Fast and Sample Efficient Multi-Task Representation Learning in Stochastic Contextual Bandits

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Abstract

We study how representation learning can improve the learning efficiency of contextual bandit problems. We study the setting where we play T contextual linear bandits with dimension d simultaneously, and these T bandit tasks collectively share a common linear representation with a dimensionality of $r \ll d$. We present a new algorithm based on alternating projected gradient descent (GD) and minimization estimator to recover a low-rank feature matrix. Using the proposed estimator, we present a multi-task learning algorithm for linear contextual bandits and prove the regret bound of our algorithm. We presented experiments and compared the performance of our algorithm against benchmark algorithms.

1. Introduction

Contextual Bandits (CB) represent an online learning problem wherein sequential decisions are made based on observed contexts, aiming to optimize rewards in a dynamic environment with immediate feedback. In CBs, the environment presents a context in each round, and in response, the agent selects an action that yields a reward. The agent's objective is to choose actions to maximize cumulative reward over N rounds. This introduces the exploration-exploitation dilemma, as the agent must balance exploratory actions to estimate the environment's reward function and exploitative actions that maximize the overall return (Bubeck & Cesa-Bianchi, 2012; Lattimore & Szepesvári, 2020). CB algorithms find applications in various fields, including robotics (Srivastava et al., 2014), clinical trials (Aziz et al., 2021), communications (Anandkumar et al., 2011), and recommender systems (Li et al., 2010).

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Proceedings of the 41st International Conference on Machine Learning, Vienna, Austria. PMLR 235, 2024. Copyright 2024 by the author(s).

Multi-task representation learning is the problem of learning a common low-dimensional representation among multiple related tasks (Caruana, 1997). Multi-task learning enables models to tackle multiple related tasks simultaneously, leveraging common patterns and improving overall performance (Zhang & Yang, 2018; Wang et al., 2016; Thekumparampil et al., 2021). By sharing knowledge across tasks, multi-task learning can lead to more efficient and effective models, especially when data is limited or expensive. Multi-task bandit learning has gained interest recently (Deshmukh et al., 2017; Fang & Tao, 2015; Cella et al., 2023; Hu et al., 2021; Yang et al., 2020; Lin & Moothedath, 2024). Many applications of CBs, such as recommending movies or TV shows to users and suggesting personalized treatment plans for patients with various medical conditions, involve related tasks. These applications can significantly benefit from this approach, as demonstrated in our empirical analysis in Section 6. This paper investigates the benefit of using representation learning in CBs theoretically and experimentally.

While representation learning has demonstrated remarkable success across various applications (Bengio et al., 2013), its theoretical understanding still remains underexplored. A prevalent assumption in the literature is the presence of a shared common representation among different tasks. (Maurer et al., 2016) introduced a general approach to learning data representation in both multi-task supervised learning and learning-to-learn scenarios. (Du et al., 2020) delved into few-shot learning through representation learning, making assumptions about a common representation shared between source and target tasks. (Tripuraneni et al., 2021) specifically addressed the challenge of multi-task linear regression with low-rank representation, presenting algorithms with robust statistical rates. Related parallel works address the same mathematical problem (referred to as low-rank column-wise compressive sensing) or its generalization (called low-rank phase retrieval) and provide better - sample-efficient and faster - solutions (Nayer et al., 2019; Nayer & Vaswani, 2021; 2023; Collins et al., 2021; Vaswani, 2024).

Motivated by the outcomes of multi-task learning in supervised learning, numerous recent works have explored the advantages of representation learning in the context of sequential decision-making problems, including reinforcement learning (RL) and bandit learning. Our paper studies

the multi-task bandit problem similar to the setting in (Hu et al., 2021; Yang et al., 2020; Cella et al., 2023). We consider T tasks of d-dimensional (infinte-arm) linear bandits are concurrently learned for N rounds. The expected reward for choosing an arm x for a context c and task t is $\phi_t(x,c)^\top \theta_t^\star$, where $\theta_t^\star$ is an unknown linear parameter and $\phi_t(x,c) \in \mathbb{R}^d$ is the feature vector. To take advantage of the multi-task representation learning framework, we assume that $\theta_t^\star$'s lie in an unknown r-dimensional subspace of $\mathbb{R}^d$, where r is much smaller compared to d and T (Hu et al., 2021; Yang et al., 2020). The dependence on the tasks makes it possible to achieve a regret bound better than solving each task independently. A naive adaptation of the optimism in the face of uncertainty principle (OFUL) algorithm in (Abbasi-Yadkori et al., 2011) will lead to an $\tilde{O}(Td\sqrt{N})$ regret for solving the T tasks individually. By leveraging the common representation structure of these tasks, we propose an alternating projected GD and minimization-based estimator to solve the multi-task CB problem. We provide the convergence guarantee for our estimator and the regret bound of the multi-task learning algorithm and, through extensive simulations, validate the advantage of the proposed approach over the state-of-the-art approaches.

2. Problem Setting

Notations: For any positive integer n, the set $[n]$ represents $\{1, 2, \cdots, n\}$. For any vector x, we use $\|x\|$ to denote its ℓ_2 norm. For a matrix A, we use $\|A\|$ to denote the 2-norm of A, $\|A\|_F$ to denote the Frobenius norm, and $\|A\|_{\max} = \max_{i,j} |A_{i,j}|$ to denote the max-norm. $\top$ denotes the transpose of a matrix or vector, while $|x|$ represents the element-wise absolute value of a vector x. The symbol I_n (or sometimes just I) represents the $n \times n$ identity matrix. We use e_k to denote the $k-$th canonical basis vector, i.e., the $k-$th column of I. For any matrix A, a_k denotes its $k-$th column.

2.1. Problem Formulation

This section introduces the standard linear bandit problem and extends it to our specific setting: representation learning in linear bandits with a low-rank structure. We denote the action set as $\mathcal{A}$ and the context set as $\mathcal{C}$. The environment interacts through a fixed but unknown reward function $y : \mathcal{X} \times \mathcal{C} \to \mathbb{R}$. In standard linear bandits, at each round $n \in [N]$, the agent observes a context $c_n \in \mathcal{C}$ and chooses an action $x_n \in \mathcal{X}$. For every combination of context and action (x, c), there is a corresponding feature vector $\phi(x, c) \in \mathbb{R}^d$. When the agent chooses an action x_n for a given context c_n, it receives a reward $y_n \in \mathbb{R}$, defined as

$$y_n := \langle \phi(x_n, c_n), \theta^\star \rangle + \eta_n,$$

where $\theta^\star \in \mathbb{R}^d$ represents the unknown but fixed reward parameter, and η_n denotes a zero-mean σ-Gaussian additive noise. The term $\langle \phi(x_n, c_n), \theta^\star \rangle$ represents the expected reward for selecting action x_n in context c_n at round n, i.e., $r_n = \mathbb{E}[y_n] = \langle \phi(x_n, c_n), \theta^\star \rangle$. The goal of the agent is to choose the best action $x_n^\star$ at each round $n \in [N]$ to maximize the cumulative reward $\sum_{n=1}^N y_n$, or in other words, to minimize the cumulative regret:

$$\mathcal{R}_N = \sum_{n=1}^N \langle \phi(x_n^\star, c_n), \theta^\star \rangle - \sum_{n=1}^N \langle \phi(x_n, c_n), \theta^\star \rangle,$$

where $x_n^\star$ represents the best action at round n for context c_n, and x_n denotes the action chosen by the agent.

This paper explores representation learning in linear bandits with a low-rank structure. We consider a scenario where $t \in [T]$ tasks deal with related sequential decision-making problems. For every round $n \in [N]$, each task t observes a context $c_{n,t} \in \mathcal{C}_{n,t}$ and chooses an action $x_{n,t} \in \mathcal{A}_{n,t}$. After an action is chosen for each task t at round n, the environment provides a reward $y_{n,t}$, where $y_{n,t} := \langle \phi(x_{n,t}, c_{n,t}), \theta_t^\star \rangle + \eta_{n,t}$ and $\theta_t^\star \in \mathbb{R}^d$ is the unknown reward parameter for task t. The goal is to choose the best action $x_{n,t}$ for each task $t \in [T]$ and each round $n \in [N]$ to maximize the cumulative reward $\sum_{n=1}^N \sum_{t=1}^T y_{n,t}$, which is equivalent to minimizing the cumulative (pseudo) regret

$$\mathcal{R}_{N,T} = \sum_{n=1}^N \sum_{t=1}^T \langle \phi(x_{n,t}^\star, c_{n,t}), \theta_t^\star \rangle - \sum_{n=1}^N \sum_{t=1}^T \langle \phi(x_{n,t}, c_{n,t}), \theta_t^\star \rangle,$$

where $x_{n,t}^\star$ denotes the best action for task t at round n given context $c_{n,t}$. We assume that $\Theta^\star = [\theta_1^\star \cdots \theta_T^\star]$ is a rank-r matrix where $r \ll \min\{d, T\}$. This low-rank structure improves collaborative learning among the agents, which enhances the overall learning efficiency.

2.2. Preliminaries

Let $\Theta^\star \overset{\text{SVD}}{=} B^\star \Sigma V^\star := B^\star W^\star$ denote its reduced (rank r) SVD, i.e., $B^\star$ and ${V^\star}^\top$ are matrices with orthonormal columns *(basis matrices)*, $B^\star$ is $d \times r$, $V^\star$ is $r \times T$, and Σ is an $r \times r$ diagonal matrix with non-negative entries (singular values). We let $W^\star := \Sigma V^\star$. We use $\sigma_{\max}^\star$ and $\sigma_{\min}^\star$ to denote the maximum and minimum singular values of Σ and we define its condition number as $\kappa := \sigma_{\max}^\star / \sigma_{\min}^\star$. We now detail the assumptions we use in our analysis.

Assumption 2.1. (Gaussian design and noise) We assume $\phi(x_{n,t}, c_{n,t})$ follows an i.i.d. standard Gaussian distribution. Moreover, the additive noise variables $\eta_{n,t}$ follow i.i.d. Gaussian distribution with a zero mean and variance σ_η^2.

Throughout, we work in the setting of random design linear regression, and in this context, Assumption 2.1 is standard

(Cella et al., 2023; Cella & Pontil, 2021). We note that while the assumption on $\phi(x_{n,t}, c_{n,t})$ holds for the first epoch in our algorithm, i.e., during random exploration, it is restrictive for future epochs. Let the feature vector $\phi(x_{n,t}, c_{n,t})$ has a mean value of $\mu_{x_{n,t},c_{n,t}}$. The reward $y_{n,t}$ is given by the equation $y_{n,t} = (\phi(x_{n,t}, c_{n,t}) - \mu_{x_{n,t},c_{n,t}})^\top \theta_t^\star + \eta_{n,t} + \mu_{x_{n,t},c_{n,t}} \theta_t^\star$, where the noise term is $\eta_{n,t} + \mu_{x_{n,t},c_{n,t}} \theta_t^\star$. Such a (re)formulation allows us to relax Assumption 2.1 into a scenario where $\phi(x_{n,t}, c_{n,t})$ follows an independent Gaussian distribution with a mean of $\mu_{x_{n,t},c_{n,t}}$.

Assumption 2.2 (Incoherence of right singular vectors)**.** We assume that $\|w_t^\star\|^2 \leqslant \mu^2 \frac{r}{T} {\sigma_{\max}^\star}^2$ for a constant $\mu \geqslant 1$.

Recovering the feature matrix is impossible without any structural assumption. Notice that y_ts are not global functions of $\Theta^\star$: no $y_{n,t}$ is a function of the entire matrix $\Theta^\star$. We thus need an assumption that enables correct interpolation across the different columns. The following incoherence (w.r.t. the canonical basis) assumption on the right singular vectors suffices for this purpose. Such an assumption on both left and right singular vectors was first introduced in (Candes & Recht, 2008) and used in recent works on representation learning (Tripuraneni et al., 2021).

Assumption 2.3. (Common Feature Extractor). There exists a linear feature extractor denoted as $B^\star \in \mathbb{R}^{d \times r}$, along with a set of linear coefficients $\{w_t\}_{t=1}^T$ such that the expected reward of the t-th task at the n-th round is given by $\mathbb{E}[y_{n,t}] = \langle \phi(x_{n,t}, c_{n,t}), B^\star w_t^\star \rangle$, where $\Theta^\star = B^\star W^\star$.

Assumption 2.3 is our main assumption, which assumes the existence of a common feature extractor for the reward parameter $\Theta^\star$. Because of this we can write $\Theta^\star = B^\star W^\star$, where $W^\star = [w_1^\star, w_2^\star, \ldots, w_T^\star]$. This assumption is used in many earlier works on representation learning, including (Yang et al., 2020; Du et al., 2020; Hu et al., 2021).

2.3. Contributions

In this paper, we proposed a novel alternating GD and minimization estimator for representation learning in linear bandits in the presence of a common feature extractor. Our algorithm builds upon the recently introduced technique known as alternating gradient descent and minimization (altGDmin) for low-rank matrix learning (Nayer & Vaswani, 2023; Vaswani, 2024). Our work introduces two key extensions: (i) We adapt the AltGDMin approach to address sequential learning problems, specifically bandit learning, departing from static learning scenarios. Hence our focus is on optimizing the selection of actions in addition to learning unknown parameters from observed data. (ii) We account for noisy observed data, a common model in learning models, rather than non-noisy observations.

While there have been many recent works on multi-task learning for linear bandits (Hu et al., 2021; Yang et al., 2020; Cella et al., 2023; Tripuraneni et al., 2021), those works either assume an optimal estimator that can solve the non-convex cost function in Eq. (1) or considers a convex relaxation of the original cost function (Du et al., 2020; Cella et al., 2023). We propose a sample and time-efficient estimator to learn the feature matrix. Our approach is GD-based, which is known to be much faster than convex relaxation methods (Cella et al., 2023; Du et al., 2020) and provides a sample-efficient estimation with guarantees. We prove that the alternating GD and minimization estimator achieves $\epsilon-$optimal convergence with the number of samples of the order of $(d+T)r^3 \log(1/\epsilon)$ and order $NTdr \log(1/\epsilon)$ time, provided the noise-to-signal ratio is bounded. Using the proposed estimator, we propose a multitask bandit learning algorithm. We provide the regret bound for our algorithm. We validated the advantage of our algorithm through numerical simulations on synthetic and real-world MNIST datasets and illustrated the advantage of our algorithm over existing state-of-the-art benchmarks.

3. Related Work

Multi-task supervised learning. Multi-task representation learning is a well-studied problem that dates back to at least (Caruana, 1997; Thrun & Pratt, 1998; Baxter, 2000). Empirically, representation learning has shown its great power in various domains (Bengio et al., 2013). The linear setting (multi-task linear regression or multi-task linear representation learning with a low rank model on the regression coefficients) was introduced in (Maurer et al., 2016; Tripuraneni et al., 2021; Du et al., 2020)

The above works required more samples per task than the feature vector length, even while assuming a low-rank model on the regression coefficients' matrix. In interesting parallel works (Nayer & Vaswani, 2023; Collins et al., 2021), a fast and communication-efficient GD-based algorithm that was referred to as AltGDmin and FedRep, respectively, was introduced for solving the same mathematical problem that multi-task linear regression or multi-task linear representation learning solves. Follow-up work (Vaswani, 2024) improved the guarantees for AltGDmin while also simplifying the proof. AltGDmin and FedRep algorithms are identical except for the initialization step. AltGDmin uses a better initialization and hence also has a better sample complexity by a factor of r. A phaseless measurements generalization of this problem, referred to as low-rank phase retrieval, was studied in (Nayer et al., 2019; 2020; Nayer & Vaswani, 2021; 2023). These works were motivated by applications in dynamic MRI (Babu et al., 2023) and dynamic Fourier ptychography (Jagatap et al., 2020).

The primary emphasis of all the above works is on the statistical rate for multi-task supervised learning and does not address the exploration problem in online sequential

decision-making problems such as bandits and RL.

Multi-task RL learning. Multi-task learning in RL domains is studied in many works, including (Taylor & Stone, 2009; Parisotto et al., 2015; D'Eramo et al., 2024; Arora et al., 2020). (D'Eramo et al., 2024) demonstrated that representation learning has the potential to enhance the rate of the approximate value iteration algorithm. (Arora et al., 2020) proved that representation learning can reduce the sample complexity of imitation learning. Both works require a probabilistic assumption similar to that in (Maurer et al., 2016) and the statistical rates are of similar forms as those in (Maurer et al., 2016).

Multi-task bandit learning. The most closely related works to ours are the recent papers on multitask bandit learning (Hu et al., 2021; Yang et al., 2020; Cella et al., 2023). (Hu et al., 2021) considered a concurrent learning setting with T linear bandits with dimension d that share a common r-dimensional linear representation. They proposed an optimism in the face of uncertainty principle (OFUL) algorithm that leverages the shared representation to achieve a $\tilde{O}(T\sqrt{drN} + d\sqrt{rNT})$ regret bound, where N is the number of rounds per task. The algorithm in (Hu et al., 2021) requires solving a least-squares problem; however, the problem is nonconvex due to the rank condition ($r \ll \min\{d, T\}$). (Yang et al., 2020) considered the finite and infinite action case and proposed explore-then-commit algorithms. For the finite case, they utilize the estimator from (Du et al., 2020), and in the infinite case, they proposed a MoM-based estimator with $\tilde{O}(Tr\sqrt{N} + d^{1.5}r\sqrt{NT})$ regret bound. However, these works assumed that the representation learning problem can be solved. (Du et al., 2020) mentions that it should be possible to solve the original nonconvex problem (Eq. (1)) by solving a trace norm-based convex relaxation of it. (Cella et al., 2023) proposed a low-rank matrix estimation-based algorithm using trace-norm regularization and achieved $\tilde{O}(T\sqrt{rN} + \sqrt{rNTd})$ regret bound under a restricted strong convexity condition when the rank is unknown. However, there are no known guarantees to ensure that the trace norm-based relaxation solution is indeed also a solution to the original low-rank representation learning problem. The regret analysis in these works assumed solvability and optimality of the nonconvex problem. We focus on GD-based solutions since these are known to be much faster than convex relaxation methods (Cella et al., 2023; Du et al., 2020) and provide a sample-efficient estimator with guarantees.

Low rank and sparse bandits. Some previous works also studied low rank and sparse bandits (Kveton et al., 2017). (Lale et al., 2019) considered a setting where the context vectors share a low-rank structure. Specifically, in their setting, the context vectors consist of two parts, i.e. $\hat{\phi} = \phi + \psi$, so that ϕ is from a hidden low-rank subspace and is i.i.d. drawn from an isotropic distribution. Works (Lu et al., 2021; Jun et al., 2019) studied bilinear bandits with low rank structure. They focus on estimating a low rank matrix $\Theta^\star$ when the reward function is given by $x^\top \Theta^\star y$, where x, y denote the two actions chosen at each round. Sparse interactive learning settings (e.g., bandits and reinforcement learning) are also studied in the literature (Cella & Pontil, 2021; Calandriello et al., 2014; Hao et al., 2020; 2021).

Algorithm 1 LRRL-AltGDMin Algorithm

Let $M = \lceil \log_2 \log_2 N \rceil$, $\mathcal{G}_0 = 0$, $\mathcal{G}_M = N$, $\mathcal{G}_m = N^{1-2^{-m}}$ for $1 \leqslant m \leqslant M-1$, let $\widehat{\theta}_t^{(0)} \leftarrow 0$
for $m \leftarrow 1, \cdots, M$ **do**
  for $n \leftarrow \mathcal{G}_{m-1}+1, \cdots, \mathcal{G}_m$ **do**
    For each task $t \in [T]$: choose action $x'_{n,t} = \arg\max_{\phi(x,c_{n,t}) \in \Psi_t} \phi(x_{n,t}, c_{n,t})^\top \widehat{\theta}_t^{(\text{m-1})}$, obtain $y_{n,t}$, where $\Psi_t = \{\phi(x, c_{n,t}) : x \in \mathcal{X}\}$, where $\phi(x, c_{n,t}) \sim \mathcal{N}(\mu_{x,c}, I)$.
  end for
  Compute $Y_t^{(\text{m})} = [y_{\mathcal{G}_{m-1}+1,t}, \cdots, y_{\mathcal{G}_m,t}]^\top$, $\Phi_t^{(\text{m})} = [\phi(x'_{\mathcal{G}_{m-1}+1,t}, c_{\mathcal{G}_{m-1}+1}), \cdots, \phi(x'_{\mathcal{G}_m,t}, c_{\mathcal{G}_m})]^\top$ for $t \in [T]$
  if $m = 1$ **then**
    Sample-split: Partition the measurements and measure matrices into $2L+1$ equal-sized disjoint sets: one for initialization and $2L$ sets for the iterations. Denote these by $Y_{t,\tau}^{(\text{m})}$, $\Phi_{t,\tau}^{(\text{m})}$, $\tau = 00, 01, \cdots 2L$.
    Initialize $\widehat{B}^{(0)}$ using Algorithm 2
    Compute $\widehat{B}^{(\text{m})}$ and $\widehat{W}^{(\text{m})}$ using Algorithm 3
  end if
  if $m \geqslant 2$ **then**
    Sample-split: Partition the measurements and measure matrices into $2L$ equal-sized disjoint sets. Denote these by $Y_{t,\tau}^{(\text{m})}$, $\Phi_{t,\tau}^{(\text{m})}$, $\tau = 01, \cdots 2L$.
    Compute $\widehat{B}^{(\text{m})}$ and $\widehat{W}^{(\text{m})}$ using Algorithm 3
  end if
  For each task $t \in [T]$: let $\widehat{\theta}_t^{(\text{m})} = \widehat{B}^{(\text{m})} \widehat{w}_t^{(\text{m})}$
end for

4. The Proposed Algorithm: LRRL-AltGDMin

This section presents our proposed algorithm (see Algorithm 1). We refer to it as the Alternating Gradient Descent (GD) and Minimization algorithm for Low-Rank Representation Learning in linear bandits (LRRL-AltGDMin). This builds on the AltGDmin algorithm of (Nayer & Vaswani, 2023) mentioned earlier. Our algorithm uses a doubling schedule rule (Gao et al., 2019; Han et al., 2020; Simchi-Levi & Xu, 2019). We update our estimation of $\Theta^\star$ only after completing an epoch, utilizing solely the samples collected within that epoch. Our algorithm consists of three

Algorithm 2 Spectral Initialization for LRRL-AltGDMin

Input: $Y^{(1)}_{t,00}$, $\Phi^{(1)}_{t,00}$, for $t \in [T]$
Parameters: Multiplier in specifying α for init step, $\tilde{C}$
Using $Y^{(1)}_t \equiv Y^{(1)}_{t,00}$, $\Phi^{(1)}_t \equiv \Phi^{(1)}_{t,00}$, set $\alpha = \frac{\tilde{C}}{\mathcal{G}_1 T}\sum_{n=1,t=1}^{\mathcal{G}_1,T} y^2_{n,t}$
$y_{t,trunc}(\alpha) := Y^{(1)}_t \circ \mathbb{1}_{\{|Y^{(1)}_t| \leqslant \sqrt{\alpha}\}}$
$\widehat{\Theta}_0 := \frac{1}{\mathcal{G}_1}\sum_{t=1}^{T} {\Phi^{(1)}_t}^\top y_{t,trunc}(\alpha) e_t^\top$
Set $\widehat{B}^{(0)} \leftarrow$ top-r-singular-vectors of $\widehat{\Theta}_0$

Algorithm 3 GD-Minimization for LRRL-AltGDMin

Input: $Y^{(\mathrm{m})}_{t,\tau}$, $\Phi^{(\mathrm{m})}_{t,\tau}$ for $t \in [T]$, $\tau = 01, \cdots 2L$, $\widehat{B}^{(\mathrm{m-1})}$ (from Algorithm 1)
Parameters: GD step size, γ; Number of iterations, L
Set $B_0 \leftarrow \widehat{B}^{(\mathrm{m-1})}$
for $\ell = 1$ to L **do**
  Let $B \leftarrow B_{\ell-1}$
  Update $w_{t,\ell}, \theta_{t,\ell}$**:** For each $t \in [T]$, set $w_{t,\ell} \leftarrow (\Phi^{(\mathrm{m})}_{t,l} B)^\dagger Y^{(\mathrm{m})}_{t,l}$ and set $\theta_{t,\ell} \leftarrow B w_{t,\ell}$
  Gradient w.r.t B**:** With $Y^{(\mathrm{m})}_t \equiv Y^{(\mathrm{m})}_{t,L+\ell}$, $\Phi^{(\mathrm{m})}_t \equiv \Phi^{(\mathrm{m})}_{t,L+\ell}$, compute $\nabla_B f(B, W_\ell) = \sum_{t=1}^{T} {\Phi^{(\mathrm{m})}_t}^\top (\Phi^{(\mathrm{m})}_t B w_{t,\ell} - Y^{(\mathrm{m})}_t) w_{t,\ell}^\top$
  GD step: Set $\widehat{B}^+ \leftarrow B - \frac{\gamma}{\mathcal{G}_m - \mathcal{G}_{m-1}} \nabla_B f(B, W_\ell)$
  Projection step: Compute $\widehat{B}^+ \overset{QR}{=} B^+ R^+$
  Set $B_\ell \leftarrow B^+$
end for
Set $\widehat{B}^{(\mathrm{m})} \leftarrow B_L$ and set $\widehat{W}^{(\mathrm{m})} \leftarrow W_L$

main components: an exploration phase (data collection), initialization, and alternating GD and minimization steps. The pseudocode of our algorithm is presented in Algorithm 1.

We partition the learning horizon N into $M+1$ epochs, $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_M$, where $\mathcal{G}_0 = 0$ and $\mathcal{G}_M = N$. Our algorithm is based on a greedy strategy. At each round n, each task $t \in [T]$ independently chooses an action $x'_{n,t} = \arg\max_{\phi(x,c_{n,t}) \in \Psi_t} \phi(x_{n,t}, c_{n,t})^\top \widehat{\theta}^{(\mathrm{m-1})}_t$, which effectively aiming to maximize the expected reward. After choosing these actions, each task receives a corresponding reward $y_{n,t}$. After completing $(\mathcal{G}_m - \mathcal{G}_{m-1})$ rounds to collect data, the algorithm then proceeds to update the estimated parameters, which is achieved by finding a matrix $\widehat{\Theta} = \widehat{B}\widehat{W}$ that minimizes the cost function, defined as

$$f_m(\widehat{B}, \widehat{W}) = \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \sum_{t=1}^{T} \left\| y_{n,t} - \phi(x_{n,t}, c_n)\widehat{B}\widehat{w}_t \right\|^2. \quad (1)$$

Here $\widehat{B} \in \mathbb{R}^{d \times r}$ and $\widehat{W} = [\widehat{w}_1, \ldots, \widehat{w}_T] \in \mathbb{R}^{r \times T}$, and $\widehat{\Theta} = \widehat{B}\widehat{W}$ is the estimate of the parameter $\Theta^\star$ in the m-the epoch. This process effectively enhances the accuracy of future action selections.

Because of the non-convexity of the cost function $f_m(\widehat{B}, \widehat{W})$, our approach needs careful initialization. We draw inspiration from the spectral initialization idea. The process begins by calculating the top r singular vectors of

$$\begin{aligned}\widehat{\Theta}_{0,full} &= \frac{1}{\mathcal{G}_1}[({\Phi^{(1)}_1}^\top Y^{(1)}_1), \cdots, ({\Phi^{(1)}_T}^\top Y^{(1)}_T)] \\ &= \frac{1}{\mathcal{G}_1}\sum_{t=1}^{T}\sum_{n=1}^{\mathcal{G}_1} \phi(x_{n,t}, c_n) y_{n,t} e_t^\top\end{aligned}$$

Here, $\Phi^{(\mathrm{m})}_t$, for $t \in [T]$, is the feature matrix obtained by stacking the feature vectors corresponding to task t in the m-the epoch, i.e., $\Phi^{(\mathrm{m})}_t = [\phi(x_{\mathcal{G}_{m-1}+1,t}, c_{\mathcal{G}_{m-1}+1}), \cdots, \phi(x_{\mathcal{G}_m,t}, c_{\mathcal{G}_m})]^\top$. Upon careful analysis of this matrix, it can be observed that the expected value of its $t-$th task equals $\theta^\star_t$ and $\mathbb{E}[\widehat{\Theta}_{0,full}] = \Theta^\star$. However, the large magnitude of the sum of independent sub-exponential random variables, which is defined by a maximum sub-exponential norm $\max_t \|\theta^\star_t\| \leqslant \mu\sqrt{\frac{r}{T}}\sigma^\star_{\max}$, causes a challenge. This magnitude limits the ability to bound $\left\|\widehat{\Theta}_{0,full} - \Theta^\star\right\|$ within the desired sample complexity. In order to solve this issue, we apply a truncation strategy borrowed from (Nayer & Vaswani, 2023). This involves initializing $\widehat{B}^{(0)}$ as the top r left singular vectors of

$$\begin{aligned}\widehat{\Theta}_0 &= \frac{1}{\mathcal{G}_1}\sum_{t=1}^{T}\sum_{n=1}^{\mathcal{G}_1} \phi(x_{n,t}, c_n) y_{n,t} e_t^\top \mathbb{1}_{\{y^2_{t,n} \leqslant \alpha\}} \\ &= \frac{1}{\mathcal{G}_1}\sum_{t=1}^{T} \phi(x_{n,t}, c_n) y_{t,trunc}(\alpha) e_t^\top\end{aligned}$$

where $\alpha = \frac{\tilde{C}}{\mathcal{G}_1 T}\sum_{n=1,t=1}^{\mathcal{G}_1,T} y^2_{n,t}$, $\tilde{C} = 9\kappa^2\mu^2$, and $y_{t,trunc}(\alpha) := Y^{(1)}_t \circ \mathbb{1}_{\{|Y^{(1)}_t| \leqslant \sqrt{\alpha}\}}$. Using Singular Value Decomposition (SVD), we extract the top r left singular vectors from $\widehat{\Theta}_0$ to obtain our initial estimate $\widehat{B}^{(0)}$. This method efficiently filters large values while preserving others and provides a good starting point that ensures a robust guarantee in parameter estimation.

After finding a good initial point, the algorithm performs the Alternating Gradient Descent optimization method to update the estimated parameter. The goal is to minimize the squared-loss cost function $f(B, W) = \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m}\sum_{t=1}^{T} \|y_{n,t} - \phi(x_{n,t}, c_n) B w_t\|^2$ by optimizing the estimated reward parameter matrix for all tasks. The process proceeds in the following manner. At each new iteration ℓ,

- **Min-W:** Given that w_t appears only in the $t-$th term of $f(B, W)$, optimizing each w_t for the function $w_t \leftarrow \arg\min_{\widetilde{w}_t} \left\|Y^{(\mathrm{m})}_t - \Phi^{(\mathrm{m})}_t B \widetilde{w}_t\right\|^2$ individually is

much simpler than optimizing W for the function $W \leftarrow \arg\min_{\widetilde{W}} \left\| f(B, \widetilde{W}) \right\|$. Consequently, we update the estimate w_t by calculating $w_t = (\Phi_t^{(\mathrm{m})} B)^\dagger Y_t^{(\mathrm{m})}$ for every task $t \in [T]$.

- **ProjGD-B:** a single step of the Gradient Descent (GD) is performed to update B, which is given by $\widehat{B}^+ \leftarrow B - \gamma \nabla_B f(B, W)$. The updated matrix B^+ is obtained using QR decomposition, represented as $\widehat{B}^+ \overset{QR}{=} B^+ R^+$.

Through this iterative process, the algorithm efficiently updates the estimated parameters, guaranteeing an optimized solution.

5. Analysis of LRRL-AltGDMin

We have the following guarantee for our initialization algorithm presented in Algorithm 2.

Theorem 5.1. *Assume that Assumptions 2.1 and 2.2 hold. Assume that $\sigma_\eta^2 \leqslant c \frac{\delta_0^2}{r^2 \kappa^4 \mathcal{G}_1} \|\theta_t^\star\|^2$. Then with probability at least $1 - \exp(\log T - c\mathcal{G}_1) - \exp(d - \frac{c\delta_0^2 \mathcal{G}_1 T}{r^2 \mu^2 \kappa^4})$, we have*

$$\mathrm{SD}(\widehat{B}^{(0)}, B^\star) \leqslant \delta_0.$$

Observe that Theorem 5.1 needs the noise-to-signal (NSR) ratio $\frac{\sigma_\eta^2}{\|\theta_t^\star\|^2} \leqslant \frac{c}{r^2}$, where $c < 1$. This is necessary to demonstrate that the spectral initialization in Algorithm 2 produces a sufficiently good initialization.

In order to show that $\widehat{B}^{(0)}$ is a good enough initialization, we need to show that $\mathrm{SD}(\widehat{B}^{(0)}, B^\star) \leqslant \delta_0$ for a constant $\delta_0 < 1$ that is small enough. This is typically done using a $\sin\Theta$ theorem, e.g., Davis-Kahan or Wedin (Chen et al., 2020), which uses a bound on the error between $\widehat{\Theta}_0$ and a matrix whose span of top r singular vectors equals that of $B^\star$. Such a matrix may be $\mathbb{E}[\widehat{\Theta}_0]$ or something else that can be shown to be close to $\widehat{\Theta}_0$. For our approach, it is not easy to compute $\mathbb{E}[\widehat{\Theta}_0]$ because the threshold, α, used in the indicator function depends on all the $y_{n,t}^2$. Our approach to solving this by using the sample-splitting idea: use a different independent set of measurements to compute α than those used for the rest of $\widehat{\Theta}_0$. Since this is a one-time step, it does not change the sample complexity order. We present the proof of Theorem 5.1 in Appendix B.2.

Theorem 5.2. *Assume that Assumptions 2.1 and 2.2 hold, $\mathrm{SD}(B, B^\star) \leqslant \delta_\ell$, and $\sigma_\eta^2 \leqslant \frac{r}{T} \delta_\ell^2 {\sigma_{\min}^\star}^2$. If $\delta_\ell \leqslant \frac{0.02}{\sqrt{r}\kappa^2}$, $\gamma = \frac{c_\gamma}{{\sigma_{\max}^\star}^2}$ with $c_\gamma \leqslant 0.5$, and if*

$$(\mathcal{G}_m - \mathcal{G}_{m-1}) T \geqslant C \kappa^4 \mu^2 d r \quad \text{and}$$

$$(\mathcal{G}_m - \mathcal{G}_{m-1}) \gtrsim \max(\log d, \log T, r),$$

then with probability at least $O(1 - d^{-10})$,

$$\mathrm{SD}(B^+, B^\star) \leqslant \delta_{\ell+1} := (1 - \frac{0.4 \mu c_\gamma}{\kappa^2}) \delta_\ell.$$

The above result proves that the error decays exponentially. We present the proof in Appendix B.1. Using Theorems 5.1 and 5.2, we have the guarantee below on estimation error.

Theorem 5.3. *Assume that Assumptions 2.1 and 2.2 hold and $\sigma_\eta^2 \leqslant \frac{c\|\theta_t^\star\|^2}{r^3 \kappa^6 \mathcal{G}_1}$. Set $\gamma = \frac{0.4}{{\sigma_{\max}^\star}^2}$ and $L = C\kappa^2 \log(\frac{1}{\max(\epsilon, \epsilon_{noise})})$. If*

$$(\mathcal{G}_m - \mathcal{G}_{m-1}) T \geqslant C \kappa^6 \mu^2 (d + T) r (\kappa^2 r^2 + \log(\frac{1}{\max(\epsilon, \epsilon_{noise})}))$$

and

$$\mathcal{G}_m - \mathcal{G}_{m-1} \geqslant C \max(\log d, \log T, r) \log(\frac{1}{\max(\epsilon, \epsilon_{noise})}),$$

then with probability at least $O(1 - d^{-10})$,

$$\mathrm{SD}(B, B^\star) \leqslant \max(\epsilon, \epsilon_{noise}) \quad \textit{and}$$
$$\|\widehat{\theta}_t - \theta_t^\star\| \leqslant \max(\epsilon, \epsilon_{noise}) \|\theta_t^\star\| \quad \textit{for all} \quad t \in [T],$$

where $\epsilon_{noise} = C\kappa^2 \sqrt{NSR}$, $NSR := \frac{\sigma_\eta^2}{\min_t \|\theta_t^\star\|^2}$. The time complexity is $(\mathcal{G}_m - \mathcal{G}_{m-1}) T d r \cdot L = C\kappa^2 (\mathcal{G}_m - \mathcal{G}_{m-1}) T d r \log(\frac{1}{\max(\epsilon, \epsilon_{noise})})$. The communication complexity is dr per node per iteration.

This result shows that the error decays exponentially until it reaches the (normalized) "noise-level" ${\sigma_\eta}^2 / \|\theta_t^\star\|^2$, but saturates after that. We present the proof in Appendix B.3.

Sample complexity. To understand the necessary lower bound on $(\mathcal{G}_m - \mathcal{G}_{m-1})T$, it is crucial to consider it in terms of the sample complexity. This can be performed by assuming that $d \approx T$ approximately. When logarithmic factors are ignored and considering κ and μ as constant values, our results indicate that an order value of r^3 samples per epoch is sufficient. Without making the low-rank assumption and without using our algorithm, if we were to perform matrix inversion for $\Phi_t^{(\mathrm{m})}$ in order to extract each vector $\theta_t^\star$ from $Y_t^{(\mathrm{m})}$, we would need at least $\mathcal{G}_m - \mathcal{G}_{m-1} \geq d$ samples per epoch, instead of just r^3. If the low-rank assumption holds and $r \ll d$ (e.g., $r = \log d$), our approach significantly lowers the amount of sample complexity needed in comparison to the requirement for inverting $\Phi_t^{(\mathrm{m})}$.

Time and communication complexity. When analyzing the time complexity of a given m-th epoch, we start by calculating the computation time needed for the initialization step. To calculate Θ_0, it is necessary to give a time of order $(\mathcal{G}_m - \mathcal{G}_{m-1})Td$. Furthermore, the time complexity of the r-SVD step dTr times the number of iterations required. An important observation is that to obtain

an initial estimate of the span of B^* that is δ_0-accurate, where $\delta_0 = \frac{c}{\kappa^2}$, it is sufficient to use an order $\log(\kappa)$ number of iterations. Therefore, the total complexity of this initialization phase can be expressed as $O(dT((\mathcal{G}_m - \mathcal{G}_{m-1}) + r)\log(\kappa)) = O((\mathcal{G}_m - \mathcal{G}_{m-1})Td\log\kappa)$, given that $(\mathcal{G}_m - \mathcal{G}_{m-1}) \geqslant r$. The time required for each gradient computation is $(\mathcal{G}_m - \mathcal{G}_{m-1})Tdr$. The QR decomposition process requires a time complexity of order dr^2. Additionally, the time required to update the columns of matrix W using the least squares method is $O((\mathcal{G}_m - \mathcal{G}_{m-1})Tdr)$. The number of iterations of these steps for each epoch can be expressed as $L = O(\kappa^2 \log(\frac{1}{\max(\epsilon,\epsilon_{noise})}))$. In summary, the overall time complexity for the process can be determined as $O((\mathcal{G}_m - \mathcal{G}_{m-1})Td\log(\kappa) + \max((\mathcal{G}_m - \mathcal{G}_{m-1})Tdr, dr^2, (\mathcal{G}_m - \mathcal{G}_{m-1})Tdr) \cdot M \cdot L) = O(\kappa^2 M(\mathcal{G}_m - \mathcal{G}_{m-1})Tdr\log(\frac{1}{\max(\epsilon,\epsilon_{noise})})\log(\kappa))$.

The communication complexity for each task in each iteration is of the order of dr. Hence, the total is $O(dr \cdot \kappa\log(\frac{1}{\max(\epsilon,\epsilon_{noise})}))$.

We now present the regret bound for our algorithm.

Theorem 5.4. *Assume that Assumptions 2.1 and 2.2 hold and* $\sigma_\eta^2 \leqslant \frac{c\|\theta_t^\star\|^2}{r^3\kappa^6\mathcal{G}_1}$. *Set* $\gamma = \frac{0.4}{{\sigma_{\max}^\star}^2}$ *and* $L = C\kappa^2\log(\frac{1}{\max(\epsilon,\epsilon_{noise})})$. *If*

$$(\mathcal{G}_m - \mathcal{G}_{m-1})T \geqslant C\kappa^6\mu^2(d+T)r(\kappa^2 r^2 + \log(\frac{1}{\max(\epsilon,\epsilon_{noise})}))$$

and

$$\mathcal{G}_m - \mathcal{G}_{m-1} \geqslant C\max(\log d, \log T, r)\log(\frac{1}{\max(\epsilon,\epsilon_{noise})}),$$

then with probability at least $O(1-\delta-d^{-10})$, *the upper bound of cumulative regret for Algorithm 1 is*

$$\mathcal{R}_{N,T} \leqslant 2\mu\sigma_{\max}^\star \max(\epsilon,\epsilon_{noise})\sqrt{rNT\log\frac{1}{\delta}}(1+\log\log N).$$

Proof of Theorem 5.4 and supporting results are presented in Appendix C. Our sample complexity on source task scales sublinearly with T and improves the linear dependence in (Yang et al., 2020), while the target sample complexity scales with k same as in (Yang et al., 2020).

6. Simulations

In this section, we present the experimental results of our LRRL-AltGDMin algorithm on both synthetic and real-world MNIST datasets. We performed a comparative analysis of our algorithm with the Method-of-Moments (MoM) algorithm proposed in (Yang et al., 2020; Tripuraneni et al., 2021), the trace-norm convex relaxation-based approach in (Cella et al., 2023), along with a baseline naive approach. The naive approach utilizes the Thompson Sampling (TS) algorithm to solve T tasks independently. All experiments were conducted using Python.

6.1. Datasets

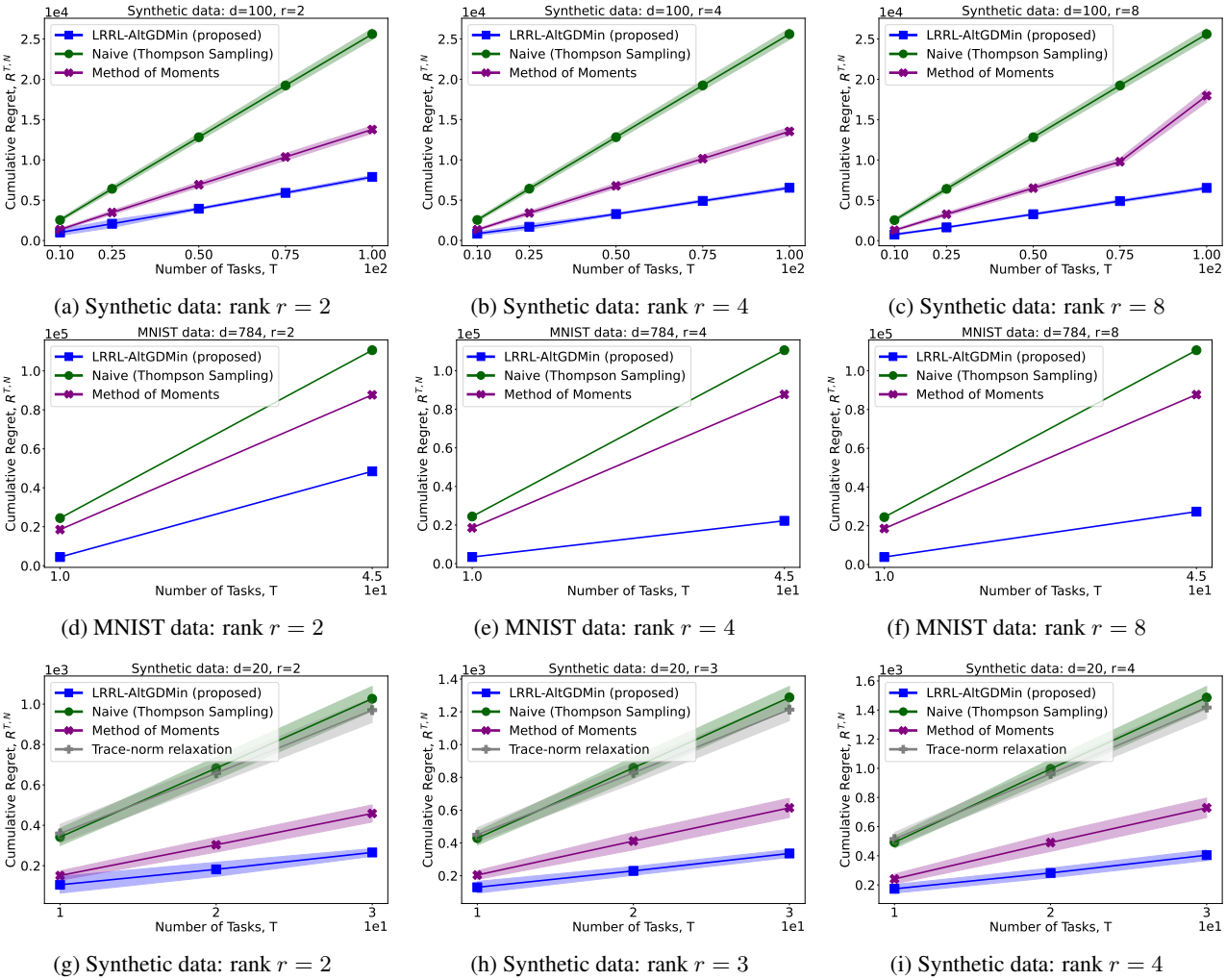
Synthetic data: We set the parameters as $d = 100$, and $K = 5$. We generate the entries of $B^\star$ by orthonormalizing an i.i.d standard Gaussian matrix. The entries of $W^\star \in \mathbb{R}^{r\times T}$ are generated from an i.i.d. Gaussian distribution. The matrices Φ_ts were i.i.d. standard Gaussian. We considered a noise model with a mean of 0 and a variance of 10^{-6} for the bandit feedback noise. The experiments were averaged over 100 independent trials. The plots also include the variance over the trials. In the synthetic experiment, we also considered another dataset with a smaller problem dimension $d = 20$ and $K = 5$.

MNIST data: We used the MNIST dataset to validate the performance of our algorithm when implemented with real-world data. We set the number of actions $K = 2$ and created a total of $T = \binom{10}{2}$ tasks similar to (Yang et al., 2020). Each task is characterized by a distinct pair (i, j), where $0 \leqslant i < j \leqslant 9$. The set of MNIST images that represent the digit i is denoted as D_i. For each round $n \in [N]$, we randomly choose one image from the set D_i and another image from the set D_j for every task (i, j). The algorithm is presented with two images, and it assigns a reward of 1 to the image with the larger digit value and a reward of 0 to the other image. The feature matrix of each image is transformed into a feature vector $\phi \in \mathbb{R}^{784}$ through vectorization. In order to calculate the estimated reward, we add random Gaussian noise with a mean of 0 and a variance of 10^{-6}.

6.2. Results and Discussions

Estimation error. We compared the estimation performance of our proposed LRRL-AltGDMin estimator with three existing approaches: (i) an alternating GD (LRRL-AltGD) estimator, (ii) Method-of-Moments (MoM) based estimator, and (iii) trace-norm convex relaxation-based estimator. The LRRL-AltGD is based on the alternating gradient descent algorithm proposed in (Yi et al., 2016) for solving the low-rank matrix completion problem. LRRL-AltGD alternatively solves for $\widehat{B}$ and $\widehat{W}$ in Eq. (1). The MoM estimator estimates the matrix $\widehat{B}$ using the top-r Singular Value Decomposition (SVD) of $\widehat{\Theta} = \frac{1}{NT}\sum_{n,t} y_{n,t}^2 \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top$. Then, it proceeds to calculate the estimated matrix $\widehat{W}$ through the method of least squares estimator in order to determine the values of $\widehat{\Theta}$. The trace-norm technique relaxes the rank constraint to a trace-norm convex constraint and then iteratively solves for the estimate $\widehat{\Theta}$ and the regularizing parameter λ. We initialized the LRRL-AltGD algorithm using our proposed spectral initialization approach (Algorithm 2). This is because spectral initialization guarantees a good initialization for solving the nonconvex problem.

We plot the empirical average of $\|\Theta - \Theta^\star\|_F / \|\Theta^\star\|_F$ at



(a) Synthetic data: rank $r = 2$ (b) Synthetic data: rank $r = 4$ (c) Synthetic data: rank $r = 8$

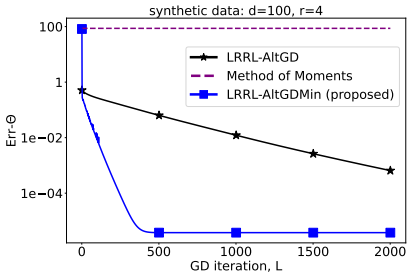
(d) MNIST data: rank $r = 2$ (e) MNIST data: rank $r = 4$ (f) MNIST data: rank $r = 8$

(g) Synthetic data: rank $r = 2$ (h) Synthetic data: rank $r = 3$ (i) Synthetic data: rank $r = 4$

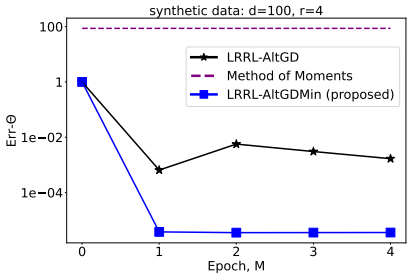
Figure 1: **Synthetic data 1:** We set the parameters as $d = 100$, $K = 5$, $N = 200$, and noise variance $= 10^{-6}$. We considered $M = 4$ epochs each with 50 data samples each. We varied the number of tasks as $T = 10, 25, 50, 75, 100$. We also varied the rank of the feature matrix as $r = 2, 4, 8$. As shown in the plots (Figures 1a, 1b, and 1c), our proposed approach outperforms the existing benchmarks. **MNIST data:** Parameters are $d = 784$, $K = 2$, $N = 5000$, and noise variance $= 10^{-6}$. We considered $M = 5$ epochs each with 1000 data samples each. We varied the number of tasks as $T = 10, 45$. We also varied the rank of the feature matrix as $r = 2, 4, 8$. The plots for MNIST data are presented in Figures 1d, 1e, and 1f. **Synthetic data 2:** We consider a smaller problem dimension here and also compare with the trace-norm relaxation method. In Figures 1g, 1h, and 1i, we set $d = 20$, $K = 5$, $N = 40$. We considered $M = 4$ epochs each with 10 data samples each, thus $N = 40$.

each iteration ℓ (Err-Θ in the plots) on the y-axis and the iteration by the algorithm until GD iteration ℓ on the x-axis. Averaging is over a 100 trials. We note that while LRRL-AltGD, trace-norm, and LRRL-AltGDMin are iterative algorithms, the MoM estimator is non-iterative. To showcase the baselines in our plots, we also show the error achieved by the MoM estimator. Figure 2 presents the error plot. Figure 2a presents the Err-Θ vs. GD iteration for the first epoch. In Figure 2b, we present the Err-Θ vs. epoch. We set a total of 5 epochs, including the zeroth epoch, which is the initialization step. From the plots, we notice that the proposed LRRL-AltGDMin estimator outperforms both the benchmark approaches. Further, the estimation error saturates close to 10^{-6}. This can be explained using our result, Theorem 5.3, which shows that the error decays exponentially until it reaches the (normalized) "noise-level" ${\sigma_\eta}^2/\|\theta_t^\star\|^2$, but saturates after that. Although the error in the trace-norm approach improves as the iteration progresses, the improvement is very minimal.

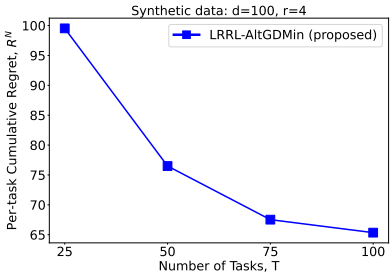
Cumulative regret. We compared the performance of our proposed algorithm against three benchmarks: the Method-of-Moments (MoM)-based representation learning algorithm for bandits in (Yang et al., 2020; Tripuraneni et al., 2021), a Thompson Sampling (TS) algorithm that solves the T tasks separately, and the trace-norm relaxation-based



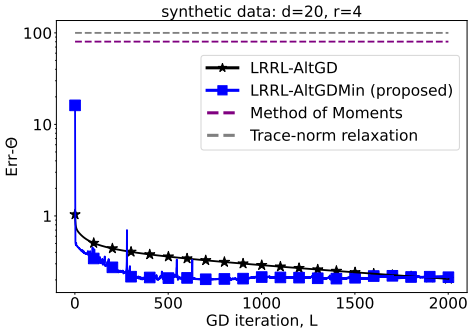
(a) Error vs. GD iteration for epoch-1



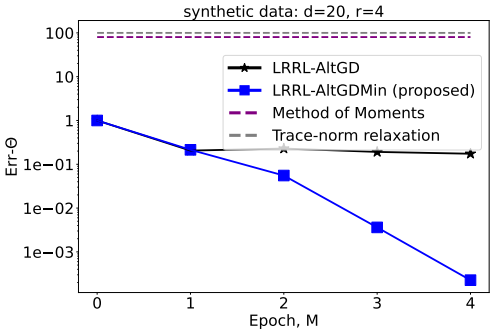
(b) Error vs. Epoch



(c) Error vs. Number of tasks



(d) Error vs. GD iteration for epoch-1



(e) Error vs. Epoch

Figure 2: **Synthetic data 1:** In Figures 2a and 2b, we set the parameters as $d = 100, T = 100$, $K = 5$, $N = 200$, and noise variance $= 10^{-6}$. We ran for $L = 2000$ GD iterations. We considered $M = 4$ epochs each with 50 data samples each. In Figure 2c, we separately present the per-task regret vs. number of task plot for $d = 100$, $K = 5$, $N = 100$ (also shown in figure 1i) to showcase the sublinear decay. **Synthetic data 2:** We consider a smaller problem dimension and also compare with the trace-norm relaxation method. In Figures 2e and 2d, we set the parameters as $d = 20, T = 30$, $K = 5$, $N = 40$, and noise variance $= 10^{-6}$. We ran for $L = 2000$ GD iterations. We considered $M = 4$ epochs each with 10 data samples each, thus $N = 40$. As expected, the estimation error for our proposed algorithm saturates close to the noise.

approach in (Cella et al., 2023). As noted, the MoM-based algorithm only estimates the unknown feature matrix $\widehat{\Theta}$ in the first epoch. In subsequent epochs, this $\widehat{\Theta}$ is consistently used to choose actions. On the other hand, the naive approach implements the TS method separately to determine the estimate of $\theta_t^\star$ for each task $t \in [T]$. The approach in (Cella et al., 2023) considered a trace-norm convex relaxation of the original non-convex cost function (Eqs. (4) and (11) in (Cella et al., 2023)). Figure 1 presents the cumulative regret plots for the different algorithms. We varied the number of tasks and the rank of the feature matrix and compared the results of our proposed algorithm with the MoM-based, trace-norm relaxation, and TS-based algorithms. Our plot demonstrates that as the number of tasks increases, the advantage of the proposed LRRL-AltGDMin algorithm increases compared to the naive approach, the MoM, and the trace-norm relaxation approaches. We varied the rank r and compared the performances. The performance of the TS algorithm is unaltered by varying the rank. This is expected since the regret of the TS algorithm does not depend on the rank. In all experiments, our algorithm consistently outperforms the benchmarks, validating its effectiveness.

7. Conclusion and Future Work

In this work, we introduced an alternating gradient descent and minimization algorithm for multi-task representation learning in linear contextual bandits. Leveraging this estimator, we developed a bandit algorithm and established its regret bound for low dimensional contextual bandits. Our approach consistently outperformed existing methods in numerical experiments. Inspired by (Hu et al., 2021), as part of our future work, we plan to extend our algorithm to an upper confidence bound-based approach by computing the confidence interval. Further, one of the very interesting future directions is to relax the i.i.d standard Gaussian assumption on the feature vectors. While this assumption holds for the initial epoch during the random exploration, it becomes restrictive when we perform greedy exploration in subsequent epochs. As part of our future work, we intend to explore methods for relaxing the i.i.d assumption for epochs after the first one. One potential direction is to fix the B estimate and solve only for W after epoch one, similar to few-shot learning and online subspace tracking (Babu et al., 2023).

Acknowledgements

We would like to thank the anonymous reviewers for the helpful comments and suggestions that helped improve the paper. S. Moothedath and N.Vaswani acknowledge funding from NSF Award-2213069.

Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none which we feel must be specifically highlighted here.

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A. Preliminaries

Proposition A.1 (Theorem 2.8.1, (Vershynin, 2018)). *Let $X_1, \cdots, X_N$ be independent, mean zero, sub-exponential random variables. Then, for every $g \geqslant 0$, we have*

$$\mathbb{P}\Big\{|\sum_{i=1}^{N} X_i \geqslant g|\Big\} \leqslant 2\exp\left[-c\min\left(\frac{g^2}{\sum_{i=1}^{N}\|X_i\|_{\psi_1}^2}, \frac{g}{\max_i \|X_i\|_{\psi_1}}\right)\right],$$

where $c > 0$ is an absolute constant.

Proposition A.2 (Chernoff bound for Gaussian). *Let $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$, then*

$$\mathbb{P}\Big\{X - \mu_x \geqslant g\Big\} \leqslant \exp(-\frac{g^2}{2\sigma_x^2}).$$

B. Guarantees for LRRL-AltGDMin Estimator

Define

$$\begin{aligned}
G &:= B^\top \Theta^\star \\
P &:= I - B^\star {B^\star}^\top \\
\text{GradB} &:= \nabla_B f(B, W) = \sum_{t=1}^{T} \Phi_t^{(\text{m})}(\Phi_t^{(\text{m})} B w_k - Y_t^{(\text{m})}) w_k^\top \\
&= \sum_{t=1}^{T} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} (y_{n,t} - \phi(x_{n,t}, c_n)^\top B w_t)\phi(x_{n,t}, c_n) w_t^\top \\
&= \sum_{t=1}^{T} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top (\theta_t - \theta_t^\star) w_t^\top + \eta_{n,t}\phi(x_{n,t}, c_n) w_t^\top \\
\text{GradB}' &= \sum_{t=1}^{T} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top (\theta_t^\star - \theta_t) w_t^\top.
\end{aligned}$$

and $g_t = B^\top \theta_t^\star$ for all $t \in [T]$, $\text{SD}(B_1, B_2) = \|(I - B_1 B_1^\top) B_2\|_F$ as the Subspace Distance (SD) measure for basis matrices B_1, B_2. Here, GradB represents the gradient that includes noise, while GradB' represents the gradient without noise.

Proposition B.1. *Assume $\text{SD}(B, B^\star) \leqslant \delta_\ell$. Then, with probability at least $O(1 - T\exp(r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$, it holds that*

$$\|M^{-1}\| \leqslant \frac{1}{0.9(\mathcal{G}_m - \mathcal{G}_{m-1})} \quad \textit{and} \quad \|M^{-1} B^\top {\Phi_t^{(m)}}^\top \Phi_t^{(m)} (I - BB^\top)\theta_t^\star\| \leqslant 1.2\epsilon_3 \delta_\ell \|w_t^\star\|,$$

where $M = B^\top {\Phi_t^{(m)}}^\top \Phi_t^{(m)} B$.

Proof. To demonstrate the upper bound of $\|M^{-1}\|$, let's consider a fixed $z \in \mathcal{S}_r$. We then have

$$z^\top B^\top {\Phi_t^{(\text{m})}}^\top \Phi_t^{(\text{m})} B z = \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} z^\top B^\top \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top B z.$$

Furthermore, we find that

$$\mathbb{E}[\langle B^\top \phi(x_{n,t}, c_n), z\rangle^2] = \mathbb{E}[z^\top B^\top \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top B z] = z^\top B^\top \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top] B z = 1,$$

and also

$$\begin{aligned}
\mathbb{E}[z^\top B^\top \phi(x_{n,t}, c_n)] &= 0 \\
\mathrm{Var}[z^\top B^\top \phi(x_{n,t}, c_n)] &= \mathbb{E}[z^\top B^\top \phi(x_{n,t}, c_n)]^2 \\
&= \mathbb{E}[z^\top B^\top \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top Bz] \\
&= z^\top B^\top \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top]Bz \\
&= 1.
\end{aligned}$$

The summands are independent sub-exponential random variables with norm $K_n \leqslant 1$. We apply the sub-exponential Bernstein inequality stated in Proposition A.1 by setting $g = \epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})$. In order to implement this, we show that

$$\begin{aligned}
\frac{g^2}{\sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} K_n^2} &\geqslant \frac{\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1})^2}{(\mathcal{G}_m - \mathcal{G}_{m-1})} = \epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1}) \\
\frac{g}{\max_n K_n} &\geqslant \frac{\epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})}{\max_n 1} = \epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})
\end{aligned}$$

Therefore, for a fixed $z \in \mathcal{S}_r$, with probability at least $1 - \exp(-c\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1}))$,

$$z^\top B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})} Bz - (\mathcal{G}_m - \mathcal{G}_{m-1})I \geqslant -\epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1}).$$

Using epsilon-net over all $z \in \mathcal{S}_r$ adds a factor of $\exp(r)$. Thus, with probability at least $1 - \exp(r - c\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1}))$, we have $\min_{z\in\mathcal{S}_r} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} z^\top B^\top \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top Bz \geqslant (1 - \epsilon_2)(\mathcal{G}_m - \mathcal{G}_{m-1})$. Setting $\epsilon_2 = 0.1$, we obtain

$$\begin{aligned}
\|M^{-1}\| &= \|(B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})} B)^{-1}\| \\
&= \frac{1}{\sigma_{\min}(B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})} B)} \\
&= \frac{1}{\min_{z\in\mathcal{S}_r} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} \langle B^\top \phi(x_{n,t}, c_n), z\rangle^2} \\
&\leqslant \frac{1}{0.9(\mathcal{G}_m - \mathcal{G}_{m-1})}.
\end{aligned}$$

To demonstrate the upper bound of $\|M^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}(I - BB^\top)\theta_t^\star\|$, it is necessary to first determine the upper bound of $\|B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}(I - BB^\top)\theta_t^\star\|$. Consider a fixed $z \in \mathcal{S}_r$, we have

$$z^\top B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}(I - BB^\top)\theta_t^\star = \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} (\phi(x_{n,t}, c_n)^\top Bz)^\top (\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star).$$

Furthermore, we find that

$$\begin{aligned}
\mathbb{E}[(\phi(x_{n,t}, c_n)^\top Bz)^\top (\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star)] &= z^\top B^\top \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top](I - BB^\top)\theta_t^\star \\
&= z^\top B^\top (I - BB^\top)\theta_t^\star \\
&= 0,
\end{aligned}$$

and also we have

$$\begin{aligned}
\mathbb{E}[(\phi(x_{n,t}, c_n)^\top Bz)^\top] &= 0 \\
\mathrm{Var}((\phi(x_{n,t}, c_n)^\top Bz)^\top) &= \mathbb{E}[(\phi(x_{n,t}, c_n)^\top Bz)^\top]^2 \\
&= \mathbb{E}[z^\top B^\top \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top Bz] \\
&= z^\top B^\top \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top]Bz \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star] &= 0 \\
\mathrm{Var}(\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star) &= \mathbb{E}[\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star]^2 \\
&= \mathbb{E}[\theta_t^{\star\top} (I - BB^\top)^\top \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star] \\
&= \theta_t^{\star\top} (I - BB^\top)^\top \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top](I - BB^\top)\theta_t^\star \\
&= \|(I - BB^\top)\theta_t^\star\|^2
\end{aligned}$$

The summands are independent sub-exponential random variables with norm $K_n \leqslant \|(I - BB^\top)\theta_t^\star\|$. We apply the sub-exponential Bernstein inequality stated in Proposition A.1 by setting $g = \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\|$. In order to implement this, we show that

$$\begin{aligned}
\frac{g^2}{\sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} K_n^2} &\geqslant \frac{\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})^2\|(I - BB^\top)\theta_t^\star\|^2}{(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\|^2} = \epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1}) \\
\frac{g}{\max_n K_n} &\geqslant \frac{\epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\|}{\max_n \|(I - BB^\top)\theta_t^\star\|} = \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})
\end{aligned}$$

Therefore, for a fixed $z \in \mathcal{S}_r$, with probability at least $1 - \exp(-c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1}))$,

$$z^\top B^\top \Phi_t^{(\mathrm{m})\top} \Phi_t^{(\mathrm{m})} Bz \leqslant \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\|.$$

Using epsilon-net over all $z \in \mathcal{S}_r$ adds a factor of $\exp(r)$. Thus, with probability at least $1 - \exp(r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1}))$, we have $\max_{z\in\mathcal{S}_r} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} (\phi(x_{n,t}, c_n)^\top Bz)^\top (\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star) \leqslant \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\|$. Therefore, we have

$$\begin{aligned}
\|B^\top \Phi_t^{(\mathrm{m})\top} \Phi_t^{(\mathrm{m})} (I - BB^\top)\theta_t^\star\| &= \max_{z\in\mathcal{S}_r} z^\top B^\top \phi(x_{n,t}, c_n)^\top \phi(x_{n,t}, c_n)(I - BB^\top)\theta_t^\star \\
&= \max_{z\in\mathcal{S}_r} \sum_{n=\mathcal{G}_{m-1}}^{\mathcal{G}_m} (\phi(x_{n,t}, c_n)^\top Bz)^\top (\phi(x_{n,t}, c_n)^\top (I - BB^\top)\theta_t^\star) \\
&\leqslant \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\|
\end{aligned}$$

By combining these results and using a union bound over all T vectors, we conclude that with probability at least $O(1 - T\exp(r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$,

$$\begin{aligned}
\|M^{-1}B^\top \Phi_t^{(\mathrm{m})\top} \Phi_t^{(\mathrm{m})} (I - BB^\top)\theta_t^\star\| &\leqslant \|M^{-1}\| \times \|B^\top \Phi_t^{(\mathrm{m})\top} \Phi_t^{(\mathrm{m})} (I - BB^\top)\theta_t^\star\| \\
&\leqslant \frac{1}{0.9(\mathcal{G}_m - \mathcal{G}_{m-1})} \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\|(I - BB^\top)\theta_t^\star\| \\
&\leqslant 1.2\epsilon_3\|(I - BB^\top)\theta_t^\star\| \\
&\leqslant 1.2\epsilon_3\delta_\ell\|w_t^\star\|.
\end{aligned}$$

□

Lemma B.2. *Assume $\sigma_\eta^2 \leqslant \frac{r}{T}\delta_\ell^2 {\sigma_{\max}^\star}^2$, and $\mathrm{SD}(B, B^\star) \leqslant \delta_\ell$, if $\delta_\ell \leqslant \frac{0.02}{\sqrt{r}\kappa}$, and if $m \geqslant C\max(\log T, \log d, r)$, then with probability at least $O(1 - \exp(\log T + r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$, the following bounds hold:*

1. $\|w_t - g_t\| \leqslant 0.4\delta_\ell\sqrt{\frac{r}{T}}\sigma_{\max}^\star$
2. $\|w_t\| \leqslant 1.1\mu\sqrt{\frac{r}{T}}\sigma_{\max}^\star$
3. $\|W - G\|_F \leqslant 0.4\delta_\ell\sqrt{r}\sigma_{\max}^\star$
4. $\|\theta_t - \theta_t^\star\| \leqslant 1.4\mu\delta_\ell\sqrt{\frac{r}{T}}\sigma_{\max}^\star$

5. $\|\Theta_l - \Theta^\star\|_F \leqslant 1.4\mu\delta_\ell\sqrt{r}\sigma^\star_{\max}$

6. $\sigma_{\min}(W) \geqslant 0.9\sigma^\star_{\min}$

7. $\sigma_{\max}(W) \leqslant 1.1\sigma^\star_{\max}$

Proof. Consider the expression for w_t, we obtain that

$$\begin{aligned}
w_t &= (\Phi_t^{(\mathrm{m})}B)^\dagger Y_t^{(\mathrm{m})} \\
&= ((\Phi_t^{(\mathrm{m})}B)^\top(\Phi_t^{(\mathrm{m})}B))^{-1}(\Phi_t^{(\mathrm{m})}B)^\top Y_t^{(\mathrm{m})} \\
&= (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top Y_t^{(\mathrm{m})} \\
&= (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}(B^\top {\Phi_t^{(\mathrm{m})}}^\top)\Phi_t^{(\mathrm{m})}BB^\top\theta_t^\star + (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}(B^\top {\Phi_t^{(\mathrm{m})}}^\top)\Phi_t^{(\mathrm{m})}(I-BB^\top)\theta_t^\star \\
&+ (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}(B^\top {\Phi_t^{(\mathrm{m})}}^\top)\eta_t^{(\mathrm{m})} \\
&= (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}(B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)B^\top\theta_t^\star + (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}(B^\top {\Phi_t^{(\mathrm{m})}}^\top)\Phi_t^{(\mathrm{m})}(I-BB^\top)\theta_t^\star \\
&+ (B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B)^{-1}(B^\top {\Phi_t^{(\mathrm{m})}}^\top)\eta_t^{(\mathrm{m})} \\
&= g_t + M^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}(I-BB^\top)\theta_t^\star + M^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})},
\end{aligned}$$

where $M = B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}B$. Consequently, $w_t - g_t = M^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top \Phi_t^{(\mathrm{m})}(I-BB^\top)\theta_t^\star + M^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})}$. The first term is bounded in Proposition B.1. To bound the second term, let's consider a fixed $z \in \mathcal{S}_r$. We analyze $z^\top B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})} = \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m}(Bz)^\top\phi(x_{n,t},c_n)\eta_{n,t}$, leading to $\mathbb{E}[(Bz)^\top\phi(x_{n,t},c_n)\eta_{n,t}] = 0$ and

$$\begin{aligned}
\mathrm{Var}((Bz)^\top\phi(x_{n,t},c_n)) &= \mathbb{E}[(Bz)^\top\phi(x_{n,t},c_n)]^2 - (\mathbb{E}[(Bz)^\top\phi(x_{n,t},c_n)])^2 \\
&= \mathbb{E}[(Bz)^\top\phi(x_{n,t},c_n)]^2 \\
&= \mathbb{E}[z^\top B^\top\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top Bz] \\
&= z^\top B^\top \mathbb{E}[\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top]Bz \\
&= z^\top B^\top Bz \\
&= I.
\end{aligned}$$

Given $\eta_{n,t} \overset{\text{iid}}{\sim} \mathcal{N}(0,\sigma_\eta^2)$, we have $\mathrm{Var}(\eta_{n,t}) = \sigma_\eta^2$. Thus, $z^\top B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})}$ is a sum of $(\mathcal{G}_m - \mathcal{G}_{m-1})$ subexponential random variables with parameter $K_n = \sigma_\eta$. We apply the sub-exponential Bernstein inequality stated in Proposition A.1 by setting $g = \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta$. In order to implement this, we show that

$$\begin{aligned}
\frac{g^2}{\sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} K_n^2} &\geqslant \frac{\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})^2\sigma_\eta^2}{(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta^2} = \epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1}) \\
\frac{g}{\max_n K_n} &\geqslant \frac{\epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta}{\sigma_\eta} = \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})
\end{aligned}$$

Therefore, for a fixed $z \in \mathcal{S}_r$, with probability at least $1 - \exp(-c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1}))$, $z^\top B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})} \leqslant \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta$. Using epsilon-net over all z adds a factor of $\exp(r)$. Thus, with probability at least $1 - \exp(r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta)$, we have $B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})} \leqslant \epsilon_3(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta$. Then, the above holds for all $t \in [T]$ with probability at least $1 - \exp(\log T + r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1}))$. According to Proposition B.1, with probability at least $O(1 - T\exp(r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$, we have $\|M^{-1}\| \leqslant \frac{1}{0.9(\mathcal{G}_m - \mathcal{G}_{m-1})}$. Combining these results, it follows that with probability at least $O(1 - T\exp(r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$, $\|M^{-1}B^\top {\Phi_t^{(\mathrm{m})}}^\top \eta_t^{(\mathrm{m})}\| \leqslant \frac{\epsilon_3\sigma_\eta}{0.9}$. Combining with bound on the first term, we then determine that with probability at least $O(1 - \exp(\log T + r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$,

$$\|w_t - g_t\| \leqslant 1.2\epsilon_3\delta_\ell\|w_t^\star\| + \frac{\epsilon_3\sigma_\eta}{0.9}.$$

Given that $\sigma_\eta \leqslant \sqrt{\frac{r}{T}}\delta_\ell \sigma^\star_{\max}$, we then have

$$\|w_t - g_t\| \leqslant 2.4\epsilon_3 \delta_\ell \max(\|w_t^\star\|, \sqrt{\frac{r}{T}}\sigma^\star_{\max}).$$

Applying the Incoherence of right singular vectors in Assumption 2.2 and set $\epsilon_3 = \frac{0.4}{2.4\mu}$, we have

$$\|w_t - g_t\| \leqslant 2.4\epsilon_3 \delta_\ell \max(\mu\sqrt{\frac{r}{T}}\sigma^\star_{\max}, \sqrt{\frac{r}{T}}\sigma^\star_{\max}) \leqslant 2.4\mu\epsilon_3\delta_\ell\sqrt{\frac{r}{T}}\sigma^\star_{\max} \leqslant 0.4\delta_\ell\sqrt{\frac{r}{T}}\sigma^\star_{\max}. \tag{2}$$

This proves 1).

Eq. (2) implies

$$\|W - G\|_F \leqslant 0.4\delta_\ell\sqrt{r}\sigma^\star_{\max}.$$

This completes the proof of 3).

To bound $\|w_t\|$, we use $\|g_t\| \leqslant \|w_t^\star\|$, and then find

$$\begin{aligned}
\|w_t\| &= \|w_t - g_t + g_t\| \\
&\leqslant \|w_t - g_t\| + \|w_t^\star\| \\
&\leqslant 0.4\delta_\ell\sqrt{\frac{r}{T}}\sigma^\star_{\max} + \mu\sqrt{\frac{r}{T}}\sigma^\star_{\max} \\
&\leqslant 1.1\mu\sqrt{\frac{r}{T}}\sigma^\star_{\max}.
\end{aligned}$$

This completes the proof of 2).

For $\|\theta_t - \theta_t^\star\|$, we derive

$$\begin{aligned}
\|\theta_t - \theta_t^\star\| &= \|Bg_t + (I - BB^\top)\theta_t^\star - Bw_t\| \\
&= \|B(g_t - w_t) + (I - BB^\top)\theta_t^\star\| \\
&\leqslant \|g_t - w_t\| + \|(I - BB^\top)B^\star w_t^\star\| \\
&\leqslant 0.4\delta_\ell\sqrt{\frac{r}{T}}\sigma^\star_{\max} + \mu\delta_\ell\sqrt{\frac{r}{T}}\sigma^\star_{\max} \\
&\leqslant 1.4\mu\delta_\ell\sqrt{\frac{r}{T}}\sigma^\star_{\max}.
\end{aligned}$$

This implies that

$$\|\Theta_l - \Theta^\star\|_F \leqslant 1.4\mu\delta_\ell\sqrt{r}\sigma^\star_{\max}.$$

This proves 4) and 5).

Furthermore,

$$\begin{aligned}
\sigma_{\min}(W) &= \sigma_{\min}(G - (G - W)) \\
&\geqslant \sigma_{\min}(G) - \|W - G\| \\
&\geqslant \sigma_{\min}(G) - \|W - G\|_F
\end{aligned}$$

we have

$$\begin{aligned}
\sigma_{\min}(G) &= \sigma_{\min}(G^\top) \\
&= \sigma_{\min}(W^{\star\top}B^{\star\top}B) \\
&\geqslant \sigma^\star_{\min}\sigma_{\min}(B^{\star\top}B),
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\min}({B^\star}^\top B) &= \sqrt{\lambda_{\min}(B^\top B^\star {B^\star}^\top B))} \\
&= \sqrt{\lambda_{\min}(B^\top (I-P) B)} \\
&= \sqrt{\lambda_{\min}(I - B^\top P B)} \\
&= \sqrt{\lambda_{\min}(I - B^\top P^2 B)} \\
&= \sqrt{1 - \lambda_{\max}(B^\top P^2 B)} \\
&= \sqrt{1 - \|PB\|^2} \\
&\geqslant \sqrt{1-\delta_\ell^2}.
\end{aligned}$$

Combining the above three bounds, if $\delta_\ell < \frac{0.02}{\sqrt{r}\kappa}$, we then have

$$\sigma_{\min}(W) \geqslant \sqrt{1-\delta_\ell^2}\sigma^\star_{\min} - 0.4\delta_\ell\sqrt{r}\sigma^\star_{\max} \geqslant 0.9\sigma^\star_{\min}$$

and

$$\begin{aligned}
\sigma_{\max}(W) &= \sigma_{\max}(G - (G-W)) \\
&\leqslant \sigma_{\max}(G) + \sigma_{\max}(G-W) \\
&= \sigma_{\max}(B^\top B^\star W^\star) + \sigma_{\max}(G-W) \\
&\leqslant \sigma_{\max}(B^\top B^\star)\sigma_{\max}(W^\star) + \|G-W\|_F \\
&\leqslant \sigma^\star_{\max} + 0.4\delta_\ell\sqrt{r}\sigma^\star_{\max} \\
&= 1.1\sigma^\star_{\max}
\end{aligned}$$

Thus, the proof is complete. □

Proposition B.3. *Assume* $\mathrm{SD}(B, B^\star) \leqslant \delta_\ell$. *The following statements are true:*

- $\mathbb{E}[\mathrm{GradB}'] = (\mathcal{G}_m - \mathcal{G}_{m-1})(\Theta^\star - \Theta)W^\top$
- $\|\mathbb{E}[\mathrm{GradB}']\| \leqslant 1.6(\mathcal{G}_m - \mathcal{G}_{m-1})\mu\delta_\ell\sqrt{r}{\sigma^\star_{\max}}^2$
- *If* $\delta_\ell < \frac{c}{\sqrt{r}\kappa}$, *then, with probability at least* $O(1-\exp(C(d+r) - c\frac{\epsilon_1^2(\mathcal{G}_m-\mathcal{G}_{m-1})T}{2.4\mu^2 r\kappa^4}) - \exp(\log T + r - c(\mathcal{G}_m - \mathcal{G}_{m-1})))$, *the inequality* $\|\mathrm{GradB}' - \mathbb{E}[\mathrm{GradB}']\| \leqslant \epsilon_1\delta_\ell(\mathcal{G}_m - \mathcal{G}_{m-1}){\sigma^\star_{\min}}^2$ *holds*

where $\mathrm{GradB}' = \sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top(\theta^\star_t - \theta_t)w_t^\top$.

Proof. By using independence of $\Phi_t^{(\mathrm{m})}$ and $\{B, w_t\}$, we can derive

$$\begin{aligned}
\mathbb{E}[\mathrm{GradB}'] &= \mathbb{E}\left[\sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top(\theta^\star_t - \theta_t)w_t^\top\right] \\
&= \sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top](\theta^\star_t - \theta_t)w_t^\top \\
&= \sum_{t=1}^{T}(\mathcal{G}_m - \mathcal{G}_{m-1})(\theta^\star_t - \theta_t)w_t^\top \\
&= (\mathcal{G}_m - \mathcal{G}_{m-1})(\Theta^\star - \Theta)W^\top.
\end{aligned}$$

Utilizing the upper bound from Lemma B.2, if $\delta_\ell < \frac{c}{\sqrt{r}\kappa}$, with probability at least $O(1-\exp(\log T + r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$,

$$\begin{aligned}
\|\mathbb{E}[\text{GradB}']\| &= \|\sum_{t=1}^{T}(\mathcal{G}_m - \mathcal{G}_{m-1})(\theta_t^\star - \theta_t)w_t^\top\| \\
&= (\mathcal{G}_m - \mathcal{G}_{m-1})\|(\Theta^\star - \Theta)W^\top\| \\
&\leqslant (\mathcal{G}_m - \mathcal{G}_{m-1})\|\Theta^\star - \Theta\| \cdot \|W\| \\
&\leqslant (\mathcal{G}_m - \mathcal{G}_{m-1})\|\Theta^\star - \Theta\|_F \cdot \|W\| \\
&\leqslant 1.6(\mathcal{G}_m - \mathcal{G}_{m-1})\mu\delta_\ell\sqrt{r}{\sigma_{\max}^\star}^2
\end{aligned}$$

To bound $\|\text{GradB}' - \mathbb{E}[\text{GradB}']\| = \max_{\|z\|=1,\|v\|=1} z^\top(\sum_{t=1}^{T}\sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m}\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)w_t^\top - \mathbb{E}[\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)w_t^\top])v$, we consider fixed unit norm vectors z, v, applying the sub-exponential Berstein inequality as stated in Proposition A.1 and extend the bound to all unit norm vectors z, v using a standard epsilon-net argument. For fixed unit norm z, v, we consider

$$\sum_{t=1}^{T}\sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m}\left((z^\top\phi(x_{n,t},c_n))(w_t^\top v)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t) - \mathbb{E}[z^\top\phi(x_{n,t},c_n))(w_t^\top v)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)]\right)$$

The analysis shows that

$$\begin{aligned}
&\mathbb{E}\left[(z^\top\phi(x_{n,t},c_n))(w_t^\top v)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t) - \mathbb{E}[z^\top\phi(x_{n,t},c_n))(w_t^\top v)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)]\right] \\
&= \left(\mathbb{E}[z^\top\phi(x_{n,t},c_n))(w_t^\top v)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)] - \mathbb{E}[z^\top\phi(x_{n,t},c_n))(w_t^\top v)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)]\right) \\
&= 0,
\end{aligned}$$

and also we have that

$$\begin{aligned}
\mathbb{E}[(z^\top\phi(x_{n,t},c_n))(w_t^\top v)] &= 0, \\
\text{Var}((z^\top\phi(x_{n,t},c_n))(w_t^\top v)) &= \mathbb{E}[(z^\top\phi(x_{n,t},c_n))(w_t^\top v)(w_t^\top v)\phi(x_{n,t},c_n)^\top z] \\
&= (w_t^\top v)^2 z^\top\mathbb{E}[\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top]z \\
&= (w_t^\top v)^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)] &= 0, \\
\text{Var}(\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)) &= \mathbb{E}[(\theta_t^\star - \theta_t)^\top\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top(\theta_t^\star - \theta_t)] \\
&= (\theta_t^\star - \theta_t)^\top\mathbb{E}[\phi(x_{n,t},c_n)\phi(x_{n,t},c_n)^\top](\theta_t^\star - \theta_t) \\
&= \|\theta_t^\star - \theta_t\|^2
\end{aligned}$$

Based on the analysis provided, we determine that the summands are independent, zero mean, sub-exponential random variables with sub-exponential norm $K_{n,t} \leqslant |w_t^\top v|\|\theta_t^\star - \theta_t\|$. We apply the sub-exponential Bernstein inequality stated in

Proposition A.1, with $g = \epsilon_1 \delta_\ell (\mathcal{G}_m - \mathcal{G}_{m-1}) {\sigma^\star_{\min}}^2$. We have

$$\begin{aligned}
\frac{g^2}{\sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} K_{n,t}^2} &\geqslant \frac{\epsilon_1^2 \delta_\ell^2 (\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma^\star_{\min}}^4}{(\mathcal{G}_m - \mathcal{G}_{m-1}) \sum_{t=1}^T |w_t^\top v|^2 \|\theta_t^\star - \theta_t\|^2} \\
&\geqslant \frac{\epsilon_1^2 \delta_\ell^2 (\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma^\star_{\min}}^4}{(\mathcal{G}_m - \mathcal{G}_{m-1}) \max_t \|\theta_t^\star - \theta_t\|^2 \sum_{t=1}^T |w_t^\top v|^2} \\
&= \frac{\epsilon_1^2 \delta_\ell^2 (\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma^\star_{\min}}^4}{(\mathcal{G}_m - \mathcal{G}_{m-1}) \max_t \|\theta_t^\star - \theta_t\|^2 \|v^\top W\|^2} \\
&\geqslant \frac{\epsilon_1^2 \delta_\ell^2 (\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma^\star_{\min}}^4 T}{1.4^2 (\mathcal{G}_m - \mathcal{G}_{m-1}) \mu^2 \delta_\ell^2 r {\sigma^\star_{\max}}^2 \|W\|^2} && (3) \\
&\geqslant \frac{\epsilon_1^2 \delta_\ell^2 (\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma^\star_{\min}}^4 T}{2.4 (\mathcal{G}_m - \mathcal{G}_{m-1}) \mu^2 \delta_\ell^2 r {\sigma^\star_{\max}}^4} && (4) \\
&= \frac{\epsilon_1^2 (\mathcal{G}_m - \mathcal{G}_{m-1}) T}{2.4 \mu^2 r \kappa^4} \\
\frac{g}{\max_{n,t} K_{n,t}} &\geqslant \frac{\epsilon_1 \delta_\ell (\mathcal{G}_m - \mathcal{G}_{m-1}) {\sigma^\star_{\min}}^2}{\max_{n,t} |w_t^\top v| \|\theta_t^\star - \theta_t\|} \\
&\geqslant \frac{\epsilon_1 \delta_\ell (\mathcal{G}_m - \mathcal{G}_{m-1}) {\sigma^\star_{\min}}^2}{\max_t \|\theta_t^\star - \theta_t\| \max_t \|w_t\|} && (5) \\
&\geqslant \frac{\epsilon_1 \delta_\ell (\mathcal{G}_m - \mathcal{G}_{m-1}) {\sigma^\star_{\min}}^2 T}{1.4 \mu^2 \delta_\ell r {\sigma^\star_{\max}}^2} && (6) \\
&= \frac{\epsilon_1 (\mathcal{G}_m - \mathcal{G}_{m-1}) T}{1.4 \mu^2 r \kappa^2}
\end{aligned}$$

where Eq. (3) follows from $\sum_{t=1}^T |w_t^\top v|^2 = \|v^\top w_t\|^2 \leqslant \|W\|^2$ and the upper bound of $\|\theta_t^\star - \theta_t\|$ resulting from Lemma B.2. Eq. (4) follows from the upper bound $\|W\| \leqslant 1.1\sigma^\star_{\max}$ obtained from Lemma B.2. Eq. (5) follows from $|w_t^\top v| \leqslant \|w_t\|$. Eq. (6) follows from the upper bound $\|W\| \leqslant 1.1\sigma^\star_{\max}$ derived from Lemma B.2 and the inequality $\|w_t\| \leqslant \mu\sqrt{\frac{r}{T}}\sigma^\star_{\max}$ from the Assumption 2.2. Consequently, with probability at least $O(1 - \exp(-c\frac{\epsilon_1^2(\mathcal{G}_m - \mathcal{G}_{m-1})T}{2.4\mu^2 r\kappa^4}) - \exp(\log T + r - c(\mathcal{G}_m - \mathcal{G}_{m-1})))$, for a given z, v,

$$z^\top (\mathrm{GradB}' - \mathbb{E}[\mathrm{GradB}'])v \leqslant \epsilon_1 \delta_\ell (\mathcal{G}_m - \mathcal{G}_{m-1}) {\sigma^\star_{\min}}^2.$$

Applying a standard epsilon-net argument to bound the maximum of the above over all unit norm z, v. We conclude that

$$\|\mathrm{GradB}' - \mathbb{E}[\mathrm{GradB}']\| \leqslant \epsilon_1 \delta_\ell (\mathcal{G}_m - \mathcal{G}_{m-1}) {\sigma^\star_{\min}}^2$$

with probability at least $O(1 - \exp(C(d+r) - c\frac{\epsilon_1^2(\mathcal{G}_m - \mathcal{G}_{m-1})T}{2.4\mu^2 r\kappa^4}) - \exp(\log T + r - c(\mathcal{G}_m - \mathcal{G}_{m-1})))$. The probability factor of $\exp(C(d+r))$ arises from the epsilon-net over z and that over v: z is an d-length unit norm vector while v is an r-length unit norm vector. The size of the smallest epsilon net that covers the hyper-sphere of all ws is $(1 + \frac{2}{\epsilon_{net}})^d$, where $\epsilon_{net} = c$. Similarly, the size of the epsilon net that covers v is C^r. Applying the union bound over both results in a factor of C^{d+r}. This completes the proof. □

Now we have the following lemma for the gradient when noise is considerd.

Lemma B.4. *Assume that* $\mathrm{SD}(B, B^\star) \leqslant \delta_\ell$*, and* $\sigma_\eta^2 \leqslant \frac{r}{T}\delta_\ell^2 {\sigma^\star_{\min}}^2$*. The following statements are true:*

- $\mathbb{E}[\mathrm{GradB}] = (\mathcal{G}_m - \mathcal{G}_{m-1})(\Theta - \Theta^\star)W^\top = (\mathcal{G}_m - \mathcal{G}_{m-1})(BWW^\top - \Theta^\star W^\top)$
- $\|\mathbb{E}[\mathrm{GradB}]\| \leqslant 1.6(\mathcal{G}_m - \mathcal{G}_{m-1})\mu\delta_\ell\sqrt{r}{\sigma^\star_{\max}}^2$
- *If* $\delta_\ell < \frac{c}{\sqrt{r}\kappa}$*, then with probability at least* $O(1 - \exp(C(d+r) - c\frac{\epsilon_1^2(\mathcal{G}_m - \mathcal{G}_{m-1})T}{2.4\mu^2 r\kappa^4}) - \exp(\log T + r - c(\mathcal{G}_m - \mathcal{G}_{m-1})))$*,*

$$\|\mathrm{GradB} - \mathbb{E}[\mathrm{GradB}]\| \leqslant 2\epsilon_1 (\mathcal{G}_m - \mathcal{G}_{m-1})\delta_\ell {\sigma^\star_{\min}}^2.$$

Proof. Recall the definition of GradB.

$$\begin{aligned}\mathrm{GradB} &= \sum_{t=1}^{T} \Phi_t^{(\mathrm{m})}(\Phi_t^{(\mathrm{m})} B w_k - Y_t^{(\mathrm{m})}) w_t^\top \\ &= \sum_{t=1, n=\mathcal{G}_{m-1}+1}^{T, \mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top (\theta_t - \theta_t^\star) w_t^\top - \eta_{n,t}\phi(x_{n,t}, c_n) w_t^\top .\end{aligned}$$

From this we obtain

$$\begin{aligned}\mathbb{E}[\mathrm{GradB}] &= \mathbb{E}\left[\sum_{t=1, n=\mathcal{G}_{m-1}+1}^{T, \mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top (\theta_t - \theta_t^\star) w_t^\top + \eta_{n,t}\phi(x_{n,t}, c_n) w_t^\top\right] \\ &= (\mathcal{G}_m - \mathcal{G}_{m-1}) \sum_{t=1}^{T} (\theta_t - \theta_t^\star) w_t^\top \\ &= (\mathcal{G}_m - \mathcal{G}_{m-1})(\Theta - \Theta^\star) W^\top\end{aligned}$$

Applying bounds on $\|W\|$ and $(\Theta^\star - \Theta)$ from Lemma B.2, we have

$$\begin{aligned}\|\mathbb{E}[\mathrm{GradB}]\| &= \|(\mathcal{G}_m - \mathcal{G}_{m-1})(\Theta - \Theta^\star) W^\top\| \\ &\leqslant (\mathcal{G}_m - \mathcal{G}_{m-1})\|\Theta - \Theta^\star\|_F \|W\| \\ &\leqslant 1.6(\mathcal{G}_m - \mathcal{G}_{m-1})\mu\delta_\ell \sqrt{r}{\sigma_{\max}^\star}^2.\end{aligned}$$

Subsequently, we finish the proof of the bound for $\|\mathbb{E}[\mathrm{GradB}]\|$. Considering unit vectors v, z, we need to bound $\sum_{t=1, n=\mathcal{G}_{m-1}+1}^{T, \mathcal{G}_m} \eta_{n,t} v^\top \phi(x_{n,t}, c_n) w_t^\top z$. This implies $\mathbb{E}[\eta_{n,t} v^\top \phi(x_{n,t}, c_n) w_t^\top z] = 0$ and

$$\begin{aligned}\mathrm{Var}(v^\top \phi(x_{n,t}, c_n) w_t^\top z) &= \mathbb{E}[v^\top \phi(x_{n,t}, c_n) w_t^\top z]^2 - (\mathbb{E}[v^\top \phi(x_{n,t}, c_n) w_t^\top z])^2 \\ &= \mathbb{E}[v^\top \phi(x_{n,t}, c_n) w_t^\top z z^\top w_t \phi(x_{n,t}, c_n)^\top v] \\ &= |w_t^\top z|^2 v^\top \mathbb{E}[\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top] v \\ &= |w_t^\top z|^2\end{aligned}$$

Given $\eta_{n,t} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\eta^2)$, we have $\mathrm{Var}(\eta_{n,t}) = \sigma_\eta^2$. Therefore, $\eta_{n,t} v^\top \phi(x_{n,t}, c_n) w_t^\top z$ is a sum of subexponential random variables with parameter $K_{n,t} \leqslant |w_t^\top z|\sigma_\eta$. Setting $g = \epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_{\min}^\star \sigma_\eta \sqrt{\frac{T}{r}}$, we obtain

$$\begin{aligned}\frac{g^2}{\sum_{t=1, n=\mathcal{G}_{m-1}+1}^{T, \mathcal{G}_m} K_{n,t}^2} &\geqslant \frac{\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma_{\min}^\star}^2 \sigma_\eta^2 \frac{T}{r}}{\sigma_\eta^2 \sum_{t=1, n=\mathcal{G}_{m-1}+1}^{T, \mathcal{G}_m} (w_t^\top z)^2} \\ &\geqslant \frac{\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma_{\min}^\star}^2 \sigma_\eta^2 \frac{T}{r}}{\sigma_\eta^2 (\mathcal{G}_m - \mathcal{G}_{m-1})\|W\|^2} \\ &\geqslant \frac{\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1})^2 {\sigma_{\min}^\star}^2 \sigma_\eta^2 \frac{T}{r}}{1.3(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_\eta^2 {\sigma_{\max}^\star}^2} \\ &= \frac{\epsilon_2^2(\mathcal{G}_m - \mathcal{G}_{m-1})T}{1.3 r\kappa^2},\end{aligned}$$

$$\begin{aligned}\frac{g}{\max_{n,t} K_{n,t}} &\geqslant \frac{\epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_{\min}^\star \sigma_\eta \sqrt{\frac{T}{r}}}{\sigma_\eta \max_{n,t} \|w_t\|} \\ &\geqslant \frac{\epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma_{\min}^\star \sigma_\eta \sqrt{\frac{T}{r}}}{\sigma_\eta \mu \sqrt{\frac{r}{T}} \sigma_{\max}^\star} \\ &\geqslant \frac{\epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})T}{\mu r \kappa}.\end{aligned}$$

Consequently, with probability at least $1 - \exp(-c\frac{\epsilon_2^2(\mathcal{G}_m-\mathcal{G}_{m-1})T}{\mu r\kappa^2})$, for fixed v, z, $\sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} |\eta_{n,t} v^\top \phi(x_{n,t}, c_n) w_t^\top z| \leqslant \epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma^\star_{\min}\sigma_\eta\sqrt{\frac{T}{r}}$. Utilizing an epsilon-net to maximize over all unit vectors v, z. This will give a factor of $\exp(d + r)$ in probability. Thus, with probability at least $1 - \exp((d + r) - c\frac{\epsilon_2^2(\mathcal{G}_m-\mathcal{G}_{m-1})T}{\mu r\kappa^2})$,

$$\left\| \sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} \eta_{n,t}\phi(x_{n,t}, c_n) w_t^\top \right\| \leqslant \epsilon_2(\mathcal{G}_m - \mathcal{G}_{m-1})\sigma^\star_{\min}\sigma_\eta\sqrt{\frac{T}{r}}.$$

Recall $\text{GradB}' = \sum_{t=1,n=\mathcal{G}_{m-1}+1}^{T,\mathcal{G}_m} \phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top(\theta_t^\star - \theta_t)w_t^\top$. From Proposition B.3, if $\delta_\ell < \frac{c}{\sqrt{r}\kappa}$, then, with probability at least $O(1 - \exp(C(d + r) - c\frac{\epsilon_1^2(\mathcal{G}_m-\mathcal{G}_{m-1})T}{2.4\mu^2 r\kappa^4}) - \exp(\log T + r - c(\mathcal{G}_m - \mathcal{G}_{m-1})))$, it holds that $\|\text{GradB}' - \mathbb{E}[\text{GradB}']\| \leqslant \epsilon_1\delta_\ell(\mathcal{G}_m - \mathcal{G}_{m-1}){\sigma^\star_{\min}}^2$. By combining both and setting $\epsilon_2 = \epsilon_1$, we conclude that with probability at least $O(1 - \exp(C(d + r) - c\frac{\epsilon_1^2(\mathcal{G}_m-\mathcal{G}_{m-1})T}{2.4\mu^2 r\kappa^4}) - \exp(\log T + r - c(\mathcal{G}_m - \mathcal{G}_{m-1})))$,

$$\|\text{GradB} - \mathbb{E}[\text{GradB}]\| \leqslant \epsilon_1(\mathcal{G}_m - \mathcal{G}_{m-1})(\delta_\ell\sigma^\star_{\min} + \sigma_\eta\sqrt{\frac{T}{r}})\sigma^\star_{\min}.$$

Thus, if $\sigma_\eta^2 \leqslant \frac{r}{T}\delta_\ell^2{\sigma^\star_{\min}}^2$, then we have

$$\|\text{GradB} - \mathbb{E}[\text{GradB}]\| \leqslant 2\epsilon_1(\mathcal{G}_m - \mathcal{G}_{m-1})\delta_\ell{\sigma^\star_{\min}}^2.$$

This completes the proof. □

B.1. Proof of Theorem 5.2

Consider the Projected GD step for B: $\widehat{B}^+ = B - \frac{\gamma}{(\mathcal{G}_m-\mathcal{G}_{m-1})}\text{GradB}$ and $\widehat{B}^+ \overset{QR}{=} B^+R^+$. Given that $B^+ = \widehat{B}^+(R^+)^{-1}$ and $\|(R^+)^{-1}\| = \frac{1}{\sigma_{min}(R^+)} = \frac{1}{\sigma_{\min}(\widehat{B}^+)}$, it follows that $\text{SD}(B^+, B^\star) = \|PB^+\|$ can be bound as

$$\text{SD}(B^+, B^\star) \leqslant \frac{\|P\widehat{B}^+\|}{\sigma_{\min}(\widehat{B}^+)} \leqslant \frac{\|P\widehat{B}^+\|}{\sigma_{\min}(B) - \frac{\gamma}{(\mathcal{G}_m-\mathcal{G}_{m-1})}\|\text{GradB}\|}. \tag{7}$$

By considering the numerator and performing adding and subtracting of $\mathbb{E}[\text{GradB}]$, left multiplying both sides by P, and utilizing the result from Lemma B.4, we derive

$$\widehat{B}^+ = B - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\mathbb{E}[\text{GradB}] + \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}(\mathbb{E}[\text{GradB}] - \text{GradB}).$$

Consequently,

$$\begin{aligned} P\widehat{B}^+ &= PB - \gamma PBWW^\top + \gamma P\Theta^\star W^\top + \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}P(\mathbb{E}[\text{GradB}] - \text{GradB}) \\ &= PB - \gamma PBWW^\top + \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}P(\mathbb{E}[\text{GradB}] - \text{GradB}) \end{aligned}$$

where the last step follows by $P\Theta^\star = (I - B^\star{B^\star}^\top)\Theta^\star = \Theta^\star - B^\star{B^\star}^\top B^\star W^\star = 0$. Thus,

$$\|P\widehat{B}^+\| \leqslant \|PB\|\|I - \gamma WW^\top\| + \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\mathbb{E}[\text{GradB}] - \text{GradB}\|. \tag{8}$$

Applying the result stated in Lemma B.2, we obtain

$$\lambda_{\min}(I - \gamma WW^\top) = 1 - \gamma\|W\|^2 \geqslant 1 - 1.21\gamma{\sigma^\star_{\max}}^2.$$

Therefore, for $\gamma < \frac{0.5}{{\sigma^\star_{\max}}^2}$, then the matrix mentioned above is a positive semidefinite. Furthermore, this along with Lemma B.2, leads to that

$$\|I - \gamma WW^\top\| = \lambda_{\max}(I - \gamma WW^\top) \leqslant 1 - 0.81\gamma{\sigma^\star_{\min}}^2.$$

Based on the result mentioned above, Eq. (8), and the bound on $\|\mathbb{E}[\text{GradB}] - \text{GradB}\|$ from Lemma B.4, we conclude the following: If $\gamma < \frac{0.5}{{\sigma^\star_{\max}}^2}$ and $\delta_\ell \leqslant \frac{c}{\sqrt{r}\kappa}$, then with probability at least $O(1-\exp(C(d+r) - \frac{c\epsilon_1^2(\mathcal{G}_m - \mathcal{G}_{m-1})T}{2.4r\mu^2\kappa^4}) - \exp(\log T + r - c\epsilon_3^2(\mathcal{G}_m - \mathcal{G}_{m-1})))$,

$$\begin{aligned}\|P\widehat{B}^+\| &\leqslant \|PB\|\|I - \gamma WW^\top\| + \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\mathbb{E}[\text{GradB}] - \text{GradB}\| \\ &\leqslant (1 - 0.81\gamma{\sigma^\star_{\min}}^2)\delta_\ell + 2\epsilon_1\gamma\delta_\ell{\sigma^\star_{\min}}^2. \end{aligned} \tag{9}$$

This probability is at least $O(1 - d^{-10})$ if $(\mathcal{G}_m - \mathcal{G}_{m-1})T \geqslant C\kappa^4\mu^2 dr$ and $(\mathcal{G}_m - \mathcal{G}_{m-1}) \gtrsim \max(\log d, \log T, r)$. Subsequently, we use Eq. (9) with $\epsilon_1 = \epsilon_2 = 0.1$ and Lemma B.4 in Eq. (7), and setting $\gamma = \frac{c_\gamma}{{\sigma^\star_{\max}}^2}$. If $c_\gamma \leqslant 0.5$, if $\delta_\ell \leqslant \frac{c}{\sqrt{r}\kappa^2}$, and lower bounds on $(\mathcal{G}_m - \mathcal{G}_{m-1})$ from above hold, then Eq. (7) implies that with high probability,

$$\begin{aligned}
\text{SD}(B^+, B^\star) &\leqslant \frac{\|P\widehat{B}^+\|}{\sigma_{\min}(\widehat{B}^+)} \\
&\leqslant \frac{\|P\widehat{B}^+\|}{\sigma_{\min}(B) - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\text{GradB}\|} \\
&= \frac{\|P\widehat{B}^+\|}{\sigma_{\min}(B) - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\text{GradB} - \mathbb{E}[\text{GradB}] + \mathbb{E}[\text{GradB}]\|} \\
&\leqslant \frac{\|PB\|\|I - \gamma WW^\top\| + \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\mathbb{E}[\text{GradB}] - \text{GradB}\|}{1 - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\mathbb{E}[\text{GradB}]\| - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\text{GradB} - \mathbb{E}[\text{GradB}]\|} \\
&\leqslant \frac{(1 - (0.81 - 0.2)\gamma{\sigma^\star_{\min}}^2)\delta_\ell}{1 - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\mathbb{E}[\text{GradB}]\| - \frac{\gamma}{(\mathcal{G}_m - \mathcal{G}_{m-1})}\|\text{GradB} - \mathbb{E}[\text{GradB}]\|} \\
&\leqslant \frac{(1 - 0.5\gamma{\sigma^\star_{\min}}^2)\delta_\ell}{1 - \gamma\delta_\ell\sqrt{r}{\sigma^\star_{\max}}^2(1.6\mu + \frac{0.2}{\kappa^2\sqrt{r}})} \\
&\leqslant \frac{(1 - 0.5\gamma{\sigma^\star_{\min}}^2)\delta_\ell}{1 - 1.8\mu\gamma\delta_\ell\sqrt{r}{\sigma^\star_{\max}}^2} && (10) \\
&\leqslant (1 - 0.5\gamma{\sigma^\star_{\min}}^2)(1 + 1.8\mu\gamma\delta_\ell\sqrt{r}{\sigma^\star_{\max}}^2)\delta_\ell && (11) \\
&\leqslant (1 - 0.5\gamma{\sigma^\star_{\min}}^2 + 1.8\gamma\mu\delta_\ell\sqrt{r}{\sigma^\star_{\max}}^2)\delta_\ell \\
&= (1 - \gamma{\sigma^\star_{\min}}^2(0.5 - 1.8\mu\delta_\ell\sqrt{r}\kappa^2))\delta_\ell \\
&\leqslant (1 - \gamma{\sigma^\star_{\min}}^2(0.5 - 0.036\mu))\delta_\ell && (12) \\
&\leqslant (1 - 0.4\mu\gamma{\sigma^\star_{\max}}^2/\kappa^2)\delta_\ell \\
&= (1 - 0.4\mu\frac{c_\gamma}{\kappa^2})\delta_\ell && (13)
\end{aligned}$$

where Eq. (10) follows from $\kappa^2\sqrt{r} > 1$. Eq. (11) follows from $(1-x)^{-1} < (1+x)$ if $|x| < 1$. Eq. (12) follows from $\delta_\ell \leqslant \frac{0.02}{\sqrt{r}\kappa^2}$. Eq. (13) follows from $\gamma = \frac{c_\gamma}{{\sigma^\star_{\max}}^2}$. This completes the proof. □

B.2. Proof of Theorem 5.1

We analyze the initialization process by computing $\widehat{B}^{(0)}$ as top r singular vectors of $Y_B = \sum_{t=1}^T\sum_{n=1}^{\mathcal{G}_1} y_{n,t}^2\phi(x_{n,t}, c_n)\phi(x_{n,t}, c_n)^\top \mathbb{1}_{\{y_{n,t}^2 \leqslant C_0\sum_{t=1}^T\sum_{n=1}^{\mathcal{G}_1}\frac{y_{n,t}^2}{\mathcal{G}_1 T}\}}$. Subsequently, we use Claim B.15 from (Nayer & Vaswani, 2021) to analyze this. Claim B.15 shows that if

$$\|\eta_t^{(\text{m})}\|^2 \leqslant c\frac{\delta_0^2}{r^2\kappa^4}\|\theta_t^\star\|^2,$$

then with probability at least $1 - \exp(d - \frac{c\delta_0\mathcal{G}_1 T}{r^2\mu^2\kappa^4})$,

$$\text{SD}(\widehat{B}^{(0)}, B^\star) \leqslant \delta_0.$$

In order to determine an upper bound for $\|\eta_t^{(\mathrm{m})}\|^2 = \sum_{n=1}^{\mathcal{G}_1} \eta_{n,t}^2$, we observe that $\eta_{n,t} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\eta^2)$. Thus, $\|\eta_t^{(\mathrm{m})}\|^2$ is a sum of subexponential random variables with parameter $K_n \leqslant \sigma_\eta \cdot \sigma_\eta = \sigma_\eta^2$. We apply the sub-exponential Bernstein inequality stated in Proposition A.1, with $g = 0.1\mathcal{G}_1\sigma_\eta^2$. We have

$$\frac{g^2}{\sum_{n=1}^{\mathcal{G}_1} K_n^2} \geqslant \frac{0.01\mathcal{G}_1^2\sigma_\eta^4}{\mathcal{G}_1\sigma_\eta^4} = 0.01\mathcal{G}_1$$
$$\frac{g}{\max_n K_n} \geqslant \frac{0.1\mathcal{G}_1\sigma_\eta^2}{\sigma_\eta^2} = 0.1\mathcal{G}_1$$

Since $\mathbb{E}[\eta_{n,t}] = \sigma_\eta^2$, it can be proved that with probability at least $1 - \exp(c\mathcal{G}_1)$, $\sum_{n=1}^{\mathcal{G}_1} \eta_{n,t}^2 \leqslant 0.1\mathcal{G}_1\sigma_\eta^2$. Thus, we can determine that with probability at least $1 - \exp(c\mathcal{G}_1)$,

$$\begin{aligned}\|\eta_t^{(\mathrm{m})}\|^2 &= \sum_{n=1}^{\mathcal{G}_1} \eta_{n,t}^2 \\ &\leqslant \sum_{n=1}^{\mathcal{G}_1} \mathbb{E}[\eta_{n,t}^2] + 0.1\mathcal{G}_1\sigma_\eta^2 \\ &= 1.1\mathcal{G}_1\sigma_\eta^2\end{aligned}$$

By utilizing a union bound over all T vectors, we conclude that with probability at least $1 - \exp(\log T - c\mathcal{G}_1)$, $\|\eta_t^{(\mathrm{m})}\| \leqslant 1.1\mathcal{G}_1\sigma_\eta^2$. By combining the results from (Nayer & Vaswani, 2021), we complete the proof. □

B.3. Proof of Theorem 5.3

From Theorem 5.1, we know that at the initialization round, we need

$$\sigma_\eta^2 \leqslant c\frac{\delta_0^2}{r^2\kappa^4\mathcal{G}_1}\|\theta_t^\star\|^2.$$

At GD round ℓ, we assume that $\mathrm{SD}(B, B^\star) \leqslant \delta_\ell$, and we need $\sigma_\eta^2 \leqslant \frac{r}{T}\delta_\ell^2{\sigma_{\min}^\star}^2$. By using Assumption 2.2, this holds if

$$\sigma_\eta^2 \leqslant c\frac{\delta_\ell^2}{\kappa^2}\|\theta_t^\star\|^2.$$

This implies that for the algorithm to converge to error level δ_ℓ, we need noise below this level. In other words, the error cannot go below the noise level. All rounds $\ell > 0$ also need $\delta_\ell \leqslant \frac{0.02}{\sqrt{r}\kappa^2}$. This is satisfied by setting $\delta_0 = \frac{0.02}{\sqrt{r}\kappa^2}$. Thus, the initialization round needs

$$\sigma_\eta^2 \leqslant \frac{c\|\theta_t^\star\|^2}{r^3\kappa^6\mathcal{G}_1}.$$

In summary, let $\epsilon_{noise} = C\kappa^2\sqrt{NSR}$, where $NSR := \frac{\sigma_\eta^2}{\min_t \|\theta_t^\star\|^2}$. From Theorem 5.2, we haven shown that if $\delta_\ell \leqslant \frac{0.02}{\sqrt{r}\kappa^2}$, $\gamma = \frac{c_\gamma}{{\sigma_{\max}^\star}^2}$ with $c_\gamma \leqslant 0.5$, and if $(\mathcal{G}_m - \mathcal{G}_{m-1})T \geqslant C\kappa^6\mu^2(d+r)r(\kappa^2r^2 + \log(\frac{1}{\max(\epsilon,\epsilon_{noise})}))$ and $(\mathcal{G}_m - \mathcal{G}_{m-1}) \gtrsim \max(\log d, \log T, r)\log(\frac{1}{\max(\epsilon,\epsilon_{noise})})$, then with probability at least $O(1 - d^{-10})$, at each round ℓ,

$$\mathrm{SD}(B, B^\star) \leqslant \delta_\ell := (1 - \frac{0.4\mu c_\gamma}{\kappa^2})^\ell\delta_0 = (1 - \frac{0.4\mu c_\gamma}{\kappa^2})^\ell\frac{0.02}{\sqrt{r}\kappa^2}.$$

Thus, to guarantee $\mathrm{SD}(B_L, B^\star) \leqslant \epsilon_{noise}$, we need

$$L = C\kappa^2\log(\frac{1}{\max(\epsilon, \epsilon_{noise})}),$$

where it follows by using $\log(1-x) < 1-x$ for $|x| < 1$ and using $\kappa^2\sqrt{r} \geqslant 1$. Thus, setting $c_\gamma = 0.4$, our sample complexity become $(\mathcal{G}_m - \mathcal{G}_{m-1})T \geqslant C\kappa^6\mu^2r(\kappa^2r^2 + \log(\frac{1}{\max(\epsilon,\epsilon_{noise})}))$, and $\mathcal{G}_m - \mathcal{G}_{m-1} \geqslant C\max(\log d\log T, r)\log(\frac{1}{\epsilon_{noise}})$. □

C. Regret Analysis Proofs

Now, our goal is to bound the per-epoch regret. In order to minimize overall regret, we must ensure that the regret incurred in each epoch is not too large because the overall regret is dominated by the epoch that has the largest regret (Han et al., 2020). To the end, we need to choose the epoch length in such away that the total; number of epochs $M = \lceil \log_2 \log_2 N \rceil$.

Guided by this observation, we can see intuitively an optimal way of selecting the grid must ensure that each batch's regret is the same (at least orderwise in terms of the dependence of T and d): for otherwise, there is a way of reducing the regret order in one batch and increasing the regret order in the other, and the sum of the two will still have a smaller regret order than before (which is dominated by the batch that has a larger regret order). As we shall see later, the following grid choice satisfies this equal-regret-across-batches requirement.

Let $\mathcal{R}_m = \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \sum_{t=1}^{T} \langle \phi(x_{n,t}^\star, c_{n,t})\theta_t^\star \rangle - \langle \phi(x_{n,t}, c_{n,t})\theta_t^\star \rangle$ denotes the cumulative regret incurred for all tasks during the $m-$th epoch. We will utilize this definition to determine its upper bound.

Lemma C.1. *Assume that Assumptions 2.1 and 2.2 hold and $\sigma_\eta^2 \leqslant \frac{c\|\theta_t^\star\|^2}{r^3\kappa^6\mathcal{G}_1}$. Set $\gamma = \frac{0.4}{{\sigma_{\max}^\star}^2}$ and $L = C\kappa^2 \log(\frac{1}{\max(\epsilon, \epsilon_{noise})})$. If*

$$(\mathcal{G}_m - \mathcal{G}_{m-1})T \geqslant C\kappa^6\mu^2(d+T)r(\kappa^2 r^2 + \log(\frac{1}{\max(\epsilon, \epsilon_{noise})}))$$

and

$$\mathcal{G}_m - \mathcal{G}_{m-1} \geqslant C \max(\log d, \log T, r) \log(\frac{1}{\max(\epsilon, \epsilon_{noise})}),$$

then for any epoch $m \in [M]$, with probability at least $O(1 - \delta - d^{-10})$ that

$$\mathcal{R}_m \leqslant 2\mu\sigma_{\max}^\star \max(\epsilon, \epsilon_{noise})\sqrt{rNT \log \frac{1}{\delta}}.$$

Proof. For any epoch $m \in [M]$, any task t, it follows that

$$\begin{aligned}
&\sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x_{n,t}^\star, c_{n,t})^\top \theta_t^\star - \phi(x_{n,t}, c_{n,t})^\top \theta_t^\star \\
&= \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x_{n,t}^\star, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) - \phi(x_{n,t}, c_{n,t})^\top \theta_t^\star + \phi(x_{n,t}^\star, c_{n,t})^\top \widehat{\theta}_{m-1,t} \\
&\leqslant \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x_{n,t}^\star, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) - \phi(x_{n,t}, c_{n,t})^\top \theta_t^\star + \phi(x_{n,t}, c_{n,t})^\top \widehat{\theta}_{m-1,t} \\
&= \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x_{n,t}^\star, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) - \phi(x_{n,t}, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t})
\end{aligned}$$

Since $\phi(x_{n,t}, c_{n,t})$ follows an i.i.d standard Gaussian distribution, we can determine that $\sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x_{n,t}^\star, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) - \phi(x_{n,t}, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) \sim \mathcal{N}(0, 2(\mathcal{G}_m - \mathcal{G}_{m-1}) \left\|\theta_t^\star - \widehat{\theta}_{m-1,t}\right\|_2^2)$. By utilizing the Chernoff bound for Gaussian stated in Proposition A.2, with probability at least $1 - \delta$,

$$\sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x_{n,t}^\star, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) - \phi(x_{n,t}, c_{n,t})^\top (\theta_t^\star - \widehat{\theta}_{m-1,t}) \leqslant 2\sqrt{(\mathcal{G}_m - \mathcal{G}_{m-1}) \log \frac{1}{\delta}} \left\|\theta_t^\star - \widehat{\theta}_{m-1,t}\right\|_2$$

Using a union bound and combining the result with Theorem 5.3, we can find that with probability at least $O(1 - \delta - d^{-10})$,

we have

$$\begin{aligned}
\mathcal{R}_m &= \sum_{t=1}^{T} \sum_{n=\mathcal{G}_{m-1}+1}^{\mathcal{G}_m} \phi(x^\star_{n,t}, c_{n,t})^\top \theta^\star_t - \phi(x_{n,t}, c_{n,t})^\top \theta^\star_t \\
&\leqslant 2 \sum_{t=1}^{T} \sqrt{(\mathcal{G}_m - \mathcal{G}_{m-1}) \log \frac{1}{\delta}} \left\| \theta^\star_t - \widehat{\theta}_{m-1,t} \right\|_2 \\
&= 2 \sqrt{(\mathcal{G}_m - \mathcal{G}_{m-1}) \log \frac{1}{\delta}} \sum_{t=1}^{T} \left\| \theta^\star_t - \widehat{\theta}_{m-1,t} \right\|_2 \\
&\leqslant 2 \sqrt{(\mathcal{G}_m - \mathcal{G}_{m-1}) \log \frac{1}{\delta}} \cdot T \cdot \max(\epsilon, \epsilon_{noise}) \mu \sqrt{\frac{r}{T}} \sigma^\star_{\max} \qquad (14) \\
&\leqslant 2\mu\sigma^\star_{\max} \max(\epsilon, \epsilon_{noise}) \sqrt{rNT \log \frac{1}{\delta}} \qquad (15)
\end{aligned}$$

where Eq. (14) is derived from Theorem 5.3 and Assumption 2.2, Eq. (15) from $\mathcal{G}_m - \mathcal{G}_{m-1} \leqslant N$. □

Proof of Theorem 5.4. By applying the result of Lemma C.1, we can demonstrate that with probability at least $O(1-\delta-d^{-10})$,

$$\begin{aligned}
\mathcal{R}_{N,T} &= \sum_{m=1}^{M} \mathcal{R}_m \\
&\leqslant M 2\mu\sigma^\star_{\max} \max(\epsilon, \epsilon_{noise}) \sqrt{rNT \log \frac{1}{\delta}} \\
&\leqslant 2\mu\sigma^\star_{\max} \max(\epsilon, \epsilon_{noise}) \sqrt{rNT \log \frac{1}{\delta}} (1 + \log\log N) \\
&= O(\max(\epsilon, \epsilon_{noise}) r^{\frac{1}{2}} N^{\frac{1}{2}} T^{\frac{1}{2}} \sqrt{\log \frac{1}{\delta}} \log\log N) \\
&= \widetilde{O}(\max(\epsilon, \epsilon_{noise}) r^{\frac{1}{2}} N^{\frac{1}{2}} T^{\frac{1}{2}})
\end{aligned}$$

where the last inequality is derived from $M = \lceil \log_2 \log_2 N \rceil$. □