EFFECT OF TOP-DOWN CONNECTIONS IN HIERARCHICAL SPARSE CODING

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Paper under double-blind review

ABSTRACT

Hierarchical Sparse Coding (HSC) is a powerful model to efficiently represent multi-dimensional, structured data such as images. The simplest solution to solve this computationally hard problem is to decompose it into independent layerwise subproblems. However, neuroscientific evidence would suggest inter-connecting these subproblems as in the Predictive Coding (PC) theory, which adds top-down connections between consecutive layers. In this study, a new model called Sparse Deep Predictive Coding (SDPC) is introduced to assess the impact of this inter-layer feedback connection. In particular, the SDPC is compared with a Hierarchical Lasso (Hi-La) network made out of a sequence of Lasso layers. A 2-layered SDPC and a Hi-La networks are trained on 3 different databases and with different sparsity parameters on each layer. First, we show that the overall prediction error generated by SDPC is lower thanks to the feedback mechanism as it transfers prediction error between layers. Second, we demonstrate that the inference stage of the SDPC is faster to converge than for the Hi-La model. Third, we show that the SDPC also accelerates the learning process. Finally, the qualitative analysis of both models dictionaries, supported by their activation probability, show that the SDPC features are more generic and informative.

1 INTRODUCTION

Finding a "efficient" representation to model a given signal in a concise and efficient manner is an inverse problem that has always been central to the machine learning community. Sparse Coding (SC) has proven to be one of the most successful methods to achieve this goal. SC holds the idea that signals (e.g. images) can be encoded as a linear combination of few features (called atoms) drawn from a bigger set called the dictionary (Elad, 2010). The pursuit of optimal coding is usually decomposed into two complementary subproblems: inference (coding) and dictionary learning. Inference consists in finding an accurate sparse representation of the input data considering the dictionaries are fixed, it could be performed using algorithms like ISTA & FISTA (Beck & Teboulle, 2009), Matching Pursuit (Mallat & Zhang, 1993), Coordinate Descent (Li & Osher, 2009), or ADMM (Heide et al., 2015). Once the representation is inferred, one can learn the atoms from the data using methods like gradient descent (Rubinstein et al., 2010; Kreutz-Delgado et al., 2003; Sulam et al., 2018), or online dictionary learning (Mairal et al., 2009a). Consequently, SC offers an unsupervised framework to learn simultaneously basis vectors (e.g. atoms) and the corresponding input representation. SC has been applied with success to image restoration (Mairal et al., 2009b), feature extraction (Szlam et al., 2010) and classification (Yang et al., 2011; Perrinet & Bednar, 2015). Interestingly, SC is also a field of interest for computational neuroscientists. Olshausen & Field (1997) first demonstrated that adding a sparse prior to a shallow neural network was sufficient to account for the emergence of neurons whose Receptive Fields (RFs) are spatially localized, band-pass and oriented filters, analogous to those found in the primary visual cortex (V1) of mammals (Hubel & Wiesel, 1962). Because most of the SC algorithms are limited to single-layer network, they cannot model the hierarchical structure of the visual cortex. However, few solutions have been proposed to tackle Hierarchical Sparse Coding (HSC) as a global optimization problem (Sulam et al., 2018; Makhzani & Frey, 2013; 2015; Aberdam et al., 2019; Sulam et al., 2019). These methods are looking for an optimal solution of HSC without considering their plausibility in term of neuronal implementation. Consequently, the quest for reliable HSC formulation that is compatible with a neural implementation remains open.

Rao & Ballard (1999) introduce the Predictive Coding (PC) to model the effect of the interaction of cortical areas in the visual cortex. PC intends to solve the inverse problem of vision by combining feedforward and feedback connections. In PC, feedback connection carries prediction of the neural activity of the lower cortical area while feedforward pass prediction error to the higher cortical area. In such a framework, neural population are updated to minimize the unexpected component of the neural signal (Friston, 2010). PC has been applied for supervised object recognition (Wen et al., 2018; Han et al., 2018; Spratling, 2017) or unsupervised prediction of future video frames (Lotter et al., 2016). Interestingly, PC is flexible enough to introduce a sparse prior to each layer. Therefore, one can consider PC as a bio-plausible formulation of the HSC problem. This formulation is to confront with the other bio-plausible HSC formulation that consists of a stack of independent Lasso problems (Sun et al., 2017). To the best of our knowledge, no study has compared these two mathematically different formulations of the same problem of optimizing the Hierarchical Sparse Coding of images. What is the effect of top-down connection of PC? What are the consequences in term of computations and convergence? What are the qualitative differences concerning the learned atoms?

The objective of this study is to experimentally answer these questions and to show that the PC framework could be successfully used for improving solutions to HSC problems. We start our study by defining the two different mathematical formulations to solve the HSC problem: the Hierarchical Lasso (Hi-La) that consists in stacking Lasso sub-problems, and the Sparse Deep Predictive Coding (SDPC) that leverages PC into a deep and sparse network of bi-directionally connected layers. To experimentally compare both models, we train 2-layered SDPC and Hi-La networks on 3 different databases and we vary the sparsity of each layer. First, we compare the overall prediction error of the two models and we break it down to understand its distribution among layers. Second, we analyze the number of iterations needed for the state variables of each network to reach their stability. Third, we compare the convergence of both models during the dictionary learning stage. Finally, we discuss the qualitative differences between the features learned by both networks in light of their activation probability.

2 METHODS

2.1 BACKGROUND

In our mathematical description, italic letters are used as symbols for *scalars*, bold lower case letters for column $\boldsymbol{vectors}$, bold uppercase letters for $\mathbf{MATRICES}$ and $\nabla_x \mathcal{L}$ denotes the gradient of $\mathcal{L}$ w.r.t. to x. The core objective of Hierarchical Sparse Coding (HSC) is to infer the internal state variables $\{\boldsymbol{\gamma}_i\}_{i=1}^L$ (also called sparse map) and to learn the parameters $\{\mathbf{D}_i\}_{i=1}^L$ that solved the inverse problem formulated in Eq. 1.

$$\begin{cases} \boldsymbol{x} = \mathbf{D}_1^T \boldsymbol{\gamma}_1 + \boldsymbol{\epsilon}_1 & \text{s.t. } \|\boldsymbol{\gamma}_1\|_0 < \alpha_1 \quad \text{and } \boldsymbol{\gamma}_1 > 0 \\ \boldsymbol{\gamma}_1 = \mathbf{D}_2^T \boldsymbol{\gamma}_2 + \boldsymbol{\epsilon}_2 & \text{s.t. } \|\boldsymbol{\gamma}_2\|_0 < \alpha_2 \quad \text{and } \boldsymbol{\gamma}_2 > 0 \\ .. \\ \boldsymbol{\gamma}_{L-1} = \mathbf{D}_L^T \boldsymbol{\gamma}_L + \boldsymbol{\epsilon}_L & \text{s.t. } \|\boldsymbol{\gamma}_L\|_0 < \alpha_L \quad \text{and } \boldsymbol{\gamma}_L > 0 \end{cases} \tag{1}$$

We refer to $\boldsymbol{x}$ as the input (i.e. image), L as the number of layers and $\boldsymbol{\epsilon}_i$ as the layer i prediction error. The map $\boldsymbol{\gamma}_i$ could be viewed as the projection of $\boldsymbol{\gamma}_{i-1}$ in the basis described by the atoms (denoted also basis vector) composing $\mathbf{D}_i$. The sparsity of the internal state variables, specified by the ℓ_0 pseudo-norm, is constrained by the scalar α_i. Furthermore, we impose a 2-dimensional convolutional structure to the parameters $\{\mathbf{D}_i\}_{i=1}^L$. Replaced in a biological context $\mathbf{D}_i$ could be interpreted as the synaptic weights between two neural populations whose activity is represented by $\boldsymbol{\gamma}_{i-1}$ and $\boldsymbol{\gamma}_i$ respectively.

2.2 FROM HIERARCHICAL LASSO ...

One possibility to solve Eq. 1 while keeping the locality of the processing required by neural implementation, is to minimize a loss for each layer corresponding to the addition of the squared ℓ_2-norm of the prediction error with a sparsity penalty. To obtain a convex cost, we relax the ℓ_0 constraint into a ℓ_1-penalty. It defines, therefore, a loss function for each layer in the form of a standard Lasso problem (Eq. 2), that could be minimized using gradient-based methods:

$$\mathcal{F}\big(\mathbf{D}_i, \boldsymbol{\gamma}_i\big) = \frac{1}{2}\|\boldsymbol{\gamma}_{i-1} - \mathbf{D}_i^T \boldsymbol{\gamma}_i\|_2^2 + \lambda_i \|\boldsymbol{\gamma}_i\|_1 \tag{2}$$

In particular, we use the Iterative Shrinkage Thresholding Algorithm (ISTA) to minimize $\mathcal{F}$ w.r.t. $\boldsymbol{\gamma}_i$ (Eq. 3) as it is proven to be computationally cheap (Beck & Teboulle, 2009). In practice, we use an accelerated version of the ISTA algorithm called FISTA (Chalasani et al., 2013).

$$\boldsymbol{\gamma}_i^{t+1} = \mathcal{T}_{\eta_{c_i}\lambda_i}\big(\boldsymbol{\gamma}_i^t - \eta_{c_i}\nabla_{\boldsymbol{\gamma}_i^t}\mathcal{F}\big) = \mathcal{T}_{\eta_{c_i}\lambda_i}\big(\boldsymbol{\gamma}_i^t + \eta_{c_i}\mathbf{D}_i(\boldsymbol{\gamma}_{i-1}^t - \mathbf{D}_i^T\boldsymbol{\gamma}_i^t)\big) \tag{3}$$

In Eq. 3, $\mathcal{T}_\alpha(\cdot)$ denotes the non-negative soft-thresholding operator, η_{c_i} is the learning rate of the inference process and $\boldsymbol{\gamma}_i^t$ is the state variable $\boldsymbol{\gamma}_i$ at time t. Interestingly, one can interpret Eq. 3 as one loop of a recurrent layer that we will call the Lasso layer (Gregor & LeCun, 2010). Finally, Lasso layers can be stacked together to form a Hierarchical Lasso (Hi-La) network (see Fig. 1). The inference of the overall Hi-La network consists in updating recursively all the sparse maps until they have reached a stable point.

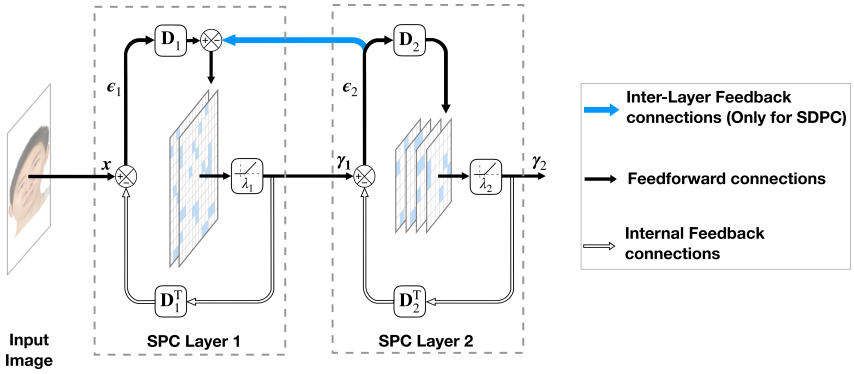


Figure 1: Inference update scheme for the 2-layered SDPC network. $\boldsymbol{x}$ is the input image, and λ_i tunes the sparseness level of $\boldsymbol{\gamma}_i$. The encoding and decoding dictionaries ($\mathbf{D}_i$ and $\mathbf{D}_i^T$, respectively) are reciprocal. The sparse maps ($\boldsymbol{\gamma}_i$) are updated through a bi-directional dynamical process (plain and empty arrows). This recursive process alone describes the Hi-La network. If we add the top-down influence, called inter-layer feedback connection (blue arrow), it then becomes an SDPC network.

2.3 … TO HIERARCHICAL PREDICTIVE CODING

Another alternative to solve Eq. 1 is to use the Predictive Coding (PC) theory. Unlike the Lasso loss function, PC is not only minimizing the bottom-up prediction error, but it also adds a top-down prediction error that takes into consideration the influence of the upper-layer on the current layer (see Eq. 4). In other words, finding the $\boldsymbol{\gamma}_i$ that minimizes $\mathcal{L}$ consists in finding a trade-off between a representation that best predicts the lower level activity and another one that is best predicted by the upper-layer.

$$\mathcal{L}(\mathbf{D}_i, \boldsymbol{\gamma}_i) = \mathcal{F}(\mathbf{D}_i, \boldsymbol{\gamma}_i) + \frac{1}{2}\|\boldsymbol{\gamma}_i - \mathbf{D}_{i+1}^T\boldsymbol{\gamma}_{i+1}\|_2^2 \tag{4}$$

For consistency, we also use the ISTA algorithm to minimize $\mathcal{L}$ w.r.t $\boldsymbol{\gamma}_i$. The update scheme is described in Eq. 5:

$$\boldsymbol{\gamma}_i^{t+1} = \mathcal{T}_{\eta_{c_i}\lambda_i}\big(\boldsymbol{\gamma}_i^t - \eta_{c_i}\nabla_{\boldsymbol{\gamma}_i^t}\mathcal{L}\big) = \mathcal{T}_{\eta_{c_i}\lambda_i}\big(\boldsymbol{\gamma}_i^t + \eta_{c_i}\mathbf{D}_i(\boldsymbol{\gamma}_{i-1}^t - \mathbf{D}_i^T\boldsymbol{\gamma}_i^t) - \eta_{c_i}(\boldsymbol{\gamma}_i^t - \mathbf{D}_{i+1}^T\boldsymbol{\gamma}_{i+1}^t)\big) \tag{5}$$

Fig. 1 shows how we can interpret this update scheme as a loop of a recurrent layer. This recurrent layer, called Sparse Predictive Coding (SPC) layer, forms the building block of the Sparse Deep Predictive Coding (SDPC) network (see Algorithm 2 in Appendix for the detailed implementation of the SDPC inference). The only difference with the Hi-La architecture is that the SDPC includes an inter-layer feedback connection to materialize the influence coming from upper-layers (see the blue arrow in Fig. 1).

2.4 CODING STOPPING CRITERION AND UNSUPERVISED LEARNING

For both networks, the inference process is finalized once the relative variation of $\boldsymbol{\gamma}_i^t$ w.r.t to $\boldsymbol{\gamma}_i^{t-1}$ is below a threshold denoted T_{stab}. In practice, the number of iterations needed to reach the stopping

criterion is between 30 to 100 (see Fig.4 for more details). Once the convergence is achieved, we update the dictionaries using gradient descent (see Algorithm. 1). It was demonstrated by Sulam et al. (2018) that this alternation of inference and learning offers reasonable convergence guarantee. For both models, dictionaries are randomly initialized using the standard normal distribution (mean 0 and variance 1) and all the sparse maps are initialized to zero at the beginning of the inference process. After every dictionary update, we ℓ_2-normalize each atom of the dictionary to avoid any redundant solution. Interestingly, although the inference update scheme is different for the two models, the dictionary learning loss is the same in both cases since the top-down prediction error term in $\mathcal{L}$ does not depend on $\boldsymbol{D}_i$. This loss is then a good evaluation point to assess the impact of both SDPC and Hi-La inference process on the layer prediction error $\boldsymbol{\epsilon}_i$. We used PyTorch 1.0 to implement, train, and test all the models described above. The code of the two models and the simulations of this paper are available at www.github.com/XXX/XXX.

Algorithm 1: Alternation of inference and learning

while *convergence not reached* **do**
  for $i = 1$ **to** L **do**
    $\boldsymbol{\gamma}_i^{t+1} = \mathcal{T}_{\eta_{c_i}\lambda_i}\left(\boldsymbol{\gamma}_i^t - \eta_{c_i}\nabla_{\boldsymbol{\gamma}_i^t}\mathcal{L}\right)$ # inference

for $i = 1$ **to** L **do**
  $\mathbf{D}_i^{t+1} = \mathbf{D}_i^t - \eta_i \nabla_{\boldsymbol{D}_i^t}\mathcal{L}$ # learning

2.5 Experimental settings: datasets and parameters

We use 3 different databases to train both networks.

AT&T. The AT&T database (ATT, 1994) is made of 400 grayscale images of size 92×112 pixels (px) representing faces of 40 distinct subjects with different lighting conditions, facial expressions, and details. This set is partitioned into batches of 20 images. The training set composed of 330 images (33 subjects) and the testing set id composed of 70 images (7 subjects).

CFD. The Chicago Face Database (CFD) (Ma et al., 2015) consists of $1,804$ high-resolution ($2,444 \times 1,718$ px), color, standardized photographs of male and female faces of varying ethnicity between the ages of 18 and 40 years. We re-sized the pictures to 170×120 px to keep reasonable computational time. The CFD database is partitioned into batches of 10 images. This dataset is split into a training set composed of 721 images and a testing set of 486 images.

MNIST. MNIST (LeCun, 1998) is composed of 28×28 px, $70,000$ grayscale images representing handwritten digits. We decomposed this dataset into batches of 32 images. This dataset is split into a training set composed of $60,000$ digits and a testing set of $10,000$ digits.

All these databases are pre-processed using Local Contrast Normalization (LCN) and whitening. LCN is inspired by neuroscience and consists in a local subtractive and divisive normalization (Jarrett et al., 2009). In addition, we use whitening to reduce dependency between pixels.

To draw a fair comparison between the SDPC and Hi-La models, we train both models using the same set of parameters. All these parameters are summarized in Table 1, for each database. Note that the parameter η_{c_i} is omitted in the table because it is computed as the inverse of the largest eigenvalue of $\mathbf{D}_i^T\mathbf{D}_i$ (Beck & Teboulle, 2009). To learn the dictionary $\mathbf{D}_i$, we use stochastic gradient descent on the training set only, with a learning rate η_{L_i} and a momentum equal to 0.9. In this study, we consider only 2-layered networks and we vary the sparsity parameters of each layer (λ_1 and λ_2) to assess their effect on both SDPC and Hi-La networks.

3 Results

For cross-validation, we run 7 times all the simulations presented in this section, each time with a different random dictionary initialization. We define the central tendency of our curves by the median of the runs, and its variation by the Median Absolute Deviation (MAD) (Pham-Gia & Hung, 2001). We prefer this measure to the classical mean $\pm$ standard deviation because a few measures did not exhibit a normal distribution. All presented curved are obtained on the testing set.

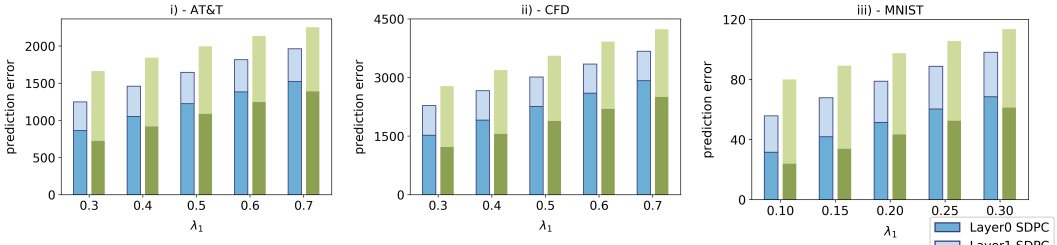
Table 1: Network architectures, training and simulation parameters. The size of the convolutional kernels are shown in the format: [# features, # channels, width, height] (stride). To describe the range of explored parameters during simulations, we use the format $[0.3 : 0.7 :: 0.1, 0.5]$,which means that we vary λ_1 from 0.3 to 0.7 by step of 0.1 while λ_2 is fixed to 0.5.

		DataBases		
		AT&T	CFD	MNIST
network param.	$\boldsymbol{D}_1$ size	[64, 1, 9, 9] (3)	[64, 3, 9, 9] (3)	[32, 1, 5, 5] (2)
	$\boldsymbol{D}_2$ size	[128, 64, 9, 9] (1)	[128, 64, 9, 9] (1)	[64, 32, 5, 5] (1)
	T_{stab}	5e-4	5e-3	5e-4
training param.	# epochs	1000	250	100
	η_{L_1}	1e-4	1e-4	5e-2
	η_{L_2}	5e-3	5e-3	1e-3
simu. param.	λ_1 range	$[0.3 : 0.7 :: 0.1, 1]$	$[0.3 : 0.7 :: 0.1, 1.8]$	$[0.1 : 0.3 :: 0.05, 0.3]$
	λ_2 range	$[0.5, 0.6 : 1.6 :: 0.2]$	$[0.5, 1 : 1.8 :: 0.2]$	$[0.2, 0.2 : 0.4 :: 0.05]$

3.1 SDPC CONVERGES TO A LOWER PREDICTION ERROR

As a first analysis we report the global prediction error, as computed as the sum of the prediction error ($\boldsymbol{\epsilon}_i$) over all layers (see Fig. 3), and its decomposition among layer (see Fig. 2) for different value of sparsity. For scaling reasons, and because the error bars are small we cannot display them on Fig. 2, we thus include them in Appendix Fig. 7. For all the simulations shown in Fig. 2 and Fig. 3, we observed that the global prediction error is lower for the SDPC than for the Hi-La model. As expected, in both models the prediction errors increase similarly when we increase λ_1 or λ_2. For

(a) Distribution of the prediction error among layers when varying λ_1.



(b) Distribution of the prediction error among layers when varying λ_2.

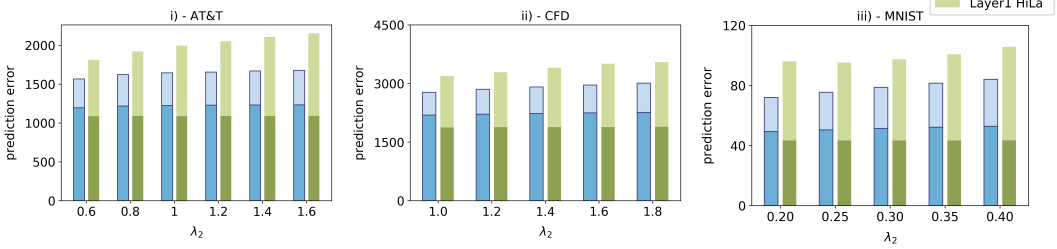


Figure 2: Evolution of all layers prediction error, evaluated on the testing set, for both SDPC and Hi-La networks and trained on 3 databases. We vary the first layer sparsity in the top 3 graphs (**a**) and the second layer sparsity in the bottom 3 graphs (**b**).

all databases and sparsity parameters, Fig. 2 shows that the first layer prediction error of the SDPC is always higher, and the second layer prediction error is always lower than the corresponding Hi-La prediction error. This is expected: while the Hi-La first layer is fully specialized in minimizing the prediction error with the lower level, the SDPC finds a trade-off between lower and higher level prediction errors. In addition, when λ_1 is increased, the Hi-La first layer prediction error increases faster ($+93\%$ for AT&T, $+105\%$ for CFD and $+157\%$ for MNIST) than the SDPC first layer prediction error ($+76\%$ for AT&T, $+90\%$ for CFD, and $+117\%$ for MNIST). This suggests that a part

of the penalty induced by the increase of λ_1 is absorbed by the second layer in the case of the SDPC. This is supported by the fact that the second layer prediction error of the SDPC is in general increasing slightly faster than the Hi-La second layer error. When λ_2 is increased, the prediction error of the first layer of the Hi-La model is not varying whereas the SDPC first layer prediction error is slightly increasing ($+3.3\%$ for AT&T, $+3\%$ for CFD and $+7.1\%$ for MNIST). The explanation here is straightforward: while the first-layer loss of the SDPC includes the influence of the upper-layer, the Hi-La doesn't have such a mechanism. It suggests that the inter-layer feedback connection of the SDPC transfers a part of the extra-penalty coming from the increase of λ_2 in the first layer. Fig.3 i) and ii) show the mapping of the global prediction error when we vary the sparsity of each layer for the SDPC and Hi-La, respectively. These heatmaps confirm what has been observed in Fig.2 and extend it to a larger range of sparsity values: both models' losses are more sensitive to a variation of λ_1 than to a change in λ_2. Fig.3 iii) is a heatmap of the relative difference between the SDPC and the Hi-La global losses. It shows that the minimum relative difference between SDPC and Hi-La (10.6%) is reached when λ_1 is maximal and λ_2 is minimal, and the maximum relative difference (19.99%) is reached when both λ_1 and λ_2 are minimal. It suggests that the previously observed mitigation mechanism originated by the feedback connection is more efficient when the sparsity of the first layer is lower. All these observations point in the same direction: the SDPC

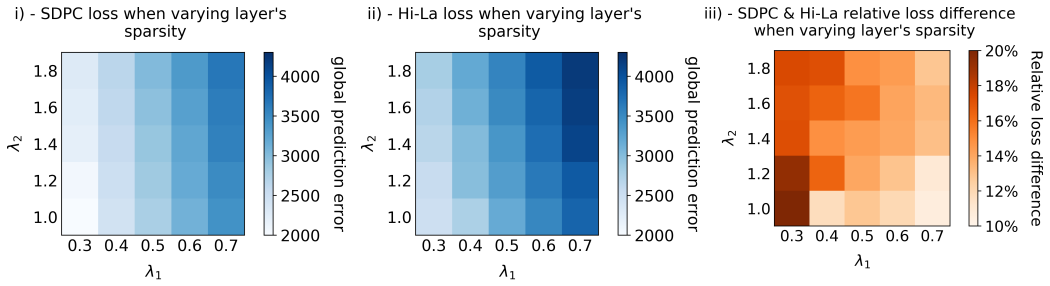


Figure 3: Heatmaps of the global loss error when varying layers' sparsity for SDPC (i) and Hi-La (ii) on CFD database. (iii) shows the heatmap of the relative difference between the Hi-La and the SDPC global losses when varying the layers' sparsity

framework mitigates the global prediction error thanks to a better distribution of the prediction error among layers. This mechanism is even more pronounced when the sparsity of the first layer is lower. Surprisingly, while the inter-layer feedback connection of the SDPC imposes more constraints on the state variables, it also happens to generate less global prediction error.

3.2 SDPC HAS A FASTER INFERENCE PROCESS

One may wonder if this lower prediction error is not achieved at the cost of a slower inference process. To address this concern, we report for both models the number of iterations needed by the inference process to converge towards a stable state on the testing set. Fig. 4 shows the evolution of this quantity, for the 3 datasets, when varying both layers' sparsity. For all the simulations, the SDPC needs less iteration than the Hi-La model to converge towards a stable state. We also observe that the data dispersion is, in general, more pronounced for the Hi-La model. In addition to converging to lower prediction error, the SDPC is also decreasing the number of iterations in the inference process to converge towards a stable state.

3.3 SDPC LEARNS FASTER

Fig. 5 shows the evolution of the global prediction error during the dictionary learning stage and evaluated on the testing set. For the 3 databases, the SDPC model reaches its minimal prediction error before the Hi-La model. The convergence rate of both models is comparable, but the SDPC has a much lower prediction error in the very first epochs. The inter-layer feedback connection of the SDPC pushes the network towards lower prediction error since the very beginning of the learning.

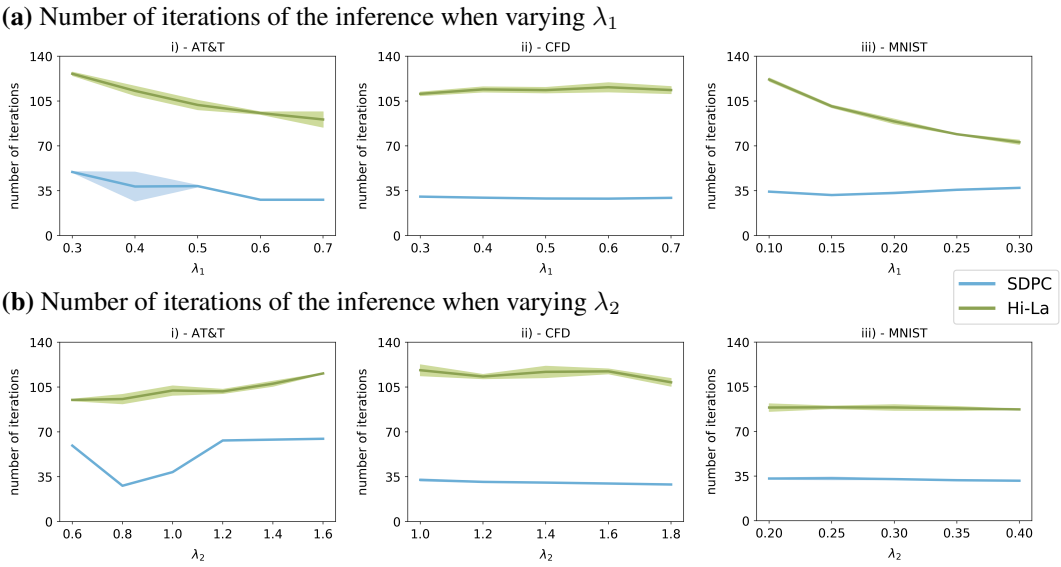


Figure 4: Evolution of the number of iterations needed to reach stability criterium for both SDPC and Hi-La networks on the testing set of 3 different database. We vary the first layer sparsity in the top 3 graphs (**a**) and the second layer sparsity in the bottom 3 graphs (**b**). Shaded areas correspond to mean absolute deviation on 7 runs. Sometimes the dispersion is so small that it looks like there is no shade.

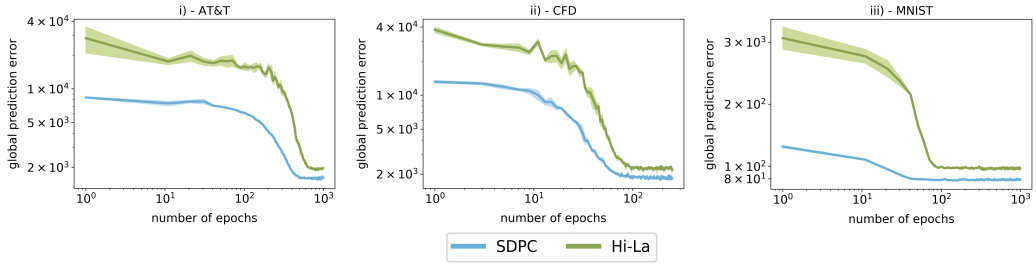


Figure 5: Evolution of the global prediction error during the training for the 3 databases testing set. Shaded areas correspond to mean absolute deviation on 7 runs. All graphs have a logarithmic scale in both x and y-axis.

3.4 Qualitative analysis of the features

Another way to grasp the impact of the inter-layer feedback connection is to visualize its effect on the dictionaries. To make human-readable visualizations of the learned dictionaries, we back-project them into the image space using a cascade of transposed convolution (see Appendix Fig.8). These back-projection are called Receptive Fields (RFs). Fig. 6 shows some of the RFs of the 2 layers and the second layer activation probability histogram for both models when they are trained on the CFD database. In general, first layer RFs are oriented Gabor-like filters, and second layer RFs are more specific and represent more abstract concepts (curvatures, eyes, mouth, nose...). Second layer RFs present longer curvatures in the SDPC than in the Hi-La model: they cover a bigger part of the input image, and include more contextual and informative details. In some extreme cases, the Hi-La second layer RFs are over-fitted to specific faces and do not describe the generality of the concept of face. The red-framed RFs highlights one of these cases: the corresponding activation probabilities are 0.25% and 0.69% for Hi-La and SDPC respectively. This is supported by the fact that the lowest activation probability of the second layer's atoms is higher for the SDPC than for the Hi-La (0.30% versus 0.16%). This phenomenon is even more striking when we sort all the features by activation probabilities in descending order (see Appendix Figures 10). We filter out the highest

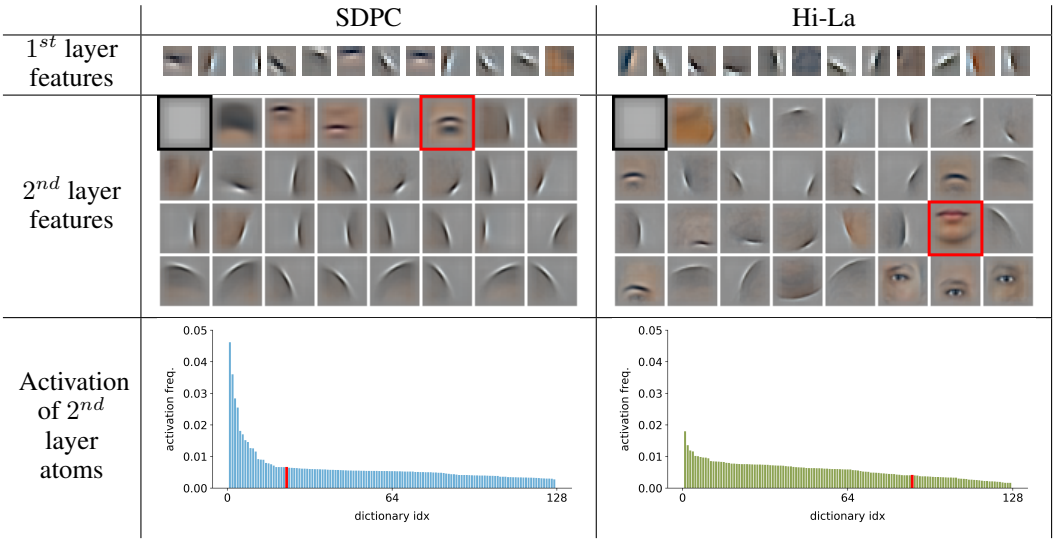


Figure 6: Hi-La and SDPC RFs obtained on the CFD database (with $\lambda_1 = 0.3$, $\lambda_2 = 1.8$) and their associated second layer activation probability histogram. The first and second layer RFs have a size of 9×9 px and 33×33 px respectively. For the first layer RFs, we randomly selected 12 out of 64 atoms. For the second layer RFs, we sub-sampled 32 out of 128 atoms ranked by their activation probability in descending order. For readability, we removed the most activated filter (RF framed black) in SDPC and Hi-La second layer activation histogram. The activation probability of the RFs framed in red are shown as a red bar in the corresponding histogram.

activation probability (corresponding to the low-frequency filters highlighted by black square) of both Hi-La and SDPC to keep good readability of the histograms. All the filters are displayed in Appendix Fig. 9, Fig. 10, Fig. 11, for AT&T, CFD, and MNIST RFs respectively. The atoms' activation probability confirm the qualitative analysis of the RFs: the features learned by the SDPC are more generic and informative as they describe a wider range of images.

4 Conclusion

What are the computational advantages of inter-layer feedback connections in hierarchical sparse coding algorithms? We answered this question by comparing the Hierarchical Lasso (Hi-La) and the Sparse Deep Predictive Coding (SDPC) models. Both are identical in every respect, except that the SDPC brings inter-layer feedback connections. This extra-connection forces the internal state variables of the SDPC to converge toward a trade-off between on one hand an accurate prediction passed by the lower-layer and on the other hand a facilitated predictability by the upper-layer. Experimentally, we demonstrated on 3 different databases and for a 2-layered network that the inter-layer feedback top-down connection (i) mitigates the overall prediction error by distributing it among layers, (ii) accelerates the convergence towards a stable internal state and (iii) accelerates the learning process. Besides, we qualitatively observed that top-down connections bring contextual information that helps to extract more informative and less over-fitted features.

Even if our results are robust as they hold for 3 different databases and with a large spectrum of first and second layer sparsity, further work will be conducted to generalize our results to deeper networks and different sparse coding algorithms such as Coordinate Descent or ADMM. Further studies will show that our SDPC framework could be used for practical applications like image inpainting, denoising, or image super-resolution.

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A SDPC PSEUDO CODE

Algorithm 2: SDPC inference algorithm

input : image: $\boldsymbol{x}$, dictionaries: $\{\boldsymbol{D}_i\}_{i=1}^L$, ℓ_1-penalty parameters: $\{\lambda_i\}_{i=1}^L$, stability threshold: T_{stab}

$\boldsymbol{\gamma}_0^t = \boldsymbol{x}$
$\{\boldsymbol{\gamma}_i^0\}_{i=1}^L = \mathbf{0}, \quad \{\boldsymbol{\gamma}_{m_i}^1\}_{i=1}^L = \mathbf{0}$ # Initializing layer state variables and FISTA momentum
$\alpha^1 = 1$ # Initializing momentum strength
$\eta_{c_i} = \dfrac{1}{max(eigen_value(\boldsymbol{D}_i^T \boldsymbol{D}_i))}$
$Stable = False$ # Initializing the stability criterion
$t = 0$

while $Stable == False$ **do**
  $t \mathrel{+}= 1$
  $\alpha^{t+1} = \dfrac{1 + \sqrt{1 + 4(\alpha^t)^2}}{2}$
  for $i = 1$ **to** L **do**
    # Update lower-layer error
    $\boldsymbol{\epsilon}_{LL} = \boldsymbol{\gamma}_{m_{i-1}}^t - \boldsymbol{D}_i^T \boldsymbol{\gamma}_{m_i}^t$
    # Update the upper-layer error
    if $i \neq L$ **then**
      $\boldsymbol{\epsilon}_{UL} = \boldsymbol{\gamma}_{m_i}^t - \boldsymbol{D}_{i+1}^T \boldsymbol{\gamma}_{m_{i+1}}^t$
    else
      $\boldsymbol{\epsilon}_{UL} = \mathbf{0}$
    $\boldsymbol{\gamma}_i^t = \mathcal{T}_{\eta_{c_i}\lambda_i}\big(\boldsymbol{\gamma}_{m_i}^t + \eta_{c_i} \boldsymbol{D}_i \boldsymbol{\epsilon}_{LL} - \eta_{c_i} \boldsymbol{\epsilon}_{UL}\big)$ # Update layer state variables
    $\boldsymbol{\gamma}_{m_i}^{t+1} = \mathcal{T}_0\Big(\boldsymbol{\gamma}_i^t + \big(\dfrac{\alpha^t - 1}{\alpha^{t+1}}\big)\big(\boldsymbol{\gamma}_i^t - \boldsymbol{\gamma}_i^{t-1}\big)\Big)$ # Update the FISTA momentum
  if $\bigwedge_{i=1}^{L} \Big(\dfrac{\|\boldsymbol{\gamma}_i^t - \boldsymbol{\gamma}_i^{t-1}\|_2}{\|\boldsymbol{\gamma}_i^t\|_2} < T_{stab}\Big)$ **then**
    $Stable = True$ # Update the stability creterion

return $\{\boldsymbol{\gamma}_i^t\}_{i=1}^L$

Note: $\mathcal{T}_\alpha(\cdot)$ denotes the element-wise non-negative soft-thresholding operator. A fortiori, $\mathcal{T}_0(\cdot)$ is a rectified linear unit operator. # comments are comments.

B EVOLUTION OF THE GLOBAL PREDICTION ERROR WITH ERROR BAR

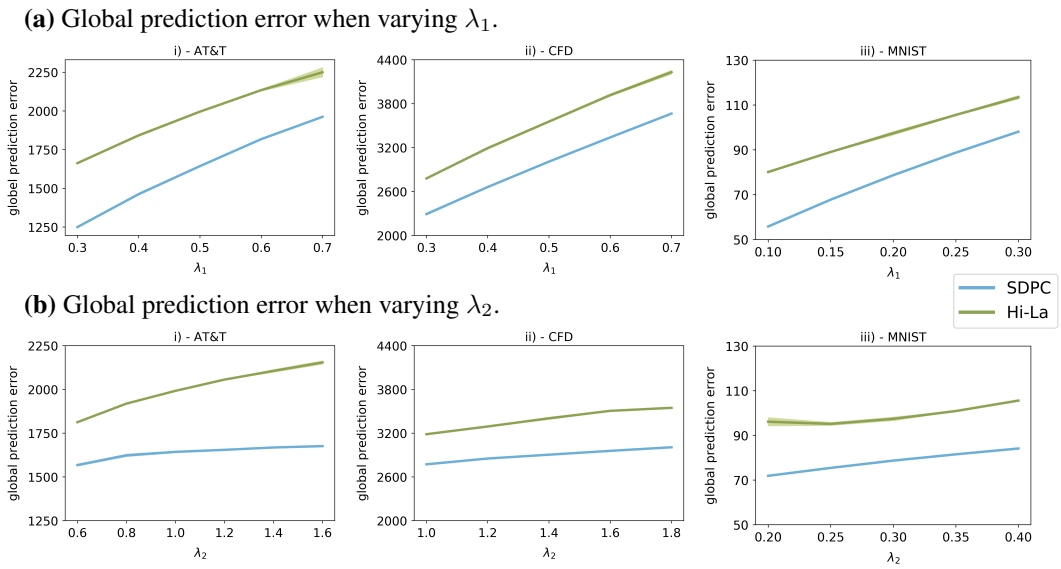


Figure 7: Evolution of the global prediction error evaluated on the testing set for both SDPC and Hi-La networks. We vary the first layer sparsity in the top 3 graphs (**a**) and the second layer sparsity in the bottom 3 graphs (**b**). All experiments have been conducted on 3 databases. Shaded areas correspond to mean absolute deviation on 7 runs. Sometimes the dispersion is so small that it looks like there is no shade.

C ILLUSTRATION OF THE BACK-PROJECTION MECHANISM

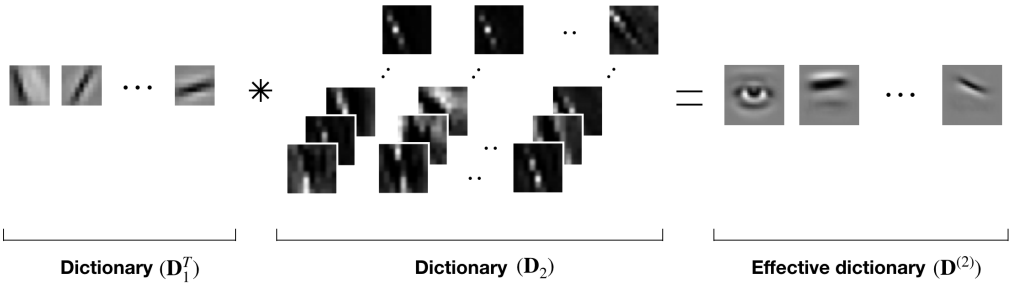
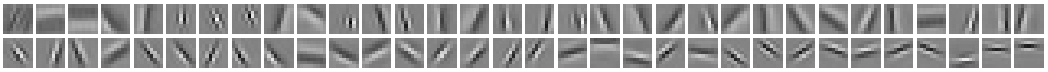


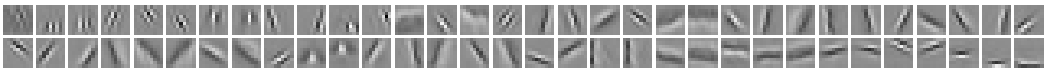
Figure 8: Generation of the second-layer effective dictionary. The result of this back-projection is called effective dictionary and could be assimilate to the notion of preferred stimulus in neuroscience. In a general case, the effective dictionary at layer i is computed as follow: $\mathbf{D}_i^{\text{eff},T} = \mathbf{D}_0^T..\mathbf{D}_{i-1}^T\mathbf{D}_i^T$ (Sulam et al., 2018).

D SDPC AND HI-LA RFS ON AT&T

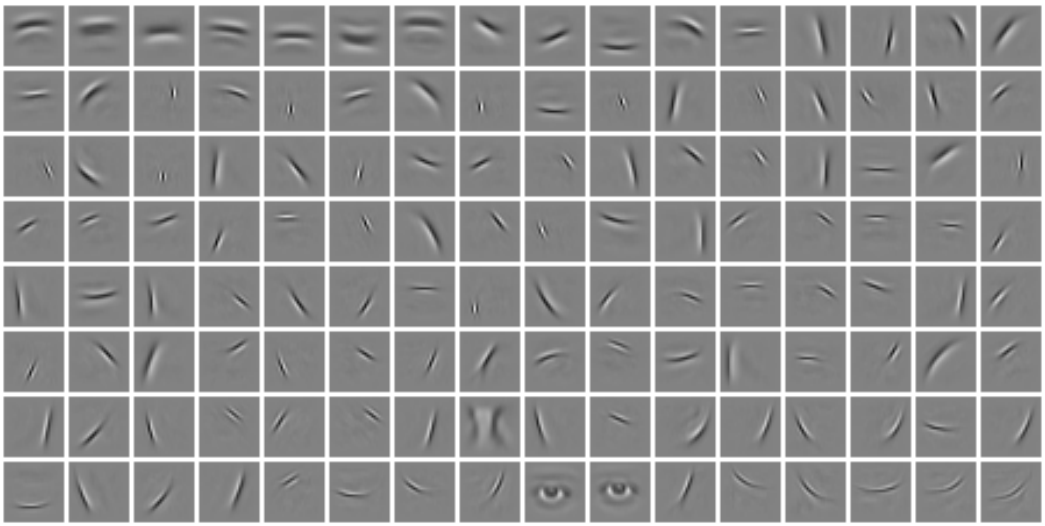
(a) SDPC first layer RFs



(b) Hi-La first layer RFs



(c) SDPC second layer RFs



(d) Hi-La second layer RFs

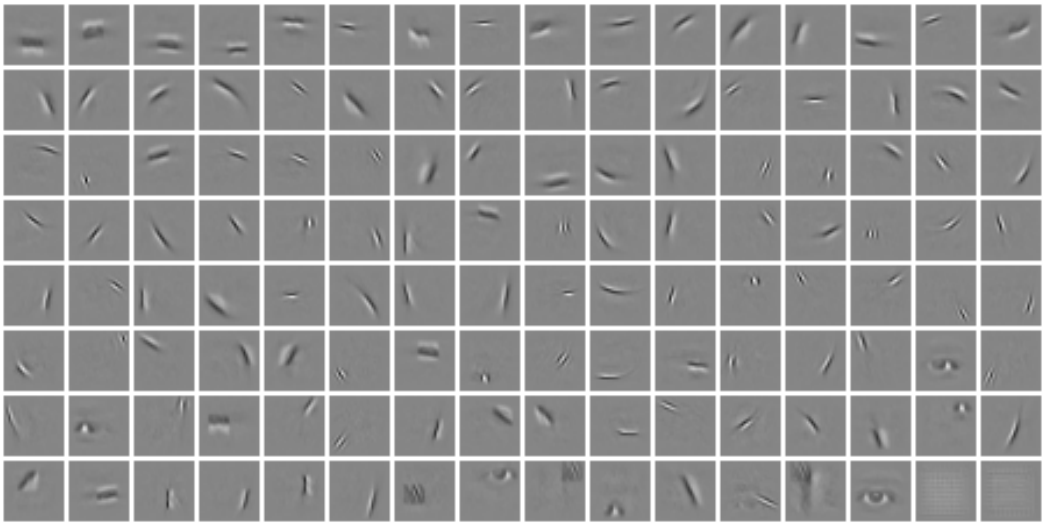


Figure 9: SDPC (**a** & **c**) and Hi-La (**b** & **d**) effective dictionaries obtained on the AT&T database, with sparsity parameter: (λ_1=0.5,λ_2=1). All other parameters are those described in Table 1 for the AT&T database.Atoms are sorted by activation probabilities in a descending order. The visualization shown here is the projection of the dictionaries into the input space. First layer effective dictionaries have a size of 9×9 px (**a** & **b**) and second layer RFs have a size of 26×26 (**c** & **d**) px respectively.

E SDPC AND HI-LA RFS ON CFD

(a) SDPC first layer RFs

(b) Hi-La first layer RFs

(c) SDPC second layer RFs

(d) Hi-La second layer RFs

Figure 10: SDPC (**a** & **c**) and Hi-La (**b** & **d**) effective dictionaries obtained on the CFD database, with sparsity parameter: (λ_1=0.3,λ_2=1.8). All other parameters are those described in Table 1 for the CFD database. Atoms are sorted by activation probabilities in a descending order. The visualization shown here is the projection of the dictionaries into the input space. First layer effective dictionaries have a size of 9×9 px (**a** & **b**) and second layer RFs have a size of 33×33 (**c** & **d**) px respectively.

F SDPC AND HI-LA RFS ON MNIST

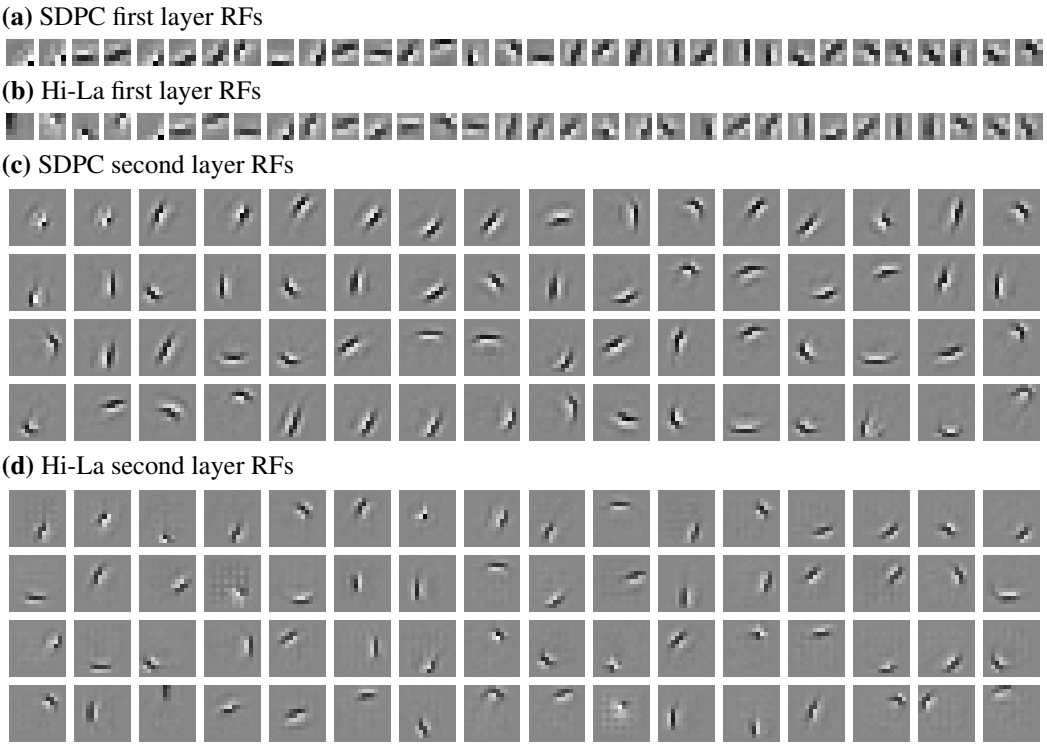


Figure 11: SDPC (**a** & **c**) and Hi-La (**b** & **d**) effective dictionaries obtained on the MNIST database, with sparsity parameter: (λ_1=0.2,λ_2=0.3). Atoms are sorted by activation probabilities in a descending order. All other parameters are those described in Table 1 for the MNIST database. The visualization shown here is the projection of the dictionaries into the input space. First layer effective dictionaries have a size of 5×5 px (**a** & **b**) and second layer RFs have a size of 14×14 (**c** & **d**) px respectively.