SCALABLE PRIVATE LEARNING WITH PATE

Anonymous authors
Paper under double-blind review

ABSTRACT

The rapid adoption of machine learning has increased concerns about the privacy implications of machine learning models trained on sensitive data, such as medical records or other personal information. To address those concerns, one promising approach is *Private Aggregation of Teacher Ensembles*, or PATE, which transfers to a "student" model the knowledge of an ensemble of "teacher" models, with intuitive privacy provided by training teachers on disjoint data and strong privacy guaranteed by noisy aggregation of teachers' answers. However, PATE has so far been evaluated only on simple classification tasks like MNIST, leaving unclear its utility when applied to larger-scale learning tasks and real-world datasets.

In this work, we show how PATE can scale to learning tasks with large numbers of output classes and uncurated, imbalanced training data with errors. For this, we introduce new noisy aggregation mechanisms for teacher ensembles that are more selective and add less noise, and prove their tighter differential-privacy guarantees. Our new mechanisms build on two insights: the chance of teacher consensus is increased by using more concentrated noise and, lacking consensus, no answer need be given to a student. The consensus answers used are more likely to be correct, offer better intuitive privacy, and incur lower-differential privacy cost. Our evaluation show our mechanisms to improve on the original PATE on all measures, and scale to larger tasks with both high utility and very strong privacy ($\varepsilon \approx 1.0$).

1 INTRODUCTION

Many attractive applications of modern machine-learning techniques involve training models using highly sensitive data. For example, models trained on people's personal messages or detailed medical information can offer invaluable insights into real-world language usage or the diagnoses and treatment of human diseases (McMahan et al., 2017; Liu et al., 2017). A key challenge in such applications is to prevent models from revealing inappropriate details of the sensitive data—a non-trivial task, since models are known to implicitly memorize such details during training and also to inadvertently reveal them during inference (Zhang et al., 2017; Shokri et al., 2016).

Recently, two promising, new model-training approaches have offered the hope that practical, high-utility machine learning may be compatible with strong privacy-protection guarantees for sensitive training data (Abadi et al., 2017). This paper revisits one of these approaches, *Private Aggregation of Teacher Ensembles*, or PATE (Papernot et al., 2017), and develops techniques that improve its scalability and practical applicability. PATE has the advantage of being able to learn from the aggregated consensus of separate "teacher" models trained on disjoint data, in a manner that both provides intuitive privacy guarantees and is agnostic to the underlying machine-learning techniques (cf. the other approach of differentially-private stochastic gradient descent (Abadi et al., 2016)). The PATE approach assumes multiple teacher are trained on disjoint sensitive data (e.g., different users' data), and uses the teachers' aggregate, consensus answers in a black-box fashion to supervise the training of a "student" model. By publishing only the student model (keeping the teachers private) and by adding carefully-calibrated Laplacian noise to the aggregate answers used to train the student, the original PATE work showed how to establish rigorous (ε, δ) differential-privacy guarantees (Papernot et al., 2017)—a gold standard of privacy (Dwork et al., 2006). However, to date, PATE has been applied to only simple tasks, like MNIST, without any realistic, larger-scale evaluation.

The techniques presented in this paper allow PATE to be applied on a larger scale to build more accurate models, in a manner that improves both on PATE's intuitive privacy-protection due to the teachers' independent consensus as well as its differential-privacy guarantees. As shown in our experiments, the result is a gain in privacy, utility, and practicality—an uncommon joint improvement.

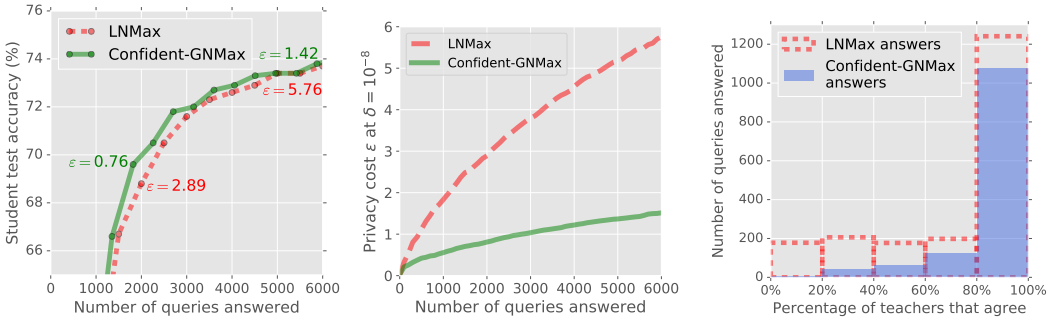

Figure 1: Our contributions are techniques (Confident-GNMax) that improve on the original PATE (LNMax) on all measures. *Left:* Accuracy is higher throughout training, despite greatly improved privacy (more in Table 1). *Middle:* The ε differential-privacy bound on privacy cost is quartered, at least (more in Figure 4). *Right:* Intuitive privacy is also improved, since students are trained on answers with a much stronger consensus among the teachers (more in Figure 4). These are results for a character-recognition task, using the most favorable LNMax parameters for a fair comparison.

The primary technical contributions of this paper are new mechanisms for aggregating teachers' answers that are more selective and add less noise. On all measures, our techniques improve on the original PATE mechanisms when evaluated on the same tasks using the same datasets, as described in Section 4. Furthermore, we evaluate both variants of PATE on a new, large-scale character recognition task with 150 output classes, inspired by MNIST. The results show that PATE can be successfully utilized even to uncurated datasets—with significant class imbalance as well as erroneous class labels—and that our new aggregation mechanisms improve both privacy and model accuracy.

To be more selective, our new mechanisms leverage some pleasant synergies between privacy and utility in PATE aggregation. For example, when teachers disagree, and there is no real consensus, the privacy cost is much higher; however, since such disagreement also implies that the teachers may give no good answer, the answer may simply be omitted. Similarly, teachers may avoid giving an answer where the student already is confidently predicting the right answer.

To add less noise, our new PATE aggregation mechanisms sample Gaussian noise, since the tails of that distribution diminish far more rapidly than those of the Laplacian noise used in the original PATE work. This reduction greatly increases the chance that the noisy aggregation of teachers' votes results in a correct consensus answer, which is especially important when PATE is scaled to learning tasks with large numbers of output classes. However, changing the sampled noise requires redoing the entire PATE privacy analysis, from scratch (see Section 3 and details in Appendix C).

Finally, of independent interest are the details of our evaluation extending that of the original PATE work. In particular, we find that the virtual adversarial training (VAT) of Miyato et al. (2017) is a good basis for semi-supervised learning on tasks with many classes, outperforming the improved GANs by Salimans et al. (2016) used in the original PATE work. Furthermore, we explain how to tune the PATE approach to achieve very strong privacy ($\varepsilon \approx 1.0$) along with high utility, for our real-world character recognition learning task.

This paper is structured simply, with Section 2 giving an overview of PATE, Section 3 describing our improved aggregation mechanisms, Section 4 detailing our experimental evaluation, and Section 5 offering conclusions, with further details, discussion, and proofs in the Appendices.

2 Overview

We introduce essential components of our approach towards a generic and flexible framework for machine learning with provable privacy guarantees for training data.

2.1 THE PATE FRAMEWORK

The PATE framework (see Figure 5 in Appendix A) due to Papernot et al. (2017) comprises three principal components: (1) an ensemble of *teachers*, (2) an *aggregation mechanism*, and (3) a *student*.

Teachers. Each teacher is a model trained independently on a subset of the data whose privacy one wishes to protect. The data is partitioned to ensure no pair of teachers will have trained on overlapping data. Any learning technique suitable for the data can be used for any teacher.

Aggregation Mechanism. The role of this mechanism is to be the intermediary between the student, who is never directly exposed to sensitive data, and the teachers, each of which has unfettered access to a data partition. This step enforces privacy by tracking the privacy budget spent (see below).

Student. The student is another model with access to public unlabeled data. During learning, the student queries the aggregator with public data and receives private labels. Yet, each of these labels increases the exposure of the teachers' sensitive training data. Hence, the student and aggregator are designed to minimize the querying needed to transfer knowledge from the teachers to the student.

2.2 DIFFERENTIAL PRIVACY AND RÉNYI DIFFERENTIAL PRIVACY FOR PATE

Differential privacy (Dwork et al., 2006) is a gold standard of privacy for statistical algorithms. It requires that the sensitivity of the distribution of an algorithm's output to small perturbations of its input be limited. The following variant of the definition captures this intuition formally:

Definition 1. *A randomized mechanism $\mathcal{M}$ with domain $\mathcal{D}$ and range $\mathcal{R}$ satisfies (ε, δ)-differential privacy if for any two adjacent inputs $D, D' \in \mathcal{D}$ and for any subset of outputs $S \subseteq \mathcal{R}$ it holds that:*

$$\mathbf{Pr}[\mathcal{M}(D) \in S] \leq e^{\varepsilon} \cdot \mathbf{Pr}[\mathcal{M}(D') \in S] + \delta. \tag{1}$$

For our application of differential privacy to ML, adjacent inputs are defined as two datasets that only differ by one training example and the randomized mechanism $\mathcal{M}$ would be the model training algorithm. Composition theorems for differential privacy (Dwork & Roth, 2014) lead to the following natural interpretation of the parameters from its definition: ε is an upper bound on the loss of privacy that accumulates over time and with each query to the dataset ("the privacy budget"), and δ is the probability with which this guarantee may not hold.

To better facilitate tracking privacy budget across multiple queries, we use Rényi Differential Privacy (RDP) recently introduced by Mironov (2017). RDP is closely related to (zero)-Concentrated Differential Privacy (Dwork & Rothblum, 2016; Bun & Steinke, 2016) and the moments accountant (Abadi et al., 2016)—also used by Papernot et al. (2017) in the original PATE proposal.

2.3 PATE AGGREGATION MECHANISMS

The aggregation step is a crucial component of PATE. It enables knowledge transfer from the teachers to the student while enforcing privacy. We improve the LNMax mechanism used by Papernot et al. (2017) which adds Laplace noise to teacher votes and outputs the class with the highest votes.

First, we add Gaussian noise with an accompanying privacy analysis in the RDP framework. This modification effectively reduces the noise needed to achieve the same privacy cost per student query.

Second, the aggregation mechanism is now *selective*: teacher votes are analyzed to decide which student queries are *worth* answering. This takes into account both the privacy cost of each query and its payout in improving the student's utility. Surprisingly, our analysis shows that these two metrics are not at odds and in fact align with each other: the privacy cost is the smallest when teachers agree, and when teachers agree, the label is more likely to be correct thus being more useful to the student.

Third, we propose and study an *interactive* mechanism that takes into account not only teacher votes on a queried example but possible student predictions on that query. Now, queries worth answering are those where the teachers agree on a class but the student is not confident in its prediction on that class. This third modification aligns the two metrics discussed above even further: queries where the student already agrees with the consensus of teachers are not worth expending our privacy budget on, but queries where the student is less confident are useful and answered at a small privacy cost.

2.4 DATA-DEPENDENT PRIVACY IN PATE

A direct privacy analysis of the aggregation mechanism, for reasonable values of the noise parameter, allows answering only few queries before the privacy cost becomes prohibitive. The original PATE proposal used a data-dependent analysis, exploiting the fact that when the teachers have large agreement, the privacy cost is usually much smaller than the data-dependent bound would suggest.

In our work, we perform a data-dependent privacy analysis of the aggregation mechanism with Gaussian noise. This change of noise distribution turns out be technically much more challenging than the Laplace noise case and we defer the details to Appendix C. This increased complexity of the analysis however does not make the algorithm any more complicated and thus allows us to improve the privacy-utility tradeoff.

An additional challenge with data-dependent privacy analyses arises from the fact that the privacy cost itself is now a function of the private data. Further, the data-dependent bound on the privacy cost has large global sensitivity (a metric used in differential privacy to calibrate the noise injected) and is therefore difficult to sanitize. To remedy this, we use the smooth-sensitivity framework, detailed in Section C.3.

3 IMPROVED AGGREGATION MECHANISMS FOR PATE

Guarantees provided by PATE stem from the design and analysis of the aggregation step. Here, we detail our improvements to the mechanism used by Papernot et al. (2017). As outlined in Section 2.3, we first replace the Laplace noise added to teacher votes with Gaussian noise, adapting the data-dependent privacy analysis. Next, we describe the Confident and Interactive Aggregators that select queries worth answering in a privacy-preserving way: the privacy budget is shared between the query selection and answer computation. The aggregators use different heuristics to select queries: the former does not take into account student predictions, while the latter does.

3.1 THE GNMAX AGGREGATOR AND ITS PRIVACY GUARANTEE

This section uses the following notation. For a sample x and classes 1 to m, let $f_j(x) \in [m]$ denote the j-th teacher model's prediction on x and $n_i(x)$ denote the vote count for the i-th class (i.e., $n_i(x) = |\{j : f_j(x) = i\}|$). We define a Gaussian NoisyMax (GNMax) aggregation mechanism as:

$$\mathcal{A}_\sigma(x) := \operatorname*{argmax}_i \left\{ n_i(x) + \mathcal{N}(0, \sigma^2) \right\},$$

where $\mathcal{N}(\mu, \sigma^2)$ denotes the Gaussian distribution with mean μ and variance σ^2. The aggregator outputs the class with noisy plurality after adding Gaussian noise to each vote count. In what follow, *plurality* more generally refers to the highest number of teacher votes assigned among the classes.

The Gaussian distribution is more concentrated than the Laplace distribution used by Papernot et al. (2017). This concentration directly improves the aggregation's utility when the number of classes m is large. The GNMax mechanism satisfies $(\lambda, \lambda/\sigma^2)$-RDP (see Lemma 8 in Appendix C) which holds for all inputs and all $\lambda \geq 1$. A straightforward application of composition theorems leads to loose privacy bounds. As an example, the standard advanced composition theorem applied to experiments in rows 5 and 6 in Table 1 would give us $\varepsilon = 8.42$ and $\varepsilon = 10.14$ resp. at $\delta = 10^{-8}$ for the Glyph dataset.

To refine these, we work out a careful *data-dependent* analysis that yields ε's close to 1 for the same δ. The following theorem translates standard RDP guarantees for higher orders into a data-dependent RDP guarantee for a smaller order λ. Its precise statement and proof appear in Appendix C. We use it in conjunction with Lemma 3 to bound the privacy cost of each query to the GNMax algorithm as a function of $\tilde{q}$, the probability that the most common answer will not be output by the mechanism.

Theorem 2 (informal). *Let $\mathcal{M}$ be a randomized algorithm with (μ_1, ε_1)-RDP and (μ_2, ε_2)-RDP guarantees and suppose that given a dataset D, there exists* a likely *outcome i^* such that $\mathbf{Pr}\left[\mathcal{M}(D) \neq i^*\right] \leq \tilde{q}$. Then the* data-dependent *Rényi differential privacy for $\mathcal{M}$ of order $\lambda \leq \mu_1, \mu_2$ at D is bounded by a function of $\tilde{q}$, μ_1, ε_1, μ_2, ε_2, which approaches 0 as $\tilde{q} \to 0$.*

The new bound improves on the data-independent privacy for λ as long as the distribution of the algorithm's output *on that input* has a strong peak (i.e., $\tilde{q} \ll 1$). Values of $\tilde{q}$ close to 1 could result in a looser bound. Therefore, in practice we take the minimum between this bound and λ/σ^2 (the data-independent one). The theorem generalizes Theorem 3 from Papernot et al. (2017), where it was shown for a mechanism satisfying ε-differential privacy (i.e., $\mu_1 = \mu_2 = \infty$ and $\varepsilon_1 = \varepsilon_2$).

The final step in our analysis uses the following lemma to bound the probability $\tilde{q}$ when i^* corresponds to the class with the true plurality of teacher votes.

Lemma 3. *For any $i^* \in [m]$, we have $\mathbf{Pr}\left[\mathcal{A}_\sigma(D) \neq i^*\right] \leq \frac{1}{2}\sum_{i \neq i^*} \operatorname{erfc}\left(\frac{n_{i^*} - n_i}{2\sigma}\right)$, where* erfc *is the complementary error function.*

In Appendix C, we detail how these results translate to privacy bounds. In short, for each query to the GNMax aggregator, given teacher votes n_i and the class i^* with maximal support, Lemma 3 gives us the value of $\tilde{q}$ to use in Theorem 2. We optimize over μ_1 and μ_2 to get a data-dependent RDP guarantee for any order λ. Finally, we use composition properties of RDP to analyze a sequence of queries, and translate the RDP bound back to an (ε, δ)-DP bound.

Expensive queries. This data-dependent privacy analysis leads us to the concept of an *expensive* query in terms of its privacy cost. When teacher votes largely disagree, some $n_{i^*} - n_i$ values may be small leading to a large value for $\tilde{q}$: i.e., the lack of consensus amongst teachers indicates that the aggregator is likely to output the wrong label. Thus expensive queries from a privacy perspective are often bad for training too. Conversely, queries with strong consensus enable tight privacy bounds. This synergy motivates the aggregation mechanisms discussed in the following sections: they evaluate the strength of the consensus before answering a query.

3.2 The Confident-GNMax Aggregator

In this section, we propose a refinement of the GNMax aggregator that enables us to filter out queries for which teachers do not have a sufficiently strong consensus. This filtering enables the teachers to avoid answering expensive queries. We also take note to do this selection step itself in a private manner.

The proposed *Confident Aggregator* is described in Algorithm 1. To select queries with overwhelming consensus, the algorithm checks if the plurality vote crosses a threshold T. To enforce privacy in this step, the comparison is done after adding Gaussian noise with a large deviation σ_1. Then, for queries that pass this noisy threshold check, the aggregator proceeds with the usual GNMax mechanism with a different deviation σ_2. For queries that do not pass the noisy threshold check, the aggregator simply returns $\perp$ and the student discards this example in its training.

In practice, we often choose significantly higher values for σ_1 compared to σ_2. The reason for doing is that we pay the cost of the noisy threshold check *always*, and without the benefit of knowing that the consensus is strong. We pick T so that queries where the plurality gets less than half the votes (often very expensive) are unlikely to pass the threshold after adding noise, but we still have a high enough yield amongst the queries with a strong consensus. This tradeoff leads us to look for T's between $0.6\times$ to $0.8\times$ the number of teachers.

The privacy cost of this aggregator is intuitive: we pay for the threshold check for every query, and for the GNMax step only for queries that pass the check. In the work of Papernot et al. (2017), the mechanism paid a privacy cost for every query, expensive or otherwise. In comparison, the Confident Aggregator expends a much smaller privacy cost to check against the threshold, and by answering a significantly smaller fraction of expensive queries, it expends a lower privacy cost overall.

3.3 The Interactive-GNMax Aggregator

While the Confident Aggregator exclude expensive queries, it ignores the possibility that the student might receive labels that contribute little to learning, and in turn to its utility. By incorporating the student's current predictions for its public training data, we design an *Interactive Aggregator* that discards queries where the student already confidently predicts the same label as the teachers.

Given a set of queries, the Interactive Aggregator (Algorithm 2) selects those answered by comparing student predictions to teacher votes for each class. Similar to Step 1 in the Confident Aggregator,

Algorithm 1 – Confident GNMax Aggregator: given a query, consensus among teachers is first estimated in a privacy-preserving way to then only reveal confident teacher predictions.

Input: input x, threshold T, noise parameters σ_1 and σ_2

if $\max_i\{n_j(x)\} + \mathcal{N}(0, \sigma_1^2) \geq T$ **then** ▷ Privately check for consensus
  return $\operatorname{argmax}_j \left\{n_j(x) + \mathcal{N}(0, \sigma_2^2)\right\}$ ▷ Run the usual max-of-Gaussian
else
  return $\perp$
end if

Algorithm 2 – Interactive GNMax Aggregator: the protocol first compares student predictions to the teacher votes in a privacy-preserving way to then either (a) reinforce the student prediction for the given query or (b) provide the student with a new label predicted by the teachers.

Input: input x, confidence γ, threshold T, noise parameters σ_1 and σ_2, total number of teachers M

Ask the student to provide prediction scores $\mathbf{p}(x)$
if $\max_j\{n_j(x) - Mp_j(x)\} + \mathcal{N}(0, \sigma_1^2) \geq T$ **then** ▷ Student does not agree with teachers
  return $\operatorname{argmax}_j\{n_j(x) + \mathcal{N}(0, \sigma_2^2)\}$ ▷ Teachers provide new label
else if $\max\{p_i(x)\} > \gamma$ **then** ▷ Student agrees with teachers and is confident
  return $\arg\max_j p_j(x)$ ▷ Reinforce student's prediction
else
  return $\perp$ ▷ No output given for this label
end if

queries where the plurality of these noised differences crosses a threshold are answered with GNMax. This noisy threshold suffices to enforce privacy of the first step because student predictions can be considered public information (the student is trained in a differentially private manner).

For queries that fail this check, the mechanism reinforces the predicted student label if the student is confident enough and does this without looking at teacher votes again. This limited form of supervision comes at a small privacy cost. Moreover, the order of the checks ensures that a student falsely confident in its predictions on a query is not accidentally reinforced if it disagrees with the teacher consensus. The privacy accounting is identical to the Confident Aggregator except in considering the difference between teachers and the student instead of only the teachers votes.

In practice, the Confident Aggregator can be used to start training a student when it can make no meaningful predictions and training can be finished off with the Interactive Aggregator after the student gains some proficiency.

4 Experimental Evaluation

Our goal is first to show that the improved aggregators introduced in Section 3 enable the application of PATE to uncurated data, thus departing from previous results on tasks with balanced and well-separated classes. We experiment with the Glyph dataset described below to address two aspects left open by Papernot et al. (2017): (a) the performance of PATE on a task with a larger number of classes (the framework was only evaluated on datasets with at most 10 classes) and (b) the privacy-utility trade-offs offered by PATE on data that is class imbalanced and partly mislabeled. In Section 4.2, we evaluate the improvements given by the GNMax aggregator over its Laplace counterpart (LNMax) and demonstrate the necessity of the Gaussian mechanism for uncurated tasks.

In Section 4.3, we then evaluate the performance of PATE with both the Confident and Interactive Aggregators on all datasets used to benchmark the original PATE framework, in addition to Glyph. With the right teacher and student training, the two mechanisms from Section 3 achieve high accuracy with very tight privacy bounds. Not answering queries for which teacher consensus is too low (Confident-GNMax) or the student's predictions already agree with teacher votes (Interactive-GNMax) better aligns utility and privacy: queries are answered at a significantly reduced cost.

4.1 EXPERIMENTAL SETUP

MNIST, SVHN, UCI Adult, UCI Diabetes. We evaluate with two computer vision tasks (MNIST and Street View House Numbers (Netzer et al., 2011)), census data from the UCI Adult dataset Kohavi (1996), and medical records from the UCI Diabetes Strack et al. (2014) dataset. This enables a comparative analysis of the utility-privacy trade-off achieved with our Confident-GNMax aggregator and the LNMax originally used in PATE. We replicate the experimental setup and results found in Papernot et al. (2017) thanks in part to the code and teacher votes made available online.

A detailed description of the experimental setup can be found in Papernot et al. (2017); we provide here only a brief overview. For MNIST and SVHN, teachers are convolutional networks trained on partitions of the training set. For Adult and Diabetes, each teacher is a random forest. The test set is split in two halves: the first is used as unlabeled inputs to simulate the student's public data and the second is used as a hold out to evaluate test performance. The MNIST and SVHN students are convolutional networks trained using semi-supervised learning with GANs à la Salimans et al. (2016). The students for the Adult and Diabetes datasets are fully supervised random forests.

Glyph. This optical character recognition task has *an order of magnitude more classes* than all previous applications of PATE. The Glyph dataset also possesses many characteristics shared by real-world tasks: e.g., it is imbalanced and some inputs are mislabeled. Each input is a 28×28 grayscale image containing a single glyph generated synthetically from a collection of over 500K computer fonts.[1] Samples representative of the difficulties raised by the data are depicted in Figure 2. The task is to classify inputs as one of the 150 Unicode symbols used to generate them.

This set of 150 classes results from pre-processing efforts. We discarded additional classes that had few samples; some classes had at least 50 times fewer inputs than the most popular classes, and these were almost exclusively incorrectly labeled inputs. We also merged classes that were too ambiguous for even a human to differentiate them. Nevertheless, a manual inspection of samples grouped by classes—favorably to the human observer—led to the conservative estimate that some classes remain 5 times more frequent, and mislabeled inputs represent at least 10% of the data.

To simulate the availability of private and public data (see Appendix A), we split data originally marked as the training set (about 65M points) into partitions given to the teachers. Each teacher is a ResNet (He et al., 2016) made of 32 leaky ReLU layers.[2] We train on batches of 100 inputs for 40K steps with the Momentum algorithm. The learning rate, initially set to 0.1, is decayed after 10K steps to 0.01 and again after 20K steps to 0.001. These parameters were found with a grid search.

We split holdout data in two subsets of 100K and 400K samples: the first acts as public data to train the student and the second as its testing data. The student architecture is a convolutional network learned in a semi-supervised fashion with virtual adversarial training (VAT) from Miyato et al. (2017). Using unlabeled data, we show how VAT can regularize the student by making predictions constant in *adversarial*[3] directions. Indeed, we found that GANs did not yield as much utility for Glyph as for MNIST or SVHN. We train with Adam for 400 epochs and a learning rate of $6 \cdot 10^{-5}$.

4.2 COMPARING THE LNMAX AND GNMAX MECHANISMS

Section 3.1 introduces the GNMax mechanism and the accompanying privacy analysis. With a Gaussian distribution, whose tail diminishes more rapidly than the Laplace distribution, we expect better utility when using the new mechanism (albeit with a more involved privacy analysis).

To study the trade-off between privacy and accuracy with the two mechanisms, we run experiments training several ensembles of M teachers for $M \in \{100, 500, 1000, 5000\}$ on the Glyph data. Recall that 65 million training inputs are partitioned and distributed among the M teachers with each teacher receiving between 650K and 13K inputs for the values of M above. The test data is used to query the teacher ensemble and the resulting labels (after the LNMax and GNMax mechanisms) are

[1] Glyph data is not public but similar data is available publicly as part of the notMNIST dataset.

[2] It is available in the TensorFlow models library at `https://github.com/tensorflow/models`.

[3] In this context, the adversarial component refers to the phenomenon commonly referred to as adversarial examples (Biggio et al., 2013; Szegedy et al., 2014) and not to the adversarial training approach taken in GANs.

compared with the ground truth labels provided in the dataset. This predictive performance of the teachers is essential to good student training with accurate labels and is a useful proxy for utility.

For each mechanism, we compute (ε, δ)-differential privacy guarantees. As is common in literature, for a dataset on the order of 10^8 samples, we choose $\delta = 10^{-8}$ and denote the corresponding ε as the privacy cost. The total ε is calculated on a subset of 4,000 queries, which is representative of the number of labels needed by a student for accurate training (see Section 4.3). We visualize in Figure 3 the effect of the noise distribution (left) and the number of teachers (right) on the tradeoff between privacy costs and label accuracy.

Observations. On the left of Figure 1,we compare our GNMax aggregator to the LNMax aggregator used by the original PATE proposal, on an ensemble of 1000 teachers and for varying noise scales σ. At fixed test accuracy, the GNMax algorithm consistently outperforms the LNMax mechanism in terms of privacy cost. To explain this improved performance, recall notation from Section 3.1. For both mechanisms, the data dependent privacy cost scales linearly with $\tilde{q}$—the likelihood of an answer other than the true plurality. The value of $\tilde{q}$ falls of as $\exp\left(-x^2\right)$ for GNMax and $\exp\left(-x\right)$ for LNMax, where x is the ratio $(n_{i^*} - n_i)/\sigma$. Thus, when $n_{i^*} - n_i$ is (say) 4σ, LNMax would have $\tilde{q} \approx e^{-4} = 0.018...$, whereas GNMax would have $\tilde{q} \approx e^{-16} \approx 10^{-7}$, thereby leading to a much higher likelihood of returning the true plurality. Moreover, this reduced $\tilde{q}$ translates to a smaller privacy cost for a given σ leading to a better utility-privacy trade-off.

As long as each teacher has sufficient data to learn a good-enough model, increasing the number M of teachers improves the trade-off—as illustrated on the right of Figure 3 with GNMax. The larger ensembles lower the privacy cost of answering queries by tolerating larger σ's. Combining the two observations made in this Figure, for a fixed label accuracy, we lower privacy costs by switching to the GNMax aggregator and training a larger number M of teachers.

4.3 Student Training with the GNMax Aggregation Mechanisms

As outlined in Section 2, we train a student on public data labeled by the aggregation mechanisms. We take advantage of PATE's flexibility and apply the technique that performs best on each dataset: semi-supervised learning with Generative Adversarial Networks (Salimans et al., 2016) for MNIST and SVHN, Virtual Adversarial Training (Miyato et al., 2017) for Glyph, and fully-supervised random forests for Adult and Diabetes. In addition to evaluating the total privacy cost associated with training the student model, we compare its utility to a non-private baseline obtained by training on the sensitive data (used to train teachers in PATE): we use the baselines of 99.2%, 92.8%, 85.0%, and 93.8% reported by Papernot et al. (2017) respectively for MNIST, SVHN, Adult, and Diabetes, and we measure a baseline of 82.2% for Glyph. We compute (ε, δ)-privacy bounds and denote the privacy cost as the ε value at a value of δ set accordingly to number of training samples.

Confident-GNMax Aggregator. Given a pool of 500 to 12,000 samples to learn from (depending on the dataset), the student submits queries to the teacher ensemble running the Confident-GNMax aggregator from Section 3.2. A grid search over a range of plausible values for parameters T, σ_1 and σ_2 yielded the values reported in Table 1, illustrating the tradeoff between utility and privacy achieved. We additionally measure the number of queries selected by the teachers to be answered and compare student utility to a non-private baseline.

The Confident-GNMax aggregator outperforms LNMax for the four datasets considered in the original PATE proposal: it reduces the privacy cost ε, increases student accuracy, or both simultaneously. On the uncurated Glyph data, despite the imbalance of classes and mislabeled data (as evidenced by the 82.2% baseline), the Confident Aggregator achieves 73.5% accuracy with a privacy cost of just $\varepsilon = 1.02$. Roughly 1,300 out of 12,000 queries made are not answered, indicating that several costly queries were successfully avoided. This selectivity is analyzed in more details in Section 4.4.

Interactive-GNMax Aggregator. On Glyph, we evaluate the utility and privacy of an interactive training routine that proceeds in *two rounds*. Round one runs student training with a Confident Aggregator. A grid search targeting the best privacy for roughly 3,400 answered queries (out of 6,000)—sufficient to bootstrap a student—led us to setting $(T$=3500$, \sigma_1$=1500$, \sigma_2$=100$)$ and a privacy cost of $\varepsilon \approx 0.65$.

Figure 2: **Some example inputs from the Glyph dataset along with the class they are labeled as.** Note the ambiguity (between the comma and apostrophe) and the mislabeled input.

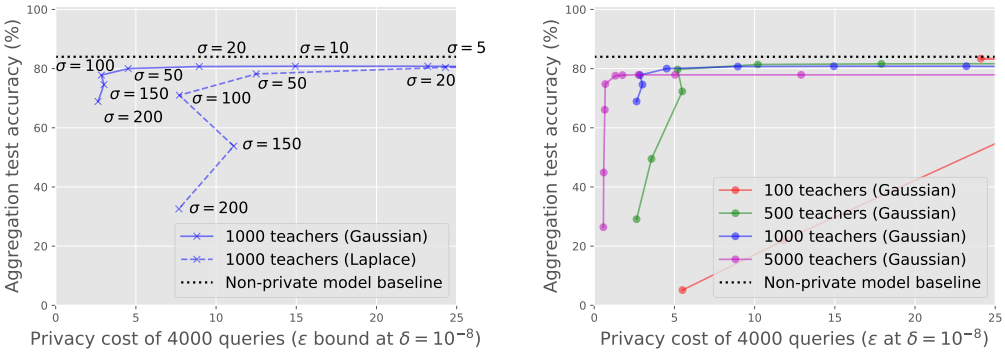

Figure 3: **Trade-off between utility and privacy for the LNMax and GNMax aggregators on Glyph:** effect of the noise distribution (left) and size of the teacher ensemble (right). The LNMax aggregator uses a Laplace distribution and GNMax a Gaussian. Smaller values of the privacy cost ε (often obtained by increasing the noise scale σ—see Section 3) and higher accuracy are better.

Dataset	Aggregator	Queries answered	Privacy bound ε	Accuracy Student	Accuracy Baseline
MNIST	LNMax (Papernot et al., 2017)	100	2.04	98.0%	99.2%
	LNMax (Papernot et al., 2017)	1,000	8.03	98.1%	
	Confident GNMax (T=200, σ_1=150, σ_2=40)	286	**1.97**	**98.5%**	
SVHN	LNMax (Papernot et al., 2017)	500	5.04	82.7%	92.8%
	LNMax (Papernot et al., 2017)	1,000	8.19	90.7%	
	Confident GNMax (T=300, σ_1=200, σ_2=40)	3,098	**4.96**	**91.6%**	
Adult	LNMax (Papernot et al., 2017)	500	2.66	83.0%	85.0%
	Confident GNMax (T=300, σ_1=200, σ_2=40)	524	**1.90**	**83.7%**	
Diabetes	LNMax (Papernot et al., 2017)	500	1.44	**93.9%**	93.8%
	Confident GNMax (T=200, σ_1=100, σ_2=15)	494	**0.7**	**93.9%**	
Glyph	LNMax	4,000	4.3	72.4%	82.2%
	Confident GNMax (T=1000, σ_1=500, σ_2=100)	10,762	2.03	**75.5%**	
	Interactive GNMax (T=3500, σ_1=2000, σ_2=200)	10,422	**0.99**	73.2%	

Table 1: **Utility and privacy of the students.** The Confident and Interactive GNMax aggregators introduced in Section 3 offer better trade-offs between privacy (characterized by the value of the bound ε) and utility (the accuracy of the student compared to a non-private baseline) than the LNMax aggregator used by the original PATE proposal on all datasets we evaluated with. For MNIST, Adult, Diabetes, and SVHN, we use the labels of ensembles of 250 teachers published by Papernot et al. (2017) and set $\delta = 10^{-5}$ to compute values of ε (to the exception of SVHN where $\delta = 10^{-6}$). All Glyph results use an ensemble of 5000 teachers and ε is computed for $\delta = 10^{-8}$.

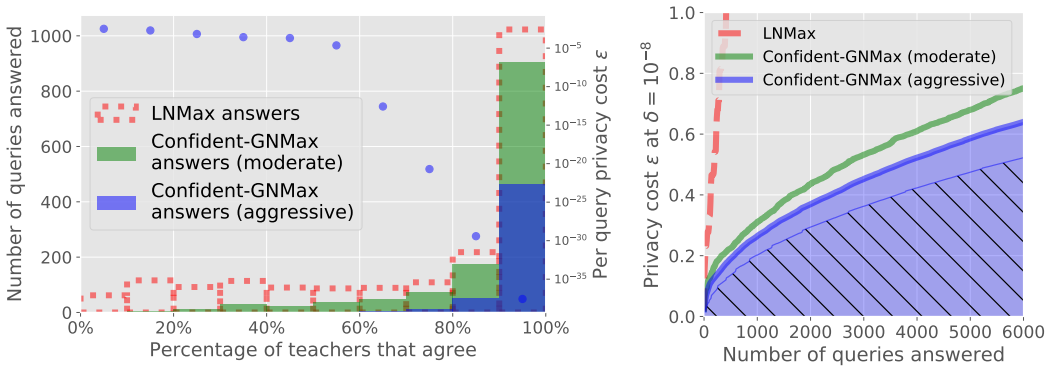

Figure 4: **Effects of the noisy threshold checking:** *Left:* The number of queries answered by LNMax, Confident-GNMax moderate (T=3500, σ_1=1500), and Confident-GNMax aggressive (T=5000, σ_1=1500). The right axis (in log scale) shows the expected cost of answering a single query in each bin (via GNMax, σ_2=100). *Right:* Privacy cost of answering all (LNMax) vs only inexpensive queries (GNMax) for a given number of answered queries. The hatched area under the curve is the cost of selecting queries; the rest is the cost of answering them.

In round two, this student was then trained with 10,000 more queries made with the Interactive-GNMax Aggregator. We computed the resulting (total) privacy cost and utility at an *exemplar* data point through another grid search of plausible parameter values. The result appears in the last row of Table 1. With just over 10,422 answered queries in total at a privacy cost of $\varepsilon = 0.99$, the trained student was able to achieve 73.2% accuracy. Note that this students required fewer answered queries compared to the Confident Aggregator. The best overall cost of student training occurred when the privacy costs for the first and second rounds of training were roughly the same. (The total ε is less than $0.65 \times 2 = 1.3$ due to better composition—via Theorems 6 and 7.)

Baseline. Note that the Glyph student's accuracy remains seven percentage points below the non-private model's accuracy achieved by training on the 65M sensitive inputs. We hypothesize that is due to the uncurated nature of the data considered. Indeed, the class imbalance naturally requires more queries to return labels from the less represented classes. For instance, a student trained on 200K queries is only 77% accurate on test data. In addition, the large fraction of mislabeled inputs are likely to have a large privacy cost: these inputs are sensitive because they are outliers of the distribution, which is reflected by the weak consensus among teachers on these inputs.

4.4 Noisy Threshold Checks and Privacy Costs

Sections 3.1 and 3.2 motivated the need for a noisy threshold checking step before having the teachers answer queries: it prevents most of the privacy budget being consumed by few queries that are expensive and also likely to be incorrectly answered. In Figure 4, we compare the privacy cost ε of answering all queries to only answering confident queries for a fixed number of queries.

We run additional experiments to support the evaluation from Section 4.3. With the votes of 5,000 teachers on the Glyph dataset, we plot in Figure 4 the histogram of the plurality vote counts (n_{i^*} in the notation of Section 3.1) across 25,000 student queries. We compare these values to the vote counts of queries that passed the noisy threshold check for two sets of parameters T and σ_1 in Algorithm 1. Smaller values imply weaker teacher agreements and consequently more expensive queries.

When (T=3500, σ_1=1500) we capture a significant fraction of queries where teachers have a strong consensus (roughly > 4000 votes) while managing to filter out many queries with poor consensus. This *moderate check* ensures that although many queries with plurality votes between 2,500 and 3,500 are answered (i.e., only 50–70% of teachers agree on a label) the expensive ones are most likely discarded. For (T=5000, σ_1=1500), queries with poor consensus are completely culled out. This selectivity comes at the expense of a noticeable drop for queries that might have had a strong consensus and little-to-no privacy cost. Thus, this *aggressive check* answer fewer queries with very

strong privacy guarantees. We reiterate that this threshold checking step itself is done in a private manner. Empirically, in our Interactive Aggregator experiments, we expend about a third to a half of our privacy budget on this step, which still yields a very small cost *per query* across 6,000 queries.

5 CONCLUSIONS

The key insight motivating the addition of a noisy thresholding step to the two aggregation mechanisms proposed in our work is that there is a form of synergy between the privacy and accuracy of labels output by the aggregation: *labels that come at a small privacy cost also happen to be more likely to be correct*. As a consequence, we are able to provide more quality supervision to the student by choosing not to output labels when the consensus among teachers is too low to provide an aggregated prediction at a small cost in privacy. This observation was further confirmed in some of our experiments where we observed that if we trained the student on either private or non-private labels, the former almost always gave better performance than the latter—for a fixed number of labels.

Complimentary with these aggregation mechanisms is the use of a Gaussian (rather than Laplace) distribution to perturb teacher votes. In our experiments with Glyph data, these changes proved essential to preserve the accuracy of the aggregated labels—because of the large number of classes. The analysis presented in Section 3 details the delicate but necessary adaptation of analog results for the Laplace NoisyMax.

As was the case for the original PATE proposal, semi-supervised learning was instrumental to ensure the student achieves strong utility given a limited set of labels from the aggregation mechanism. However, we found that virtual adversarial training outperforms the approach from Salimans et al. (2016) for the Glyph dataset. Thus, future work may continue to investigate other techniques for learning the student in these particular settings with restricted supervision.

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A APPENDIX: BACKGROUND ON PATE

Here, we provide an overview of the PATE framework. To protect the privacy of training data during learning, PATE transfers knowledge from an ensemble of teacher models trained on partitions of the data to a student model. Privacy guarantees may be understood intuitively and expressed rigorously in terms of differential privacy.

Illustrated in Figure 5, the PATE framework consists of three key parts: (1) an ensemble of n teacher models, (2) an aggregation mechanism and (3) a student model.

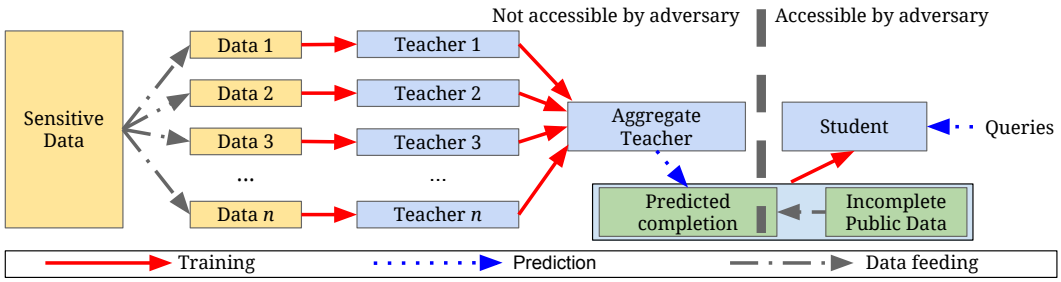

Figure 5: Overview of the approach: (1) an ensemble of teachers is trained on disjoint subsets of the sensitive data, (2) a student model is trained on public data labeled using the ensemble. Figure reproduced from Papernot et al. (2017) with permission.

Teacher models: Training each teacher on a *partition* of the sensitive data produces n different models solving the same task. At inference, teachers independently predict labels. When there is a strong consensus among teachers, the label they almost all agree on does not depend on the model learned by any given teacher. Hence, this collective decision is intuitively private with respect to any given training point—because such a point could have been included only in one of the teachers' training set.

Aggregation mechanism: The aggregation mechanism of the original PATE framework counts votes assigned to each class, adds carefully calibrated Laplacian noise to the resulting vote histogram, and outputs the class with the most noisy votes as the ensemble's prediction. This mechanism is referred to as the max-of-Laplacian mechanism going forward.

For samples x and classes $1, \ldots, m$, let $f_j(x) \in [m]$ denote the j-th teacher model's prediction and n_i denote the vote count for the i-th class (i.e., $n_i \triangleq |f_j(x) = i|$). The output of the mechanism is $\mathcal{A}(x) \triangleq \operatorname{argmax}_i \left(n_i(x) + \operatorname{Lap}\left(1/\gamma\right)\right)$. Through a rigorous analysis of this mechanism, the PATE framework provides a differentially private API: the privacy cost of each aggregated prediction made by the teacher ensemble is known.

Student model: PATE's final step involves the training of a student model by knowledge transfer from the teacher ensemble using access to public—but *unlabeled*—data. To limit the privacy cost of labeling them, queries are only made to the aggregation mechanism for a subset of public data to train the student in a semi-supervised way using a fixed number of queries. The authors note that every additional ensemble prediction increases the privacy cost spent and thus cannot work with unbounded queries. Fixed queries fixes privacy costs as well as diminishes the avlue of attacks analyzing model parameters to recover training data (Zhang et al., 2017). The student only sees public data and privacy-preserving labels.

B APPENDIX: RELATED WORK

Differential privacy is by now the gold standard of privacy. It offers a rigorous framework whose threat model makes few assumptions about the adversary's capabilities, allowing differentially private algorithms to effectively cope against strong adversaries. This is not the case of all privacy definitions, as demonstrated by successful attacks against anonymization techniques (Aggarwal, 2005; Narayanan & Shmatikov, 2008; Bindschaedler et al., 2017).

The first learning algorithms adapted to provide differential privacy with respect to their training data were often linear and convex (Pathak et al., 2010; Chaudhuri et al., 2011; Song et al., 2013; Bassily et al., 2014; Hamm et al., 2016). More recently, successful developments in deep learning called for differentially private stochastic gradient descent algorithms (Abadi et al., 2016), some of which have been tailored to learn in federated (McMahan et al., 2017) settings.

Differentially private selection mechanisms like GNMax are commonly used in hypothesis testing, frequent itemset mining, and as building blocks of more complicated private mechanisms. The most commonly used differentially private selection mechanisms are exponential mechanism (McSherry & Talwar, 2007) and LNMax (Bhaskar et al., 2010). Recent works offer lower bounds on sample complexity of such problem (Steinke & Ullman, 2017; Bafna & Ullman, 2017).

The Confident and Interactive Aggregator proposed in our work use the intuition that selecting samples under certain constraints could result in better training than using samples uniformly at random. In Machine Learning Theory, active learning (Cohn et al., 1994) has been shown to allow learning from fewer labeled examples than the passive case (see e.g. Hanneke (2014)). Similarly, in model stealing (Tramèr et al., 2016), a goal is to learn a model from limited access to a teacher network. There is previous work in differential privacy literature (Hardt & Rothblum, 2010; Roth & Roughgarden, 2010) where the mechanism first *decides* whether or not to answer a query, and then privately answers the queries it chooses to answer using a traditional noise-addition mechanism. In these cases, the sparse vector technique (Dwork & Roth, 2014, Chapter 3.6) helps bound the privacy cost in terms of the number of answered queries. This is in contrast to our work where a constant *fraction* of queries get answered and the sparse vector technique does not seem to help reduce the privacy cost.

C Appendix: Rényi Differential Privacy and Privacy Proofs

C.1 Rényi Differential Privacy

Papernot et al. (2017) note that the natural approach to bound PATE's privacy loss—by first having a bound for the privacy cost of each label queried and then using strong composition (Dwork et al., 2010) to derive the total cost—yields loose privacy guarantees. Instead, their approach uses *data-dependent* privacy analysis. This takes advantage of the fact that when the consensus among the teachers is very strong, the plurality outcome has overwhelming likelihood leading to a very small privacy cost whenever the consensus occurs. To capture this effect quantitatively, Papernot et al. (2017) rely on the *moments accountant*, introduced by Abadi et al. (2016) and building on previous work (Bun & Steinke, 2016; Dwork & Rothblum, 2016).

In this section, we restate the privacy analysis framework of PATE in the language of Rényi Differential Privacy or RDP (Mironov, 2017). RDP generalizes pure differential privacy ($\delta = 0$) and is closely related to the moments accountant. We choose to use RDP as a more natural analysis framework when dealing with our mechanisms that use Gaussian noise. Defined below, the RDP of a mechanism is stated in terms of the Rényi divergence.

Definition 4 (Rényi Divergence)**.** *The Rényi divergence of order λ between two distributions P and Q is defined as:*

$$D_\lambda(P\|Q) \triangleq \frac{1}{\lambda-1}\log \mathbb{E}_{x\sim Q}\left[(P(x)/Q(x))^\lambda\right] = \frac{1}{\lambda-1}\log \mathbb{E}_{x\sim P}\left[(P(x)/Q(x))^{\lambda-1}\right].$$

Definition 5 (Rényi Differential Privacy (RDP))**.** *A randomized mechanism $\mathcal{M}$ is said to guarantee (λ,ε)-RDP with $\lambda \geq 1$ if for any neighboring datasets D and D',*

$$D_\lambda(\mathcal{M}(D)\|\mathcal{M}(D')) = \frac{1}{\lambda-1}\log \mathbb{E}_{x\sim\mathcal{M}(D)}\left[\left(\frac{\mathbf{Pr}\left[\mathcal{M}(D)=x\right]}{\mathbf{Pr}\left[\mathcal{M}(D')=x\right]}\right)^{\lambda-1}\right] \leq \varepsilon.$$

RDP generalizes pure differential privacy in the sense that ε-differential is equivalent to (∞,ε)-RDP. Mironov (2017) proves the following key facts that allow easy composition of RDP guarantees and their conversion to (ε,δ)-differential privacy bounds.

Theorem 6 (Composition). *If a mechanism $\mathcal{M}$ consists of a sequence of adaptive mechanisms $\mathcal{M}_1, \ldots, \mathcal{M}_k$ such that for any $i \in [k]$, $\mathcal{M}_i$ guarantees (λ, ε_i)-RDP, then $\mathcal{M}$ guarantees $(\lambda, \sum_{i=1}^k \varepsilon_i)$-RDP.*

Theorem 7 (From RDP to DP). *If a mechanism $\mathcal{M}$ guarantees (λ, ε)-RDP, then $\mathcal{M}$ guarantees $(\varepsilon + \frac{\log 1/\delta}{\lambda - 1}, \delta)$-differential privacy for any $\delta \in (0, 1)$.*

While both (ε, δ)-differential privacy and RDP are relaxations of pure ε-differential privacy, the two main advantages of RDP are as follows. First, it composes nicely; second, it captures the privacy guarantee of Gaussian noise in a much cleaner manner compared to (ε, δ)-differential privacy. This lets us do a careful privacy analysis of the GNMax mechanism as stated in Theorem 2. While the analysis of Papernot et al. (2017) leverages the first aspect of such frameworks with the Laplace noise (LNMax mechanism), our analysis of the GNMax mechanism relies on both.

C.2 Privacy Proofs

In this section, we provide the proofs of Theorem 2 and Lemma 3. Moreover, we present Lemma 9, which provides optimal values of μ_1 and μ_2 to apply towards Theorem 2 for the GNMax mechanism. We start off with a lemma about the Rényi differential privacy guarantee of the GNMax.

Lemma 8. *The GNMax aggregator $\mathcal{A}_\sigma$ guarantees $\left(\lambda, \lambda/\sigma^2\right)$-RDP for all $\lambda \geq 1$.*

Proof. The result follows from observing that $\mathcal{A}_\sigma$ can be decomposed into applying the argmax operator to a noisy histogram resulted from adding Gaussian noise to each dimension of the original histogram. The Gaussian mechanism satisfies $(\lambda, \lambda/2\sigma^2)$-RDP (Mironov, 2017), and since each teacher may change two counts (incrementing one and decrementing the other), the overall RDP guarantee is as claimed. ∎

Lemma 3. *For any $i^* \in [m]$, we have $\mathbf{Pr}\left[\mathcal{A}_\sigma(D) \neq i^*\right] \leq \frac{1}{2} \sum_{i \neq i^*} \operatorname{erfc}\left(\frac{n_{i^*} - n_i}{2\sigma}\right)$.*

Proof. For any $i^* \in [n]$, we have

$$\begin{aligned}
\mathbf{Pr}(\mathcal{M}(D) \neq i^*) = \mathbf{Pr}\left[\exists i, n_i + Z_i > n_{i^*} + Z_{i^*}\right] &\leq \sum_{i \neq i^*} \mathbf{Pr}\left[n_i + Z_i > n_{i^*} + Z_{i^*}\right] \\
&= \sum_{i \neq i^*} \mathbf{Pr}\left[Z_i - Z_{i^*} > n_{i^*} - n_i\right] \\
&= \sum_{i \neq i^*} \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{n_{i^*} - n_i}{2\sigma}\right)\right).
\end{aligned}$$

where the last equality follows from the fact that $Z_i - Z_j$ is a Gaussian random variable with mean zero and variance $2\sigma^2$. ∎

We now present a precise statement of Theorem 2.

Theorem 2. *Let $\mathcal{M}$ be a randomized algorithm with (μ_1, ε_1)-RDP and (μ_2, ε_2)-RDP guarantees and suppose that there exists* a likely *outcome i^* given a dataset D such that $\tilde{q} \geq \mathbf{Pr}\left[\mathcal{M}(D) \neq i^*\right]$. Let $\lambda \leq \mu_1$ and $\tilde{q} \leq e^{(\mu_2 - 1)\varepsilon_2} / \left(\frac{\mu_1}{\mu_1 - 1} \cdot \frac{\mu_2}{\mu_2 - 1}\right)^{\mu_2}$. Then, for any neighboring dataset D' of D, we have:*

$$D_\lambda(\mathcal{M}(D) \| \mathcal{M}(D')) \leq \frac{1}{\lambda - 1} \log\left((1 - \tilde{q}) \cdot \boldsymbol{A}(\tilde{q}, \mu_2, \varepsilon_2)^{\lambda - 1} + \tilde{q} \cdot \boldsymbol{B}(\tilde{q}, \mu_1, \varepsilon_1)^{\lambda - 1}\right) \tag{2}$$

where $\boldsymbol{A}(\tilde{q}, \mu_2, \varepsilon_2) \triangleq (1 - \tilde{q}) / \left(1 - (\tilde{q} e^{\varepsilon_2})^{\frac{\mu_2 - 1}{\mu_2}}\right)$ and $\boldsymbol{B}(\tilde{q}, \mu_1, \varepsilon_1) \triangleq e^{\varepsilon_1} / \tilde{q}^{\frac{1}{\mu_1 - 1}}$.

Proof. Before we proceed to the proof, we introduce some simplifying notation. For a randomized mechanism $\mathcal{M}$ and neighboring datasets D and D', we define

$$\begin{aligned}
\boldsymbol{\beta}_{\mathcal{M}}(\lambda; D, D') &\triangleq D_\lambda(\mathcal{M}(D) \| \mathcal{M}(D')) \\
&= \frac{1}{\lambda - 1} \log \mathbb{E}_{x \sim \mathcal{M}(D)}\left[\left(\frac{\mathbf{Pr}\left[\mathcal{M}(D) = x\right]}{\mathbf{Pr}\left[\mathcal{M}(D') = x\right]}\right)^{\lambda - 1}\right].
\end{aligned}$$

As the proof involves working with the RDP bounds in the exponent, we set $\zeta_1 \triangleq e^{\varepsilon_1(\mu_1-1)}$ and $\zeta_2 \triangleq e^{\varepsilon_2(\mu_2-1)}$.

Finally, we define the following shortcuts:

$$q_i \triangleq \mathbf{Pr}\left[\mathcal{M}(D)=i\right] \text{ and } q \triangleq \sum_{i\neq i^*} q_i = \mathbf{Pr}\left[\mathcal{M}(D)\neq i^*\right],$$

$$p_i \triangleq \mathbf{Pr}\left[\mathcal{M}(D')=i\right] \text{ and } p \triangleq \sum_{i\neq i^*} p_i = \mathbf{Pr}\left[\mathcal{M}(D')\neq i^*\right],$$

and note that $q \leq \tilde{q}$.

From the definition of Rényi differential privacy of order μ_1, we have:

$$\exp\left(\boldsymbol{\beta}_{\mathcal{M}}(\mu_1; D, D')\right) = \left(\frac{(1-q)^{\mu_1}}{(1-p)^{\mu_1-1}} + \sum_{i\neq i*} \frac{q_i^{\mu_1}}{p_i^{\mu_1-1}}\right)^{1/(\mu_1-1)} \leq \exp\left(\varepsilon_1\right)$$

$$\implies \sum_{i>1} \frac{q_i^{\mu_1}}{p_i^{\mu_1-1}} = \sum_{i>1} q_i \left(\frac{q_i}{p_i}\right)^{\mu_1-1} \leq \zeta_1. \tag{3}$$

Since $\mu_1 > \lambda$, $f(x) \triangleq x^{\frac{\mu_1-1}{\lambda-1}}$ is convex. Applying Jensen's Inequality we have the following:

$$\left(\frac{\sum_{i\neq i^*} q_i \left(\frac{q_i}{p_i}\right)^{\lambda-1}}{q}\right)^{\frac{\mu_1-1}{\lambda-1}} \leq \frac{\sum_{i\neq i^*} q_i \left(\frac{q_i}{p_i}\right)^{\mu_1-1}}{q}$$

$$\implies \sum_{i\neq i^*} q_i \left(\frac{q_i}{p_i}\right)^{\lambda-1} \leq q \left(\frac{\sum_{i\neq i^*} q_i \left(\frac{q_i}{p_i}\right)^{\mu_1-1}}{q}\right)^{\frac{\lambda-1}{\mu_1-1}}$$

$$\overset{(3)}{\implies} \sum_{i\neq i^*} q_i \left(\frac{q_i}{p_i}\right)^{\lambda-1} \leq {\zeta_1}^{\frac{\lambda-1}{\mu_1-1}} \cdot q^{1-\frac{\lambda-1}{\mu_1-1}}. \tag{4}$$

Next, by the bound at order μ_2, we have:

$$\exp\left(\boldsymbol{\beta}_{\mathcal{M}}(\mu_2; D', D)\right) = \left(\frac{(1-p)^{\mu_2}}{(1-q)^{\mu_2-1}} + \sum_{i\neq i^*} \frac{p_i^{\mu_2}}{q_i^{\mu_2-1}}\right)^{1/(\mu_2-1)} \leq \exp\left(\varepsilon_2\right)$$

$$\implies \frac{(1-p)^{\mu_2}}{(1-q)^{\mu_2-1}} + \sum_{i\neq i^*} \frac{p_i^{\mu_2}}{q_i^{\mu_2-1}} \leq \zeta_2.$$

By the data processing inequality of Rényi divergence, we have

$$\frac{(1-p)^{\mu_2}}{(1-q)^{\mu_2-1}} + \frac{p^{\mu_2}}{q^{\mu_2-1}} \leq \zeta_2,$$

which implies $\frac{p^{\mu_2}}{q^{\mu_2-1}} \leq \zeta_2$ and thus

$$p \leq \left(q^{\mu_2-1}\zeta_2\right)^{\frac{1}{\mu_2}}. \tag{5}$$

Combining (4) and (5), we can derive a bound at λ.

$$\exp\left(\boldsymbol{\beta}_{\mathcal{M}}(\lambda, D, D')\right) = \left(\frac{(1-q)^{\lambda}}{(1-p)^{\lambda-1}} + \sum_{i\neq i^*} \frac{q_i^{\lambda}}{p_i^{\lambda-1}}\right)^{1/(\lambda-1)}$$

$$\leq \left(\frac{(1-q)^{\lambda}}{\left(1-\left(q^{\mu_2-1}\zeta_2\right)^{\frac{1}{\mu_2}}\right)^{\lambda-1}} + {\zeta_1}^{\frac{\lambda-1}{\mu_1-1}} \cdot q^{1-\frac{\lambda-1}{\mu_1-1}}\right)^{1/(\lambda-1)}. \tag{6}$$

Although Equation (6) is very close to the corresponding statement in the theorem's claim, one subtlety remains. The bound (6) applies to the exact probability $q = \mathbf{Pr}\left[\mathcal{M}(D) \neq i^*\right]$. In the theorem statement, and in practice, we can only derive an upper bound $\tilde{q}$ on $\mathbf{Pr}\left[\mathcal{M}(D) \neq i^*\right]$. The last step of the proof requires showing that the expression in Equation (6) is monotone in the range of values of q that we care about.

Monotonicity of the bound. Below we demonstrate that for the functions

$$f_1(x) \triangleq \frac{(1-x)^\lambda}{\left(1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}}\right)^{\lambda-1}} \qquad \text{and} \qquad f_2(x) \triangleq {\zeta_1}^{\frac{\lambda-1}{\mu_1-1}} \cdot x^{1-\frac{\lambda-1}{\mu_1-1}},$$

defined on $x \in [0,1]$ as above, $f_1(x) + f_2(x)$ is increasing in $\left[0, \zeta_2 / \left(\frac{\mu_1}{\mu_1-1} \cdot \frac{\mu_2}{\mu_2-1}\right)^{\mu_2}\right]$.

Taking the derivative of $f_1(x)$, we have:

$$\begin{aligned} f_1'(x) &= \frac{-\lambda(1-x)^{\lambda-1}(1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}})^{\lambda-1}}{(1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}})^{2\lambda-2}} \\ &\quad + \frac{(1-x)^\lambda(\lambda-1)(1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}})^{\lambda-2}{\zeta_2}^{\frac{1}{\mu_2}} \cdot \frac{\mu_2-1}{\mu_2} \cdot x^{-\frac{1}{\mu_2}}}{(1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}})^{2\lambda-2}} \\ &= \frac{(1-x)^{\lambda-1}}{(1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}})^{\lambda-1}} \left(-\lambda + (\lambda-1)\left(1-\frac{1}{\mu_2}\right)\frac{1-x}{1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}}}\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}}\right). \end{aligned}$$

We intend to show that:

$$f_1'(x) \geq -\lambda + (\lambda-1)\left(1-\frac{1}{\mu_2}\right)\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}}. \tag{7}$$

For $x \in \left[0, \zeta_2 / \left(\frac{\mu_1}{\mu_1-1} \cdot \frac{\mu_2}{\mu_2-1}\right)^{\mu_2}\right]$ and $y \in [1, \infty)$, define $g(x,y)$ as:

$$g(x,y) \triangleq -\lambda \cdot y^{\lambda-1} + (\lambda-1)\left(1-\frac{1}{\mu_2}\right)\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} y^\lambda.$$

We claim that $g(x,y)$ is increasing in y and therefore $g(x,y) \geq g(x,1)$, and prove it by showing the partial derivative of $g(x,y)$ with respect to y is non-negative. Take a derivative with respect to y as:

$$\begin{aligned} g_y'(x,y) &= -\lambda(\lambda-1)y^{\lambda-2} + \lambda(\lambda-1)\left(1-\frac{1}{\mu_2}\right)\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} y^{\lambda-1} \\ &= \lambda(\lambda-1)y^{\lambda-2}\left(-1 + \left(1-\frac{1}{\mu_2}\right)\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} y\right). \end{aligned}$$

To see why $g_y'(x,y)$ is non-negative in the respective ranges of x and y, note that:

$$\begin{aligned} x \leq \zeta_2 / \left(\frac{\mu_1}{\mu_1-1} \cdot \frac{\mu_2}{\mu_2-1}\right)^{\mu_2} &\implies x \leq \zeta_2 / \left(\frac{\mu_2}{\mu_2-1}\right)^{\mu_2} \\ &\implies 1 \leq \frac{\zeta_2}{x} \cdot \left(\frac{\mu_2-1}{\mu_2}\right)^{\mu_2} \\ &\implies 1 \leq \frac{\mu_2-1}{\mu_2}\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} \\ &\implies 1 \leq \frac{\mu_2-1}{\mu_2}\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} y \qquad \text{(as } y \geq 1\text{)} \\ &\implies 0 \leq -1 + \frac{\mu_2-1}{\mu_2}\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} y \\ &\implies 0 \leq g_y'(x,y). \qquad \text{(in the resp. range of } x \text{ and } y\text{)} \end{aligned}$$

Consider $\frac{1-x}{1-(x^{\mu_2-1}\zeta_2)^{1/\mu_2}}$. Since $\zeta_2 \geq 1$ and $x \leq 1$, we have $x \leq \zeta_2$ and hence

$$\frac{1-x}{1-(x^{\mu_2-1}\zeta_2)^{\frac{1}{\mu_2}}} \geq \frac{1-x}{1-(x^{\mu_2-1}x)^{\frac{1}{\mu_2}}} = 1.$$

Therefore we can set $y = \frac{1-x}{1-(x^{\mu_2-1}\zeta_2)^{1/\mu_2}}$ and apply the fact that $g(x,y) \geq g(x,1)$ for all $y \geq 1$ to get

$$f_1'(x) \geq -\lambda + (\lambda-1)\left(1-\frac{1}{\mu_2}\right)\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}},$$

as required by (7).

Taking the derivative of $f_2(x)$, we have:

$$f_2'(x) = \zeta_1{}^{\frac{\lambda-1}{\mu_1-1}} \cdot \left(1-\frac{\lambda-1}{\mu_1-1}\right) x^{-\frac{\lambda-1}{\mu_1-1}} = \left(\frac{\zeta_1}{x}\right)^{\frac{\lambda-1}{\mu_1-1}} \left(1-\frac{\lambda-1}{\mu_1-1}\right) \geq 1-\frac{\lambda-1}{\mu_1-1}.$$

Combining the two terms together, we have:

$$\begin{aligned} f'(x) &\geq -\lambda + (\lambda-1)\left(1-\frac{1}{\mu_2}\right)\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} + 1 - \frac{\lambda-1}{\mu_1-1} \\ &= (\lambda-1)\left(-\frac{\mu_1}{\mu_1-1} + \frac{\mu_2-1}{\mu_2}\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}}\right). \end{aligned}$$

For $f'(x)$ to be non-negative we need:

$$\begin{aligned} &-\frac{\mu_1}{\mu_1-1} + \frac{\mu_2-1}{\mu_2}\left(\frac{\zeta_2}{x}\right)^{\frac{1}{\mu_2}} \geq 0 \\ \iff &\left(\frac{\mu_1}{\mu_1-1}\cdot\frac{\mu_2}{\mu_2-1}\right)^{\mu_2} \leq \frac{\zeta_2}{x}. \end{aligned}$$

So $f(x)$ is increasing for $x \in \left[0, \zeta_2/\left(\frac{\mu_1}{\mu_1-1}\cdot\frac{\mu_2}{\mu_2-1}\right)^{\mu_2}\right]$. This means for $q \leq \tilde{q} \leq \zeta_2/\left(\frac{\mu_1}{\mu_1-1}\cdot\frac{\mu_2}{\mu_2-1}\right)^{\mu_2}$, we have $f(q) \leq f(\tilde{q})$. This completes the proof of the theorem. ∎

Theorem 2 yields data-dependent Rényi differential privacy bounds for *any* value of μ_1 and μ_2 larger than λ. The following lemma simplifies this search by calculating optimal higher moments μ_1 and μ_2 for the GNMax mechanism with variance σ^2.

Lemma 9. *When applying Theorem 2 and Lemma 8 for GNMax with Gaussian of variance σ^2, Eq.* (2) *is minimized at*

$$\mu_2 = \sigma \cdot \sqrt{\log(1/\tilde{q})}, \text{ and } \mu_1 = \mu_2 + 1.$$

Proof. We can minimize both terms in (2) independently. To minimize the first term in (2), we minimize $(\tilde{q}e^{\varepsilon_2})^{1-1/\mu_2}$ by considering logarithms:

$$\begin{aligned} \log\left\{(\tilde{q}e^{\varepsilon_2})^{1-1/\mu_2}\right\} &= \log\left\{\tilde{q}^{1-\frac{1}{\mu_2}}\exp\left(\frac{\mu_2-1}{\sigma^2}\right)\right\} \\ &= \left(1-\frac{1}{\mu_2}\right)\cdot\log\tilde{q} + \frac{\mu_2-1}{\sigma^2} \\ &= \frac{1}{\mu_2}\log\frac{1}{\tilde{q}} + \frac{\mu_2}{\sigma^2} - \frac{1}{\sigma^2} - \log\frac{1}{\tilde{q}}, \end{aligned}$$

which is minimized at $\mu_2 = \sigma \cdot \sqrt{\log(1/\tilde{q})}$.

To minimize the second term in (2), we minimize $e^{\varepsilon_1}/\tilde{q}^{1/(\mu_1-1)}$ as follows:

$$\begin{aligned}\log\left\{\frac{e^{\varepsilon_1}}{\tilde{q}^{1/(\mu_1-1)}}\right\} &= \log\left\{\tilde{q}^{-1/(\mu_1-1)}\exp\left(\frac{\mu_1}{\sigma^2}\right)\right\}\\ &= \frac{\mu_1}{\sigma^2}+\frac{1}{\mu_1-1}\log\frac{1}{\tilde{q}}\\ &= \frac{1}{\sigma^2}+\frac{\mu_1-1}{\sigma^2}+\frac{1}{\mu_1-1}\log\frac{1}{\tilde{q}},\end{aligned}$$

which is minimized at $\mu_1 = 1+\sigma\cdot\sqrt{\log(1/\tilde{q})}$ completing the proof. ∎

The following corollary offers a simple asymptotic expression of the privacy of GNMax for the case when there are large (relative to σ) gaps between the highest *three* vote counts.

Corollary 10. *If the top three vote counts are $n_1 > n_2 > n_3$ and $n_1 - n_2$, $n_2 - n_3 \gg \sigma$, then the mechanism GNMax with Gaussian of variance σ^2 satisfies $(\lambda, \exp\left(-2\lambda/\sigma^2\right)/\lambda)$-RDP for $\lambda = (n_1 - n_2)/4$.*

Proof. Denote the noisy counts as $\tilde{n}_i = n_i + \mathcal{N}(0,\sigma^2)$. Ignoring outputs other than those with the highest and the second highest counts, we bound $q = \mathbf{Pr}\left[\mathcal{M}(D)\neq 1\right]$ as $\mathbf{Pr}[\tilde{n}_1 < \tilde{n}_2] = \mathbf{Pr}[N(0,2\sigma^2) > n_1 - n_2] < \exp\left(-(n_1-n_2)^2/4\sigma^2\right)$, which we use as $\tilde{q}$. Plugging $\tilde{q}$ in Lemma 9, we have $\mu_1 - 1 = \mu_2 = (n_1-n_2)/2$, limiting the range of applicability of Theorem 2 to $\lambda < (n_1-n_2)/2$.

Choosing $\lambda = (n_1-n_2)/4$ ensures $\boldsymbol{A}(\tilde{q},\mu_2,\varepsilon_2)\approx 1$, which allows approximating the bound (2) as $\tilde{q}\cdot\boldsymbol{B}(\tilde{q},\mu_1,\varepsilon_1)^{\lambda-1}/(\lambda-1)$. The proof follows by straightforward calculation. ∎

C.3 Smooth Sensitivity and Publishing the Privacy Parameter

The privacy guarantees obtained for the mechanisms in this paper via Theorem 2 take as input $\tilde{q}$, an upper bound on the probability that the aggregate mechanism returns the true plurality. This means that the resulting privacy parameters computed depend on teacher votes and hence the underlying data. To avoid potential privacy breaches from simply publishing the data-dependent parameter, we need to publish a *sanitized version* of the privacy loss. This is done by adding noise to the computed privacy loss estimates using the *smooth sensitivity* algorithm proposed by Nissim et al. (2007). This section provides an overview of an algorithm that computes the smooth sensitivity of the privacy loss function.

Any dataset D defines a histogram $\bar{n} = (n_1,\ldots,n_m)\in\mathbb{R}^m$ of the teachers' votes. We have a natural notion of the distance between two histograms $\mathrm{dist}(\bar{n},\bar{n}')$ and a function $q\colon\mathbb{R}^m\to[0,1]$ on these histograms computing the bound according to Lemma 3. The value $q(\bar{n})$ can be used as $\tilde{q}$ in the application of Theorem 2. Additionally we have $n^{(i)}$ denote the i-th highest bar in the histogram.

Recall from Section 3.1 that in calculating the privacy parameter we use the lower of two bounds: a data-dependent and a data-independent one. Therefore, in what follows, for a given moment λ, we use a (rescaled) bound provided by Theorem 2 and Lemma 8.

$$\boldsymbol{\alpha}\left(q\right)\triangleq\min\left\{\log\left\{(1-q)\cdot\boldsymbol{A}(q,\mu_2,\varepsilon_2)^{\lambda-1}+q\cdot\boldsymbol{B}(q,\mu_1,\varepsilon_1)^{\lambda-1}\right\},\lambda(\lambda-1)/\sigma^2\right\}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as in the theorem statement, and the parameters $\mu_1,\mu_2,\varepsilon_1,\varepsilon_2$ are selected according to Lemmas 8 and 9.

We aim at calculating the smooth sensitivity of $\boldsymbol{\alpha}\left(q(\cdot)\right)$ whose definition we recall now.

Definition 11 (Smooth Sensitivity)**.** *Given the smoothness parameter β, a β-*smooth sensitivity *of $f(n)$ is defined as*

$$\mathrm{SS}(\bar{n})\triangleq\max_{d\geq 0}e^{-\beta d}\cdot\max_{\bar{n}':\mathrm{dist}(\bar{n},\bar{n}')=d}\tilde{\mathrm{LS}}(\bar{n}'),$$

where

$$\tilde{\mathrm{LS}}(\bar{n})\geq\max_{\bar{n}':\mathrm{dist}(\bar{n},\bar{n}')=1}|f(n)-f(n')|$$

is an upper bound on the local sensitivity.

We now describe Algorithms 3–5 computing the smooth sensitivity of $\boldsymbol{\alpha}(q(\cdot))$. Their correctness is stated in lemmas that follow. The algorithms assume the existence of the following mathematical constructs, which can be observed to be true for the GNMax algorithm or shown via numerical computations:

- There exist increasing functions $\mathrm{B_U}, \mathrm{B_L}\colon [0,1] \to [0,1]$ that are inverses of each other (i.e., $\mathrm{B_U}(\mathrm{B_L}(q)) = q$) and respectively upper bound and lower bound the value of q evaluated at any neighbor of $\bar{n}$ given $q(\bar{n})$.
- $\boldsymbol{\alpha}(\cdot)$ increases in the range $[0, q_0]$ for some value $q_0 < 1/2$ and remains fixed at $\boldsymbol{\alpha}(q_0)$ for $q > q_0$ where the data-independent—and hence q-independent—bound operates. Additionally, $q([n, 0, \ldots, 0]) \leq q_0$.
- $\Delta\boldsymbol{\alpha}(q) \triangleq \boldsymbol{\alpha}(\mathrm{B_U}(q)) - \boldsymbol{\alpha}(q)$ is increasing in $[0, \mathrm{B_L}(q_0)]$.
- For any given histogram of values, neighboring histograms that maximize the increase and decrease of q can be derived from the given histogram in a straightforward manner.

Lemma 12. *For $\boldsymbol{\alpha}(\cdot)$ defined as above and q defined in Lemma 3, the GNMax algorithm satisfies the assumptions stated above. More specifically, we can set explicitly*

$$\mathrm{B_U}(q) \triangleq \min\left\{\frac{m-1}{2}\mathrm{erfc}\left(\mathrm{erfc}^{-1}\left(\frac{2q}{m-1}\right) - \frac{1}{\sigma}\right), 1\right\},$$
$$\mathrm{B_L}(q) \triangleq \frac{m-1}{2}\mathrm{erfc}\left(\mathrm{erfc}^{-1}\left(\frac{2q}{m-1}\right) + \frac{1}{\sigma}\right),$$

where $\mathrm{erfc}(\cdot)$ *is* $1 - \mathrm{erf}(\cdot)$*, the complementary error function.*

Algorithm 3 – Local Sensitivity: use the functions $\mathrm{B_U}$ and $\mathrm{B_L}$ to compute (an upper bound) of the local sensitivity at a given q value by looking at the difference of $\boldsymbol{\alpha}(\cdot)$ evaluated on the bounds.

```
procedure L̃S(q)
    if B_L(q_0) < q < q_0 then                    ▷ Interpolate the middle part.
        q ← B_L(q_0)
    end if
    return max{α(B_U(q)) − α(q), α(q) − α(B_L(q))}
end procedure
```

Lemma 13. *Under the assumptions satisfied by the GNMax mechanism (Lemma 12), Algorithm 3 computes an upper bound on the local sensitivity of $\boldsymbol{\alpha}(\cdot)$. Moreover, $\tilde{\mathrm{LS}}(q)$ is increasing in $[0, \mathrm{B_L}(q_0)]$, constant in $(\mathrm{B_L}(q_0), q_0)$, and decreasing in $[q_0, 1]$.*

A careful analysis of Algorithms 4 and 5 which we defer to a full version of this paper enables us to conclude the following Theorem.

Theorem 14. *Algorithm 5 computes a β-smooth upper bound on $\tilde{\mathrm{LS}}$.*

Summary. For the privacy values published in this work, we compute the smooth sensitivity and estimate that we need to add typically less than 1% noise to the final privacy cost values before publishing them—values that are still conservative and widely used and recommended in practice and in literature.

Algorithm 4 – Sensitivity at a distance: given a histogram $\bar{n}$, compute the sensitivity of $\boldsymbol{\alpha}(\cdot)$ at distance d using the procedure $\tilde{\mathrm{LS}}$, functions $q(\cdot)$, $\mathrm{B_L}(\cdot)$, $\mathrm{B_U}(\cdot)$, constant q_0, and careful case analysis that estimates the neighbor at distance d with the maximum sensitivity.

procedure ATDISTANCED($\bar{n}$, d)
  $q \leftarrow q(\bar{n})$
  if $q \leq \mathrm{B_L}(q_0)$ **then** ▷ Need to increase q.
    if $\bar{n}^{(1)} - \bar{n}^{(2)} < 2d$ **then**
      $q' \leftarrow \mathrm{B_L}(q_0)$
      return $\tilde{\mathrm{LS}}(q')$, TRUE
    else
      $\bar{n}' \leftarrow \bar{n} + [-d, d, 0, \ldots, 0]$
      $q' \leftarrow q(\bar{n}')$
      if $q' > \mathrm{B_L}(q_0)$ **then**
        return $\tilde{\mathrm{LS}}(q_0)$, TRUE
      else
        return $\tilde{\mathrm{LS}}(q')$, FALSE
      end if
    end if
  else ▷ Need to decrease q.
    if $\sum_{i=2}^{d} \bar{n}^{(i)} < d$ **then**
      $\bar{n}' \leftarrow [n, 0, \ldots, 0]$
      $q' \leftarrow q(\bar{n}')$
      return $\tilde{\mathrm{LS}}(q)$, TRUE
    else
      $\bar{n}' \leftarrow \bar{n} + [d, 0, \ldots, 0]$
      for $d' = 1, \ldots, d$ **do**
        $\bar{n}'^{(2)} \leftarrow \bar{n}'^{(2)} - 1$
        Sort $\bar{n}'$ in descending order
      end for
      $q' \leftarrow q(\bar{n}')$
      if $q' < q_0$ **then**
        return $\tilde{\mathrm{LS}}(q_0)$, TRUE
      else
        return $\tilde{\mathrm{LS}}(q')$, FALSE
      end if
    end if
  end if
end procedure

Algorithm 5 – Smooth Sensitivity: Compute the β smooth sensitivity of $\boldsymbol{\alpha}(\cdot)$ via Definition 11 by looking at sensitivities at various distances and returning the maximum weighted by $e^{-\beta d}$.

procedure SMOOTHSENSITIVITY($\bar{n}$, β)
  $S \leftarrow 0$
  $d \leftarrow 0$
  repeat
    c, StoppingCondition $\leftarrow$ ATDISTANCED($\bar{n}, d$)
    $S \leftarrow \max\{S, c \cdot e^{-\beta d}\}$
    $d \leftarrow d + 1$
  until StoppingCondition
end procedure