THEORETICAL ANALYSIS OF AUTO RATE-TUNING BY BATCH NORMALIZATION

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ABSTRACT

Batch Normalization (BN) has become a cornerstone of deep learning across diverse architectures, appearing to help optimization as well as generalization. While the idea makes intuitive sense, theoretical analysis of its effectiveness has been lacking. Here theoretical support is provided for one of its conjectured properties, namely, the ability to allow gradient descent to succeed with less tuning of learning rates. It is shown that even if we fix the learning rate of scale-invariant parameters (e.g., weights of each layer with BN) to a constant (say, 0.3), gradient descent still approaches a stationary point (i.e., a solution where gradient is zero) in the rate of $T^{-1/2}$ in T iterations, asymptotically matching the best bound for gradient descent with well-tuned learning rates. A similar result with convergence rate $T^{-1/4}$ is also shown for stochastic gradient descent.

1 INTRODUCTION

Batch Normalization (abbreviated as BatchNorm or BN) (Ioffe & Szegedy, 2015) is one of the most important innovation in deep learning, widely used in modern neural network architectures such as ResNet (He et al., 2016), Inception (Szegedy et al., 2017), and DenseNet (Huang et al., 2017). It also inspired a series of other normalization methods (Ulyanov et al., 2016; Ba et al., 2016; Ioffe, 2017; Wu & He, 2018).

BatchNorm consists of standardizing the output of each layer to have zero mean and unit variance. For a single neuron, if $x_1, \ldots, x_B$ is the original outputs in a mini-batch, then it adds a BatchNorm layer which modifies the outputs to

$$\mathrm{BN}(x_i) = \gamma \frac{x - \mu}{\sigma} + \beta, \tag{1}$$

where $\mu = \frac{1}{B}\sum_{i=1}^{B} x_i$ and $\sigma^2 = \frac{1}{B}\sum_{i=1}^{B}(x_i - \mu)^2$ are the mean and variance within the mini-batch, and γ, β are two learnable parameters. BN appears to stabilize and speed up training, and improve generalization. The inventors suggested (Ioffe & Szegedy, 2015) that these benefits derive from the following:

1. By stabilizing layer outputs it reduces a phenomenon called *Internal Covariate Shift*, whereby the training of a higher layer is continuously undermined or undone by changes in the distribution of its inputs due to parameter changes in previous layers.,
2. Making the weights invariant to scaling, appears to reduce the dependence of training on the scale of parameters and enables us to use a higher learning rate;
3. By implictly regularizing the model it improves generalization.

But these three benefits are not fully understood in theory. Understanding generalization for deep models remains an open problem (with or without BN). Furthermore, in demonstration that intuition can sometimes mislead, recent experimental results suggest that BN does not reduce internal covariate shift either (Santurkar et al., 2018), and the authors of that study suggest that the true explanation for BN's effectiveness may lie in a smoothening effect (i.e., lowering of the Hessian norm) on the objective. Another recent paper (Kohler et al., 2018) tries to quantify the benefits of BN for simple machine learning problems such as regression but does not analyze deep models.

Provable quantification of Effect 2 (learning rates). Our study consists of quantifying the effect of BN on learning rates. Ioffe & Szegedy (2015) observed that without BatchNorm, a large learning rate leads to a rapid growth of the parameter scale. Introducing BatchNorm usually stabilizes the growth of weights and appears to implicitly tune the learning rate so that the effective learning rate adapts during the course of the algorithm. They explained this intuitively as follows. After BN the output of a neuron $z = \mathrm{BN}(\boldsymbol{w}^\top \boldsymbol{x})$ is unaffected when the weight $\boldsymbol{w}$ is scaled, i.e., for any scalar $c > 0$,

$$\mathrm{BN}(\boldsymbol{w}^\top \boldsymbol{x}) = \mathrm{BN}((c\boldsymbol{w})^\top \boldsymbol{x}).$$

Taking derivatives one finds that the gradient at $c\boldsymbol{w}$ equals to the gradient at $\boldsymbol{w}$ multiplied by a factor $1/c$. Thus, even though the scale of weight parameters of a linear layer proceeding a BatchNorm no longer means anything to the function represented by the neural network, their growth has an effect of reducing the learning rate.

Our paper considers the following question: *Can we rigorously capture the above intuitive behavior?* Theoretical analyses of speed of gradient descent algorithms in nonconvex settings study the number of iterations required for convergence to a stationary point (i.e., where gradient vanishes). But they need to assume that the learning rate has been set (magically) to a small enough number determined by the smoothness constant of the loss function — which in practice are of course unknown. With this tuned learning rate, the norm of the gradient reduces asymptotically as $T^{-1/2}$ in T iterations. In case of stochastic gradient descent, the reduction is like $T^{-1/4}$. Thus a potential way to quantify the rate-tuning behavior of BN would be to show that even when the learning rate is fixed to a suitable constant, say 0.1, from the start, after introducing BN the convergence to stationary point is asymptotically just as fast (essentially) as it would be with a hand-tuned learning rate required by earlier analyses. The current paper rigorously establishes such *auto-tuning* behavior of BN (See below for an important clarification about scale-invariance).

We note that a recent paper (Wu et al., 2018) introduced a new algorithm WNgrad that is motivated by BN and provably has the above auto-tuning behavior as well. That paper did not establish such behavior for BN itself, but it was a clear inspiration for our analysis of BN.

Scale-invariant and scale-variant parameters. The intuition of Ioffe & Szegedy (2015) applies for all scale-invariant parameters, but the actual algorithm also involves other parameters such as γ and β whose scale does matter. Our analysis partitions the parameters in the neural networks into two groups W (*scale-invariant*) and $\boldsymbol{g}$ (*scale-variant*). The first group, $W = \{\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(m)}\}$, consists of all the parameters whose scales does not affect the loss, i.e., scaling $\boldsymbol{w}^{(i)}$ to $c\boldsymbol{w}^{(i)}$ for any $c > 0$ do not change the loss (see Definition 2.1 for a formal definition); the second group, $\boldsymbol{g}$, consists of all other parameters that are not scale-invariant. In a feedforward neural network with BatchNorm added at each layer, the layer weights are scale-invariant. And this fact is also true for Weight Normalization (Salimans & Kingma, 2016), Layer Normalization (Ba et al., 2016), Group Normalization (Wu & He, 2018) (see Table 1 in Ba et al. (2016) for a summary).

1.1 Our contributions

In this paper, we show that the scale-invariant parameters do not require rate tuning. To illustrate this, we consider the case in which we set learning rates separately for scale-invariant parameters W and scale-variant parameters $\boldsymbol{g}$. Under some assumptions on the smoothness and the boundedness of the noise, we show that

1. In full-batch gradient descent, if the learning rate for $\boldsymbol{g}$ is set optimally, then no matter how the learning rates for W is set, $(W; \boldsymbol{g})$ converges to a first-order stationary point in the rate $O(T^{-1/2})$, which asymptomatically matches with the convergence rate of gradient descent with optimal choice of learning rates for all parameters (Theorem 3.1);
2. In stochastic gradient descent, if the learning rate for $\boldsymbol{g}$ is set optimally, then no matter how the learning rate for W is set, $(W; \boldsymbol{g})$ converges to a first-order stationary point in the rate $O(T^{-1/4} \operatorname{polylog}(T))$, which asymptomatically matches with the convergence rate of gradient descent with optimal choice of learning rates for all parameters (up to a $\operatorname{polylog}(T)$ factor) (Theorem 4.2).

In the usual case where we set a unified learning rate for all parameters, our results imply that we only need to set a learning rate that is suitable for $\boldsymbol{g}$. This means introducing scale-invariance into neural networks potentially reduces the efforts to tune learning rates, since there are less number of parameters we need to concern in order to guarantee an asymptomatically fastest convergence.

Note that we study the dynamics with assumptions that the loss is smooth. However, BN introduces non-smoothness in extreme cases due to division by zero when the input variance is zero (see equation 1). Note that the suggested implementation of BN in Ioffe & Szegedy (2015) uses a smoothening constant in the whitening step, but it does not preserve scale-invariance. In order to avoid this issue, we describe a simple modification of the smoothening that maintains scale-invariance. Also, our result cannot be applied to neural networks with ReLU, but it is applicable for its smooth approximation softplus (Dugas et al., 2001).

1.2 Related works

Previous work for understanding Batch Normalization. Only a few recent works try to theoretically understand BatchNorm. Santurkar et al. (2018) was described earlier. Kohler et al. (2018) aims to find theoretical setting such that training neural networks with BatchNorm is faster than without BatchNorm. In particular, the authors analyzed three types of shallow neural networks, but rather than consider gradient descent, the authors designed task-specific training methods when discussing neural networks with BatchNorm. Bjorck et al. (2018) observes that the higher learning rates enabled by BatchNorm improves generalization.

Convergence of adaptive algorithms. Our analysis is inspired by the proof for WNGrad (Wu et al., 2018), where the author analyzed an adaptive algorithm, WNGrad, motivated by Weight Normalization (Salimans & Kingma, 2016).Other works analyzing the convergence of adaptive methods are (Ward et al., 2018; Li & Orabona, 2018; Zou & Shen, 2018; Dongruo Zhou, 2018).

Invariance by Batch Normalization. Cho & Lee (2017) proposed to run riemmanian gradient descent on Grassmann manifold $\mathcal{G}(1,n)$ since the weight matrix is scaling invariant to the loss function. Zhang & Li (2018) observed that the effective stepsize is proportional to $\frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_t\|^2}$.

2 General framework

In this section, we introduce our general framework in order to study the benefits of scale-invariance.

2.1 Motivating examples of neural networks

Scale-invariance is common in neural networks with BatchNorm. We formally state the definition of scale-invariance below:

Definition 2.1. (Scale-invariance) Let $\mathcal{F}(\boldsymbol{w},\boldsymbol{\theta}')$ be a loss function. We say that $\boldsymbol{w}$ is a scale-invariant parameter of $\mathcal{F}$ if for all $c>0$, $\mathcal{F}(\boldsymbol{w},\boldsymbol{\theta}')=\mathcal{F}(c\boldsymbol{w},\boldsymbol{\theta}')$; if $\boldsymbol{w}$ is not scale-invariant, then we say $\boldsymbol{w}$ is a scale-variant parameter of $\mathcal{F}$.

We consider the following L-layer "fully-batch-normalized" feedforward network Φ for illustration:

$$\mathcal{L}(\boldsymbol{\theta})=\mathbb{E}_{Z\sim\mathcal{D}^B}\left[\frac{1}{B}\sum_{b=1}^{B} f_{y_b}(\mathrm{BN}(W^{(L)}\sigma(\mathrm{BN}(W^{(L-1)}\cdots\sigma(\mathrm{BN}(W^{(1)}\boldsymbol{x}_b)))))))\right]. \tag{2}$$

$Z=\{(\boldsymbol{x}_1,y_1),\ldots,(\boldsymbol{x}_B,y_B)\}$ is a mini-batch of B pairs of input data and ground-truth label from a data set $\mathcal{D}$. f_y is an objective function depending on the label, e.g., f_y could be a cross-entropy loss in classification tasks. $W^{(1)},\ldots,W^{(L)}$ are weight matrices of each layer. $\sigma:\mathbb{R}\to\mathbb{R}$ is a nonlinear activation function which processes its input elementwise (such as ReLU, sigmoid). Given a batch of inputs $\boldsymbol{z}_1,\ldots,\boldsymbol{z}_B\in\mathbb{R}^m$, $\mathrm{BN}(\boldsymbol{z}_b)$ outputs a vector $\tilde{\boldsymbol{z}}_b$ defined as

$$\tilde{z}_{b,k}:=\gamma_k\frac{z_{b,k}-\mu_k}{\sigma_k}+\beta_k, \tag{3}$$

where $\mu_k=\mathbb{E}_{b\in[B]}[z_{b,k}]$ and $\sigma_k^2=\mathbb{E}_{b\in[B]}[(z_{b,k}-\mu_k)^2]$ are the mean and variance of $\boldsymbol{z}_b$, γ_k and β_k are two learnable parameters which rescale and offset the normalized outputs to retain the

representation power. The neural network Φ is thus parameterized by weight matrices $W^{(i)}$ in each layer and learnable parameters γ_k, β_k in each BN.

BN has the property that the output is unchanged when the batch inputs $z_{1,k}, \ldots, z_{B,k}$ are scaled or shifted simultaneously. For $z_{b,k} = \boldsymbol{w}_k^\top \hat{\boldsymbol{x}}_b$ being the output of a linear layer, it is easy to see that $\boldsymbol{w}_k$ is scale-invariant, and thus each row vector of weight matrices $W^{(1)}, \ldots, W^{(L)}$ in Φ are scale-invariant parameters of $\mathcal{L}(\boldsymbol{\theta})$. In convolutional neural networks with BatchNorm, a similar argument can be done. In particular, each filter of convolutional layer normalized by BN is scale-invariant.

With a general nonlinear activation, other parameters in Φ, the scale and shift parameters γ_k and β_k in each BN, are scale-variant. When ReLU or Leaky ReLU (Maas et al., 2013) are used as the activation σ, the vector $(\gamma_1, \ldots, \gamma_m, \beta_1, \ldots, \beta_m)$ of each BN at layer $1 \leq i < L$ (except the last one) is indeed scale-invariant. This can be deduced by using the the (positive) homogeneity of ReLU and noticing that the output of internal activations is processed by a BN in the next layer. Nevertheless, we are not able to analyze either ReLU or Leaky ReLU activations because we need the loss to be smooth in our analysis. We can instead analyse smooth activations, such as sigmoid, tanh, softplus (Dugas et al., 2001), etc.

2.2 Framework

Now we introduce our general framework. Let Φ be a neural network parameterized by $\boldsymbol{\theta}$. Let $\mathcal{D}$ be a dataset, where each data point $\boldsymbol{z} \sim \mathcal{D}$ is associated with a loss function $\mathcal{F}_{\boldsymbol{z}}(\boldsymbol{\theta})$ ($\mathcal{D}$ can be the set of all possible mini-batches). We partition the parameters $\boldsymbol{\theta}$ into $(W; \boldsymbol{g})$, where $W = \{\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(m)}\}$ consisting of parameters that are scale-invariant to all $\mathcal{F}_{\boldsymbol{z}}$, and $\boldsymbol{g}$ contains the remaining parameters. The goal of training the neural network is to minimize the expected loss over the dataset:

$$\mathcal{L}(W; \boldsymbol{g}) := \mathbb{E}_{\boldsymbol{z} \sim \mathcal{D}}[\mathcal{F}_{\boldsymbol{z}}(W; \boldsymbol{g})].$$

In order to illustrate the optimization benefits of scale-invariance, we consider the process of training this neural network by stochastic gradient descent with separate learning rates for W and $\boldsymbol{g}$:

$$\boldsymbol{w}_{t+1}^{(i)} \leftarrow \boldsymbol{w}_t^{(i)} - \eta_{\mathrm{w},t} \nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t), \qquad \boldsymbol{g}_{t+1} \leftarrow \boldsymbol{g}_t - \eta_{\mathrm{g},t} \nabla_{\boldsymbol{g}_t} \mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t). \tag{4}$$

2.3 The intrinsic optimization problem

Thanks to the scale-invariant properties, the scale of each weight $\boldsymbol{w}^{(i)}$ does not affect loss values. However, the scale does affect the gradients. Let $V = \{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(m)}\}$ be the set of normalized weights, where $\boldsymbol{v}^{(i)} = \boldsymbol{w}^{(i)} / \|\boldsymbol{w}^{(i)}\|_2$. The following simple lemma can be easily shown:

Lemma 2.2 (Implied by Ioffe & Szegedy (2015))**.** *For any W and $\boldsymbol{g}$,*

$$\nabla_{\boldsymbol{w}^{(i)}} \mathcal{F}_{\boldsymbol{z}}(W; \boldsymbol{g}) = \frac{1}{\|\boldsymbol{w}^{(i)}\|_2} \nabla_{\boldsymbol{v}^{(i)}} \mathcal{F}_{\boldsymbol{z}}(V; \boldsymbol{g}), \qquad \nabla_{\boldsymbol{g}} \mathcal{F}_{\boldsymbol{z}}(W; \boldsymbol{g}) = \nabla_{\boldsymbol{g}} \mathcal{F}_{\boldsymbol{z}}(V; \boldsymbol{g}). \tag{5}$$

To make $\|\nabla_{\boldsymbol{w}^{(i)}} \mathcal{F}_{\boldsymbol{z}}(W; \boldsymbol{g})\|_2$ to be small, one can just scale the weights by a large factor. Thus there are ways to reduce the norm of the gradient that do not reduce the loss.

For this reason, we define the intrinsic optimization problem for training the neural network. Instead of optimizing W and $\boldsymbol{g}$ over all possible solutions, we focus on parameters $\boldsymbol{\theta}$ in which $\|\boldsymbol{w}^{(i)}\|_2 = 1$ for all $\boldsymbol{w}^{(i)} \in W$. This does not change our objective, since the scale of W does not affect the loss.

Definition 2.3 (Intrinsic optimization problem)**.** Let $\mathcal{U} = \{\boldsymbol{\theta} \mid \|\boldsymbol{w}^{(i)}\|_2 = 1 \text{ for all } i\}$ be the *intrinsic domain*. The *intrinsic optimization problem* is defined as optimizing the original problem in $\mathcal{U}$:

$$\min_{(W; \boldsymbol{g}) \in \mathcal{U}} \mathcal{L}(W; \boldsymbol{g}). \tag{6}$$

For $\{\boldsymbol{\theta}_t\}$ being a sequence of points for optimizing the original optimization problem, we can define $\{\tilde{\boldsymbol{\theta}}_t\}$, where $\tilde{\boldsymbol{\theta}}_t = (V_t; \boldsymbol{g}_t)$, as a sequence of points optimizing the intrinsic optimization problem.

In this paper, we aim to show that training neural network for the original optimization problem by gradient descent can be seen as training by adaptive methods for the intrinsic optimization problem, and it converges to a first-order stationary point in the intrinsic optimization problem with no need for tuning learning rates for W.

2.4 ASSUMPTIONS ON THE LOSS

We assume $\mathcal{F}_{\boldsymbol{z}}(W;\boldsymbol{g})$ is defined and twice continuously differentiable at any $\boldsymbol{\theta}$ satisfying none of $\boldsymbol{w}^{(i)}$ is 0. Also, we assume that the expected loss $\mathcal{L}(\boldsymbol{\theta})$ is lower-bounded by $\mathcal{L}_{\min}$.

Furthermore, for $V = \{\boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(m)}\}$, where $\boldsymbol{v}^{(i)} = \boldsymbol{w}^{(i)}/\|\boldsymbol{w}^{(i)}\|_2$, we assume that the following bounds on the smoothness:

$$\left\|\frac{\partial}{\partial \boldsymbol{v}^{(i)}\partial \boldsymbol{v}^{(j)}}\mathcal{F}_{\boldsymbol{z}}(V;\boldsymbol{g})\right\|_2 \le L_{ij}^{\mathrm{vv}}, \qquad \left\|\frac{\partial}{\partial \boldsymbol{v}^{(i)}\partial \boldsymbol{g}}\mathcal{F}_{\boldsymbol{z}}(V;\boldsymbol{g})\right\|_2 \le L_{i}^{\mathrm{vg}}, \qquad \left\|\nabla_{\boldsymbol{g}}^2\mathcal{F}_{\boldsymbol{z}}(V;\boldsymbol{g})\right\|_2 \le L^{\mathrm{gg}}.$$

In addition, we assume that the noise on the gradient of $\boldsymbol{g}$ in SGD is upper bounded by G_{g}:

$$\mathbb{E}\left[\|\nabla_{\boldsymbol{g}}\mathcal{F}_{\boldsymbol{z}}(V;\boldsymbol{g}) - \mathbb{E}_{\boldsymbol{z}\sim\mathcal{D}}\left[\nabla_{\boldsymbol{g}}\mathcal{F}_{\boldsymbol{z}}(V;\boldsymbol{g})\right]\|_2^2\right] \le G_{\mathrm{g}}^2.$$

Smoothed version of motivating neural networks. Note that the neural network Φ illustrated in Section 2.1 does not meet the conditions of the smooothness at all since the loss function could be non-smooth. We can make some mild modifications to the motivating example to smoothen it [1]:

(1). The activation could be non-smooth. A possible solution is to use smooth nonlinearities, e.g., sigmoid, tanh, softplus (Dugas et al., 2001), etc. Note that softplus can be seen as a smooth approximation of the most commonly used activation ReLU.

(2). The formula of BN shown in equation 3 may suffer from the problem of division by zero. To avoid this, the inventors of BN, Ioffe & Szegedy (2015), add a small smoothening parameter $\epsilon > 0$ to the denominator, i.e.,

$$\tilde{z}_{b,k} := \gamma_k \frac{z_{b,k} - \mu_k}{\sqrt{\sigma_k^2 + \epsilon}} + \beta_k, \tag{7}$$

However, when $z_{b,k} = \boldsymbol{w}_k^\top \hat{\boldsymbol{x}}_b$, adding a constant ϵ directly breaks the scale-invariance of $\boldsymbol{w}_k$. We can preserve the scale-invariance by making the smoothening term propositional to $\|\boldsymbol{w}_k\|_2$, i.e., replacing ϵ with $\epsilon\|\boldsymbol{w}_k\|_2$. By simple linear algebra and letting $\boldsymbol{u} := \mathbb{E}_{b\in[B]}[\hat{\boldsymbol{x}}_b]$, $\boldsymbol{S} := \mathrm{Var}_{b\in[B]}(\hat{\boldsymbol{x}}_b)$, this smoothed version of BN can also be written as

$$\tilde{z}_{b,k} := \gamma_k \frac{\boldsymbol{w}_k^\top(\hat{\boldsymbol{x}}_b - \boldsymbol{u})}{\|\boldsymbol{w}_k\|_{\boldsymbol{S}+\epsilon\boldsymbol{I}}} + \beta_k. \tag{8}$$

Since the variance of inputs is usually large in practice, for small ϵ, the effect of the smoothening term is negligible except in extreme cases.

Using the above two modifications, the loss function is already smooth. However, the scale of scale-variant parameters may be unbounded during training, which could cause the smoothness unbounded. To avoid this issue, we can either project scale-variant parameters to a bounded set, or use weight decay for those parameters (see Appendix C for a proof for the latter solution).

2.5 KEY OBSERVATION: THE GROWTH OF WEIGHTS

The following lemma is our key observation. It establishes a connection between the scale-invariant property and the growth of weight scale, which further implies an automatic decay of learning rates:

Lemma 2.4. *For any scale-invariant weight $\boldsymbol{w}^{(i)}$ in the network Ψ, we have:*

1. *$\boldsymbol{w}_t^{(i)}$ and $\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{L}(\boldsymbol{\theta}_t)$ are always perpendicular;*
2. *$\|\boldsymbol{w}_{t+1}^{(i)}\|_2^2 = \|\boldsymbol{w}_t^{(i)}\|_2^2 + \eta_{\mathrm{w},t}^2\|\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{F}(\boldsymbol{\theta}_t)\|_2^2 = \|\boldsymbol{w}_t^{(i)}\|_2^2 + \frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^2}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{F}(\tilde{\boldsymbol{\theta}}_t)\|_2^2.$*

Proof. Let $\boldsymbol{\theta}'$ be all the parameters other than $\boldsymbol{w}^{(i)}$. Taking derivatives with respect to c for the both sides of $\mathcal{F}_{\boldsymbol{z}}(\boldsymbol{w}^{(i)}, \boldsymbol{\theta}') = \mathcal{F}_{\boldsymbol{z}}(c\boldsymbol{w}^{(i)}, \boldsymbol{\theta}')$, we have

$$0 = \frac{\partial}{\partial c}\mathcal{F}_{\boldsymbol{z}}(c\boldsymbol{w}^{(i)}, \boldsymbol{\theta}').$$

[1] Our results to this network are rather conceptual, since the smoothness upper bound can be as large as $M^{O(L)}$, where L is the number of layers and M is the maximum width of each layer.

The right hand side equals $\nabla_{c\boldsymbol{w}^{(i)}}\mathcal{L}(c\boldsymbol{w}^{(i)},\boldsymbol{\theta}')^\top\boldsymbol{w}^{(i)}$, then the first proposition follows by taking $c=1$. Applying Pythagorean theorem and Lemma 2.2, the second proposition directly follows. □

Using Lemma 2.4, we can show that performing gradient descent for the original problem is equivalent to performing an adaptive gradient method for the intrinsic optimization problem:

Theorem 2.5. *Let Π be a projection operator which maps a vector $\boldsymbol{w}$ to $\boldsymbol{w}/\|\boldsymbol{w}\|_2$. Then*

$$\boldsymbol{v}_{t+1}^{(i)} = \Pi\left(\boldsymbol{v}_t^{(i)} - \frac{\eta_{\mathrm{w},t}}{G_t}\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{F}(\tilde{\boldsymbol{\theta}}_t)\right), \qquad G_{t+1}^{(i)} = G_t^{(i)} + \frac{\eta_{\mathrm{w},t}^2}{G_t^{(i)}}\|\nabla_{\boldsymbol{v}_t^{(i)}},\mathcal{F}(\tilde{\boldsymbol{\theta}}_t)\|_2^2 \tag{9}$$

Remark 2.6. *Wu et al. (2018) noticed that Theorem 2.5 is true for Weight Normalization by direct calculation of gradients. Inspiring by this, they proposed a new adaptive method called* `WNGrad`*. Our theorem is more general since it holds for any normalization methods as long as it induces scale-invariant properties to the network. The adaptive update rule derived in our theorem can be seen as* `WNGrad` *with projection to unit sphere after each step.*

Proof for Theorem 2.5. Let $G_t = \|\boldsymbol{w}_t^{(i)}\|_2^2$. The second equation is by Lemma 2.4. By Lemma 2.2,

$$\boldsymbol{w}_{t+1}^{(i)} = \boldsymbol{w}_t^{(i)} - \frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t^{(i)}\|_2}\nabla_{\boldsymbol{v}^{(i)}}\mathcal{F}_{\boldsymbol{z}}(\tilde{\boldsymbol{\theta}}_t) = \|\boldsymbol{w}_t^{(i)}\|_2\left(\boldsymbol{v}_t^{(i)} - \frac{\eta_{\mathrm{w},t}}{G_t}\nabla_{\boldsymbol{v}^{(i)}}\mathcal{F}_{\boldsymbol{z}}(\tilde{\boldsymbol{\theta}}_t)\right),$$

which implies equation 9. □

While popular adaptive gradient methods such as AdaGrad (Duchi et al., 2011), RMSprop (Hinton et al.), Adam (Kingma & Ba, 2014) adjust learning rates for each single coordinate, this implicit adaptive gradient method described in Theorem 2.5 sets adaptive learning rates $1/G_t^{(i)}$ for each scale-invariant parameter respectively. Sometimes we call $\eta_{\mathrm{w},t}/G_t^{(i)}$ the *effective learning rate* of $\boldsymbol{v}_t^{(i)}$ (or $\boldsymbol{w}_t^{(i)}$) in this paper.

3 Training by full-batch gradient descent

In this section, we rigorously analyze the effect related to the scale-invariant properties in training neural network by full-batch gradient descent. We use the framework introduced in Section 2.2 and assumptions from Section 2.4. We focus on the full-batch training, i.e., $\boldsymbol{z_t}$ is always equal to the whole training set and $\mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}) = \mathcal{L}(\boldsymbol{\theta})$.

3.1 Settings and main theorem

Assumptions on learning rates. We consider the case that we use fixed learning rates for both W and $\boldsymbol{g}$, i.e., $\eta_{\mathrm{w},0} = \cdots = \eta_{\mathrm{w},T-1} = \eta_{\mathrm{w}}$ and $\eta_{\mathrm{g},0} = \cdots = \eta_{\mathrm{g},T-1} = \eta_{\mathrm{g}}$. We assume that η_{g} is tuned carefully to $\eta_{\mathrm{g}} = (1-c_{\mathrm{g}})/L^{\mathrm{gg}}$ for some constant $c_{\mathrm{g}} \in (0,1)$. For η_{w}, we do not make any assumption, i.e., η_{w} can be set to any positive value.

Theorem 3.1. *Consider the process of training Φ by gradient descent with $\eta_{\mathrm{g}} = 2(1-c_{\mathrm{g}})/L^{\mathrm{gg}}$ and arbitrary $\eta_{\mathrm{w}} > 0$. Then Φ converges to a stationary point in the rate of*

$$\min_{0\le t<T}\|\nabla\mathcal{L}(V_t;\boldsymbol{g}_t)\|_2 \le \tilde{O}\left(\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{\eta_{\mathrm{w}}}} + \eta_{\mathrm{w}}^2 + \frac{1+\eta_{\mathrm{w}}}{\sqrt{\eta_{\mathrm{g}}}}\right)\right), \tag{10}$$

where $V_t = \{\boldsymbol{v}_t^{(1)},\ldots,\boldsymbol{v}_t^{(m)}\}$ with $\boldsymbol{v}_t^{(i)} = \boldsymbol{w}_t^{(i)}/\|\boldsymbol{w}_t^{(i)}\|_2$, $\tilde{O}$ suppresses polynomial factors in L_{ij}^{vv}, L_i^{vg}, L^{gg}, $\|\boldsymbol{w}_0^{(i)}\|_2$, $\|\boldsymbol{w}_0^{(i)}\|_2^{-1}$, $\mathcal{L}(\theta_0) - \mathcal{L}_{\min}$ for all i,j, and we see $L^{\mathrm{gg}} = \Omega(1)$.

This matches the asymptotic convergence rate of GD by Carmon et al. (2018).

3.2 Proof sketch

The high level idea is to use the decrement of loss function to upper bound the sum of the squared norm of the gradients. Note that $\|\nabla\mathcal{L}(V_t;\boldsymbol{g}_t)\|_2^2 = \sum_{i=1}^m\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{L}(V_t;\boldsymbol{g}_t)\|_2^2 + \|\nabla_{\boldsymbol{g}_t}\mathcal{L}(V_t;\boldsymbol{g}_t)\|_2^2$.

For the first part $\sum_{i=1}^{m} \|\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2$, we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 &= \sum_{i=1}^{m}\sum_{t=0}^{T-1} \|\boldsymbol{w}_t^{(i)}\|_2^2 \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(W_t; \boldsymbol{g}_t)\|_2^2 \\ &\leq \sum_{i=1}^{m} \|\boldsymbol{w}_T^{(i)}\|_2^2 \cdot \frac{\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2}{\eta_{\mathrm{w}}^2}\end{aligned} \tag{11}$$

Thus the core of the proof is to show that the monotone increasing $\|\boldsymbol{w}_T^{(i)}\|_2$ has an upper bound for all T. It is shown that for every $\boldsymbol{w}^{(i)}$, the whole training process can be divided into at most two phases. In the first phase, the effective learning rate $\eta_w / \|\boldsymbol{w}_t^{(i)}\|_2^2$ is larger than some threshold $\frac{1}{C_i}$ (defined in Lemma 3.2) and in the second phase it is smaller.

Lemma 3.2 (Taylor Expansion)**.** *Let* $C_i := \frac{1}{2}\sum_{j=1}^{m} L_{ij}^{\mathrm{vv}} + L_i^{\mathrm{vg}2} m/(c_{\mathrm{g}} L^{\mathrm{gg}}) = \tilde{O}(1)$. *Then*

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) - \mathcal{L}(\boldsymbol{\theta}_t) = -\sum_{i=1}^{m} \eta_{\mathrm{w}} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\boldsymbol{\theta}_t)\|_2^2 \left(1 - \frac{C_i \eta_{\mathrm{w}}}{\|\boldsymbol{w}_t^{(i)}\|_2^2}\right) - \frac{1}{2} c_{\mathrm{g}} \eta_{\mathrm{g}} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(\boldsymbol{\theta}_t)\|_2^2 . \tag{12}$$

If $\|\boldsymbol{w}_t^{(i)}\|_2$ is large enough and that the process enters the second phase, then by Lemma 3.2 in each step the loss function $\mathcal{L}$ will decrease by $\frac{\eta_{\mathrm{w}}}{2}\|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\boldsymbol{\theta}_t)\|_2^2 = \frac{\|\boldsymbol{w}_{t+1}^{(i)}\|_2^2 - \|\boldsymbol{w}_t^{(i)}\|_2^2}{2\eta_{\mathrm{w}}}$ (Recall that $\|\boldsymbol{w}_{t+1}^{(i)}\|_2^2 - \|\boldsymbol{w}_t^{(i)}\|_2^2 = \eta_{\mathrm{w},t}^2 \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\boldsymbol{\theta}_t)\|_2^2$ by Lemma 2.4). Since $\mathcal{L}$ is lower-bounded, we can conclude $\|\boldsymbol{w}_t^{(i)}\|_2^2$ is also bounded .

For the second part, we can also show that by Lemma 3.2

$$\sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 \leq \frac{2}{\eta_{\mathrm{g}} c_{\mathrm{g}}} \left(\mathcal{L}(\boldsymbol{\theta}_0) - \mathcal{L}(\boldsymbol{\theta}_T) + \sum_{i=1}^{m} \frac{C_i \|\boldsymbol{w}_T^{(i)}\|_2^2}{\|\boldsymbol{w}_0^{(i)}\|_2^2} \right)$$

Thus we can conclude $\tilde{O}(\frac{1}{\sqrt{T}})$ convergence rate of $\|\nabla \mathcal{L}(\boldsymbol{\theta}_t)\|_2$ as follows.

$$\min_{0 \leq t < T} \|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \left(\sum_{i=1}^{m} \|\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 + \sum_{i=1}^{m} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 \right) \leq \tilde{O}(\frac{1}{T})$$

The full proof is postponed to Appendix A.

4 Training by stochastic gradient descent

In this section, we analyze the effect related to the scale-invariant properties when training a neural network by stochastic gradient descent. We use the framework introduced in Section 2.2 and assumptions from Section 2.4.

4.1 Settings and main theorem

Assumptions on learning rates. As usual, we assume that the learning rate for $\boldsymbol{g}$ is chosen carefully and the learning rate for W is chosen rather arbitrarily. More specifically, we consider the case that the learning rates are chosen as

$$\eta_{\mathrm{w},t} = \eta_{\mathrm{w}} \cdot (t+1)^{-\alpha}, \qquad \eta_{\mathrm{g},t} = \eta_{\mathrm{g}} \cdot (t+1)^{-1/2}.$$

We assume that the initial learning rate η_{g} of $\boldsymbol{g}$ is tuned carefully to $\eta_{\mathrm{g}} = (1 - c_{\mathrm{g}})/L^{\mathrm{gg}}$ for some constant $c_{\mathrm{g}} \in (0, 1)$. Note that this learning rate schedule matches the best known convergence rate $O(T^{-1/4})$ of SGD in the case of smooth non-convex loss functions (Ghadimi & Lan, 2013).

For the learning rates of W, we only assume that $0 \leq \alpha \leq 1/2$, i.e., the learning rate decays equally as or slower than the optimal SGD learning rate schedule. η_{w} can be set to any positive value. Note that this includes the case that we set a fixed learning rate $\eta_{\mathrm{w},0} = \cdots = \eta_{\mathrm{w},T-1} = \eta_{\mathrm{w}}$ for W by taking $\alpha = 0$.

Remark 4.1. *Note that the auto-tuning behavior induced by scale-invariances always decreases the learning rates. Thus, if we set $\alpha > 1/2$, there is no hope to adjust the learning rate to the optimal strategy $\Theta(t^{-1/2})$. Indeed, in this case, the learning rate $1/G_t$ in the intrinsic optimization process decays exactly in the rate of $\tilde{\Theta}(t^{-\alpha})$, which is the best possible learning rate can be achieved without increasing the original learning rate.*

Theorem 4.2. *Consider the process of training Φ by gradient descent with $\eta_{\mathrm{w},t} = \eta_{\mathrm{w}} \cdot (t+1)^{-\alpha}$ and $\eta_{\mathrm{g},t} = \eta_{\mathrm{g}} \cdot (t+1)^{-1/2}$, where $\eta_{\mathrm{g}} = 2(1-c_{\mathrm{g}})/L^{\mathrm{gg}}$ and $\eta_{\mathrm{w}} > 0$ is arbitrary. Then Φ converges to a stationary point in the rate of*

$$\min_{0\le t<T} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] \le \begin{cases} \tilde{O}\left(\frac{\log T}{\sqrt{T}}\right) & 0 \le \alpha < 1/2; \\ \tilde{O}\left(\frac{(\log T)^{3/2}}{\sqrt{T}}\right) & \alpha = 1/2. \end{cases} \tag{13}$$

where $V_t = \{\boldsymbol{v}_t^{(1)}, \dots, \boldsymbol{v}_t^{(m)}\}$ with $\boldsymbol{v}_t^{(i)} = \boldsymbol{w}_t^{(i)}/\|\boldsymbol{w}_t^{(i)}\|_2$, $\tilde{O}$ suppresses polynomial factors in $\eta_{\mathrm{w}}, \eta_{\mathrm{w}}^{-1}, \eta_{\mathrm{g}}^{-1}$, L_{ij}^{vv}, L_i^{vg}, L^{gg}, $\|\boldsymbol{w}_0^{(i)}\|_2$, $\|\boldsymbol{w}_0^{(i)}\|_2^{-1}$, $\mathcal{L}(\theta_0) - \mathcal{L}_{\min}$ for all i, j, and we see $L^{\mathrm{gg}} = \Omega(1)$.

Note that this matches the asymptotic convergence rate of SGD, within a $\mathrm{polylog}(T)$ factor.

4.2 Proof sketch

We delay the full proof into Appendix B and give a proof sketch in a simplified setting where there is no $\boldsymbol{g}$ and $\alpha \in [0, \frac{1}{2})$. We also assume there's only one $\boldsymbol{w}_i$, that is, $m = 1$ and omit the index i.

By Taylor expansion, we have

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})\right] \le \mathcal{L}(\boldsymbol{\theta}_t) - \frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{v}_t\|^2}\|\nabla_{\boldsymbol{w}_t}\mathcal{L}(\boldsymbol{\theta}_t)\|_2^2 + \mathbb{E}\left[\frac{\eta_{\mathrm{w},t}^2 L^{\mathrm{vv}}}{\|\boldsymbol{w}_t\|_2^2}\|\nabla_{\boldsymbol{w}_t}\mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t)\|_2^2\right] \tag{14}$$

We can lower bound the effective learning rate $\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T\|^2}$ and upper bound the second order term respectively in the following way:

(1). For all $0 \le \alpha < \frac{1}{2}$, the effective learning rate $\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T\|^2} = \tilde{\Omega}(T^{-1/2})$;

(2). $\sum_{t=0}^{T} \frac{\eta_{\mathrm{w},t}^2 L^{\mathrm{vv}}}{\|\boldsymbol{w}_t\|_2^2}\|\nabla_{\boldsymbol{w}_t}\mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t)\|_2^2 = \tilde{O}\left(\log\left(\frac{\|\boldsymbol{w}_T^2\|}{\|\boldsymbol{w}_0^2\|}\right)\right) = \tilde{O}(\log T)$.

Taking expectation over equation 14 and summing it up, we have

$$\mathbb{E}\left[\sum_{t=0}^{T-1} \frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t\|^2}\|\nabla_{\boldsymbol{v}_t}\mathcal{L}(\boldsymbol{\theta}_t)\|_2^2\right] \le \mathcal{L}(\boldsymbol{\theta}_0) - \mathbb{E}[\mathcal{L}(\boldsymbol{\theta}_T)] + \mathbb{E}\left[\sum_{t=0}^{T-1} \frac{\eta_{\mathrm{w},t}^2 L^{\mathrm{vv}}}{\|\boldsymbol{w}_t\|_2^2}\|\nabla_{\boldsymbol{w}_t}\mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t)\|_2^2\right].$$

Plug the above bounds into the above inequality, we complete the proof.

$$\Omega(T^{-1/2}) \cdot \mathbb{E}\left[\sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{v}_t}\mathcal{L}(\boldsymbol{\theta}_t)\|_2^2\right] \le \mathcal{L}(\boldsymbol{\theta}_0) - \mathbb{E}[\mathcal{L}(\boldsymbol{\theta}_T)] + \tilde{O}(\log T).$$

5 Conclusions and future works

In this paper, we studied how scale-invariances in neural networks with BN help optimization, and showed that (stochastic) gradient descent can achieve asymptotic best convergence rate without tuning learning rates for scale-invariant parameters. Thus, our analysis implies that scale-invariance in nerual networks introduced by BN reduces the efforts for tuning learning rate.

However, our analysis only applies to smooth loss functions. In modern neural networks, ReLU or Leaky ReLU are often used, which makes the loss non-smooth. It would have more implications by showing similar results in non-smooth settings. Also, we only considered gradient descent in this paper. It can be shown that if we perform (stochastic) gradient descent with momentum, the norm of scale-invariant parameters will also be monotone increasing and the effective learning rate will decrease automatically. It would be interesting to use it to show similar convergence results for more gradient methods.

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A Proof for Full-Batch Gradient Descent

By the scale-invariant property of $\boldsymbol{w}^{(i)}$, we know that $\mathcal{L}(W;\boldsymbol{g}) = \mathcal{L}(V;\boldsymbol{g})$. Also, the following identities about derivatives can be easily obtained:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{w}^{(i)}\partial \boldsymbol{w}^{(j)}}\mathcal{L}(W;\boldsymbol{g}) &= \frac{1}{\|\boldsymbol{w}^{(i)}\|_2\|\boldsymbol{w}^{(j)}\|_2}\frac{\partial}{\partial \boldsymbol{v}^{(i)}\partial \boldsymbol{v}^{(j)}}\mathcal{L}(V;\boldsymbol{g}) \\
\frac{\partial}{\partial \boldsymbol{w}^{(i)}\partial \boldsymbol{g}}\mathcal{L}(W;\boldsymbol{g}) &= \frac{1}{\|\boldsymbol{w}^{(i)}\|_2\|\boldsymbol{w}^{(j)}\|_2}\frac{\partial}{\partial \boldsymbol{v}^{(i)}\partial \boldsymbol{g}}\mathcal{L}(V;\boldsymbol{g}) \\
\nabla^2_{\boldsymbol{g}}\mathcal{L}(W;\boldsymbol{g}) &= \nabla^2_{\boldsymbol{g}}\mathcal{L}(V;\boldsymbol{g}).
\end{aligned}$$

Thus, the assumptions on the smoothness imply

$$\left\|\frac{\partial}{\partial \boldsymbol{w}^{(i)}\partial \boldsymbol{w}^{(j)}}\mathcal{L}(W;\boldsymbol{g})\right\|_2 \le \frac{L^{\mathrm{vv}}_{ij}}{\|\boldsymbol{w}^{(i)}\|_2\|\boldsymbol{w}^{(j)}\|_2} \tag{15}$$

$$\left\|\frac{\partial}{\partial \boldsymbol{w}^{(i)}\partial \boldsymbol{g}}\mathcal{L}(W;\boldsymbol{g})\right\|_2 \le \frac{L^{\mathrm{vg}}_{i}}{\|\boldsymbol{w}^{(i)}\|_2} \tag{16}$$

$$\left\|\nabla^2_{\boldsymbol{g}}\mathcal{L}(W;\boldsymbol{g})\right\|_2 \le L^{\mathrm{gg}}. \tag{17}$$

Proof for Lemma 3.2. Using Taylor expansion, we have $\exists\gamma \in (0,1)$, such that for $\boldsymbol{w}_t^{(i')} = (1-\gamma)\boldsymbol{w}_t^{(i)} + \gamma\boldsymbol{w}_{t+1}^{(i)}$,

$$\begin{aligned}
\mathcal{L}(\theta_{t+1}) - \mathcal{L}(\theta_t) \le& \sum_{i=1}^{m}\Delta\boldsymbol{w}_t^{(i)\top}\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{L}(\theta_t) + \Delta\boldsymbol{g}_t{}^{\top}\nabla_{\boldsymbol{g}_t}\mathcal{L}(\theta_t) \\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\|\Delta\boldsymbol{w}_t^{(i)}\right\|_2\left\|\Delta\boldsymbol{w}_t^{(j)}\right\|_2\frac{L^{\mathrm{vv}}_{ij}}{\left\|\boldsymbol{w}_t^{(i')}\right\|_2\left\|\boldsymbol{w}_t^{(j')}\right\|_2} \\
&+ \sum_{i=1}^{m}\left\|\Delta\boldsymbol{w}_t^{(i)}\right\|_2\|\Delta\boldsymbol{g}_t\|_2\frac{L^{\mathrm{vg}}_{i}}{\left\|\boldsymbol{w}_t^{(i')}\right\|_2} \\
&+ \frac{1}{2}\sum_{i=1}^{m}\|\Delta\boldsymbol{g}_t\|_2^2\,L^{\mathrm{gg}}.
\end{aligned}$$

Note that $\boldsymbol{w}_{t+1}^{(i)} - \boldsymbol{w}_t^{(i)} = \eta_{\mathrm{w}} \nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\boldsymbol{\theta}_t)$ is perpendicular to $\boldsymbol{w}_t^{(i)}$, we have

$$\|\boldsymbol{w}_t^{(i')}\|_2 \geq \|\gamma(\boldsymbol{w}_{t+1}^{(i)} - \boldsymbol{w}_t^{(i)}) + \boldsymbol{w}_t^{(i)}\|_2 \geq \|\boldsymbol{w}_t^{(i)}\|_2.$$

Thus,

$$\begin{aligned}
\mathcal{L}(\theta_{t+1}) - \mathcal{L}(\theta_t) \leq & \sum_{i=1}^{m} \Delta \boldsymbol{w}_t^{(i)\top} \nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\theta_t) + \Delta \boldsymbol{g}_t{}^\top \nabla_{\boldsymbol{g}_t} \mathcal{L}(\theta_t) \\
& + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \left\| \Delta \boldsymbol{w}_t^{(i)} \right\|_2 \left\| \Delta \boldsymbol{w}_t^{(j)} \right\|_2 \frac{L_{ij}^{\mathrm{vv}}}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2 \left\| \boldsymbol{w}_t^{(j)} \right\|_2} \\
& + \sum_{i=1}^{m} \left\| \Delta \boldsymbol{w}_t^{(i)} \right\|_2 \|\Delta \boldsymbol{g}_t\|_2 \frac{L_i^{\mathrm{vg}}}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2} \\
& + \frac{1}{2} \sum_{i=1}^{m} \|\Delta \boldsymbol{g}_t\|_2^2 L^{\mathrm{gg}}.
\end{aligned}$$

By the inequality of arithmetic and geometric means, we have

$$\begin{aligned}
\left\| \Delta \boldsymbol{w}_t^{(i)} \right\|_2 \left\| \Delta \boldsymbol{w}_t^{(j)} \right\|_2 \frac{L_{ij}^{\mathrm{vv}}}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2 \left\| \boldsymbol{w}_t^{(j)} \right\|_2} &\leq \frac{1}{2} \left\| \Delta \boldsymbol{w}_t^{(i)} \right\|_2^2 \frac{L_{ij}^{\mathrm{vv}}}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2^2} + \frac{1}{2} \left\| \Delta \boldsymbol{w}_t^{(j)} \right\|_2^2 \frac{L_{ij}^{\mathrm{vv}}}{\left\| \boldsymbol{w}_t^{(j)} \right\|_2^2} \\
\left\| \Delta \boldsymbol{w}_t^{(i)} \right\|_2 \|\Delta \boldsymbol{g}_t\|_2 \frac{L_i^{\mathrm{vg}}}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2} &\leq \left\| \Delta \boldsymbol{w}_t^{(i)} \right\|_2^2 \frac{L_i^{\mathrm{vg}2} m/(c_{\mathrm{g}} L^{\mathrm{gg}})}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2^2} + \frac{1}{4} c_{\mathrm{g}} \|\Delta \boldsymbol{g}_t\|_2^2 \frac{L^{\mathrm{gg}}}{m}.
\end{aligned}$$

Taking $\Delta \boldsymbol{w}_t^{(i)} = -\eta_{\mathrm{w}} \nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\theta_t), \Delta \boldsymbol{g}_t = -\eta_{\mathrm{g}} \nabla_{\boldsymbol{g}_t} \mathcal{L}(\theta_t)$, we have

$$\begin{aligned}
\mathcal{L}(\theta_{t+1}) - \mathcal{L}(\theta_t) \leq & -\sum_{i=1}^{m} \eta_{\mathrm{w}} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\theta_t)\|_2^2 - \eta_{\mathrm{g}} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(\theta_t)\|_2^2 \\
& + \frac{1}{2} \sum_{i=1}^{m} \left(\sum_{j=1}^{m} L_{ij}^{\mathrm{vv}} + 2 L_i^{\mathrm{vg}2} m/(c_{\mathrm{g}} L^{\mathrm{gg}}) \right) \frac{\eta_{\mathrm{w}}^2}{\left\| \boldsymbol{w}_t^{(i)} \right\|_2^2} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(\theta_t)\|_2^2 \\
& + \frac{1}{2} (1 + c_{\mathrm{g}}/2) \eta_{\mathrm{g}}^2 \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(\theta_t)\|_2^2 L^{\mathrm{gg}}.
\end{aligned}$$

We can complete the proof by replacing $\frac{1}{2} \sum_{j=1}^{m} L_{ij}^{\mathrm{vv}} + L_i^{\mathrm{vg}2} m/(c_{\mathrm{g}} L^{\mathrm{gg}})$ with C_i. □

Using the assumption on the smoothness, we can show that the gradient with respect to $\boldsymbol{w}^{(i)}$ is essentially bounded:

Lemma A.1. *For any W and $\boldsymbol{g}$, we have*

$$\|\nabla_{\boldsymbol{w}^{(i)}} \mathcal{L}(W; \boldsymbol{g})\|_2 \leq \frac{\pi L_{ii}^{\mathrm{vv}}}{\|\boldsymbol{w}^{(i)}\|_2}. \tag{18}$$

Proof. A.1 Fix all the parameters except $\boldsymbol{w}^{(i)}$. Then $\mathcal{L}(W; \boldsymbol{g})$ can be written as a function $f(\boldsymbol{w}^{(i)})$ on the variable $\boldsymbol{w}^{(i)}$. Let $S = \{\boldsymbol{w}^{(i)} \mid \|\boldsymbol{w}^{(i)}\|_2 = 1\}$. Since f is continuous and S is compact, there must exist $\boldsymbol{v}_{\min}^{(i)} \in S$ such that $f(\boldsymbol{v}_{\min}^{(i)}) \leq f(\boldsymbol{w}^{(i)})$ for all $\boldsymbol{w}^{(i)} \in S$. Note that $\boldsymbol{w}^{(i)}$ is scale-invariant, so $\boldsymbol{w}_{\min}^{(i)}$ is also a minimum in the entire domain and $\nabla f(\boldsymbol{w}_{\min}^{(i)}) = 0$.

For an arbitrary $\boldsymbol{w}^{(i)}$, let $\boldsymbol{v}^{(i)} = \boldsymbol{w}^{(i)}/\|\boldsymbol{w}^{(i)}\|_2$. Let $h : [0, 1] \to S$ be a curve such that $h(0) = \boldsymbol{v}_{\min}^{(i)}$, $h(1) = \boldsymbol{v}^{(i)}$, and h goes along the geodesic from $\boldsymbol{v}_{\min}^{(i)}$ to $\boldsymbol{v}^{(i)}$ on the unit sphere S with constant speed. Let $H(\tau) = \nabla f(h(\tau))$. By Taylor expansion, we have

$$\nabla f(\boldsymbol{v}^{(i)}) = H(1) = H(0) + H'(\xi) = \nabla^2 f(h(\xi)) h'(\xi).$$

Thus, $\|\nabla f(\boldsymbol{v}^{(i)})\|_2 \leq \pi L_i^{\mathrm{vv}}$ and $\|\nabla_{\boldsymbol{w}^{(i)}} \mathcal{L}(W; \boldsymbol{g})\|_2 \leq \frac{\pi L_{ii}^{\mathrm{vv}}}{\|\boldsymbol{w}^{(i)}\|_2}$. □

The following lemma gives an upper bound to the weight scales.

Lemma A.2. *For all $T \geq 0$, we have*

1. *for $i = 1, \ldots, m$, let t_i be the maximum time $0 \leq \tau < T$ satisfying $\|\boldsymbol{w}_\tau^{(i)}\|_2^2 \leq 2C_i\eta_{\mathrm{w}}$ ($t_i = -1$ if no such τ exists), then $\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2$ can be bounded by*

$$\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2 \leq \|\boldsymbol{w}_0^{(i)}\|_2^2 + \eta_{\mathrm{w}}\left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right); \tag{19}$$

2. *the following inequality on weight scales at time T holds:*

$$\sum_{i=1}^{m} \frac{\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2}{2\eta_{\mathrm{w}}} + \frac{1}{2}c_{\mathrm{g}}\eta_{\mathrm{g}} \sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{g}_t}\mathcal{L}(\theta_t)\|_2^2 \leq \mathcal{L}(\theta_0) - \mathcal{L}_{\min} + \sum_{i=1}^{m} K_i, \tag{20}$$

where $K_i = \left(\frac{C_i\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2} + \frac{1}{2}\right)\left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right) = \tilde{O}(\eta_{\mathrm{w}}^2 + 1)$.

Proof. A.2 For every $i = 1, \ldots, m$, let $S_t^{(i)} := -\eta_{\mathrm{w}} \sum_{\tau=0}^{t-1} \|\nabla_{\boldsymbol{w}_\tau^{(i)}}\mathcal{L}(\theta_\tau)\|_2^2 \left(1 - \frac{C_i\eta_{\mathrm{w}}}{\|\boldsymbol{w}_\tau^{(i)}\|_2^2}\right)$. Also let $G_t := -(1/2)c_{\mathrm{g}}\eta_{\mathrm{g}} \sum_{\tau=0}^{t-1} \|\nabla_{\boldsymbol{g}_t}\mathcal{L}(\theta_t)\|_2^2$. By Lemma 3.2 we know that $\mathcal{L}(\theta_T) \leq \mathcal{L}(\theta_0) + \sum_{i=1}^{m} S_T^{(i)} + G_T$.

Upper Bound for Weight Scales at $t_i + 1$. If $t_i = -1$, then $\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2 = \|\boldsymbol{w}_0^{(i)}\|_2^2$. For $t_i \geq 0$, we can bound $\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2$ by Lemma A.1:

$$\begin{aligned}
\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2 = \|\boldsymbol{w}_{t_i}^{(i)}\|_2^2 + \eta_{\mathrm{w}}^2 \left\|\nabla_{\boldsymbol{w}_{t_i}^{(i)}}\mathcal{L}(\theta_{t_i})\right\|_2^2 &\leq 2C_i\eta_{\mathrm{w}} + \eta_{\mathrm{w}}^2 \left(\frac{\pi L_i^{\mathrm{vv}}}{\|\boldsymbol{w}_0^{(i)}\|_2}\right)^2 \\
&= \eta_{\mathrm{w}}\left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right).
\end{aligned}$$

In either case, we have $\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2 \leq \|\boldsymbol{w}_0^{(i)}\|_2^2 + \eta_{\mathrm{w}}\left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right)$.

Upper Bound for Weight Scales at T. Since $\|\boldsymbol{w}_\tau^{(i)}\|_2^2$ is non-decreasing with τ and $\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2 > 2C_i\eta_{\mathrm{w}}$,

$$\begin{aligned}
S_T^{(i)} - S_{t_i+1}^{(i)} &= -\eta_{\mathrm{w}} \sum_{\tau=t_i'+1}^{T-1} \|\nabla_{\boldsymbol{w}_\tau^{(i)}}\mathcal{L}(\theta_\tau)\|_2^2 \left(1 - \frac{C_i\eta_{\mathrm{w}}}{\|\boldsymbol{w}_\tau^{(i)}\|_2^2}\right) \\
&\leq -\frac{\eta_{\mathrm{w}}}{2} \sum_{\tau=t_i'+1}^{T-1} \|\nabla_{\boldsymbol{w}_\tau^{(i)}}\mathcal{L}(\theta_\tau)\|_2^2 \\
&= -\frac{1}{2\eta_{\mathrm{w}}}\left(\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_{t_i'+1}^{(i)}\|_2^2\right) \\
&\leq -\frac{1}{2\eta_{\mathrm{w}}}\|\boldsymbol{w}_T^{(i)}\|_2^2 + \frac{1}{2\eta_{\mathrm{w}}}\left(\|\boldsymbol{w}_0^{(i)}\|_2^2 + \eta_{\mathrm{w}}\left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right)\right) \\
&\leq -\frac{\|\boldsymbol{w}_T^{(i)}\|_2^2}{2\eta_{\mathrm{w}}} + \frac{\|\boldsymbol{w}_0^{(i)}\|_2^2}{2\eta_{\mathrm{w}}} + \frac{1}{2}\left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right),
\end{aligned}$$

where we use the fact that $\|\boldsymbol{w}_t^{(i)}\|_2^2 = \|\boldsymbol{w}_0^{(i)}\|_2^2 + \eta_{\mathrm{w}}^2 \sum_{\tau=0}^{t-1} \|\nabla_{\boldsymbol{w}_\tau^{(i)}} \mathcal{L}(\theta_\tau)\|_2^2$ at the third line. We can bound $S_{t_i+1}^{(i)}$ by

$$
\begin{aligned}
S_{t_i+1}^{(i)} \leq \eta_{\mathrm{w}} \left(\sum_{\tau=0}^{t_i} \|\nabla_{\boldsymbol{w}_\tau^{(i)}} \mathcal{L}(\theta_\tau)\|_2^2 \right) \frac{C_i \eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2} &\leq (\|\boldsymbol{w}_{t_i+1}^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2) \cdot \frac{C_i}{\|\boldsymbol{w}_0^{(i)}\|_2^2} \\
&\leq \eta_{\mathrm{w}} \left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2} \right) \cdot \frac{C_i}{\|\boldsymbol{w}_0^{(i)}\|_2^2}.
\end{aligned}
$$

Combining them together, we have

$$
\begin{aligned}
\frac{\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2}{2\eta_{\mathrm{w}}} &\leq -S_T^{(i)} + S_{t_i+1}^{(i)} + \frac{1}{2} \left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2} \right) \\
&\leq -S_T^{(i)} + \left(\frac{C_i \eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2} + \frac{1}{2} \right) \left(2C_i + (\pi L_i^{\mathrm{vv}})^2 \frac{\eta_{\mathrm{w}}}{\|\boldsymbol{w}_0^{(i)}\|_2^2} \right) \\
&\leq -S_T^{(i)} + K_i.
\end{aligned}
$$

Taking sum over all $i = 1, \ldots, m$ and also subtracting G_T on the both sides, we have

$$
\begin{aligned}
\sum_{i=1}^{m} \frac{\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2}{2\eta_{\mathrm{w}}} + (1/2) c_{\mathrm{g}} \eta_{\mathrm{g}} \sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(\theta_t)\|_2^2 &\leq -\sum_{i=1}^{m} S_T^{(i)} - G_T + \sum_{i=1}^{m} K_i \\
&\leq \mathcal{L}(\theta_0) - \mathcal{L}_{\min} + \sum_{i=1}^{m} K_i.
\end{aligned}
$$

where $\mathcal{L}_{\min} \leq \mathcal{L}(\theta_T) \leq \mathcal{L}(\theta_0) + \sum_{i=1}^{m} S_T^{(i)} + G_T$ is used at the second line. □

Combining the lemmas above together, we can obtain our results.

Proof for Theorem 3.1. By Lemma A.2, we have

$$
\begin{aligned}
\sum_{i=1}^{m} \sum_{t=0}^{T-1} \left\| \nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}(V_t; \boldsymbol{g}_t) \right\|_2^2 &= \sum_{i=1}^{m} \sum_{t=0}^{T-1} \|\boldsymbol{w}_t^{(i)}\|_2^2 \left\| \nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}(W_t; \boldsymbol{g}_t) \right\|_2^2 \\
&\leq \sum_{i=1}^{m} \|\boldsymbol{w}_T^{(i)}\|_2^2 \cdot \frac{\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2}{\eta_{\mathrm{w}}^2} \\
&\leq \max_{1 \leq i \leq m} \left\{ \frac{\|\boldsymbol{w}_T^{(i)}\|_2^2}{\eta_{\mathrm{w}}} \right\} \cdot \sum_{i=1}^{m} \frac{\|\boldsymbol{w}_T^{(i)}\|_2^2 - \|\boldsymbol{w}_0^{(i)}\|_2^2}{\eta_{\mathrm{w}}} \\
&\leq 4 \left(\mathcal{L}(\theta_0) - \mathcal{L}_{\min} + \sum_{i=1}^{m} K_i + \sum_{i=1}^{m} \frac{\|\boldsymbol{w}_0^{(i)}\|_2^2}{\eta_{\mathrm{w}}} \right) \left(\mathcal{L}(\theta_0) - \mathcal{L}_{\min} + \sum_{i=1}^{m} K_i \right) \\
&\leq \tilde{O} \left(\frac{1}{\eta_{\mathrm{w}}} + \eta_{\mathrm{w}}^4 \right). \\
\sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 &\leq \frac{2}{c_{\mathrm{g}} \eta_{\mathrm{g}}} \left(\mathcal{L}(\theta_0) - \mathcal{L}_{\min} + \sum_{i=1}^{m} K_i \right) \\
&\leq \tilde{O} \left(\frac{1 + 1\eta_{\mathrm{w}}^2}{\eta_{\mathrm{g}}} \right).
\end{aligned}
$$

Combining them together we have

$$
\begin{aligned}
& T \cdot \min_{0 \le t < T} \left\{ \sum_{i=1}^{m} \left\| \nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}(V_t; \boldsymbol{g}_t) \right\|_2^2 + \|\boldsymbol{g}_t\|_2^2 \right\} \\
& \le \sum_{i=1}^{m} \sum_{t=0}^{T-1} \left\| \nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}(V_t; \boldsymbol{g}_t) \right\|_2^2 + \sum_{t=0}^{T-1} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2 \\
& \le \tilde{O}\left(\frac{1}{\eta_{\mathrm{w}}} + \eta_{\mathrm{w}}^4 + \frac{1+\eta_{\mathrm{w}}^2}{\eta_{\mathrm{g}}} \right).
\end{aligned}
$$

Thus $\min_{0 \le t < T} \|\nabla \mathcal{L}(V_t, g_t)\|_2$ converges in the rate of

$$
\min_{0 \le t < T} \|\nabla \mathcal{L}(V_t, g_t)\|_2 = \tilde{O}\left(\frac{1}{\sqrt{T}} \left(\frac{1}{\sqrt{\eta_{\mathrm{w}}}} + \eta_{\mathrm{w}}^2 + \frac{1+\eta_{\mathrm{w}}}{\sqrt{\eta_{\mathrm{g}}}} \right) \right).
$$

□

B Proof for Stochastic Gradient Descent

Let $\mathscr{F}_t = \sigma\{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{t-1}\}$ be the filtration, where $\sigma\{\cdot\}$ denotes the sigma field.

We use $\mathcal{L}_t := \mathcal{L}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t), \mathcal{F}_t := \mathcal{F}_{\boldsymbol{z}_t}(\boldsymbol{\theta}_t)$ for simplicity. As usual, we define $\boldsymbol{v}_t^{(i)} = \boldsymbol{w}_t^{(i)} / \|\boldsymbol{w}_t^{(i)}\|_2$. We use the notations $\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}_t := \nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}_{\boldsymbol{z}_t}(V_t, \boldsymbol{g}_t)$ and $\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{F}_t := \nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{F}_{\boldsymbol{z}_t}(V_t, \boldsymbol{g}_t)$ for short. Let $\Delta \boldsymbol{w}_t^{(i)} = -\eta_{\mathrm{w},t} \nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{F}_t$, $\Delta \boldsymbol{g}_t = -\eta_{\mathrm{g},t} \nabla_{\boldsymbol{g}_t} \mathcal{F}_t$.

Lemma B.1. *For any $a_0, \ldots, a_T \in [0, B]$ with $a_0 > 0$,*

$$
\sum_{t=1}^{T} \frac{a_t}{\sum_{\tau=0}^{t-1} a_\tau} \le \log_2 \left(\sum_{t=0}^{T-1} a_t / a_0 \right) + 1 + 2B/a_0
$$

Proof. Let t_i be the minimum $1 \le t \le T$ such that $\sum_{\tau=0}^{t-1} a_\tau \ge a_0 \cdot 2^i$. Let k be the maximum i such that t_i exists. Let $t_{k+1} = T+1$. Then we know that

$$
\sum_{t=t_i}^{t_{i+1}-1} \frac{a_t}{\sum_{\tau=0}^{t-1} a_\tau} \le \frac{a_0 \cdot 2^i + B}{a_0 \cdot 2^i} \le 1 + \frac{B}{a_0} \cdot 2^{-i}.
$$

Thus, $\sum_{t=1}^{T} \frac{a_t}{\sum_{\tau=0}^{t-1} a_\tau} \le \sum_{i=0}^{k} \left(1 + \frac{B}{a_0} \cdot 2^{-i}\right) = k + 1 + \frac{2B}{a_0} \le \log_2 \left(\sum_{t=0}^{T-1} a_t / a_0 \right) + 1 + 2B/a_0$.

□

Lemma B.2. *Fix $T > 0$. Let*

$$
\begin{aligned}
C_i &:= \frac{1}{2} \sum_{j=1}^{m} L_{ij}^{\mathrm{vv}} + L_i^{\mathrm{vg}2} m / (c_{\mathrm{g}} L^{\mathrm{gg}}) \\
U &:= \frac{1 + c_{\mathrm{g}}/2}{2} L^{\mathrm{gg}} G_{\mathrm{g}}^2 \\
S_i &:= -\sum_{t=0}^{T-1} \frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t^{(i)}\|_2^2} \|\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}_t\|_2^2 + C_i \sum_{t=0}^{T-1} \frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^4} \|\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{F}_t\|_2^2 \\
R &:= -\frac{1}{2} c_{\mathrm{g}} \sum_{t=0}^{T-1} \eta_{\mathrm{g},t} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}_t\|_2^2 + U \sum_{t=0}^{T-1} \eta_{\mathrm{g},t}^2 G_{\mathrm{g}}
\end{aligned}
$$

Then $\mathbb{E}[\mathcal{L}_T] - \mathcal{L}_0 \le \sum_{i=1}^{m} \mathbb{E}[S_i] + \mathbb{E}[R]$.

Proof. Conditioned on $\mathscr{F}_t$, by Taylor expansion, we have

$$\mathbb{E}[\mathcal{L}_{t+1} \mid \mathscr{F}_t] \leq \mathcal{L}_t - \eta_{\mathrm{w},t} \sum_{i=1}^{m} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{L}_t\|_2^2 - \eta_{\mathrm{g},t} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}_t\|_2^2 + \mathbb{E}[Q_t \mid \mathscr{F}_t] \tag{21}$$

where Q_t is

$$\begin{aligned} Q_t = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{L_{ij}^{\mathrm{vv}}}{\left\|\boldsymbol{w}_t^{(i)}\right\|_2 \left\|\boldsymbol{w}_t^{(j)}\right\|_2} \|\Delta \boldsymbol{w}_t^{(i)}\|_2 \|\Delta \boldsymbol{w}_t^{(j)}\|_2 \\ + \sum_{i=1}^{m} \frac{L_i^{\mathrm{vg}}}{\left\|\boldsymbol{w}_t^{(i)}\right\|_2} \|\Delta \boldsymbol{w}_t^{(i)}\|_2 \|\Delta \boldsymbol{g}_t\|_2 + \frac{1}{2} L^{\mathrm{gg}} \|\Delta \boldsymbol{g}_t\|_2^2 \end{aligned}$$

By the inequality $\sqrt{ab} \leq \frac{1}{2}a + \frac{1}{2}b$, we have

$$\begin{aligned} \|\Delta \boldsymbol{w}_t^{(i)}\|_2 \|\Delta \boldsymbol{w}_t^{(j)}\|_2 \frac{L_{ij}^{\mathrm{vv}}}{\|\boldsymbol{w}_t^{(i)}\|_2 \|\boldsymbol{w}_t^{(j)}\|_2} &\leq \frac{1}{2} \|\Delta \boldsymbol{w}_t^{(i)}\|_2^2 \frac{L_{ij}^{\mathrm{vv}}}{\|\boldsymbol{w}_t^{(i)}\|_2^2} + \frac{1}{2} \|\Delta \boldsymbol{w}_t^{(j)}\|_2^2 \frac{L_{ij}^{\mathrm{vv}}}{\|\boldsymbol{w}_t^{(j)}\|_2^2} \\ \|\Delta \boldsymbol{w}_t^{(i)}\|_2 \|\Delta \boldsymbol{g}_t\|_2 \frac{L_i^{\mathrm{vg}}}{\|\boldsymbol{w}_t^{(i)}\|_2} &\leq \|\Delta \boldsymbol{w}_t^{(i)}\|_2^2 \frac{L_i^{\mathrm{vg}2} m/(c_{\mathrm{g}} L^{\mathrm{gg}})}{\|\boldsymbol{w}_t^{(i)}\|_2^2} + \frac{1}{4} c_{\mathrm{g}} \|\Delta \boldsymbol{g}_t\|_2^2 \frac{L^{\mathrm{gg}}}{m}. \end{aligned}$$

Note that $\mathbb{E}[\|\Delta \boldsymbol{g}_t\|_2^2 \mid \mathscr{F}_t] \leq \eta_{\mathrm{g},t}^2 \left(\|\nabla_{\boldsymbol{g}_t} \mathcal{F}_t\|_2^2 + G_{\mathrm{g}}^2\right)$. Thus,

$$\begin{aligned} \mathbb{E}[Q_t \mid \mathscr{F}_t] &\leq \sum_{i=1}^{m} \frac{1}{2} \left(\sum_{j=1}^{m} L_{ij}^{\mathrm{vv}} + 2 L_i^{\mathrm{vg}2} m/(c_{\mathrm{g}} L^{\mathrm{gg}}) \right) \frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^2} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{F}_t\|_2^2 \\ &\quad + \frac{1 + c_{\mathrm{g}}/2}{2} L^{\mathrm{gg}} \cdot \eta_{\mathrm{g},t}^2 \left(\|\nabla_{\boldsymbol{g}_t} \mathcal{F}_t\|_2^2 + G_{\mathrm{g}}^2\right) \\ &\leq \sum_{i=1}^{m} C_i \frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^2} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{F}_t\|_2^2 + (1 - c_{\mathrm{g}}/2) \eta_{\mathrm{g},t} \|\nabla_{\boldsymbol{g}_t} \mathcal{F}_t\|_2^2 + U \eta_{\mathrm{g},t}^2 \end{aligned}$$

Taking this into equation 21 and summing up for all t, we have

$$\begin{aligned} \mathbb{E}[\mathcal{L}_T] - \mathcal{L}_0 \leq - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \mathbb{E} \left[\frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t^{(i)}\|_2^2} \|\nabla_{\boldsymbol{v}_t^{(i)}} \mathcal{L}_t\|_2^2 \right] - \frac{1}{2} c_{\mathrm{g}} \sum_{t=0}^{T-1} \mathbb{E} \left[\eta_{\mathrm{g},t} \|\nabla_{\boldsymbol{g}_t} \mathcal{L}_t\|_2^2 \right] \\ + \sum_{i=1}^{m} C_i \sum_{t=0}^{T-1} \mathbb{E} \left[\frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^2} \|\nabla_{\boldsymbol{w}_t^{(i)}} \mathcal{F}_t\|_2^2 \right] + U \sum_{t=0}^{T-1} \eta_{\mathrm{g},t}^2 G_{\mathrm{g}}, \end{aligned}$$

and the right hand side can be expressed as $\sum_{i=1}^{m} \mathbb{E}[S_i] + \mathbb{E}[R]$ by definitions. □

Lemma B.3. *For any $T \geq 0$, $1 \leq i \leq m$, we have*

$$\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T^{(i)}\|_2^2} \geq \begin{cases} \tilde{\Omega}(T^{-1/2}) & \text{if } 0 \leq \alpha < 1/2; \\ \tilde{\Omega}((T \log T)^{-1/2}) & \text{if } \alpha = 1/2. \end{cases}$$

Proof. By Lemma 2.4, we have

$$\begin{aligned}
\|\boldsymbol{w}_{t+1}^{(i)}\|_2^4 &= \left(\|\boldsymbol{w}_t^{(i)}\|_2^2 + \frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^2}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{F}_t\|_2^2\right)^2 \\
&= \|\boldsymbol{w}_t^{(i)}\|_2^4 + 2\eta_{\mathrm{w},t}^2\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{F}_t\|_2^2 + \frac{\eta_{\mathrm{w},t}^4}{\|\boldsymbol{w}_t^{(i)}\|_2^4}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{F}_t\|_2^4 \\
&\le \|\boldsymbol{w}_t^{(i)}\|_2^4 + 2\eta_{\mathrm{w}}^2(t+1)^{-2\alpha}\left(\pi L_{ii}^{\mathrm{vv}}\right)^2 + \eta_{\mathrm{w}}^4(t+1)^{-4\alpha}\left(\frac{\pi L_{ii}^{\mathrm{vv}}}{\|\boldsymbol{w}_0^{(i)}\|_2}\right)^4 \\
&\le \|\boldsymbol{w}_t^{(i)}\|_2^4 + (t+1)^{-2\alpha}\left(2\eta_{\mathrm{w}}^2\left(\pi L_{ii}^{\mathrm{vv}}\right)^2 + \eta_{\mathrm{w}}^4\left(\frac{\pi L_{ii}^{\mathrm{vv}}}{\|\boldsymbol{w}_0^{(i)}\|_2}\right)^4\right) \\
&\le \|\boldsymbol{w}_0^{(i)}\|_2^4 + \left(\sum_{\tau=0}^{t}\frac{1}{(\tau+1)^{2\alpha}}\right)\left(2\eta_{\mathrm{w}}^2\left(\pi L_{ii}^{\mathrm{vv}}\right)^2 + \eta_{\mathrm{w}}^4\left(\frac{\pi L_{ii}^{\mathrm{vv}}}{\|\boldsymbol{w}_0^{(i)}\|_2}\right)^4\right).
\end{aligned}$$

For $0 \le \alpha < 1/2$, $\sum_{\tau=0}^{T-1}\frac{1}{(\tau+1)^{2\alpha}} = O(T^{1-2\alpha})$, so $\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T^{(i)}\|_2^2} = \tilde{\Omega}(T^{-1/2})$; for $\alpha = 1/2$, $\sum_{\tau=0}^{T-1}\frac{1}{(\tau+1)^{2\alpha}} = O(\log T)$, so $\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T^{(i)}\|_2^2} = \tilde{\Omega}((T\log T)^{-1/2})$. □

Lemma B.4. *For $T > 0$,*

- *S_i can be bounded by $S_i \le -\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T^{(i)}\|_2^2}\sum_{t=0}^{T-1}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{L}_t\|_2^2 + \tilde{O}(\log T)$;*
- *R can be bounded by $R \le -\tilde{\Omega}(T^{-1/2})\sum_{t=0}^{T-1}\|\nabla_{\boldsymbol{g}_t}\mathcal{L}_t\|_2^2 + \tilde{O}(\log T)$.*

Proof. Fix $i \in \{1, \ldots, m\}$. First we bound S_i. Recall that

$$S_i := -\sum_{t=0}^{T-1}\frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t^{(i)}\|_2^2}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{L}_t\|_2^2 + C_i\sum_{t=0}^{T-1}\frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{v}_t^{(i)}\|_2^2}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{F}_t\|_2^2.$$

Note that $\frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t^{(i)}\|_2^2}$ is non-increasing, so

$$-\sum_{t=0}^{T-1}\frac{\eta_{\mathrm{w},t}}{\|\boldsymbol{w}_t^{(i)}\|_2^2}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{L}_t\|_2^2 \le -\frac{\eta_{\mathrm{w},T}}{\|\boldsymbol{w}_T^{(i)}\|_2^2}\sum_{t=0}^{T-1}\|\nabla_{\boldsymbol{v}_t^{(i)}}\mathcal{L}_t\|_2^2. \tag{22}$$

Also note that $\|\boldsymbol{w}_t^{(i)}\|_2^2 \ge \|\boldsymbol{w}_0^{(i)}\|_2^2 + \sum_{\tau=0}^{t-1}\eta_{\mathrm{w},\tau}^2\|\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{F}_t\|_2^2$. By Lemma B.1,

$$\begin{aligned}
C_i\sum_{t=0}^{T-1}\frac{\eta_{\mathrm{w},t}^2}{\|\boldsymbol{w}_t^{(i)}\|_2^2}\|\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{F}_t\|_2^2 &\le C_i\sum_{t=0}^{T-1}\frac{\eta_{\mathrm{w},t}^2\|\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{F}_t\|_2^2}{\|\boldsymbol{w}_0^{(i)}\|_2^2 + \sum_{\tau=0}^{t-1}\eta_{\mathrm{w},\tau}^2\|\nabla_{\boldsymbol{w}_t^{(i)}}\mathcal{F}_t\|_2^2} \\
&\le C_i\left(\log_2\left(\frac{\|\boldsymbol{w}_T^{(i)}\|_2^2}{\|\boldsymbol{w}_0^{(i)}\|_2^2}\right) + 1 + \frac{2\eta_{\mathrm{w}}^2(\pi L_{ii}^{\mathrm{vv}})^2}{\|\boldsymbol{w}_0^{(i)}\|_2^4}\right) = \tilde{O}(\log T).
\end{aligned} \tag{23}$$

We can get the bound for S_i by combining equation 22 and equation 23.

Now we bound R. Recall that

$$R := -\frac{1}{2}c_{\mathrm{g}}\sum_{t=0}^{T-1}\eta_{\mathrm{g},t}\|\nabla_{\boldsymbol{g}_t}\mathcal{L}_t\|_2^2 + U\sum_{t=0}^{T-1}\eta_{\mathrm{g},t}^2 G_{\mathrm{g}}.$$

The first term can be bounded by $-\tilde{\Omega}(T^{-1/2})\sum_{t=0}^{T-1}\|\nabla_{\boldsymbol{g}_t}\mathcal{L}_t\|_2^2$ by noticing that $\eta_{\mathrm{g},t} \ge \eta_{\mathrm{g},T} = \Omega(T^{-1/2})$. The second term can be bounded by $\tilde{O}(\log T)$ since $\sum_{t=0}^{T-1}\eta_{\mathrm{g},t}^2 = \sum_{t=0}^{T-1}\tilde{O}(1/t) = \tilde{O}(\log T)$. □

Proof for Theorem 4.2. Combining Lemma B.2 and Lemma B.4, for $0 \leq \alpha < 1/2$, we have

$$\mathcal{L}_{\min} - \mathcal{L}_0 \leq \mathbb{E}[\mathcal{L}_T] - \mathcal{L}_0 \leq -\tilde{\Omega}(T^{-1/2}) \sum_{t=0}^{T-1} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] + \tilde{O}(\log T).$$

Thus,

$$\min_{0 \leq t < T} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] \leq \tilde{O}\left(\frac{\log T}{\sqrt{T}}\right).$$

Similarly, for $\alpha = 1/2$, we have

$$\mathcal{L}_{\min} - \mathcal{L}_0 \leq \mathbb{E}[\mathcal{L}_T] - \mathcal{L}_0 \leq -\tilde{\Omega}((T \log T)^{-1/2}) \sum_{t=0}^{T-1} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] + \tilde{O}(\log T).$$

Thus,

$$\min_{0 \leq t < T} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\|\nabla \mathcal{L}(V_t; \boldsymbol{g}_t)\|_2^2\right] \leq \tilde{O}\left(\frac{(\log T)^{3/2}}{\sqrt{T}}\right).$$

□

C PROOF FOR THE SMOOTHNESS OF THE MOTIVATING NEURAL NETWORK

In this section we prove that the modified version of the motivating neural network does meet the assumptions in Section 2.4. More specifically, we assume:

- We use the network structure Φ in Section 2.1 with the smoothed variant of BN as described in Section 2.4;
- The objective $f_y(\cdot)$ is twice continuously differentiable, lower bounded by $f_{\min}$ and Lipschitz ($|f_y'(\hat{y})| \leq \alpha_f$);
- The activation $\sigma(\cdot)$ is twice continuously differentiable and Lipschitz ($|f_y'(\hat{y})| \leq \alpha_\sigma$);
- We add an extra weight decay (L2 regularization) term $\frac{\lambda}{2}\|\boldsymbol{g}\|_2^2$ to the loss in equation 2 for some $\lambda > 0$.

First, we show that $\boldsymbol{g}_t$ (containing all scale and shift parameters in BN) is bounded during the training process. Then the smoothness follows compactness using Extreme Value Theorem.

We use the following lemma to calculate back propagation:

Lemma C.1. *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_B$ be a set of vectors. Let $\boldsymbol{u} := \mathbb{E}_{b \in [B]}[\boldsymbol{x}_b]$ and $\boldsymbol{S} := \mathrm{Var}_{b \in [B]}(\boldsymbol{x}_b)$. Let*

$$z_b := \gamma \frac{\boldsymbol{w}^\top (\boldsymbol{x}_b - \boldsymbol{u})}{\|\boldsymbol{w}\|_{S+\epsilon I}} + \beta.$$

Let $y := f(z_1, \ldots, z_B) = f(\boldsymbol{z})$ for some function f. If $\|\nabla f(\boldsymbol{z})\|_2 \leq G$, then

$$\left|\frac{\partial y}{\partial \gamma}\right| \leq G\sqrt{B} \qquad \left|\frac{\partial y}{\partial \beta}\right| \leq G\sqrt{B} \qquad \|\nabla_{\boldsymbol{x}_b} y\|_2 \leq \frac{3\gamma}{\epsilon} G\sqrt{B}.$$

Proof. Let $\tilde{\boldsymbol{x}} \in \mathbb{R}^B$ be the vector where $\tilde{x}_b := (\boldsymbol{w}^\top(\boldsymbol{x}_b - \boldsymbol{u}))/\|\boldsymbol{w}\|_{S+\epsilon I}$. It is easy to see $\|\tilde{\boldsymbol{x}}\|_2^2 \leq B$. Then

$$\left|\frac{\partial y}{\partial \gamma}\right| = \left|\nabla f(\boldsymbol{z})^\top \tilde{\boldsymbol{x}}\right| \leq G\|\tilde{\boldsymbol{x}}\|_2 \leq G\sqrt{B}.$$

$$\left|\frac{\partial y}{\partial \beta}\right| \leq \|\nabla f(\boldsymbol{z})\|_1 \leq G\sqrt{B}.$$

For $\boldsymbol{x}_b$, we have

$$\nabla_{\boldsymbol{x}_b}\boldsymbol{u} = \frac{1}{B}\boldsymbol{I}$$

$$\nabla_{\boldsymbol{x}_b}\|\boldsymbol{w}\|^2_{S+\epsilon\boldsymbol{I}} = \nabla_{\boldsymbol{x}_b}\left(\frac{1}{B}\sum_{b'=1}^{B}(\boldsymbol{x}_{b'}^\top\boldsymbol{w})^2 - \left(\frac{1}{B}\sum_{b'=1}^{B}\boldsymbol{x}_{b'}^\top\boldsymbol{w}\right)^2\right)$$

$$= \frac{2}{B}\boldsymbol{w}\boldsymbol{w}^\top(\boldsymbol{x}_b - \boldsymbol{u}).$$

Then

$$\nabla_{\boldsymbol{x}_b} z_{b'} = \gamma\frac{(\mathbf{1}_{b=b'} - 1/B)\|\boldsymbol{w}\|_{S+\epsilon I}\boldsymbol{w} - \boldsymbol{w}^\top(\boldsymbol{x}_b - \boldsymbol{u})\cdot\frac{1}{\|\boldsymbol{w}\|_{S+\epsilon I}}\cdot\frac{2}{B}\boldsymbol{w}\boldsymbol{w}^\top(\boldsymbol{x}_b - \boldsymbol{u})}{\|\boldsymbol{w}\|^2_{S+\epsilon I}}$$

Since $(\boldsymbol{w}^\top(\boldsymbol{x}_b - \boldsymbol{u}))^2 \leq B\|\boldsymbol{w}\|^2_{S+\epsilon I}$,

$$\|\nabla_{\boldsymbol{x}_b} z_{b'}\|_2 \leq \gamma\left(\frac{1}{\|\boldsymbol{w}\|_{S+\epsilon I}}(\mathbf{1}_{b=b'} - 1/B)\|\boldsymbol{w}\|_2 + \frac{1}{\|\boldsymbol{w}\|_{S+\epsilon I}}\cdot 2\|\boldsymbol{w}\|_2\right) \leq \frac{3\gamma}{\epsilon}.$$

Thus,

$$\|\nabla_{\boldsymbol{x}_b} y\|_2 \leq \left(\sum_{b'=1}^{B}\left|\frac{\partial y}{\partial z_b}\right|^2\right)^{1/2}\left(\sum_{b'=1}^{B}\|\nabla_{\boldsymbol{x}_b} z_{b'}\|_2^2\right)^{1/2} \leq \frac{3\gamma}{\epsilon}G\sqrt{B}.$$

□

Lemma C.2. *If $\|\boldsymbol{g}_0\|_2$ is bounded by a constant, there exists some constant K such that $\|\boldsymbol{g}_t\|_2 \leq K$.*

Proof. Fix a time t in the training process. Consider the process of back propagation. Define

$$R_i = \sum_{b=1}^{B}\sum_{k=1}^{m_i}\left(\frac{\partial}{\partial x_{b,k}^{(i)}}\mathcal{F}_{\boldsymbol{z}}(\theta)\right)^2,$$

where $x_{b,k}^{(i)}$ is the output of the k-th neuron in the i-th layer in the b-th data sample in the batch. By the Lipschitzness of the objective, R_L can be bounded by a constant. If R_i can be bounded by a constant, then by the Lipschitzness of σ and Lemma C.1, the gradient of γ and β in layer i can also be bounded by a constant. Note that

$$g_{t+1,k} = g_{t,k} - \eta_{\mathrm{g},t}\frac{\partial}{\partial g_{t,k}}\mathcal{F}_{\boldsymbol{z}}(\boldsymbol{\theta}_t) - \lambda\eta_{\mathrm{g},t}g_{t,k}$$

Thus γ and β in layer i can be bounded by a constant since

$$|g_{t+1,k}| \leq (1 - \eta_{\mathrm{g},t}\lambda)|g_{t,k}| + \eta_{\mathrm{g},t}\lambda\cdot\frac{1}{\lambda}\left|\frac{\partial}{\partial g_{t,k}}\mathcal{F}_{\boldsymbol{z}}(\boldsymbol{\theta}_t)\right|.$$

Also Lemma C.1 and the Lipschitzness of σ imply that R_{i-1} can be bounded if R_i and γ in the layer i can be bounded by a constant. Using a simple induction, we can prove the existence of K for bounding the norm of $\|\boldsymbol{g}_t\|_2$ for all time t. □

Theorem C.3. *If $\|\boldsymbol{g}_0\|_2$ is bounded by a constant, then Φ satisfies the assumptions in Section 2.4.*

Proof. Let C be the set of parameters $\boldsymbol{\theta}$ satisfying $\|\boldsymbol{g}\| \leq K$ and $\|\boldsymbol{w}^{(i)}\|_2 = 1$ for all $1 \leq i \leq m$. By Lemma C.2, C contains the set of $\tilde{\boldsymbol{\theta}}$ associated with the points lying between each pair of $\boldsymbol{\theta}_t$ and $\boldsymbol{\theta}_{t+1}$ (including the endpoints).

It is easy to show that $\mathcal{F}_{\boldsymbol{z}}(\tilde{\boldsymbol{\theta}})$ is twice continously differentiable. Since C is compact, by the Extreme Value Theorem, there must exist such constants $L_{ij}^{\mathrm{vv}}, L_i^{\mathrm{vg}}, L^{\mathrm{gg}}, G_{\mathrm{g}}$ for upper bounding the smoothness and the difference of gradients. □