An Information-Theoretic Framework for Deep Learning

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Abstract

Each year, deep learning demonstrates new and improved empirical results with deeper and wider neural networks. Meanwhile, with existing theoretical frameworks, it is difficult to analyze networks deeper than two layers without resorting to counting parameters or encountering sample complexity bounds that are exponential in depth. Perhaps it may be fruitful to try to analyze modern machine learning under a different lens. In this paper, we propose a novel information-theoretic framework with its own notions of regret and sample complexity for analyzing the data requirements of machine learning. We use this framework to study the sample complexity of learning from data generated by deep ReLU neural networks and deep networks that are infinitely wide but have a bounded sum of weights. We establish that the sample complexity of learning under these data generating processes is at most linear and quadratic, respectively, in network depth.

1 Introduction

In modern machine learning, the apparent capabilities of empirical methods have rapidly outpaced what is soundly understood theoretically. Modern neural network architectures have scaled immensely in both parameter count and depth. GPT3, for example, encodes about 175 billion parameters in 96 decoder blocks, each with many layers within. Yet, contrary to traditional intuition, these deep neural networks with gargantuan parameter counts are able to generalize well and produce useful models with tractable amounts of data. This gap between what has been shown theoretically versus empirically makes it quite enticing to develop a coherent framework that could potentially explain this phenomenon.

Existing theory breaks down when trying to explain learning under models that are simultaneously very deep (many layers) and wide (many hidden units per layer). Classical results such as those of Haussler [1992] and Bartlett et al. [1998] can potentially handle the deep but narrow case. These results bound the sample complexity of a learning a neural network function in terms of the number of parameters and the depth. More recently, Harvey et al. [2017] established a general result that suggests that for neural networks with piecewise-linear activation units, the sample complexity grows linearly in the product of parameter count and depth.

However, when we consider neural networks with arbitrary width, these bounds become vacuous. As an alternative to parameter count methods, researchers have produced sample complexity bounds

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that depend on the product of norms of realized weight matrices. Bartlett et al. [2017] and Neyshabur et al. [2018], for example, establish sample complexity bounds that scale with the product of spectral norms. Neyshabur et al. [2015] and Golowich et al. [2018] establish similar bounds that instead scale in the product of Frobenius norms. While this line of work provides sample complexity bounds that are *width*-independent, they pay for it via an exponential dependence on *depth*, which is also inconsistent with empirical results.

A large line of work has tried to ameliorate this exponential depth dependence via so-called *data-dependent* quantities Dziugaite and Roy [2017], Arora et al. [2018], Nagarajan and Kolter [2018], Wei and Ma [2019]. Among these, the most relevant to our work is Wei and Ma [2019], which bounds sample complexity as a function of depth and statistics of trained neural network weights. While difficult to interpret due to dependence on complicated data-dependent statistics, their bound suggests a nonic dependence on depth. Arora et al. [2018] also utilize concepts of compression in their analysis, which we generalize and expand upon. While they establish a sample complexity bound that suggests quadratic dependence on depth, further dependence may be hidden in data-dependent terms.

We suspect that the looseness of these results in comparison to empirical findings are due to a worst-case analysis framework. In this paper, we study an average-case notion of regret and sample complexity that is motivated by information theory. Our information-theoretic framework generalizes that developed by Haussler et al. [1994], which provided a basis for understanding the relationship between prediction error and information. In a similar vein, Russo and Zou [2019] introduced tools that establish general relationships between mutual information and error. Using these results, Xu and Raginsky [2017] established upper bounds on the generalization error of learning algorithms with countably infinite hypothesis spaces. We extend these results in several directions to enable analysis of data generating processes related to deep learning. For example, the results of Haussler et al. [1994] do not address noisy observations, and all three aforementioned papers do not accommodate continuous parameter spaces, let alone nonparametric data generating processes. A distinction of our work is that it builds on rate-distortion theory to address these limitations. While Nokleby et al. [2016] also use rate-distortion theory to study bayes risk, these results are again limited to parametric classification and only offer lower bounds. The rate-distortion function that we study is equivalent to one defined the *information bottleneck* Tishby et al. [2000]. However, instead of using it as a basis for *optimization methods* as do Shwartz-Ziv and Tishby [2017], we develop tools to study *sample complexity* and arrive at concrete and novel results.

In this paper, we consider contexts in which an agent learns from an iid sequence of data pairs. We establish tight upper and lower bounds for the average regret and sample complexity that depend on the rate-distortion function. With these information-theoretic tools, we analyze two deep neural network data generating processes and quantify the number of samples required to arrive at a useful model. **We establish novel sample complexity bounds for ReLU neural network data generating processes that are roughly linear in the parameter count** (as opposed to linear in the product of parameter count and depth as in Harvey et al. [2017]). **For a multilayer process with arbitrary width but bounded sum of weights, we establish sample complexity bounds with only a quadratic depth dependence** as opposed to exponential or high-order polynomial dependence as in Bartlett et al. [2017] and Wei and Ma [2019] respectively. The **main contribution** of this paper is an **elegant and intuitive information-theoretic framework for analyzing machine learning problems**. We demonstrate its promise by deriving the two aforementioned theoretical results pertaining to deep neural networks which were not possible to show with existing analysis methods.

2 Prediction and Error

We begin by introducing the structure of data generating processes we consider in our framework and defining our notions of error and sample complexity.

2.1 Data Generating Process

We consider a stochastic process that generates a sequence $((X_t, Y_{t+1}) : t = 0, \ldots, T-1)$ of data pairs. We refer to each X_t as an *input* and each Y_{t+1} as an *output*. We define these and all other random variables we will consider with respect to a probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

Elements of the sequence $(X_t : t = 0, \ldots, T-1)$ are independent and identically distributed. Denote the history of data generated through time t by $H_t = (X_0, Y_1, \ldots, X_{t-1}, Y_t, X_t)$. We assume that each (X_t, Y_{t+1}) pair is exchangeable. As a result of de Finetti's theorem, there exists a latent variable $\mathcal{E}$ for which the pairs (X_t, Y_{t+1}) are iid conditioned on $\mathcal{E}$. Here, $\mathcal{E}$ (the environment) is a random function that specifies a conditional output distribution $\mathbb{P}(Y_{t+1} \in \cdot | \mathcal{E}, X_t = x) = \mathcal{E}(\cdot | x)$ for each input x. Initial uncertainty about $\mathcal{E}$ is expressed by the prior distribution $\mathbb{P}(\mathcal{E} \in \cdot)$.

2.2 Prediction

We consider an agent that predict the next response Y_{t+1} given the history H_t. Rather than a point estimate, the agent provides as its prediction a probability distribution P_t over possible responses. We characterize the agent in terms of a function π for which $P_t = \pi(H_t)$.

We now introduce some notation for referring to particular predictions. We will generally use P_t as a dummy variable – that is a generic prediction whose definition depends on context. We denote the prediction conditioned on the environment, which could only be produced by a prescient agent, by

$$P_t^* = \mathbb{P}(Y_{t+1} \in \cdot | \mathcal{E}, X_t) = \mathcal{E}(\cdot | X_t).$$

We will refer to this as the *target distribution* as it represents what an agent aims to learn. Finally, we use $\hat{P}_t$ to denote the posterior-predictive conditioned on H_t, which will turn out to be optimal for the objective we will define next.

$$\hat{P}_t = \mathbb{P}(Y_{t+1} \in \cdot | H_t).$$

2.3 Error

We assess the error of a prediction P_t in terms of the KL-divergence relative to P_t^*:

$$\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t) = \int P_t^*(dy) \ln \frac{dP_t^*}{dP_t}(y).$$

This quantifies mismatch between the prediction P_t and target distribution P_t^*. This notion of error is closely related to more common error notions from machine learning. When Y takes values in a countable set, this notion of error is equivalent to expected cross-entropy loss. Meanwhile when Y is continuous, this notion of error is upper and lower bounded by linear functions of mean-squared error under reasonable circumstances (refer to Appendix A).

3 Regret and Sample Complexity

We assess an agent's performance over duration T in terms of the expected cumulative error

$$\mathcal{R}_\pi(T) = \mathbb{E}\left[\sum_{t=0}^{T-1} \mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t)\right].$$

The focus of this paper is on understanding how well an *optimal* agent can perform, given particular data generating processes. We will use *regret* to refer to the optimal performance defined below.

Definition 1. (optimal regret) *For all $T \in \mathbb{Z}_+$, environments $\mathcal{E}$ and data generating processes $((X_0, Y_1), (X_1, Y_2), \ldots)$, the* ***optimal regret*** *is*

$$\mathcal{R}(T) := \inf_\pi \mathcal{R}_\pi(T).$$

We will also consider sample complexity, which we take to be the duration required to attain expected average error within some fraction $\epsilon \geq 0$ of an optimal uninformed prediction:

Definition 2. (sample complexity) *For all $\epsilon \geq 0$, environments $\mathcal{E}$ and data generating processes $((X_0, Y_1), (X_1, Y_2), \ldots)$, the **sample complexity** is*

$$T_\epsilon := \min \left\{ T : \frac{\mathcal{R}(T)}{T} \leq \epsilon \right\}.$$

Note that this definition of sample complexity measures the number of samples necessary to achieve average *cumulative* error below ϵ. Traditionally, generalization error quantifies the number of samples necessary to achieve out-of-sample error (such as $\mathbb{I}(Y_{T+1}; \mathcal{E}|H_T)$) below ϵ. Due to a simple monotonicity result, our notion of sample complexity upper bounds the one for out-of-sample error i.e $\mathbb{I}(Y_{T+1}; \mathcal{E}|H_T) \leq \frac{\mathcal{R}(T)}{T}$.

3.1 Optimal Predictions

In this paper, we focus on how well an *optimal* agent performs, rather than on how to design practical agents that economize on memory and computation. Recall that an agent is characterized by a function π, which generates predictions $P_t = \pi(H_t, Z_t)$, where Z_t represents algorithmic randomness. The following result establishes that the conditional distribution $\hat{P}_t = \mathbb{P}(Y_{t+1} \in \cdot | H_t)$ offers an optimal prediction.

Theorem 3. (optimal prediction) *For all $t \geq 0$,*

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t) \mid H_t] = \inf_\pi \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t) \mid H_t],$$

where $P_t = \pi(H_t, Z_t)$.

A proof may be found in Appendix B. A corollary that characterizes the performance shortfall of a *suboptimal* agent may also be found there. This corrolary opens future avenues regarding learning under misspecification or compute constraints. In the remainder of the paper we will study an agent that generates optimal predictions $P_t = \mathbb{P}(Y_{t+1} \in \cdot | H_t)$, as illustrated in Figure 1.

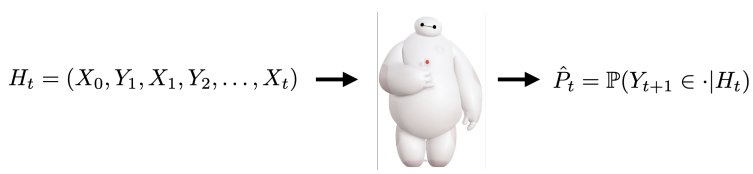

Figure 1: We consider an agent that, given a history H_t, generates an optimal prediction $\hat{P}_t$.

4 Information

We define concepts for quantifying uncertainty and the information gained from observations. The entropy $\mathbb{H}(\mathcal{E})$ of the environment quantifies the agent's initial degree of uncertainty in terms of the information required to identify $\mathcal{E}$. We will measure information in *nats*, each of which is equivalent to $1/\ln 2$ bits. For example, if $\mathcal{E}$ occupies a countable range Θ then $\mathbb{H}(\mathcal{E}) = -\sum_{\theta \in \Theta} \mathbb{P}(\mathcal{E} = \theta) \ln \mathbb{P}(\mathcal{E} = \theta)$. Uncertainty at time t can be expressed in terms of the conditional entropy $\mathbb{H}(\mathcal{E}|H_t)$, which is the number of remaining nats after observing H_t. The mutual information $\mathbb{I}(\mathcal{E}; H_t) = \mathbb{H}(\mathcal{E}) - \mathbb{H}(\mathcal{E}|H_t)$ quantifies the information about $\mathcal{E}$ gained from H_t. Proofs for all lemmas in this section may be found in Appendix B

4.1 Learning from Errors

Each data pair (X_t, Y_{t+1}) provides $\mathbb{I}(\mathcal{E}; (X_t, Y_{t+1})|H_{t-1}, Y_t)$ nats of new information about the environment. By the chain rule of mutual information, this is the sum

$$\mathbb{I}(\mathcal{E}; (X_t, Y_{t+1})|H_{t-1}, Y_t) = \mathbb{I}(\mathcal{E}; X_t|H_{t-1}, Y_t) + \mathbb{I}(\mathcal{E}; Y_{t+1}|H_t)$$

of the information gained from X_t and Y_{t+1}. The former term $\mathbb{I}(\mathcal{E}; X_t | H_{t-1}, Y_t)$ is equal to 0 because X_t is independent from both $\mathcal{E}$ and (H_{t-1}, Y_t). The latter term $\mathbb{I}(\mathcal{E}; Y_{t+1} | H_t)$ can be thought of as the level of surprise experienced by the agent upon observing Y_{t+1}. Surprise is associated with prediction error, and the following result formalizes the equivalence between error and information.

Lemma 4. (expected prediction error equals information gain) *For all* $t \in \mathbb{Z}_+$,

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)] = \mathbb{I}(\mathcal{E}; Y_{t+1} | H_t),$$

and $\mathcal{R}(t) = \mathbb{I}(\mathcal{E}; H_t)$.

The agent's ability to predict tends to improve as it learns from experience. This is formalized by the following result, which establishes that expected prediction errors are monotonically nonincreasing.

Lemma 5. (expected prediction error is monotonically nonincreasing) *For all* $t \in \mathbb{Z}_+$,

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)] \geq \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_{t+1}^* \| \hat{P}_{t+1})].$$

5 General Bounds via Rate-Distortion Theory

An *environment proxy* is a random variable $\tilde{\mathcal{E}}$ that provides information about the environment $\mathcal{E}$ but no additional information pertaining to inputs or outputs. In other words, $\tilde{\mathcal{E}} \perp (X, Y) | \mathcal{E}$. We will denote the set of environment proxies by $\tilde{\Theta}$. While an infinite amount of information must be acquired to identify the environment when $\mathbb{H}(\mathcal{E}) = \infty$, there can be a proxy $\tilde{\mathcal{E}}$ with $\mathbb{H}(\tilde{\mathcal{E}}) < \infty$ that enables accurate predictions. The minimal expected error attainable based on the proxy is achieved by a prediction $\tilde{P} = \mathbb{P}(Y \in \cdot | \tilde{\mathcal{E}}, X_t)$. This results in expected error $\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})]$.

We will establish that the expected error $\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})]$ equals the information gained, beyond that supplied by the proxy $\tilde{\mathcal{E}}$, about the environment $\mathcal{E}$ from observing Y. This is intuitive: more is learned from Y if knowledge of $\mathcal{E}$ enables a better prediction of Y than does $\tilde{\mathcal{E}}$. We quantify this information gain in terms of the difference $\mathbb{H}(Y | \tilde{\mathcal{E}}, X) - \mathbb{H}(Y | \mathcal{E}, X)$ between the uncertainty conditioned on $\tilde{\mathcal{E}}$ and that conditioned on $\mathcal{E}$. This is equal to the mutual information $\mathbb{I}(\mathcal{E}; Y | \tilde{\mathcal{E}}, X) = \mathbb{H}(Y | \tilde{\mathcal{E}}, X) - \mathbb{H}(Y | \mathcal{E}, X)$. The following result equates this with expected error.

Lemma 6. (proxy error equals information gain) *For all* $\tilde{\mathcal{E}} \in \tilde{\Theta}$,

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})] = \mathbb{I}(\mathcal{E}; Y | \tilde{\mathcal{E}}, X).$$

A proof may be found in Appendix C. $\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})]$ is the *distortion* incurred by our estimate of Y given only $\tilde{\mathcal{E}}$ as opposed to $\mathcal{E}$. For example, $\tilde{\mathcal{E}}$ may be a quantization or lossy compression of $\mathcal{E}$ an $\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})]$ is measuring how inaccurate our prediction of Y is under this compression $\tilde{\mathcal{E}}$.

Now we consider the following ϵ-optimal set:

$$\tilde{\Theta}_\epsilon = \left\{ \tilde{\mathcal{E}} \in \tilde{\Theta} : \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})] \leq \epsilon \right\},$$

$\tilde{\Theta}_\epsilon$ denotes the set of proxies that produce predictions that incur a *distortion* of no more than an ϵ. One can imagine that for environments with simple input distributions, such as those constrained to a subspace or low-dimensional manifold, this set $\tilde{\Theta}_\epsilon$ will be much larger since the average error need only be below ϵ in a small subset of the domain.

With the *distortion* component of *rate-distortion* covered, the *rate* remains. The *rate* is $\mathbb{I}(\mathcal{E}; \tilde{\mathcal{E}})$, the amount of information about the environment conveyed by the proxy. For example, a finer quantization would result in a higher rate since $\tilde{\mathcal{E}}$ would capture $\mathcal{E}$ with more bits of precision. However, with this higher rate, the *distortion* incurred by $\tilde{\mathcal{E}}$ should in turn be *lower*. A higher fidelity compression should cost more bits but result in less distortion. The *rate-distortion function* formalizes this trade-off concretely:

Definition 7. *For all $\epsilon \geq 0$ and environments $\mathcal{E}$, The* ***rate-distortion function*** *w.r.t distortion function $\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^* \| \tilde{P})]$ is*

$$\mathbb{H}_\epsilon(\mathcal{E}) := \inf_{\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon} \mathbb{I}(\mathcal{E}; \tilde{\mathcal{E}}).$$

$\mathbb{H}_\epsilon$ characterizes the minimal amount of information that a proxy must convey about the environment in order to make ϵ-accurate predictions. Even when $\mathbb{H}(\mathcal{E})$ is infinite and ϵ is small, $\mathbb{H}_\epsilon(\mathcal{E})$ can be manageable. Once again, if we consider an environment with a simple input distribution, the rate-distortion would be lower since we would be taking an infimum over a larger set $\tilde{\Theta}_\epsilon$. Depending on the structure of the input distribution and the environment, one could imagine that the reduction could be significant. As we will see in the following section, both the regret and sample complexity of learning scales with $\mathbb{H}_\epsilon(\mathcal{E})$.

5.1 Bounding Regret and Sample Complexity via Rate-Distortion

With rate-distortion in place, we produce tight bounds on regret and sample complexity. These bounds are very general, applying to *any* data generating process. The results upper and lower bound error and sample complexity in terms of rate-distortion. As such, for any particular data generating process, bounds can be produced by characterizing the associated rate-distortion function. Proofs for the following two theorems can be found in Appendix C.

The following result brackets the cumulative error of optimal predictions.

Theorem 8. (rate-distortion regret bounds) *For all T,*

$$\sup_{\epsilon \geq 0} \min\{\mathbb{H}_\epsilon(\mathcal{E}),\ \epsilon T\} \ \leq\ \mathcal{R}(T) \ \leq\ \inf_{\epsilon \geq 0} (\mathbb{H}_\epsilon(\mathcal{E}) + \epsilon T).$$

This upper bound is intuitive. Knowledge of a proxy $\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon$ enables an agent to limit prediction error to ϵ per timestep. Getting to that level of prediction error requires $\mathbb{H}_\epsilon(\mathcal{E})$ nats, and therefore, that much cumulative error. Hence, $\mathcal{R}(T) \leq \mathbb{H}_\epsilon(\mathcal{E}) + \epsilon T$.

To motivate the lower bound, note that an agent requires $\mathbb{H}_\epsilon(\mathcal{E})$ nats to attain per timestep error within $\epsilon\mathcal{R}(1)$. Obtaining those nats requires cumulative error at least $\mathbb{H}_\epsilon(\mathcal{E})$. So prior to obtaining that many nats, the agent *must* incur at least $\epsilon\mathcal{R}(1)$ error per timestep, hence the $\epsilon\mathcal{R}(1)T$ term in the minimum. Meanwhile, if at time T, the agent is able to produce predictions with error less than $\epsilon\mathcal{R}(1)$ it means that it has already accumulated at least $\mathbb{H}_\epsilon(\mathcal{E})$ nats of information about $\mathcal{E}$ (error).

Sample complexity bounds follow almost immediately from Theorem 8.

Theorem 9. (rate-distortion sample complexity bounds) *For all $\epsilon \geq 0$,*

$$\frac{\mathbb{H}_\epsilon(\mathcal{E})}{\epsilon\mathcal{R}(1)} \ \leq\ T_\epsilon \ \leq\ \inf_{\delta \in [0,\epsilon]} \left\lceil \frac{\mathbb{H}_{\epsilon-\delta}(\mathcal{E})}{\delta\mathcal{R}(1)} \right\rceil \leq \left\lceil \frac{2\mathbb{H}_{\epsilon/2}(\mathcal{E})}{\epsilon\mathcal{R}(1)} \right\rceil.$$

6 Deep Neural Network Environments

We now study the rate-distortion function of multilayer environments. As we established, sample complexity is governed by the rate-distortion function $\mathbb{H}_\epsilon(\mathcal{E})$. In this section, we will focus on characterizing this rate-distortion function, and hence the sample complexity, of two prototypical multilayer environments.

6.1 Prototypical Multilayer Environments

6.1.1 Prototypical Environment 1: (ReLU Neural Networks)

Prototypical Environment 1 mirrors the architecture of fully-connected feed-forward neural networks (refer to Figure 2 left). Let

$$U_1 = f_1(U_0) = \mathrm{ReLU}(A^{(1)}U_0 + b^{(1)}),$$

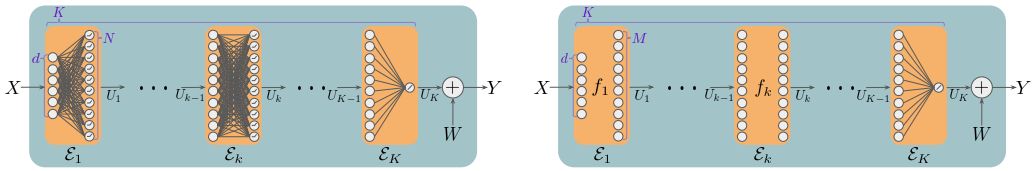

Figure 2: (Left) Prototypical environment 1, (Right) Prototypical environment 2.

where $A^{(1)} \in \Re^{N\times d}, b^{(1)} \in \Re^N$ both with independent elements each with variance $\frac{1}{d}$. For $k \in \{2,\dots,K-1\}$, let

$$U_k = f_k(U_{k-1}) = \mathrm{ReLU}(A^{(k)}U_{k-1} + b^{(k)}),$$

where $A^{(k)} \in \Re^{N\times N}, b^{(k)} \in \Re^N$, both with independent elements each with variance $\frac{1}{N}$. For the final layer, let

$$U_K = f_K(U_{K-1}) = A^{(K)\top}U_{K-1},$$

where $A^{(K)} \in \Re^N$ with independent elements, each with variance $\frac{1}{N}$. In this environment, $\mathcal{E}_k$ is identified by $(A^{(k)}, b^{(k)})$ for $k \in \{1,\dots,K-1\}$ and $\mathcal{E}_K$ is identified by $A^{(K)}$. We will refer to this environment as the *deep ReLU network*.

6.1.2 Prototypical Environment 2: (Deep Nonparametric Networks)

Prototypical Environment 2 considers a deep nonparametric neural network (refer to Figure 2 right). For $k \in \{1,\dots,K\}$, let

$$U_k = f_k(U_{k-1}) = \sum_{n=1}^{N} \alpha_n^{(k)} g_n^{(k)}(U_{k-1}),$$

where $\alpha^{(k)} = (\alpha_1^{(k)},\dots,\alpha_N^{(k)}) \sim \mathrm{dir}(N,[1,\dots,1])$. For $k \in \{2,\dots,K-1\}$, the deterministic basis functions $g_n^{(k)}$ are maps from $\Re^M \mapsto \Re^M$. $(g_n^{(1)} : n \in \{1,\dots,N\})$ are maps from $\Re^d \mapsto \Re^M$ and $(g_n^{(K)} : n \in \{1,\dots,N\})$ are maps from $\Re^M \mapsto \Re$. For regularity, we will assume that for all n and k, the basis functions satisfy $\mathbb{E}[g_n^{(k)}(U_{k-1})^2] \le 1$ and that

$$L = \sup_{x,y} \mathbb{E}\left[\frac{\|f^{(k)}(x) - f^{(k)}(y)\|_2^2}{\|x-y\|_2^2}\Big| x = x, y = y\right] \le 1.$$

The expectation here is taken over the randomness in the function $f^{(k)}$. L can be thought of as an expected squared Lipschitz constant of $f^{(k)}$. Note that this condition is also met when $f^{(k)}$ is a layer from the deep ReLU network. The set of random functions for which $L \le 1$ is much broader than say 1-Lipschitz functions because of both the square and the expectation. For example, $f(x) = Ax$ is $\|A\|_2$-Lipschitz but if say $\mathrm{cov}\,[A] = I_n$, then $L = 1$.

$\mathcal{E}_k$ is identified by $\alpha^{(k)}$. We will refer to this environment $(\mathcal{E}_{K:1})$ as the *deep nonparametric network*. For intractably large N, f_k is effectively nonparametric. In this regime, parameter count becomes a vacuous description of the data generating process's complexity. However, the complexity is still controlled by the fact that $\alpha_n \ge 0$ and $\sum_{n=1}^N \alpha_n = 1$ due to the Dirichlet prior. We will establish rate-distortion and sample complexity bounds that are actually *independent* of N and M.

A natural example of such an environment is on in which the basis functions are $g_n^{(k)}(X) = \mathrm{ReLU}(\theta_{n,k}^\top X)$ where $\|\theta\|_2 = 1$ and N may be prohibitively large i.e. infinite or exponential in the input dimension.

6.2 From Single to Multilayer Environments

We are now interested in analyzing the *depth* dependence of the rate-distortion and sample complexity of these environments. We will use $\mathcal{E}_{i:j}$ and $\tilde{\mathcal{E}}_{i:j}$ for $i \ge j$ to denote $(\mathcal{E}_i, \mathcal{E}_{i-1},\dots,\mathcal{E}_j)$ and $(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_{i-1},\dots,\tilde{\mathcal{E}}_j)$ respectively.

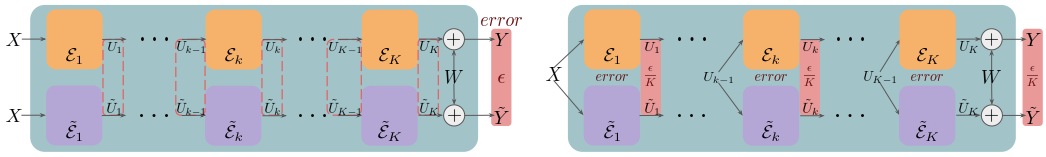

Figure 3: (Left) The error incurred by a multilayer proxy $\tilde{\mathcal{E}}_{K:1}$ measures the difference between true output Y and the prediction $\tilde{Y}$ (shown in red box). This difference is the result of error that builds up through layers of the environment (denoted by red dotted outline). (Right) A much easier system to analyze is one in which we measure the incremental error at each stage of the multilayer environment. If each layer incurs an error of $\frac{\epsilon}{K}$, the total error incurred will be ϵ.

For *multilayer* environments, the error is cumbersome to reason about. Figure 3 (left) depicts the error incurred by using a multilayer proxy $\mathcal{E}_{K:1}$ to approximate multilayer environment $\mathcal{E}_{K:1}$. Evidently, it seems difficult to manage the error propagation through the layers of the environment. Many traditional lines of analysis struggle on this front and result in sample complexity bounds that are exponential in the depth Bartlett et al. [2017], Golowich et al. [2018]. These papers consider a worst-case reasoning under which an ϵ error between the first outputs U_1 and $\tilde{U}_1$ may blow up to a $\lambda^K \epsilon$ error when passed through remaining layers of the network (where λ is a spectral radius).

It would be much simpler to instead analyze the incremental error incurred at each stage of the network. Figure 3 (right) depicts this. We consider that at each layer, we know the true input U_{k-1} and simply measure the immediate error incurred at the output U_k as opposed to the error incurred at the final output of the network Y.

Mathematically, the error incurred by the full system from using proxy $\tilde{\mathcal{E}}_{K:1}$ can be expressed as $\mathbb{I}(Y; \mathcal{E}_{K:1} | \tilde{\mathcal{E}}_{K:1}, X)$. By the chain rule, we have that

$$\mathbb{I}(Y; \mathcal{E}_{K:1} | \tilde{\mathcal{E}}_{K:1}, X) = \sum_{k=1}^{K} \mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X). \tag{1}$$

Therefore, the error incurred from layer k can be expressed as $\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X)$. This is cumbersome since we are not given the true input U_{k-1} but rather an approximation from input X and $\tilde{\mathcal{E}}_{k-1:1}$. Furthermore, we are measuring the error in the *final* output Y as opposed to the *immediate* output U_k. It would be much more simple to analyze something like the following:

$$\sum_{k=1}^{K} \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1}), \tag{2}$$

where W is independent 0-mean gaussian noise with variance σ^2 in each dimension. This sum is much easier to work with because the kth term only depends on $\mathcal{E}_k, \tilde{\mathcal{E}}_k, U_{k-1}$, and U_k. There is no inter-layer dependence.

The key insight is that in our two prototypical environments (and many others), something *akin* to the following will hold:

$$\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1}). \tag{3}$$

As a result, the cumbersome sum 1 will be *upper bounded* by the nice sum 2.

The condition in inequality 3 involves two parts:

1. $\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq \mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_k, U_{k-1})$
   - Conditioning on the true input U_{k-1} provides more information about $\mathcal{E}_k$ than conditioning on an approximation $(\tilde{\mathcal{E}}_{k-1:1}, X)$.
2. $\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_k, U_{k-1}) \leq \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1})$

- The immediate output $U_k + W$ provides more information about $\mathcal{E}_k$ than the final output Y does.

1) holds for all proxies of the form $\tilde{\mathcal{E}} = (\tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_K)$ where $\tilde{\mathcal{E}}_i \perp \tilde{\mathcal{E}}_j$ for $i \neq j$. We prove this result explicitly in Lemma 19 in Appendix E. It is rather intuitive that the pristine data pair (U_{k-1}, Y) would provide more information about $\mathcal{E}_k$ than $(U_0, \tilde{\mathcal{E}}_{k-1:1}, Y)$ would. We expect some information to be lost through the noisy reconstruction of U_{k-1} from $U_0, \tilde{\mathcal{E}}_{k-1:1}$.

2) will not hold exactly for our two environments. $\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, U_{k-1})$ will not always be $\leq \mathbb{I}(U_k + W, \mathcal{E}_k | \mathcal{E}_k, U_{k-1})$. However, $\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, U_{k-1})$ will be $\leq$ a very natural upper bound of $\mathbb{I}(U_k + W, \mathcal{E}_k | \mathcal{E}_k, U_{k-1})$, which we detail in Lemma 20 of Appendix E. Intuitively $U_k + W$ will provide more information about $\mathcal{E}_k$ than Y so long as in expectation, the layers $f_{k+1}, \ldots, f_K$ don't amplify the scale of the output. If they were to amplify the scale, then the signal to noise ratio of Y would look more favorable than that of $U_k + W$, leading to potentially information. This is precisely why we make the assumption that $L = \sup_{x,y} \mathbb{E}[\frac{\|f^{(k)}(x) - f^{(k)}(y)\|_2^2}{\|x-y\|_2^2} | x = x, y = y] \leq 1$.

For multilayer processes for which we can establish a relationship akin to inequality 3, we have the following relationship between the rate-distortion function $\mathbb{H}_\epsilon(\mathcal{E}_{K:1})$ and the individual rate-distortion functions $\mathbb{H}_\epsilon(\mathcal{E}_k)$ (proof in Appendix E).

Theorem 10. multilayer rate-distortion bound *For all $K \in \mathbb{Z}_{++}$, $\sigma^2, \epsilon \geq 0$, if $\mathcal{E}_{K:1}$ is a multilayer environment such that for all $k \in \{1, \ldots, K\}$ and $\delta \geq 0$, there exist $\tilde{\mathcal{E}}_k$ s.t.*

$$\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1}) \leq \delta,$$

where $W \sim \mathcal{N}(0, \sigma^2 I)$, then

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) \leq \sum_{k=1}^{K} \mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k).$$

6.3 Sample Complexity Bounds for Multilayer Environments

With Theorem 10 in place, we present the two main results (proofs can be found in Appendix F).

Theorem 11. (relu neural network rate-distortion and sample complexity bound) *For all $d, N, K \in \mathbb{Z}_{++}$ and $\sigma^2, \epsilon \geq 0$, if multilayer environment $\mathcal{E}_{K:1}$ is the deep ReLU network with input $X : \Omega \mapsto \Re^d$ s.t. $\mathbb{V}[X] = I_d$ and output $Y \sim \mathcal{N}(U_K, \sigma^2)$, then*

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) = \tilde{\mathcal{O}}\left(KN^2 + dN\right), \quad T_\epsilon = \tilde{\mathcal{O}}\left(\frac{KN^2 + dN}{\epsilon}\right).$$

While this is a parameter-count based bound in nature, it improves upon existing sample complexity results based on VC Dimension Bartlett et al. [1998],Harvey et al. [2017] which under our notation would be $\tilde{O}(K^2N^2 + KdN)$. Note however that our result is average case while theirs holds with high probability. Regardless, the reduction in sample complexity by a factor of K is notable.

For nonparametric deep networks, we have the following result.

Theorem 12. (nonparametric network rate-distortion and sample complexity bound) *For all $d, N, M, K \in \mathbb{Z}_{++}$, $\sigma^2, \epsilon \geq 0$, if multilayer environment $\mathcal{E}_{K:1}$ is the deep nonparametric network with input $X : \Omega \mapsto \Re^d$ and output $Y \sim \mathcal{N}(U_K, \sigma^2)$, then*

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) \leq \frac{K^2}{2\sigma^2\epsilon}, \quad T_\epsilon \leq 2\left(\frac{K}{\sigma\epsilon}\right)^2.$$

The nonparametric network's sample complexity grows only quadratically in K not exponentially or high-order polynomially as in Bartlett et al. [2017] and Wei and Ma [2019] respectively.

7 Closing Remarks

We have introduced a novel and elegant information-theoretic framework for analyzing the sample complexity of data generating processes. We demonstrate its usefulness by proving two new theoretical results that suggests that it is possible to learn efficiently from data generated by deep and nonparametric functions.

Beyond the scope of this paper, we believe that the flexibility and simplicity of our framework will allow for the analysis of machine learning systems such as semi-supervised learning, multitask learning, bandits and reinforcement learning. We also believe that many of the nuances of empirical deep learning such as batch-normalization, pooling, and structured input distributions can be analyzed through the average-case nature of information theory and powerful tools such as the data processing inequality.

Notably omitted from this paper is any analysis of practical algorithms for estimating the environment. We have assumed perfect Bayesian inference, while in practice, particular neural network architectures are used together with stochastic gradient descent. Whether practical algorithms of this sort can achieve sample complexity bounds similar to what we have established for our multi-layer environments remains an interesting subject for future research. Our discussion of suboptimal algorithms and misspecification in section 3.1 and Appendix B provide some starting points for such a pursuit.

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The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. For example:

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1. For all authors...
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2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] Either in the theorem statement or in the respective section of the paper.
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(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

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A Relation to Other Notions of Error

A.1 Connections to Cross-Entropy Loss

We establish that in the classification setting, our notion of error is equivalent to cross-entropy loss up to translations.

Example 1. (cross-entropy loss) *Suppose the set $\mathcal{Y}$ of responses is finite. Then,*

$$\mathbf{d}_{\mathrm{KL}}(P^*\|P) = \sum_{y\in\mathcal{Y}} P^*(y)\ln\frac{P^*(y)}{P(y)} = \sum_{y\in\mathcal{Y}} P^*(y)\ln P^*(y) - \sum_{y\in\mathcal{Y}} P^*(y)\ln P(y).$$

The first term does not depend on P, so minimizing KL-divergence is equivalent to minimizing the final term,

$$-\sum_{y\in\mathcal{Y}} P^*(y)\ln P(y) = -\mathbb{E}[\ln(P(Y_{t+1}))|\mathcal{E}, P, X],$$

which is exactly the expected cross-entropy loss of P, as is commonly used to assess classifiers.

A.2 Connections to Mean-Squared Error

In the regression setting we first establish a direct link between KL-divergence and mean-squared error for the case in which P_t^* and P_t are Gaussian.

Example 2. (Gaussian mean-squared error) *Fix $\mu \in \Re$ and $\sigma^2 \in \Re_{++}$. Let $\mathbb{P}(Y_{t+1} \in \cdot|\mathcal{E}, X_t) \sim \mathcal{N}(\mu, \sigma^2)$. Consider a point prediction $\hat{\mu}_t$ that is determined by H_t and a distributional prediction $P_t \sim \mathcal{N}(\hat{\mu}_t, \sigma^2)$. Then,*

$$\mathbf{d}_{\mathrm{KL}}(P_t^*\|P_t) = \frac{\mathbb{E}[(\mu-\hat{\mu}_t)^2|\mathcal{E}, H_t]}{2\sigma^2} = \frac{\mathbb{E}[(Y_{t+1}-\hat{\mu}_t)^2|\mathcal{E}, H_t] - \mathbb{E}[(Y_{t+1}-\mu_t)^2|\mathcal{E}, H_t]}{2\sigma^2}.$$

Hence, KL-divergence grows monotonically the squared error $\mathbb{E}[(Y_{t+1}-\hat{\mu}_t)^2|\mathcal{E}, H_t]$. However, while the minimal squared error $\mathbb{E}[(Y_{t+1}-\mu)^2|\mathcal{E}, H_t] = \sigma^2$ that is attainable with full knowledge of the environment remains positive, the minimal KL-divergence, which is delivered by $P_t \sim \mathcal{N}(\mu, \sigma^2)$, is zero.

Now consider a distributional prediction $P_t \sim \mathcal{N}(\hat{\mu}_t, \hat{\sigma}_t^2)$, based on a variance estimate $\hat{\sigma}_t^2 \neq \sigma^2$. Then,

$$\mathbf{d}_{\mathrm{KL}}(P_t^*\|P_t) = \frac{\mathbb{E}[(\mu-\hat{\mu}_t)^2|\mathcal{E}, H_t]}{2\hat{\sigma}_t^2} + \frac{1}{2}\left(\frac{\sigma^2}{\hat{\sigma}_t^2} - 1 - \ln\frac{\sigma^2}{\hat{\sigma}_t^2}\right).$$

Consider optimizing the choice of $\hat{\sigma}_t^2$ given H_t:

$$\min_{\hat{\sigma}_t^2} \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t) | H_t].$$

The minimum is attained by

$$\hat{\sigma}_t^2 = \underbrace{\sigma^2}_{\text{aleatoric}} + \underbrace{\mathbb{E}[(\mu - \mathbb{E}[\mu|H_t])^2|H_t]}_{\text{epistemic}} + \underbrace{\mathbb{E}[(\mathbb{E}[\mu|H_t] - \hat{\mu}_t)^2|H_t]}_{bias},$$

which differs from σ^2. While σ^2 characterizes aleatoric uncertainty, the incremental variance $\hat{\sigma}_t^2 - \sigma^2$ accounts for epistemic uncertainty and bias.

Now, for P_t^* and P_t that are not Gaussian, we have the following upper bound:

Lemma 13. *For all $t \in \mathbb{Z}_+$, if $\hat{\mu}_t = \int_{y \in \mathcal{Y}} y \, dP_t(y)$, then*

$$\mathbb{E}\left[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t)\right] \leq \frac{1}{2} \ln \left(1 + \frac{\mathbb{E}\left[(Y_{t+1} - \hat{\mu}_t)^2\right]}{\sigma^2} \right).$$

Therefore, decreasing the mean squared error will always decrease the expected KL-divergence. A corresponding lower bound holds for data generating processes for which Y_{t+1} satisfies a certain subgaussian condition:

Lemma 14. *For all $t \in \mathbb{Z}_+$, let $\hat{\mu}_t = \int_{y \in \mathcal{Y}} y \, dP_t(y)$. If $P_t(Y_{t+1} \in \cdot)$ is δ_t^2 -subgaussian conditioned on H_t w.p 1, then*

$$\mathbb{E}\left[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t)\right] \geq \frac{\mathbb{E}\left[(Y_{t+1} - \hat{\mu}_t)^2\right]}{\delta_t^2}$$

Therefore, for data generating processes that obey this subgaussian condition, we have both upper and lower bounds for expected KL divergence in terms of mean-squared error.

B Proof of Information and Error Results

Here, we provide a proof of Theorem 3.

Theorem 3. (optimal prediction) *For all $t \geq 0$,*

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t) \mid H_t] = \inf_{\pi} \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t) \mid H_t],$$

where $P_t = \pi(H_t, Z_t)$.

Proof. Let $\hat{P}_t = \mathbb{P}(Y_{t+1} \in \cdot | H_t)$. By Gibbs' inequality,

$$\inf_{P_t} \mathbf{d}_{\mathrm{KL}}(\hat{P}_t \| P_t) = \mathbf{d}_{\mathrm{KL}}(\hat{P}_t \| \hat{P}_t) = 0.$$

Let $P_t^* = \mathbb{P}(Y_{t+1} \in \cdot | \mathcal{E}, X_t)$. Then, for all P_t,

$$\begin{aligned}
\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t) =& \mathbb{E}\left[\ln \frac{dP_t^*}{dP_t}(Y_{t+1}) \middle| \mathcal{E}, H_t\right] \\
=& \mathbb{E}\left[\ln dP_t^*(Y_{t+1}) | \mathcal{E}, H_t\right] - \mathbb{E}\left[\ln dP_t(Y_{t+1}) | \mathcal{E}, H_t\right] \\
=& \mathbb{E}\left[\ln dP_t^*(Y_{t+1}) | \mathcal{E}, H_t\right] - \mathbb{E}\left[\ln d\hat{P}_t(Y_{t+1}) | \mathcal{E}, H_t\right] \\
&+ \mathbb{E}\left[\ln d\hat{P}_t(Y_{t+1}) | \mathcal{E}, H_t\right] - \mathbb{E}\left[\ln dP_t(Y_{t+1}) | \mathcal{E}, H_t\right] \\
=& \mathbb{E}\left[\ln \frac{dP_t^*}{d\hat{P}_t}(Y_{t+1}) \middle| \mathcal{E}, H_t\right] + \mathbb{E}\left[\ln \frac{d\hat{P}_t}{dP_t}(Y_{t+1}) \middle| \mathcal{E}, H_t\right] \\
=& \mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t) + \mathbb{E}\left[\ln \frac{d\hat{P}_t}{dP_t}(Y_{t+1}) \middle| \mathcal{E}, H_t\right].
\end{aligned}$$

It follows that

$$\begin{aligned}\inf_{\pi} \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t)|H_t] =& \inf_{\pi} \mathbb{E}\left[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t) + \mathbb{E}\left[\ln \frac{d\hat{P}_t}{dP_t}(Y_{t+1})\Big|\mathcal{E}, H_t\right]\Big|H_t\right] \\ =& \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)|H_t] + \inf_{\pi} \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\hat{P}_t \| P_t)|H_t] \\ =& \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)|H_t] + \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\hat{P}_t \| \hat{P}_t)|H_t] \\ =& \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)|H_t].\end{aligned}$$

□

A direct corollary of this result is the following shortfall quantification for a misspecified/suboptimal prediction.

Corollary 15. (misspecified/suboptimal prediction) *For all $t \geq 0$ and $P_t = \pi(H_t, Z_t)$,*

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| P_t) \mid H_t] = \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t) \mid H_t] + \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\hat{P}_t \| P_t) \mid H_t].$$

An interpretation of the above result is that the error of a suboptimal agent with prediction P_t is the sum of the error of an *optimal* agent $\hat{P}_t$ and the expected KL divergence between $\hat{P}_t$ and P_t. Since all the results in this paper bound the first term, in order to understand the behavior of a suboptimal agent, techniques to bound the second term would be required.

Next, we provide a proof for Lemma 4.

Lemma 4. (expected prediction error equals information gain) *For all $t \in \mathbb{Z}_+$,*

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)] = \mathbb{I}(\mathcal{E}; Y_{t+1}|H_t),$$

and $\mathcal{R}(t) = \mathbb{I}(\mathcal{E}; H_t)$.

Proof. It is well known that the mutual information $\mathbb{I}(A; B)$ between random variables A and B can be expressed in terms of the expected KL-divergence $\mathbb{I}(A; B) = \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\mathbb{P}(A \in \cdot|B) \| \mathbb{P}(A \in \cdot))]$. It follows that

$$\begin{aligned}\mathbb{I}(Y_{t+1}; \mathcal{E}|H_t) &= \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\mathbb{P}(Y_{t+1} \in \cdot|\mathcal{E}, H_t) \parallel \mathbb{P}(Y_{t+1} \in \cdot|H_t))] \\ &\overset{(a)}{=} \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\mathbb{P}(Y_{t+1} \in \cdot|\mathcal{E}, X_t) \parallel \mathbb{P}(Y_{t+1} \in \cdot|H_t))] \\ &= \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)],\end{aligned}$$

where (a) follows from the fact that $Y_{t+1} \perp H_t|(\mathcal{E}, X_t)$. We then have

$$\mathcal{R}(T) = \mathbb{E}\left[\sum_{t=0}^{T-1} \mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)\right] = \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \mathcal{E}|H_t) \overset{a}{=} \mathbb{I}(\mathcal{E}; H_T),$$

where (a) follows from the chain rule of mutual information. □

Lastly, we provide a proof for Lemma 5.

Lemma 5. (expected prediction error is monotonically nonincreasing) *For all $t \in \mathbb{Z}_+$,*

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_t^* \| \hat{P}_t)] \geq \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_{t+1}^* \| \hat{P}_{t+1})].$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^*_{t+1}\|\hat{P}_{t+1})] &\overset{(a)}{=}\mathbb{I}(\mathcal{E};Y_{t+2}|H_{t+1})\\
&=\mathbf{h}(Y_{t+2}|H_{t+1})-\mathbf{h}(Y_{t+2}|\mathcal{E},H_{t+1})\\
&\overset{(b)}{=}\mathbf{h}(Y_{t+2}|H_{t+1})-\mathbf{h}(Y_{t+2}|\mathcal{E},H_{t-1},Y_t,X_{t+1})\\
&\overset{(c)}{\leq}\mathbf{h}(Y_{t+2}|H_{t-1},Y_t,X_{t+1})-\mathbf{h}(Y_{t+2}|\mathcal{E},H_{t-1},Y_t,X_{t+1})\\
&=\mathbb{I}(\mathcal{E};Y_{t+2}|H_{t-1},Y_t,X_{t+1})\\
&\overset{(d)}{=}\mathbb{I}(\mathcal{E};Y_{t+1}|H_{t-1},Y_t,X_t)\\
&=\mathbb{I}(\mathcal{E};Y_{t+1}|H_t)\\
&\overset{(e)}{=}\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^*_t\|\hat{P}_t)],
\end{aligned}$$

where (a) follows from Lemma 4, (b) follows from the the fact that $Y_{t+2}\perp(X_t,Y_{t+1})|(\mathcal{E},X_{t+1})$, (c) follows from the fact that conditioning reduces differential entropy, (d) follows from the fact that (X_t,Y_{t+1}) and (X_{t+1},Y_{t+2}) are independent and identically distributed conditioned on (H_{t-1},Y_t), and (e) follows from the equivalence between mutual information and expected KL-divergence. □

C Proofs of Regret and Sample Complexity Results

In this section, we prove results pertaining to bounding optimal regret and sample complexity in terms of rate-distortion.

Lemma 6. (proxy error equals information gain) *For all $\tilde{\mathcal{E}}\in\tilde{\Theta}$,*

$$\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^*\|\tilde{P})]=\mathbb{I}(\mathcal{E};Y|\tilde{\mathcal{E}},X).$$

Proof. It is well known that the mutual information $\mathbb{I}(A;B)$ between random variables A and B can be expressed in terms of the expected KL-divergence $\mathbb{I}(A;B)=\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\mathbb{P}(A\in\cdot|B)\|\mathbb{P}(A\in\cdot))]$. We therefore have

$$\begin{aligned}
\mathbb{I}(\mathcal{E};Y|\tilde{\mathcal{E}},X)=&\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\mathbb{P}(Y\in\cdot|\mathcal{E},\tilde{\mathcal{E}},X)\parallel\mathbb{P}(Y\in\cdot|\tilde{\mathcal{E}},X))]\\
=&\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(\mathbb{P}(Y\in\cdot|\mathcal{E},X)\parallel\mathbb{P}(Y\in\cdot|\tilde{\mathcal{E}},X))]\\
=&\mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P^*\|\tilde{P})],
\end{aligned}$$

where the second equation follows from the fact that $(X,Y)\perp\tilde{\mathcal{E}}|\mathcal{E}$. □

Theorem 8. (rate-distortion regret bounds) *For all T,*

$$\sup_{\epsilon\geq 0}\min\{\mathbb{H}_\epsilon(\mathcal{E}),\ \epsilon T\}\ \leq\ \mathcal{R}(T)\ \leq\ \inf_{\epsilon\geq 0}(\mathbb{H}_\epsilon(\mathcal{E})+\epsilon T).$$

Proof. We begin by establishing the upper bound.

$$
\begin{aligned}
\mathcal{R}(T) &= \sum_{t=0}^{T-1} \mathbb{E}\left[\mathbf{d}_{\mathrm{KL}}(P_T^* \| \hat{P}_t)\right] \\
&\overset{(a)}{=} \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \mathcal{E} | H_t) \\
&= \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \mathcal{E}, \tilde{\mathcal{E}} | H_t) \\
&\overset{(b)}{=} \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \tilde{\mathcal{E}} | H_t) + \mathbb{I}(Y_{t+1}; \mathcal{E} | \tilde{\mathcal{E}}, H_t) \\
&\overset{(c)}{\leq} \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \tilde{\mathcal{E}} | H_t) + \mathbb{I}(Y_{t+1}; \mathcal{E} | \tilde{\mathcal{E}}, H_t) \\
&\overset{(d)}{=} \mathbb{I}(H_T; \tilde{\mathcal{E}}) + \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \mathcal{E} | \tilde{\mathcal{E}}, H_t) \\
&\overset{(e)}{\leq} \mathbb{I}(H_T; \tilde{\mathcal{E}}) + \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \mathcal{E} | \tilde{\mathcal{E}}, X_t) \\
&\overset{(f)}{\leq} \mathbb{I}(H_T; \tilde{\mathcal{E}}) + \epsilon T \\
&\overset{(g)}{\leq} \mathbb{I}(\mathcal{E}; \tilde{\mathcal{E}}) + \epsilon T
\end{aligned}
$$

where (a) follows from Lemma 4, (b) follows from the chain rule of mutual information, (c) follows from the chain rule of mutual information, (d) follows from the facts that $\mathbf{h}(Y_{t+1}|\tilde{\mathcal{E}}, H_t) \leq \mathbf{h}(Y_{t+1}|\tilde{\mathcal{E}}, X_t)$ and $\mathbf{h}(Y_{t+1}|\mathcal{E}, H_t) = \mathbf{h}(Y_{t+1}|\mathcal{E}, X_t)$, (e) holds for any $\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon$, and (f) follows from the data processing inequality. Since the above inequality holds for all $\epsilon \geq 0$ and $\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon$, the result follows.

Next, we establish the lower bound. Fix $T \in \mathbb{Z}_+$. Let $\tilde{\mathcal{E}} = (\tilde{H}_{T-2}, \tilde{Y}_{T-1})$ be independent from but distributed identically with (H_{T-2}, Y_{T-1}), conditioned on $\mathcal{E}$. In other words, $\tilde{\mathcal{E}} \perp (H_{T-2}, Y_{T-1})|\mathcal{E}$ and $\mathbb{P}(\tilde{\mathcal{E}} \in \cdot|\mathcal{E}) = \mathbb{P}((H_{T-2}, Y_{T-1}) \in \cdot|\mathcal{E})$. This implies that $\mathbb{P}((\mathcal{E}, \tilde{H}_{T-2}, \tilde{Y}_{T-1}, X_{T-1}, Y_T) \in \cdot) = \mathbb{P}((\mathcal{E}, H_{T-2}, Y_{T-1}, X_{T-1}, Y_T) \in \cdot)$, and therefore, $\mathbb{I}(\mathcal{E}; Y_T|H_{T-1}) = \mathbb{I}(\mathcal{E}; Y_T|\tilde{\mathcal{E}}, X_{T-1})$.

Fix $\epsilon \geq 0$. If $\mathcal{R}(T) < \mathbb{H}_\epsilon(\mathcal{E})$ then,

$$
\begin{aligned}
\mathcal{R}(T) &\overset{(a)}{=} \mathbb{I}(\mathcal{E}; H_T) \\
&\overset{(b)}{=} \sum_{t=0}^{T-1} \mathbb{I}(\mathcal{E}; Y_{t+1}|H_t) \\
&\overset{(c)}{\geq} \mathbb{I}(\mathcal{E}; Y_T|H_{T-1})T \\
&= \mathbb{I}(\mathcal{E}; Y_T|\tilde{\mathcal{E}}, X_{T-1})T \\
&\overset{(d)}{=} \mathbb{E}[\mathbf{d}_{\mathrm{KL}}(P_{T-1}^* \| \tilde{P}_{T-1})]T \\
&\overset{(e)}{>} \epsilon T,
\end{aligned}
$$

where (a) follows from Lemma 4, (b) follows from the chain rule of mutual information, (c) follows from Lemma 5, (d) follows from Lemma 6, and (e) follows from the fact that $\tilde{\mathcal{E}} \notin \tilde{\Theta}_\epsilon$ because $\mathcal{R}(T) = \mathbb{I}(\mathcal{E}; H_T) = \mathbb{I}(\mathcal{E}; \tilde{\mathcal{E}}) < H_\epsilon(\mathcal{E})$. Therefore,

$$
\mathcal{R}(T) \geq \min\{\mathbb{H}_\epsilon(\mathcal{E}),\ \epsilon T\}.
$$

Since this holds for any $\epsilon \geq 0$, the result follows. $\square$

Theorem 9. (rate-distortion sample complexity bounds) *For all $\epsilon \geq 0$,*

$$\frac{\mathbb{H}_\epsilon(\mathcal{E})}{\epsilon \mathcal{R}(1)} \leq T_\epsilon \leq \inf_{\delta \in [0,\epsilon]} \left\lceil \frac{\mathbb{H}_{\epsilon-\delta}(\mathcal{E})}{\delta \mathcal{R}(1)} \right\rceil \leq \left\lceil \frac{2\mathbb{H}_{\epsilon/2}(\mathcal{E})}{\epsilon \mathcal{R}(1)} \right\rceil.$$

Proof. We begin by showing the upper bound. Fix $\epsilon \geq 0$ and $\delta \in [0, \epsilon]$. Let

$$T = \left\lceil \frac{\mathbb{H}_{\epsilon-\delta}(\mathcal{E})}{\delta} \right\rceil,$$

so that $\mathbb{H}_{\epsilon-\delta}(\mathcal{E}) \leq \delta T$. We have that:

$$\begin{aligned} \mathcal{R}(T) &\overset{(a)}{\leq} \mathbb{H}_{\epsilon-\delta}(\mathcal{E}) + (\epsilon - \delta)T \\ &\overset{(b)}{\leq} \delta T + (\epsilon - \delta)T \\ &= \epsilon T, \end{aligned}$$

where (a) follows from the upper bound of Theorem 8 and (b) follows from our choice of T. Since $T_\epsilon = \min\{T : \mathcal{R}(T) \leq \epsilon \mathcal{R}(1) T\}$, it follows that $T \geq T_\epsilon$. Since the above holds for arbitrary $\delta \in [0, \epsilon]$, the result follows.

We now show the lower bound. Fix $\epsilon \geq 0$. By the definition of T_ϵ, we have

$$\mathcal{R}(T_\epsilon) \leq \epsilon T_\epsilon.$$

In the proof of the lower bound in Theorem 8, we show that for all $\epsilon \geq 0$, $\mathcal{R}(T) < \mathbb{H}_\epsilon(\mathcal{E}) \implies \mathcal{R}(T) > \epsilon T$. Therefore, using the contrapositive and the above definition of T_ϵ, we have that $\mathbb{H}_\epsilon(\mathcal{E}) \leq \mathcal{R}(T_\epsilon)$ and therefore

$$\mathbb{H}_\epsilon(\mathcal{E}) \leq \mathcal{R}(T_\epsilon) \leq \epsilon T_\epsilon.$$

The result follows. $\square$

D Proofs of Single-Layer Rate-Distortion Bounds

We have established in Theorem 10 that the rate-distortion function for a multilayer environment is a sum of the rate-distortion functions for each single-layer environment under tolerance $\frac{\epsilon}{K}$. In this section, we show the following rate-distortion bounds for a single layer of the deep ReLU network and the deep nonparametric network.

We begin by stating a well known result from information theory:

Lemma 16. (maximum differential entropy) *For all random vectors $X : \Omega \mapsto \Re^d$ with covariance K,*

$$\mathbf{h}(X) \leq \frac{1}{2} \ln \left((2\pi e)^d |K| \right),$$

with equality iff $Pr(X \in \cdot) \sim \mathcal{N}(\mu, K)$ for some $\mu \in \Re^d$.

Proof. Follows from Theorems 8.6.3 and 8.6.5 of Cover and Thomas [2006]. $\square$

Theorem 17. (single-layer relu neural network rate-distortion bound) *For all $N, k \in \mathbb{Z}_{++}$ and $\sigma^2, \epsilon \geq 0$, if $\mathcal{E}_k$ is identified by (A, b) where $A : \Omega \mapsto \Re^{N \times N}, b : \Omega \mapsto \Re^N$ both consist of independent elements each with variance $\frac{1}{N}$, $X : \Omega \mapsto \Re^N$ is a random vector with covariance I_N, and $Y \sim \mathcal{N}(\mathrm{ReLU}(AX + b), \sigma^2 I_N)$, then*

$$\mathbb{H}_\epsilon(\mathcal{E}_k) = \mathcal{O}\left(N^2 \ln \left(\frac{N}{2\sigma^2 \epsilon} \right) \right).$$

Proof. We use A_i, b_i to denote the ith rows of A, b respectively. Let $\tilde{\mathcal{E}}_k = (\tilde{A}, \tilde{b})$ where $\tilde{A} = A + V$ where $V \perp A$ and $V \sim \mathcal{N}(0, \delta^2 I_d)$ and likewise $\tilde{b} = b + Z$ where $Z \perp b$ and $Z \sim \mathcal{N}(0, \delta^2)$. $\delta^2 = \frac{\sigma^2\left(e^{\frac{2\epsilon}{N}}-1\right)}{N+1}$. We have that

$$
\begin{aligned}
\mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}}_k) &= \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}}_k) \\
&= \mathbb{I}(A; \tilde{A}) + \mathbb{I}(b; \tilde{b}) \\
&\leq \mathbf{h}(\tilde{A}) - \mathbf{h}(\tilde{A}|A) + \mathbb{I}(b; \tilde{b}) \\
&= \frac{N^2}{2} \ln\left(2\pi e\left(\delta^2 + \frac{1}{N}\right)\right) - \mathbf{h}(V) + \mathbb{I}(b; \tilde{b}) \\
&= \frac{N^2}{2} \ln\left(1 + \frac{1}{N\delta^2}\right) + \frac{N}{2} \ln\left(1 + \frac{1}{N\delta^2}\right) \\
&\leq \frac{N(N+1)}{2} \ln\left(1 + \frac{1}{N\delta^2}\right) \\
&= \mathcal{O}\left(N^2 \ln\left(1 + \frac{1}{N\delta^2}\right)\right) \\
&= \mathcal{O}\left(N^2 \ln\left(1 + \frac{1}{\sigma^2\left(e^{\frac{2\epsilon}{N}} - 1\right)}\right)\right) \\
&= \mathcal{O}\left(N^2 \ln\left(\frac{N}{2\sigma^2\epsilon}\right)\right).
\end{aligned}
$$

We now verify that our choice of $\tilde{\mathcal{E}}_k$ satisfies the distortion constraint:

$$
\begin{aligned}
\mathbb{I}(Y; \mathcal{E}_k|\tilde{\mathcal{E}}_k, X) &= \mathbb{I}(Y; A, b|\tilde{A}, \tilde{b}, S, X) \\
&= \mathbf{h}(Y|\tilde{A}, \tilde{b}, S, X) - \mathbf{h}(Y|A, b, S, X) \\
&= N\left(\mathbf{h}(Y_i|\tilde{A}, \tilde{b}, S, X) - \mathbf{h}(Y_i|A, b, S, X)\right) \\
&\overset{(a)}{=} N\left(\mathbf{h}(Y_i|\tilde{A}_i, \tilde{b}_i, S_i, X) - \mathbf{h}(W_i)\right) \\
&= N\left(\mathbf{h}(Y_i - \mathrm{ReLU}(\tilde{A}_i^\top X + \tilde{b}_i)|\tilde{\theta}_i, \tilde{b}_i, S_i, X) - \mathbf{h}(W_i)\right) \\
&\overset{(b)}{\leq} N\left(\mathbf{h}(Y_i - \mathrm{ReLU}(\tilde{A}_i^\top X + \tilde{b}_i)|X) - \mathbf{h}(W_i)\right) \\
&\overset{(c)}{\leq} \mathbb{E}\left[N\left(\frac{1}{2}\ln\left(2\pi e\left(\sigma^2 + \mathbb{V}[\mathrm{ReLU}(A_i^\top X + b_i) - \mathrm{ReLU}(\tilde{A}_i^\top X + \tilde{b}_i)|X]\right)\right) - \mathbf{h}(W_i)\right)\right] \\
&\overset{(d)}{\leq} \frac{N}{2}\ln\left(1 + \frac{\mathbb{V}[\mathrm{ReLU}(A_i^\top X + b_i) - \mathrm{ReLU}(\tilde{A}_i^\top X + \tilde{b}_i)]}{\sigma^2}\right) \\
&\overset{(e)}{\leq} \frac{N}{2}\ln\left(1 + \frac{\mathbb{V}[A_i^\top X + b_i - \tilde{A}_i^\top X - \tilde{b}_i]}{\sigma^2}\right) \\
&= \frac{N}{2}\ln\left(1 + \frac{(N+1)\delta^2}{\sigma^2}\right) \\
&= \epsilon
\end{aligned}
$$

where in (a), $W_i \sim \mathcal{N}(0, \sigma^2)$, (b) follows from the fact that conditioning reduces entropy, (c) follows from Lemma 16, (d) follows from Jensen's Inequality, and (e) follows from the fact that for all $x, y \in \Re$, $(\mathrm{ReLU}(x) - \mathrm{ReLU}(y))^2 \leq (x - y)^2$. □

Theorem 18. (single-layer nonparametric rate-distortion bound) *For all $N, M \in \mathbb{Z}_{++}$ and $\sigma^2, \epsilon \geq 0$, if $\mathcal{E}$ is identified by α where $\alpha \sim Dir(N, [1, \ldots, 1])$ and*

$$f(X) = \sum_{n=1}^{N} \alpha_n g_n(X)$$

for deterministic basis functions $(g_1, \ldots, g_N)$ mapping $\Re^M \mapsto \Re^M$ that satisfy $\mathbb{E}[g_n(X)^2] \leq 1$ for random vector $X : \Omega \mapsto \Re^N$, and $Y \sim \mathcal{N}(f(X), \sigma^2 I_M)$, then

$$\mathbb{H}_\epsilon(\mathcal{E}) \leq \frac{1 - \mathbb{E}[f(X)^2]}{2\sigma^2 \epsilon}$$

Proof. Let $\tilde{\mathcal{E}} = (h_1, \ldots, h_C)$ for $C = \frac{1-\mathbb{E}\left[f(X)^2\right]}{\sigma^2 M\left(e^{\frac{2\epsilon}{M}}-1\right)}$ and where for all $i \in \{1, \ldots, C\}$,

$$h_i \overset{iid}{\sim} \{g_n \quad w.p\ \alpha_n\ .$$

The distortion of $\tilde{\mathcal{E}}$ is

$$\begin{aligned}
\mathbb{I}(f(X) + W; \mathcal{E}|\tilde{\mathcal{E}}, X) &= \mathbf{h}(f(X) + W|\tilde{\mathcal{E}}, X) - \mathbf{h}(f(X) + W|\mathcal{E}, X) \\
&= \mathbf{h}\left(f(X) - \frac{1}{C}\sum_{i=1}^{C} h_i(X) + W \middle| \tilde{\mathcal{E}}, X\right) - \mathbf{h}(W) \\
&\overset{(a)}{\leq} \mathbf{h}\left(f(X) - \frac{1}{C}\sum_{i=1}^{C} h_i(X) + W\right) - \mathbf{h}\left(W\right) \\
&\leq \frac{M}{2}\ln\left(2\pi e\left(\sigma^2 + \mathbb{E}\left[\frac{\left\|f(X) - \frac{1}{C}\sum_{i=1}^{C} h_i(X)\right\|^2}{M}\right]\right)\right) - \mathbf{h}(W) \\
&= \frac{M}{2}\ln\left(1 + \frac{\mathbb{E}\left[\left\|f(X) - \frac{1}{C}\sum_{i=1}^{C} h_i(X)\right\|^2\right]}{\sigma^2 M}\right) \\
&= \frac{M}{2}\ln\left(1 + \frac{\mathbb{E}\left[\frac{1+(C-1)\|f(X)\|^2}{C} - \|f(X)\|^2\right]}{\sigma^2}\right) \\
&= \frac{M}{2}\ln\left(1 + \frac{\mathbb{E}\left[\frac{1-\|f(X)\|^2}{C}\right]}{\sigma^2 M}\right) \\
&= \epsilon,
\end{aligned}$$

where (a) follows from Lemma 16.

We now upper bound the rate of $\tilde{\mathcal{E}}$.

$$
\begin{aligned}
\mathbb{I}(\alpha;\tilde{\mathcal{E}}) &= \mathbb{I}(\alpha;(h_1,\dots,h_C)) \\
&\overset{(a)}{\leq} C\mathbb{I}(\alpha;g_1) \\
&= C\left(\mathbf{h}(\alpha)-\mathbf{h}(\alpha|g_1)\right) \\
&\overset{(b)}{=} C\left(\ln\left(\frac{\Gamma(1)^N}{\Gamma(N)}\right)-\ln\left(\frac{\Gamma(1)^{N-1}\Gamma(2)}{\Gamma(N+1)}\right)-\psi(N+2)+\psi(2)\right) \\
&= C\left(\ln\left(\frac{\Gamma(N+1)}{\Gamma(N)}\right)-\sum_{i=2}^{N}\frac{1}{i}\right) \\
&\leq C\left(\ln(N)-(\ln(N+1)-1)\right) \\
&\leq C\left(1+\ln\frac{N}{N+1}\right) \\
&\leq C \\
&= \frac{1-\mathbb{E}[f(X)^2]}{\sigma^2 M(e^{\frac{2}{M}\epsilon}-1)} \\
&\leq \frac{1-\mathbb{E}[f(X)^2]}{2\sigma^2\epsilon}
\end{aligned}
$$

where (a) follows from Lemma 5 and (b) follows from the differential entropy of a Dirichlet random variable and the fact that g_1 is categorical (hence $\alpha|g_1$ is also Dirichlet). The result follows. □

E Proof of Multilayer Results

Lemma 19. (more is learned with the true input) *Let $\tilde{\mathcal{E}}_{K:1}$ be a multilayer proxy. Then,*

$$\mathbb{I}(Y;\mathcal{E}_k|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_{k:1},X) \leq \mathbb{I}(Y;\mathcal{E}_k|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k,U_{k-1}).$$

Proof.

$$
\begin{aligned}
&\mathbb{I}(\mathcal{E}_k;Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_{k:1},X) \\
&= \mathbb{I}(\mathcal{E}_k;\tilde{\mathcal{E}}_{k-1:1},X,Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k)-\mathbb{I}(\mathcal{E}_k;\tilde{\mathcal{E}}_{k-1:1},X|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k) \\
&= \mathbb{I}(\mathcal{E}_k;\tilde{\mathcal{E}}_{j-1:1},X,Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k) \\
&= \mathbb{I}(\mathcal{E}_k;Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k)+\mathbb{I}(\mathcal{E}_k;\tilde{\mathcal{E}}_{k-1:1},X|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k,Y) \\
&\overset{(a)}{\leq} \mathbb{I}(\mathcal{E}_k;Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k)+\mathbb{I}(\mathcal{E}_k;\mathcal{E}_{k-1:1},X|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k,Y) \\
&= \mathbb{I}(\mathcal{E}_k;\mathcal{E}_{k-1,1},X,Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k) \\
&\overset{(b)}{=} \mathbb{I}(\mathcal{E}_k;Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k,\mathcal{E}_{k-1:1},X) \\
&\overset{(c)}{=} \mathbb{I}(\mathcal{E}_k;Y|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k,U_{k-1}),
\end{aligned}
$$

where (a) follows from the fact that $\mathcal{E}_k \perp \tilde{\mathcal{E}}_{k-1:1}|(X,Y,\mathcal{E}_{k-1:1})$ and the data processing inequality, (b) follows from the fact that $\mathbb{I}(\mathcal{E}_k;\mathcal{E}_{k-1:1},X|\mathcal{E}_{K:k+1},\tilde{\mathcal{E}}_k)=0$, and (c) follows from the fact that $Y \perp (\mathcal{E}_{k-1:1},X)|U_{k-1}$. □

Lemma 19 states that we learn more information about $\mathcal{E}_k$ when we are given the true input U_{k-1} than when we are given $(X,\tilde{\mathcal{E}}_{k-1:1})$ and have to infer U_{k-1}. This is intuitive as we should be able to recover more about $\mathcal{E}_k$ when we observe its input exactly.

To prove 2), that the immediate output U_k provides more information about $\mathcal{E}_k$ than does Y, each prototypical environment will involve slightly different analysis techniques. In essence, we expect a result like 2) to hold as a result of the data-processing inequality. No amount of post-processing of U_k into (Y) can increase the amount of available information conveyed about $\mathcal{E}_k$.

Lemma 20. *For all $K \in \mathbb{Z}_{++}, (N_0, N_1, \ldots, N_K) \in \mathbb{Z}_{++}^{K+1}$, and $\sigma^2 \geq 0$, if $N_K = 1$, and multilayer environment $\mathcal{E}_{K:1}$ consists of single-layer environments $\mathcal{E}_k$ that each identify a random function $f^{(k)}$ from $\Re^{N_{k-1}} \mapsto \Re^{N_k}$ for which*

$$L^{(k)} = \sup_{x,y \in \Re^{N_{k-1}}} \mathbb{E}\left[\frac{\|f^{(k)}(x) - f^{(k)}(y)\|_2^2}{\|x-y\|_2^2}\Big| x = x, y = y\right] \leq 1,$$

and

$$U_k = \begin{cases} X & k = 0 \\ f^{(k)}(U_{k-1}) & 1 \leq k \leq K \end{cases}$$

with $Y \sim \mathcal{N}(U_K, \sigma^2)$, then for any proxy $\tilde{\mathcal{E}}_k$,

$$\mathbb{I}(Y; \mathcal{E}_k | \tilde{\mathcal{E}}_k, \mathcal{E}_{K:k+1}, U_{k-1}) \leq \frac{1}{2} \ln\left(1 + \frac{\mathbb{E}\left[\|U_k - \mathbb{E}[U_k|\tilde{\mathcal{E}}_k, U_{k-1}]\|_2^2\right]}{\sigma^2}\right)$$

Proof.

$$\begin{aligned}
\mathbb{I}(Y; \mathcal{E}_k | \tilde{\mathcal{E}}_k, \mathcal{E}_{K:k+1}, U_{k-1}) &= \mathbf{h}(Y|\tilde{\mathcal{E}}_k, \mathcal{E}_{K:k+1}, U_{k-1}) - \mathbf{h}(Y|\mathcal{E}_K : k, U_{k-1}) \\
&= \mathbf{h}(Y|\tilde{\mathcal{E}}_k, \mathcal{E}_{K:k+1}, U_{k-1}) - \mathbf{h}(W) \\
&= \mathbf{h}\left(Y - (f_K \circ \ldots \circ f_{k+1})(\mathbb{E}[U_k|\tilde{\mathcal{E}}, U_{k-1}])|\tilde{\mathcal{E}}_k, \mathcal{E}_{K:k+1}, U_{k-1}\right) - \mathbf{h}(W) \\
&\leq \mathbf{h}\left(Y - (f_K \circ \ldots \circ f_{k+1})(\mathbb{E}[U_k|\tilde{\mathcal{E}}, U_{k-1}])\right) - \mathbf{h}(W) \\
&\leq \frac{1}{2} \ln\left(1 + \frac{\mathbb{V}\left[U_K - (f_K \circ \ldots \circ f_{k+1})(\mathbb{E}[U_k|\tilde{\mathcal{E}}, U_{k-1}])\right]}{\sigma^2}\right) \\
&\leq \frac{1}{2} \ln\left(1 + \frac{\mathbb{E}\left[(U_K - (f_K \circ \ldots \circ f_{k+1})(\mathbb{E}[U_k|\tilde{\mathcal{E}}, U_{k-1}]))^2\right]}{\sigma^2}\right) \\
&\leq \frac{1}{2} \ln\left(1 + \frac{\mathbb{E}\left[\prod_{i=k+1}^{K} L^{(i)} \left\|U_k - \mathbb{E}[U_k|\tilde{\mathcal{E}}_k, U_{k-1}]\right\|^2\right]}{\sigma^2}\right) \\
&\leq \frac{1}{2} \ln\left(1 + \frac{\mathbb{E}\left[\left\|U_k - \mathbb{E}[U_k|\tilde{\mathcal{E}}_k, U_{k-1}]\right\|^2\right]}{\sigma^2}\right)
\end{aligned}$$

□

We now provide a proof of Theorem 10.

Theorem 10. multilayer rate-distortion bound *For all $K \in \mathbb{Z}_{++}$, $\sigma^2, \epsilon \geq 0$, if $\mathcal{E}_{K:1}$ is a multilayer environment such that for all $k \in \{1, \ldots, K\}$ and $\delta \geq 0$, there exist $\tilde{\mathcal{E}}_k$ s.t.*

$$\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1}) \leq \delta,$$

where $W \sim \mathcal{N}(0, \sigma^2 I)$, then

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) \leq \sum_{k=1}^{K} \mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k).$$

Proof. Let

$$\tilde{\Theta}_\epsilon^{K:1} = \{\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon : \tilde{\mathcal{E}} = (\tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_K); \tilde{\mathcal{E}}_i \perp \tilde{\mathcal{E}}_j \wedge \tilde{\mathcal{E}}_i \perp \mathcal{E}_j \text{ for } i \neq j\},$$

We have that

$$\begin{aligned}
\inf_{\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon} \mathbb{I}(\mathcal{E}; \tilde{\mathcal{E}}) &= \inf_{\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon} \sum_{k=1}^{K} \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}} | \mathcal{E}_{K:k+1}) \\
&\overset{(a)}{\leq} \inf_{\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon^{K:1}} \sum_{k=1}^{K} \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}} | \mathcal{E}_{K:k+1}) \\
&\overset{(b)}{=} \inf_{\tilde{\mathcal{E}} \in \tilde{\Theta}_\epsilon^{K:1}} \sum_{k=1}^{K} \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}}_k) \\
&\overset{(c)}{\leq} \sum_{k=1}^{K} \inf_{\tilde{\mathcal{E}}_k \in \tilde{\Theta}_{\frac{\epsilon}{K}}^{(k)}} \mathbb{I}(\mathcal{E}_k, \tilde{\mathcal{E}}_k) \\
&= \sum_{k=1}^{K} \mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k)
\end{aligned}$$

where (a) follows from the fact that $\tilde{\Theta}_\epsilon^{K:1} \subset \tilde{\Theta}_\epsilon$, (b) follows from the fact that $\tilde{\mathcal{E}}_i \perp \mathcal{E}_j$ for $i \neq j$, and (c) follows from the fact that for $\tilde{\Theta}_{\epsilon/K}^{(k)} := \{\tilde{\mathcal{E}}_k \in \tilde{\Theta} : \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1}) \leq \frac{\epsilon}{K}\}$, $\tilde{\Theta}_{\epsilon/K}^{(1)} \times \ldots \times \tilde{\Theta}_{\epsilon/K}^{(K)} \subset \tilde{\Theta}_\epsilon^{K:1}$ since we assumed that $\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq \mathbb{I}(U_k + W; \mathcal{E}_k | \tilde{\mathcal{E}}_k, U_{k-1})$ for all k. □

F Proof of Main Results

Theorem 11. (relu neural network rate-distortion and sample complexity bound) *For all $d, N, K \in \mathbb{Z}_{++}$ and $\sigma^2, \epsilon \geq 0$, if multilayer environment $\mathcal{E}_{K:1}$ is the deep ReLU network with input $X : \Omega \mapsto \Re^d$ s.t. $\mathbb{V}[X] = I_d$ and output $Y \sim \mathcal{N}(U_K, \sigma^2)$, then*

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) = \tilde{\mathcal{O}}\left(KN^2 + dN\right), \quad T_\epsilon = \tilde{\mathcal{O}}\left(\frac{KN^2 + dN}{\epsilon}\right).$$

Proof. If we have a distortion function $d(\mathcal{E}_k, \tilde{\mathcal{E}}_k)$ for which

$$\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq d(\mathcal{E}_k, \tilde{\mathcal{E}}_k),$$

then by the same proof as in Theorem 10,

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) \leq \sum_{k=1}^{K} \mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k, d),$$

where $\mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k, d)$ denotes the rate-distortion function for random variable $\mathcal{E}_k$ under distortion function $d(\mathcal{E}_k, \tilde{\mathcal{E}}_k)$. By Lemma 20, we have the above condition for $d(\mathcal{E}_k, \tilde{\mathcal{E}}_k) = \frac{1}{2} \ln\left(1 + \frac{\mathbb{E}[\|U_k - \mathbb{E}[U_k | \tilde{\mathcal{E}}_k, U_{k-1}]\|_2^2]}{\sigma^2}\right)$. Furthermore, distortion used in the proof of Theorem 17 is exactly $d(\mathcal{E}_k; \tilde{\mathcal{E}}_k)$. As a result,

$$\begin{aligned}
\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) &\leq \sum_{k=1}^{K} \mathbb{H}_{\epsilon,d}(\mathcal{E}_k, d) \\
&\overset{(a)}{=} \mathcal{O}\left((K-2)N^2 \ln\left(\frac{NK}{2\sigma^2\epsilon}\right) + dN \ln\left(\frac{dK}{2\sigma^2\epsilon}\right) + N \ln\left(\frac{NK}{2\sigma^2\epsilon}\right)\right) \\
&= \tilde{O}(KN^2 + dN).
\end{aligned}$$

where (a) follows from Theorem 17. The sample complexity result follows from applying Theorem 9. □

Theorem 12. (nonparametric network rate-distortion and sample complexity bound) *For all $d, N, M, K \in \mathbb{Z}_{++}$, $\sigma^2, \epsilon \geq 0$, if multilayer environment $\mathcal{E}_{K:1}$ is the deep nonparametric network with input $X : \Omega \mapsto \Re^d$ and output $Y \sim \mathcal{N}(U_K, \sigma^2)$, then*

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) \leq \frac{K^2}{2\sigma^2\epsilon}, \quad T_\epsilon \leq 2\left(\frac{K}{\sigma\epsilon}\right)^2.$$

Proof. If we have a distortion function $d(\mathcal{E}_k, \tilde{\mathcal{E}}_k)$ for which

$$\mathbb{I}(Y; \mathcal{E}_k | \mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, X) \leq d(\mathcal{E}_k, \tilde{\mathcal{E}}_k),$$

then by the same proof as in Theorem 10,

$$\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) \leq \sum_{k=1}^{K} \mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k, d),$$

where $\mathbb{H}_{\frac{\epsilon}{K}}(\mathcal{E}_k, d)$ denotes the rate-distortion function for random variable $\mathcal{E}_k$ under distortion function $d(\mathcal{E}_k, \tilde{\mathcal{E}}_k)$. By Lemma 20, we have the above condition for $d(\mathcal{E}_k, \tilde{\mathcal{E}}_k) = \frac{1}{2}\ln\left(1 + \frac{\mathbb{E}[\|U_k - \mathbb{E}[U_k|\tilde{\mathcal{E}}_k, U_{k-1}]\|_2^2]}{\sigma^2}\right)$. Furthermore, distortion used in the proof of Theorem 18 is exactly $d(\mathcal{E}_k; \tilde{\mathcal{E}}_k)$. As a result,

$$\begin{aligned}
\mathbb{H}_\epsilon(\mathcal{E}_{K:1}) &\leq \sum_{k=1}^{K} \mathbb{H}_{\epsilon,d}(\mathcal{E}_k, d) \\
&\overset{(a)}{=} \sum_{k=1}^{K} \frac{K}{2\sigma^2\epsilon} \\
&= \frac{K^2}{2\sigma^2\epsilon}.
\end{aligned}$$

where (a) follows from Theorem 18. The sample complexity result follows from applying Theorem 9. □