Nonparametric Uncertainty Quantification for Single Deterministic Neural Network

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Abstract

This paper proposes a fast and scalable method for uncertainty quantification of machine learning models' predictions. First, we show the principled way to measure the uncertainty of predictions for a classifier based on Nadaraya-Watson's nonparametric estimate of the conditional label distribution. Importantly, the approach allows to disentangle explicitly *aleatoric* and *epistemic* uncertainties. The resulting method works directly in the feature space. However, one can apply it to any neural network by considering an embedding of the data induced by the network. We demonstrate the strong performance of the method in uncertainty estimation tasks on text classification problems and a variety of real-world image datasets, such as MNIST, SVHN, CIFAR-100 and several versions of ImageNet.

1 Introduction

In many machine learning applications, it is crucial to complement model predictions with uncertainty scores that reflect the degree of trust in these predictions. The total uncertainty of a prediction sums from two uncertainty types, arising from different sources: *aleatoric* and *epistemic* [10, 25]. The former reflects the irreducible noise and ambiguity in the data due to class overlap, while the latter is related to the lack of knowledge about model parameters, and can be reduced by expanding the training dataset. Disentangling epistemic uncertainty can help to identify *out-of-distribution (OOD) data* or to spot instances important for annotation during active learning [17]. The areas with high aleatoric uncertainty might contain some incorrectly labeled instances. If we quantify both types of uncertainty well, we can effectively *abstain from predictions* in unreliable areas and address a decision to a human expert [13], which is important in safe-critical field, such as medicine [38], autonomous driving [31, 15], and finance [3].

There is no universally accepted uncertainty measure, and diverse, often heuristic treatments are usually used in practice. For this purpose, one could use maximum softmax probabilities of deep neural networks (MaxProb). However, MaxProb represents only aleatoric uncertainty and the resulting measure is notorious to be overconfident for data the model did not see during training [44]. Methods based on ensembling [28] or Bayesian techniques [16] can capture both types of uncertainty and yield more reliable uncertainty estimates (UEs). However, they also introduce large computational overhead or require big modifications to a model architecture and a training procedure.

Recently, a series of UE methods based on a single deterministic neural network model has been developed [51, 32, 52]. The main idea behind these methods is to leverage proximity of instances in a vector space of neural network hidden representations that capture geometrical relationships between instances in the input space. Such methods are very perspective since they are computationally efficient and usually do not require big changes in network architectures and training procedures, which gives them versatility.

Submitted to 36th Conference on Neural Information Processing Systems (NeurIPS 2022). Do not distribute.

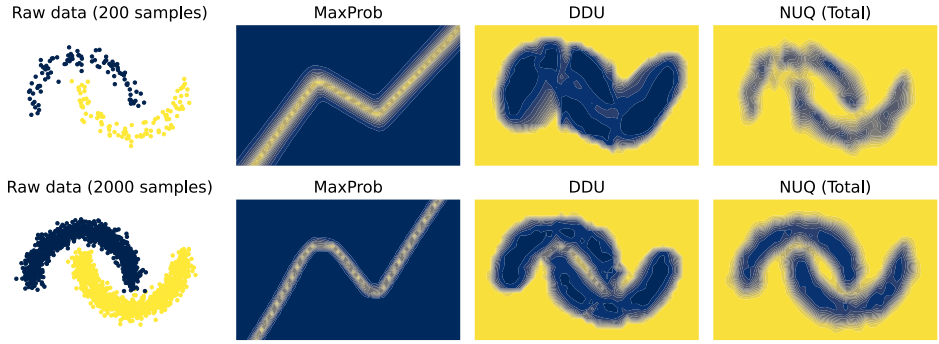


Figure 1: The first column on the left shows the raw data from the Two Moons dataset. The second column illustrates neural network prediction uncertainty obtained using MaxProb. The last two columns illustrate the UE of DDU and the sum of aleatoric and epistemic uncertainties obtained using NUQ. The lighter color indicates higher uncertainty.

Many of these methods rely on the assumption that (conditionally on the class) training instances are distributed normally in the latent space of a trained neural network. However, this assumption does not always hold. For example, a border between classes might be more complex, as well as the border between the in-domain and the out-of-domain region. This happens when training is accomplished in a low-resource regime with small amounts of labeled instances. Another problem is that some methods are hard to train and the model performance can substantially deteriorate without proper hyperparameter tuning. Finally, some methods require computing the covariance matrix of training data. This operation might be computationally unstable if the training dataset is not large enough.

In this work, we propose a new principled approach to UE that overcomes these limitations and demonstrates very robust performance both in low-resource and in high-resource regimes. We suggest to look at the pointwise Bayes risk (probability of the wrong prediction) as the natural measure of the model prediction uncertainty at a particular data point. Then we consider the Nadaraya-Watson estimator of the conditional label distribution. Its asymptotic Gaussian approximation allows deriving uncertainty estimate based on the upper bound for the risk. The resulting Nonparametric Uncertainty Quantification (NUQ) method is amenable for uncertainty disentanglement (into aleatoric and epistemic) and is implemented in a scalable manner, which allows it to be used on large datasets such as ImageNet.

To illustrate the strong points of the proposed method, we suggest to look at a toy example in Figure 1. Here, we compare NUQ with MaxProb and DDU [40], one of the strong baselines in deterministic uncertainty estimation aimed at epistemic uncertainty. MaxProb as a measure of aleatoric uncertainty gives high values only in-between classes while being very confident far from the data. We also see that for a larger data set of 2000 points NUQ and DDU show similar results. However, when the neural network model is trained only on 200 labeled instances, DDU scores become very rough. It cannot spot an unreliable area in-between classes, while NUQ-based total uncertainty measure is able to precisely spot it with high uncertainty. Importantly, the presented total uncertainty for NUQ can be also disentangled into aleatoric and epistemic ones.

Below we give a **summary of contributions** for the present paper.

1. We develop *a new and theoretically grounded* method for nonparametric uncertainty quantification (see Section 2), which is (1) applicable to any deterministic neural network model, (2) has an efficient implementation, (3) allows to disentangle aleatoric and epistemic uncertainty, (4) outperforms other techniques on image and text classification tasks.
2. We formally prove the consistency of the NUQ-based estimator when applied to the problem of classification with the reject option, see Section 3.
3. We conduct a vast empirical investigation of NUQ and other UE techniques on image and text classification tasks that supports our claims, see Section 5.

2 Nonparametric Uncertainty Quantification

2.1 Classification under Covariate Shift

In this section and below, for the sake of clarity, we provide derivations for the binary case. We derive a generalization to a multi-class setting in Supplementary Material (SM), Section A.

Let's consider the standard binary classification setup $(X, Y) \in \mathbb{R}^d \times \{0, 1\}$ with $(X, Y) \sim \mathbb{P}$. We assume that we observe the dataset $\mathcal{D} = \big\{(\mathbf{x}_i, y_i)\big\}_{i=1}^N$ of *i.i.d.* points from $\mathbb{P} = \mathbb{P}_{\text{train}}$. Here, X, Y denote random variables, while $\mathbf{x}, y$ are their realisations.

The classical problem in statistics and machine learning is to find a rule $\hat{g}$ based on the dataset $\mathcal{D}$ which approximates the optimal one:

$$g^* = \arg\min_g \mathbb{P}(g(X) \neq Y).$$

Here $g\colon \mathbb{R}^d \to \{0, 1\}$ is any classifier and the probability of wrong classification $\mathbb{P}(g(X) \neq Y)$ is usually called *risk*. The rule g^* is given by the *Bayes optimal classifier*:

$$g^*(\mathbf{x}) = \begin{cases} 1, & \eta(\mathbf{x}) \geq \frac{1}{2}, \\ 0, & \eta(\mathbf{x}) < \frac{1}{2}, \end{cases}$$

where $\eta(\mathbf{x}) = p(Y = 1 \mid X = \mathbf{x})$ which is the conditional probability of Y given $X = \mathbf{x}$ under the distribution $\mathbb{P}$.

In this work, we consider a situation when the distribution of the test samples $\mathbb{P}_{\text{test}}$ is different from the one for the training dataset $\mathbb{P}_{\text{train}}$, i.e. $\mathbb{P}_{\text{test}} \neq \mathbb{P}_{\text{train}}$. Obviously, the rule g^* obtained for $\mathbb{P} = \mathbb{P}_{\text{train}}$ might no longer be optimal if the aim is to minimize the error on the test data $\mathbb{P}_{\text{test}}(g(X) \neq Y)$.

In order to formulate a meaningful estimation problem, some additional assumptions are needed. We assume that the conditional label distribution $p(Y \mid X)$ **is the same** under both $\mathbb{P}_{\text{train}}$ and $\mathbb{P}_{\text{test}}$. This assumption has two important consequences:

1. All the difference between $\mathbb{P}_{\text{train}}$ and $\mathbb{P}_{\text{test}}$ is due to the difference between marginal distributions of X: $p_{\text{train}}(X)$ and $p_{\text{test}}(X)$. The situation when $p_{\text{test}}(X) \neq p_{\text{train}}(X)$ is known as *covariate shift*.
2. The rule g^* is still valid, i.e., optimal under $\mathbb{P}_{\text{test}}$.

However, while the classifier g^* is still optimal under covariate shift, its approximation $\hat{g}$ might be arbitrarily bad. The reason for that is that we can't expect $\hat{g}$ to approximate g^* well in the areas where we have few samples from the training set or don't have them at all. Thus, some special treatment of covariate shift is required.

2.2 Pointwise Risk and Its Estimation

We consider a classification rule $\hat{g}(X) = \hat{g}_{\mathcal{D}}(X)$ constructed based on the dataset $\mathcal{D}$. Let us start from defining pointwise risk of prediction:

$$\mathcal{R}(\mathbf{x}) = \mathbb{P}(\hat{g}(X) \neq Y \mid X = \mathbf{x}),$$

where $\mathbb{P}(\hat{g}(X) \neq Y \mid X = \mathbf{x}) \equiv \mathbb{P}_{\text{train}}(\hat{g}(X) \neq Y \mid X = \mathbf{x}) \equiv \mathbb{P}_{\text{test}}(\hat{g}(X) \neq Y \mid X = \mathbf{x})$ under the assumptions above. The value $\mathcal{R}(\mathbf{x})$ is independent of covariate distribution $p_{\text{test}}(X)$ and essentially allows to define a meaningful target of estimation which is based solely on the quantities known for the training distribution.

Let us note that the total risk value $\mathcal{R}(\mathbf{x})$ admits the following decomposition:

$$\mathcal{R}(\mathbf{x}) = \tilde{\mathcal{R}}(\mathbf{x}) + \mathcal{R}^*(\mathbf{x}),$$

where $\mathcal{R}^*(\mathbf{x}) = \mathbb{P}(g^*(X) \neq Y \mid X = \mathbf{x})$ is Bayes risk and $\tilde{\mathcal{R}}(\mathbf{x}) = \mathbb{P}(\hat{g}(X) \neq Y \mid X = \mathbf{x}) - \mathbb{P}(g^*(X) \neq Y \mid X = \mathbf{x})$ is an excess risk. Here $\mathcal{R}^*(\mathbf{x})$ corresponds to aleatoric uncertainty as it completely depends on the data distribution. Excess risk $\tilde{\mathcal{R}}(\mathbf{x})$ directly measures imperfectness of the model $\hat{g}$ and thus can be seen as a measure of epistemic uncertainty.

To proceed, we first assume that the classifier $\hat{g}$ has the standard form:

$$\hat{g}(\mathbf{x}) = \begin{cases} 1, & \hat{\eta}(\mathbf{x}) \geq \frac{1}{2}, \\ 0, & \hat{\eta}(\mathbf{x}) < \frac{1}{2}, \end{cases}$$

where $\hat{\eta}(\mathbf{x}) = \hat{p}(Y = 1 \mid X = \mathbf{x})$ is an estimate of the conditional density $\eta(\mathbf{x})$.

For such an estimate we can upper bound the excess risk via the following classical inequality [11]:

$$\tilde{\mathcal{R}}(\mathbf{x}) \leq 2|\hat{\eta}(\mathbf{x}) - \eta(\mathbf{x})|.$$

It allows us to obtain an upper bound for the total risk:

$$\mathcal{R}(\mathbf{x}) \leq \mathcal{L}(\mathbf{x}) = \mathcal{R}^*(\mathbf{x}) + 2|\hat{\eta}(\mathbf{x}) - \eta(\mathbf{x})|,$$

where $\mathcal{R}^*(\mathbf{x}) = \min\{\eta(\mathbf{x}), 1 - \eta(\mathbf{x})\}$ in the case of binary classification. While this upper bound still depends on the unknown quantity $\eta(\mathbf{x})$, we will see in the next section that $\mathcal{L}(\mathbf{x})$ allows for an efficient approximation under mild assumptions.

2.3 Nonparametric Uncertainty Quantification

2.3.1 Kernel Density Estimate and Its Asymptotic Distribution

To obtain an estimate of $\mathcal{L}(\mathbf{x})$ and, consequently, bound the risk, we need to consider some particular type of estimator $\hat{\eta}$. In this work, we choose the classical kernel-based Nadaraya-Watson estimator of the conditional label distribution as it allows for a simple description of its asymptotic properties.

Let us denote by $K_h \colon \mathbb{R}^d \mapsto \mathbb{R}$ the multi-dimensional kernel function with bandwidth h. Typically, we consider a multi-dimensional Gaussian kernel, but other choices are also available. We refer readers to SM, Section C.1 for details.

Then, the conditional probability estimate is expressed as (y_i is either 0 or 1):

$$\hat{\eta}(\mathbf{x}) = \frac{\sum_{i=1}^{N} \mathbb{1}[y_i = 1] \cdot K_h(\mathbf{x} - \mathbf{x}_i)}{\sum_{i=1}^{N} K_h(\mathbf{x} - \mathbf{x}_i)}. \tag{1}$$

The difference between $\hat{\eta}(\mathbf{x}) - \eta(\mathbf{x})$ for properly chosen bandwidth h converges in distribution as follows (see, e.g. [47]):

$$\hat{\eta}(\mathbf{x}) - \eta(\mathbf{x}) \to \mathcal{N}\left(0, \frac{1}{N}\frac{\sigma^2(\mathbf{x})}{p(\mathbf{x})}\int [K_h(\mathbf{u})]^2 d\mathbf{u}\right), \tag{2}$$

where N is the number of data points in the training set, and $\sigma^2(\mathbf{x})$ is the standard deviation of the data label at point $\mathbf{x}$.

Now we are equipped with an estimate of the distribution for $\hat{\eta}(\mathbf{x}) - \eta(\mathbf{x})$. Let us denote by $\tau(\mathbf{x})$ the standard deviation of a Gaussian from equation (2):

$$\tau^2(\mathbf{x}) = \frac{1}{N}\frac{\sigma^2(\mathbf{x})}{p(\mathbf{x})}\int [K_h(\mathbf{u})]^2 d\mathbf{u}.$$

In the following sections, we first show how to use the obtained property for uncertainty estimation, and then we will show how it can be computed.

2.3.2 Total, Aleatoric and Epistemic Uncertainty and Their Estimates

In this work, we suggest a particular uncertainty quantification procedure inspired by the derivation above which we call *Nonparametric Uncertainty Quantification (NUQ)*. More specifically, we suggest to consider the following measure of the total uncertainty:

$$\mathbf{U}_t(\mathbf{x}) = \min\big\{\eta(\mathbf{x}), 1 - \eta(\mathbf{x})\big\} + 2\sqrt{\frac{2}{\pi}}\tau(\mathbf{x}).$$

This measure is obtained by considering an asymptotic approximation of expected value of total risk upper bound:

$$\mathbb{E}_{\mathcal{D}}\mathcal{L}(\mathbf{x}) = \min\big\{\eta(\mathbf{x}), 1 - \eta(\mathbf{x})\big\} + 2\mathbb{E}_{\mathcal{D}}\big|\hat{\eta}(\mathbf{x}) - \eta(\mathbf{x})\big|$$

in a view of (2) and the fact, that $\mathbb{E}|\xi| = \text{std}(\xi)\sqrt{\frac{2}{\pi}}$ for the zero-mean normal variable ξ. The resulting estimate upper bounds the average error of estimation at point $\mathbf{x}$ and thus indeed can be used as the measure of total uncertainty.

We also can write the corresponding measures of aleatoric and epistemic uncertainties:

$$\mathbf{U}_a(\mathbf{x}) = \min\{\eta(\mathbf{x}), 1 - \eta(\mathbf{x})\}, \quad \mathbf{U}_e(\mathbf{x}) = 2\sqrt{\frac{2}{\pi}}\tau(\mathbf{x}). \tag{3}$$

Finally, the data-driven uncertainty estimates $\hat{\mathbf{U}}_a(\mathbf{x}), \hat{\mathbf{U}}_e(\mathbf{x})$ and $\hat{\mathbf{U}}_t(\mathbf{x})$ can be obtained via plug-in using estimates $\hat{\eta}(\mathbf{x})$, $\hat{\sigma}(\mathbf{x})$, $\hat{p}(\mathbf{x})$ and, consequently, $\hat{\tau}^2(\mathbf{x}) = \frac{1}{N}\frac{\hat{\sigma}^2(\mathbf{x})}{\hat{p}(\mathbf{x})}\int\big[K_h(\mathbf{u})\big]^2 d\mathbf{u}$.

We should note that despite being based on asymptotic approximation, the resulting formulas for uncertainties are very natural and make a sense for finite samples (for example, epistemic uncertainty proportional to $\sigma^2(x)/p(x)$). We also should note that even known non-asymptotic decompositions for the risk of NW-estimator still contain the same $\sigma^2(x)/p(x)$ component, see Proposition 1 in [4]. That means that the proposed estimate well captures the general uncertainty trend.

Efficient computation. We note that computation of nonparametric estimate (1) involves a sum over the whole available data. This could be intractable in practice when we are working with large datasets. However, the typical kernel K_h quickly approaches zero with the increase of the norm of the argument: $\|\mathbf{x} - \mathbf{x}_i\|$. Thus, we can use an approximation of kernel estimate: instead of the sum over all elements in the dataset, we consider the contribution of only several nearest neighbors. It requires a fast algorithm for finding the nearest neighbors. For this purpose, we use the approach of [37] based on Hierarchical Navigable Small World graphs (HNSW). It provides a fast, scalable, and easy-to-use solution to the computation of nearest neighbors.

Application to NN and Comparison with Existing Methods. The resulting NUQ method can be applied to NN in the postprocessing fashion, i.e. one can fit it on top of the embeddings of the trained NN model. One may wonder about the difference between NUQ and other embedding based methods such as, for example, DUQ [51] or DDU [40]. The difference is twofold: (i) NUQ is based on the rigorous derivation of total, aleatoric and epistemic uncertainties, while other methods usually consider more heuristic treatment and don't allow for uncertainty disentaglement; (ii) NUQ considers more flexible estimator of density in the embedding space that allows to achive better quality in small sample regime or for complicated data; see experimental evaluation in Section 5.

3 Consistency of NUQ-based Classification with Reject Option

Above we obtained uncertainty estimates that characterize the classical risk of prediction. However, they are also helpful to solve the formal problem of classification with reject option. In this problem, for any input $\mathbf{x}$ we can choose either we perform prediction or reject it. Following [5], we assume that in case of prediction we pay binary price depending whether the prediction was correct or not, while in the case of rejection we pay the constant price $\lambda \in (0, 1)$. For this task the risk function is

$$\mathcal{R}_\lambda(\mathbf{x}) = \mathcal{R}(\mathbf{x})\mathbb{1}\{\alpha(\mathbf{x}) = 0\} + \lambda\mathbb{1}\{\alpha(\mathbf{x}) = 1\},$$

where $\alpha(\mathbf{x})$ is an indicator of the rejection.

The minimizer of $\mathcal{R}_\lambda(\mathbf{x})$ is given by the optimal Bayes classifier $g^*(\mathbf{x})$ and the abstention function

$$\alpha^*(\mathbf{x}) = \begin{cases} 0, & \mathcal{R}^*(\mathbf{x}) \leq \lambda, \\ 1, & \mathcal{R}^*(\mathbf{x}) > \lambda. \end{cases}$$

To approximate $\alpha^*(\mathbf{x})$, we utilize hypothesis testing:

$$H_0\colon \mathcal{R}^*(\mathbf{x}) > \lambda \text{ vs. } H_1\colon \mathcal{R}^*(\mathbf{x}) \leqslant \lambda.$$

We choose the confidence level $\beta > 0$ and consider the statistic $\hat{\mathbf{U}}_\beta(\mathbf{x}) = \min\big\{\hat{\eta}(\mathbf{x}), 1 - \hat{\eta}(\mathbf{x})\big\} + z_{1-\beta}\hat{\tau}(\mathbf{x})$,where $z_{1-\beta}$ is $1 - \beta$ quantile of the standard normal distribution. The statistic $\hat{\mathbf{U}}_\beta(\mathbf{x})$ combines the plug-in estimate of the Bayes risk $\min\big\{\hat{\eta}(\mathbf{x}), 1 - \hat{\eta}(\mathbf{x})\big\}$ and the term $z_{1-\beta}\hat{\tau}(\mathbf{x})$ accounting for the confidence of estimation. The resulting abstention rule is given by:

$$\hat{\alpha}_\beta(\mathbf{x}) = \begin{cases} 0, & \hat{\mathbf{U}}_\beta(\mathbf{x}) \leq \lambda, \\ 1, & \hat{\mathbf{U}}_\beta(\mathbf{x}) > \lambda. \end{cases}$$

Finally, if we consider the pair of kernel classifier $\hat{g}(\mathbf{x})$ and $\hat{\alpha}_\beta(\mathbf{x})$ then we can prove the consistency result for the corresponding risk $\hat{\mathcal{R}}_\lambda(\mathbf{x})$ under standard assumptions on nonparametric densities, see SM, Section B for details.

Theorem 3.1. *Suppose that assumptions B.1, B.3 hold and $p(\mathbf{x}) > 0$, the bandwidth $h \to 0$ and $Nh^d \to \infty$ as N tends to infinity. Then, for any $\beta < 1/2$:*

$$\mathbb{E}_{\mathcal{D}} \hat{\mathcal{R}}_{\lambda}(\mathbf{x}) - \mathcal{R}^*_{\lambda}(\mathbf{x}) \underset{N \to \infty}{\longrightarrow} 0.$$

This result shows the validity of the NUQ-based abstention procedure. Interesting future work is to obtain a precise convergence rate for the method. It should be possible based on the finite sample bounds provided in SM, Section B. We also experimentally illustrate the benefits of the proposed estimator in SM, Section B.1.

4 Related Work

The notion of uncertainty naturally appears in Bayesian statistics [19], and, thus, Bayesian methods are often used for uncertainty quantification. Exact Bayesian inference is computationally intractable, and approximations are used. Two popular ideas are Markov Chain Monte Carlo sampling (MCMC; [42]) and Variational Inference (VI; [2]). MCMC has theoretical guarantees to be asymptotically unbiased but high computational cost. VI-based approaches [48, 12, 45, 26] are more scalable, they are biased and at least double the number of parameters. That is why some alternatives are considered, such as the Bayesian treatment of Monte-Carlo dropout [16].

Deep Ensemble [28] is usually considered as a quite strong yet expensive approach. A series of papers developed ways of approximating the distribution obtained using an ensemble of models by a single probabilistic model [35, 36, 50]. These methods require changing the training procedure and need more parameters to train.

Recently, a series of uncertainty quantification approaches for a single deterministic neural network model was proposed. In DUQ [51], an RBF layer is added to the network with a custom procedure to adjust the centroid points (in the embedding space). The downside of the method is its inability to distinguish aleatoric and epistemic uncertainty. Another approach to capture epistemic uncertainty was proposed in DDU [40]. It uses a Gaussian mixture model to estimate the density of objects in the embedding space of a trained neural network. The density values are then used as a confidence measure. SNGP [32] and DUE [52] are similar but use a Gaussian process as the final layer, requiring estimating covariance with the use of inducing points or RFF expansion.

There is a wide range of papers discussing classification with the reject option. Most likely, the problem was firstly studied by Chow in papers [6, 5]. Moreover, in [5], he introduced a risk function used across this paper. Herbei et al. [24] studied an optimal procedure for this risk and provided a plug-in rule. In the following works, empirical risk minimization among a class of hypotheses (see [1, 7]) or other types of risk (see [9, 14, 30]) were investigated. Besides, a number of practical works have been presented, see, for example, [20, 18, 41].

5 Experiments

We conduct a series of experiments on image and text classification datasets. In each experiment, (1) we train a parametric model – a neural network, which we call a *base model*; (2) fit NUQ on the training data using the embeddings obtained from the base model. We use logits as extracted features, if not explicitly stated otherwise. However, other options are also possible; see Section E.4.

Following SNGP and DDU, we use spectral normalization to train the base model to achieve bi-Lipschitz property and avoid the feature collapse problem. However, NUQ works sufficiently good even without this regularization (see Table 6 in Section E.4).

5.1 How NUQ Affects Model Predictions?

One may ask whether nonparametric classification method used in NUQ, trained on some embedding from the base model, has any relation to the original neural network. To reassure the reader, we provide an argument that it well approximates the predictions of the base model and NUQ-based uncertainty estimates can be used for the base model as well. Specifically, we compute the agreement between predictions obtained from the Bayes classifier based on kernel estimate (i.e. the one used in NUQ) and base models' predictions. This metric formally can be defined as

OOD dataset	MaxProb*	Entropy*	Dropout	Ensemble	TTA	Energy*	DUQ*	SNGP*	DDU*	NUQ*
SVHN	79.7±1.3	81.1±1.6	77.6±2.5	82.9±0.9	81.6±1.2	62.0±1.7	88.7±6.3	86.2±7.4	**89.6±1.6**	**89.7±1.6**
LSUN	81.5±2.0	83.0±2.1	76.8±5.1	86.5±0.8	85.0±2.7	82.7±0.1	90.8±6.7	83.7±8.6	**92.1±0.6**	**92.3±0.6**
Smooth	76.6±3.5	77.8±5.2	63.3±3.8	83.7±1.2	73.2±10.8	71.5±4.6	91.1±8.4	60.9±12.5	**97.1±3.1**	**96.8±3.8**

Table 1: OOD detection for CIFAR-100 in-distribution dataset with ResNet-50 neural network. The top two results are shown in bold. Evaluation is done for three models trained with different seeds to estimate the standard deviation. Methods requiring a single pass over the data to compute uncertainty estimates are marked with *.

OOD dataset	MaxProb*	Entropy*	TTA	Energy*	Ensemble	DDU*	DUQ*	SNGP*	NUQ*
ImageNet-R	80.4	83.6	85.8	78.34	84.4	80.1	73.3	85.0	**99.5**
ImageNet-O	28.2	29.1	30.5	60.0	51.9	74.1	71.4	75.8	**82.4**

Table 2: ROC-AUC score for ImageNet out-of-distribution detection tasks for different methods. Methods requiring a single pass over the data to compute uncertainty estimates are marked with *.

agreement$(\hat{p}, p) = \frac{1}{n}\sum_{i=1}^{n} I\left[\arg\max_j \hat{p}(y = j \mid \mathbf{x}_i) = \arg\max_j p(y = j \mid \mathbf{x}_i)\right]$. For CIFAR-100 (see experiments with this dataset in Section 5.2.1), this metric gives us the agreement of 0.975, which tells that the approach is accurate. Additionally, we computed the aleatoric uncertainty for all test objects and found the average percentile of the objects with disagreement is 94.94 ± 4.75. Thus, disagreement appears for high uncertainty points as expected.

5.2 Image Classification

The main experiments with image classification are conducted on CIFAR-100 [27] and ImageNet [8]. However, several additional experiments with other image classification datasets such as MNIST and SVHN can be found in SM, Section E.

We compare NUQ with popular UE methods, which do not require significant modifications to model architectures and training procedures. More specifically, we consider Maximum probability (MaxProb) of softmax response of NN, Entropy of predictive distribution, Monte-Carlo (MC) dropout [16], ensemble of models trained with different random seeds (deep ensemble), Test-Time Augmentation (TTA) [34], DDU [40], SNGP [32], DUQ [51], and an Energy-based approach [33].

For Monte-Carlo dropout, Ensembles, and TTA, we first compute average predicted class probabilities and then compute their entropy. More details can be found in SM, Section C. For Deep Ensembles, we fixed the number of models to 5. Additional experiments where we changed the number of models are presented in SM, Section E.5.

5.2.1 CIFAR-100

In this experiment, we test UE methods on the out-of-distribution detection task. We treat the OOD detection as binary classification (OOD/not-OOD) using only uncertainty score. Following the setup from the recent works [51, 52, 49], we use SVHN, LSUN [53], and Smooth [21] as OOD datasets. The reported metric is ROC-AUC.

As a base model, we train a ResNet-50 model from scratch on the CIFAR-100 dataset. NUQ was applied to the features from the penultimate layer of the model, and the density estimate is given by GMM, as it provides the best results (see the results for other choices of hyperparameters in Section E.4).

The results are presented in Table 1. We can clearly see that NUQ and DDU show close results while outperforming the competitors with a significant margin.

5.2.2 ImageNet

To demonstrate the applicability of NUQ to large-scale data, we evaluate it in the OOD detection task on ImageNet [8]. As OOD data, we use the ImageNet-O [23] and ImageNet-R [22] datasets. ImageNet-O consists of images from classes not found in the standard ImageNet dataset. ImageNet-R contains different artistic renditions of ImageNet classes.

a) Low uncertainty b) Medium uncertainty c) High uncertainty

Figure 2: Typical OOD images from ImageNet-R. Image ranged from low uncertainty (bottom 10%) (a) to high uncertainty (top 90%) (c) by NUQ predictions. We can clearly see that low-uncertainty samples resemble real-world objects from vanilla ImageNet.

In contrast to the previous experiment, we found that for NUQ, it is more beneficial to use KDE as a density estimator $p(x)$, rather than GMM. Importantly, it took us approximately 5 minutes to receive uncertainties over all ImageNet datasets with CPU, i.e. our NUQ implementation is readily applicable to large-scale data.

The results are summarized in Table 2. We see that for ImageNet-O, many methods show good OOD detection quality, but NUQ achieves an almost perfect result. For ImageNet-R, simple approaches completely fail while DDU, SNGP, and DUQ perform well, and NUQ shows the best result with a large margin.

Note that unlike in the CIFAR-100 experiment, for ImageNet, NUQ significantly outperforms DDU. We conjecture that GMM struggles to approximate density here as an embedding structure is much more complicated for ImageNet compared to CIFAR-100 (see some visualizations in Section E.3). NUQ is beneficial in this case as KDE is much more flexible than GMM and provides a better result.

Additionally, we looked at some typical samples from ImageNet-R with low, moderate, and high levels of uncertainty as assigned by NUQ, see Figure 2. Here, low, medium, and high uncertainties correspond to 10, 50, and 90% quantiles of epistemic uncertainty distribution for images from the ImageNet-R dataset. We observe that uncertainty values corresponds well to these images' intuitive degree of complexity compared to the original ImageNet data.

5.3 Text Classification

Experiments on textual data are performed in low-resource settings, where we train models on small subsamples of original datasets. This regime can be challenging for many UE methods, while NUQ is naturally adapted to it. We compare NUQ to the best performing methods on image classification: DDU and deep ensemble, and to the standard baselines MC dropout and MaxProb.

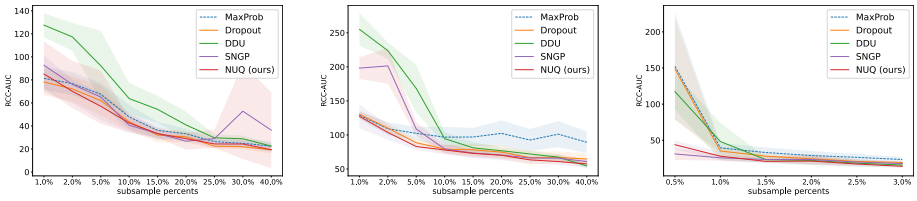
We use a pre-trained ELECTRA model with 110 million parameters. The features for NUQ and DDU are taken from the penultimate layer. The model details, fine-tuning hyperparameters, and other details of UE methods are presented in SM (Section F).

Classification with rejection experiments are conducted on SST-2, MRPC, and CoLA. The evaluation metric is RCC-AUC [13]. Experiments with OOD detection, are conducted on ROSTD and CLINC, which contain instances marked as OOD, and a benchmark composed from SST-2 (in-domain), 20 News Groups, TREC-10, WMT-16, Amazon (sports and outdoors categories), MNLI, and RTE. SST-2 is used as an in-domain dataset, while the rest as OOD datasets. The final score is averaged across all OOD datasets. Evaluation metric is ROC-AUC as in the image classification task. The dataset statistics is presented in SM (Section F).

5.3.1 Classification with Rejection

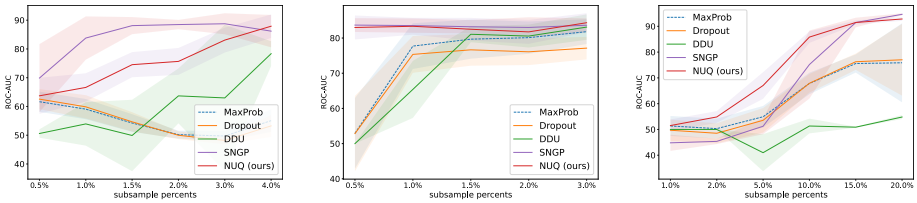
Figure 3 presents the results for classification with rejection. For the MRPC dataset, we can note that NUQ, SNGP, and MC dropout show similar results to the MaxProb baseline. DDU stands out, demonstrating substantially worse performance in settings with small amount of training data.

A similar dynamic for DDU can be noted on the CoLA dataset, where it works substantially worse than the baseline till 10% of training data is unlocked. SNGP in this experiment has the same pattern as DDU with slightly better results. On contrary, NUQ is always on par or better than the



a) MRPC b) CoLA c) SST-2

Figure 3: RCC AUC↓ of classification with rejection depending on fraction of unlocked training data.



a) ROSTD b) SST-2 c) CLINC

Figure 4: ROC-AUC↑ of OOD detection depending on the fraction of unlocked training data.

baseline outperforming all other computationally efficient methods and has similar performance as computationally expensive MC dropout.

On SST-2, NUQ works similar to SNGP, while substantially outperforming DDU, the MaxProb baseline, and MC dropout in the extremely low resource setting. Starting from 1.5%, all methods work similarly with small advantage over the baseline.

Overall, we can conclude that in the settings with a small amount of training data, NUQ can be the best choice for estimating uncertainty for the misclassification detection task: it works similar to other methods when there is much training data and does not deteriorate in the low-resource regime, demonstrating much better results than others. SNGP is able to approach NUQ on SST-2. However, its performance is not stable, which is illustrated by poor results on CoLA. DDU always fails to outperform the baseline with small amount of training data, which might be due to its reliance on the assumption that training data has a Gaussian distribution, which does not hold in this setting.

5.3.2 Out of Distribution Detection

Figure 4 presents the results of OOD detection. We see that NUQ confidently outperforms MC dropout and DDU, especially for extremely low-resource settings. SNGP has some advantage over NUQ on ROSTD, but it notably falls behind on CLINC. However, unlike SNGP, NUQ always outperforms the MaxProb baseline, therefore, it might be a better choice for OOD detection in low-resource regimes. Finally, DDU works poorly, often failing to outperform the MaxProb baseline, which stems from incorrect assumption about the Gaussian distribution of training data.

6 Conclusions

This work proposes NUQ, a new principled uncertainty estimation method that applies to a wide range of neural network models. It does not require retraining the model and acts as a postprocessing step working in the embedding space induced by the neural network. NUQ significantly outperforms the competing approaches with only recently proposed DDU method [40] showing comparable results. Importantly, in the most practical example of OOD detection for ImageNet data, NUQ shows the best results with a significant margin. NUQ is also superior to DDU and other methods on text classification datasets in both OOD detection and classification with rejection. The code to reproduce the experiments is available online at `http://github.com/omitted/to/preserve/anonymity`

We hope that our work opens a new perspective on uncertainty quantification methods for deterministic neural networks. We also believe that NUQ is suitable for in-depth theoretical investigation, which we defer to future work.

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Checklist

The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. For example:

- Did you include the license to the code and datasets? [Yes] See Section ...
- Did you include the license to the code and datasets? [No] The code and the data are proprietary.
- Did you include the license to the code and datasets? [N/A]

Please do not modify the questions and only use the provided macros for your answers. Note that the Checklist section does not count towards the page limit. In your paper, please delete this instructions block and only keep the Checklist section heading above along with the questions/answers below.

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [No] Our proposed method is widely applicable, almost any modern neural network can be equipped with this mechanism. Potential social impact will highly depend on the specific domain and task.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] Proofs for our results are in Supplementary Material, see Section B
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] We provide the link to our code in Conclusion.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] We discuss this in Supplementary Material, Sections C.1, C, F
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] It is not included in the main paper, but will be addressed in Supplementary material.
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] We provide references to the datasets in the 5 section. For our comparisons with existing approaches we have used the author's implementations that were provided in the respective original papers that we cite.
   (b) Did you mention the license of the assets? [No] We use only open source code and datasets.
   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] We provide the link to our code in Conclusion.
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A] All the assets used are free for research purposes.
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No] We use publicly available datasets to benchmark our results and show only numeric values and abstract plots. To the best of our knowledge we have not exposed any personal information or offensive content.

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]