.6. *Solving general nonlinear equations* (72) *under Assumption F.1 and F.2 by proposed quasi-Newton methods (Algorithm 5, 6 and 7) with* $M = 8\varkappa^2 L_2/L$ *and* $\mathbf{G}_0 = 4L^2\mathbf{I}$*, if the initial point* $\mathbf{z}_0$ *is sufficiently close to the solution* $\mathbf{z}^*$ *such that*

$$\frac{M\lambda_0}{\mu} \leq \frac{\ln 2}{64\sqrt{2}\varkappa^2},$$

with probability $1 - \delta$ *for any* $\delta \in (0, 1)$*, we have the following results.*

1. *For random Broyden family method (Algorithm 5), we have*

$$\lambda_{k_0+k} \leq \left(1 - \frac{1}{8d\varkappa^2 + 1}\right)^{\frac{k(k-1)}{2}} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{32\varkappa^2}\right)^{k_0} \lambda_0$$

   for all $k > 0$ *and* $k_0 = \mathcal{O}\left(d\varkappa^2 \ln(d\varkappa/\delta)\right)$.

2. *For random BFGS/SR1 method (Algorithm 6, 7), we have*

$$\lambda_{k_0+k} \leq \left(1 - \frac{1}{d+1}\right)^{\frac{k(k-1)}{2}} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{32\varkappa^2}\right)^{k_0} \lambda_0$$

   for all $k > 0$ *and* $k_0 = \mathcal{O}\left(\max\{d, \varkappa^2\} \ln(d\varkappa/\delta)\right)$.

Proof. We denote

$$L' \stackrel{\text{def}}{=} 2L, \qquad \mu' \stackrel{\text{def}}{=} \sqrt{2}\mu \qquad \text{and} \qquad \varkappa' \stackrel{\text{def}}{=} \frac{L'}{\mu'} = \frac{2\sqrt{2}L}{\mu}.$$

The initial condition on λ_0 and Lemma F.5 means all the points $\mathbf{z}_k$ generated from our algorithms are located in Ω^*. Thus we can directly use the results of 3.18 by replacing $\varkappa$ and μ of the Corollary 3.18 into $\varkappa' = \frac{L'}{\mu'} = 2\sqrt{2}\varkappa$ and $\mu' = \frac{1}{\sqrt{2}}\mu$ here. □

Discussion Recently, Lin et al. [23], Ye et al. [46] showed Broyden's methods have explicit local superlinear convergence rate for solving nonlinear equations, we compare our methods with theirs as follow.

- For the assumptions, they suppose $\|\hat{\mathbf{H}}(\mathbf{z}) - \hat{\mathbf{H}}(\mathbf{z}^*)\| \leq L_2 \|\mathbf{z} - \mathbf{z}^*\|$ for any $\mathbf{z} \in \mathbb{R}^d$, which is weaker than Assumption F.1 of ours.
- For the convergence rate, we compare our results with Lin et al. [23], Ye et al. [46] in Table 2. The superlinear convergence rate of Lin et al. [23] is $\mathcal{O}((1/\sqrt{k})^k)$, which is worse than ours. The one of Ye et al. [46] is $\mathcal{O}((1 - 1/(d+1))^{k(k-1)/4})$, which is comparable to ours.

Table 2: Comparison of Upper Bound. The measure of Lin et al. [23], Ye et al. [46] is the Euclidean distance $\|\mathbf{z}_k - \mathbf{z}^*\|$. We denote $c = \left\|\left(\hat{\mathbf{H}}^*\right)^{-1}\right\|$ and $\sigma_0 = \left\|\left(\hat{\mathbf{H}}^*\right)^{-1}(\hat{\mathbf{G}}_0 - \hat{\mathbf{H}}^*)\right\|$. The measure of ours is the gradient norm $\|\mathbf{g}(\mathbf{z}_k)\|$.

Algorithms	Upper Bound of $\lambda_{k+k_0}/\lambda_0$
[23, Theorem 4.4]	$\left(\frac{7(\sigma_0 + \sqrt{cL_2\|\mathbf{z}_0 - \mathbf{z}^*\|})}{\sqrt{k+k_0}}\right)^{k+k_0}$
[46, Theorem 4.5]	$\left(\frac{4d^2 e}{\delta}\right)^{k+k_0}\left(1 - \frac{1}{d+1}\right)^{(k+k_0)(k+k_0-1)/4}$
Theorem F.6 of this paper	$\left(1 - \frac{1}{d+1}\right)^{k(k-1)/2}\left(\frac{1}{2}\right)^{k}\left(1 - \frac{1}{32\varkappa^2}\right)^{k_0}$

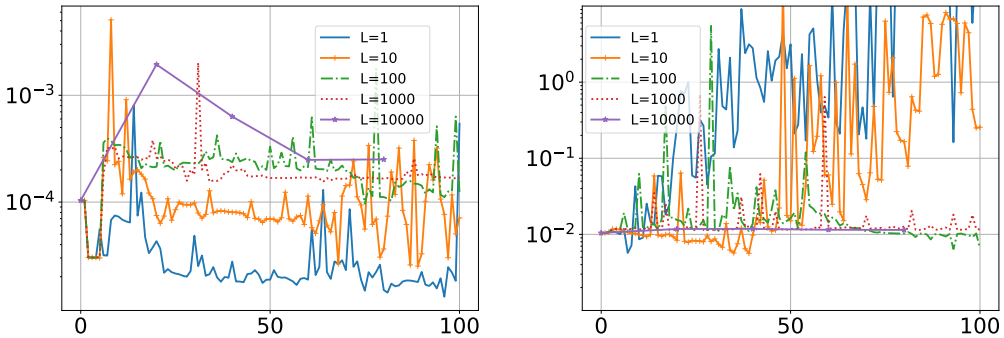
Table 3: Comparison of Initial Condition

Reference	Initial Condition
[23, Theorem 4.4]	$48L_2\left\|\left(\hat{\mathbf{H}}^*\right)^{-1}\right\|\left\|\mathbf{z}_0 - \mathbf{z}^*\right\| + \left\|\left(\hat{\mathbf{H}}^*\right)^{-1}(\hat{\mathbf{G}}_0 - \hat{\mathbf{H}}^*)\right\|_F \leq \frac{1}{3}$
[46, Theorem 4.3]	$48L_2\left\|\left(\hat{\mathbf{H}}^*\right)^{-1}\right\|\left\|\mathbf{z}_0 - \mathbf{z}^*\right\| + \left\|\left(\hat{\mathbf{H}}^*\right)^{-1}\right\|\left\|\hat{\mathbf{G}}_0 - \hat{\mathbf{H}}(\mathbf{z}_0)\right\| \leq \frac{1}{3}$
Theorem F.6 of this paper	$\lambda_0 \leq \frac{\ln(2)\mu^5}{512\sqrt{2}L^3L_2}$
[37, Theorem 4.7] (min)	$\langle \nabla f(\mathbf{z}_0), \nabla^2 f(\mathbf{z}_0)^{-1}\nabla f(\mathbf{z}_0)\rangle^{1/2} \leq \frac{\ln(3/2)\mu^{5/2}}{4L_2L}$
[24, Corollary 21] (min)	$\langle \nabla f(\mathbf{z}_0), \nabla^2 f(\mathbf{z}_0)^{-1}\nabla f(\mathbf{z}_0)\rangle^{1/2} \leq \frac{\ln(3/2)\mu^{5/2}}{4L_2L}$

- As for the initial condition, we compare of us with Lin et al. [23], Ye et al. [46] and quasi-Newton methods for minimization problem Lin et al. [24], Rodomanov and Nesterov [37] in Table 3. Our algorithms only require $\mathbf{z}_0$ be sufficiently close to the solution, while Lin et al. [23], Ye et al. [46] need stronger initial condition. Concretely, Lin et al. [23] requires $\hat{\mathbf{G}}_0$ be sufficiently close to $\hat{\mathbf{H}}^*$ and Ye et al. [46] requires $\hat{\mathbf{G}}_0$ be sufficiently close to $\hat{\mathbf{H}}(\mathbf{z}_0)$.

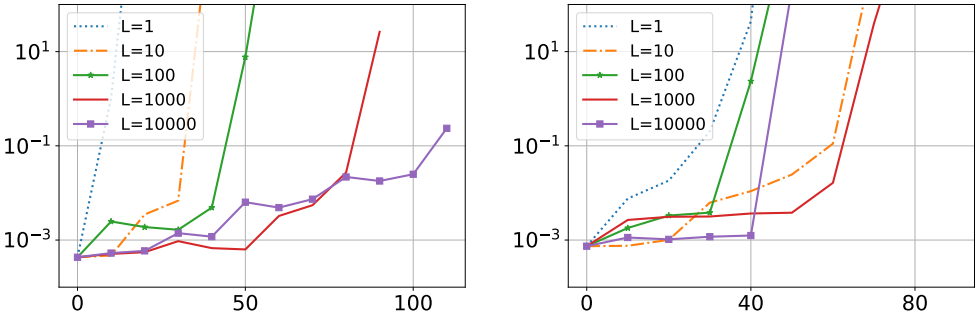
In practice, there is no general method to achieve an initial matrix $\mathbf{G}_0$ that is sufficiently closed to the exact Jacobian at $\mathbf{z}_0$ or $\mathbf{z}^*$ and computing the inverse of $\mathbf{G}_0$ always requires huge computation. Another way is using the matrix of the form $\mathbf{G}_0 = L\mathbf{I}$ as initialization, however, this strategy does not always work. we their methods on adversarial debiasing model of Section 5.2.

We tune the stepsize of EG from $\{0.01, 0.05, 0.1, 0.5\}$ and run it with 10000 and 20000 iterations for the "adults" and "law school" respectively as warm up to obtain $\mathbf{z}_0$ as initial point. Start with such initial point $\mathbf{z}_0$, our methods always converge as is shown in Figure 2 while the results in Figure 4 and 5 show that Broyden methods of Lin et al. [23] and Ye et al. [46] fail to converge.



(a) "adults" (b) "law school"

Figure 4: We demonstrate iteration numbers vs. $\|\mathbf{g}(\mathbf{z})\|_2$ for Broyden methods of Lin et al. [23] for adversarial debiasing model on datasets "adults" ($d = 123$, $n = 32561$) and "law school" ($d = 380$, $n = 20427$) with $\mathbf{G}_0 = L\mathbf{I}$. We choose L from $\{1, 10, 100, 1000, 10000\}$



(a) "adults" (b) "law school"

Figure 5: We demonstrate iteration numbers vs. $\|\mathbf{g}(\mathbf{z})\|_2$ for Broyden methods of Ye et al. [46] for adversarial debiasing model on datasets "adults" ($d = 123$, $n = 32561$) and "law school" ($d = 380$, $n = 20427$) with $\mathbf{G}_0 = L\mathbf{I}$. We choose L from $\{1, 10, 100, 1000, 10000\}$