# Structuring Representations Using Group Invariants

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## Abstract

A finite set of invariants can identify many interesting transformation groups. For example, distances, inner products and angles are preserved by Euclidean, Orthogonal and Conformal transformations, respectively. In an equivariant representation, the group invariants should remain constant on the embedding as we transform the input. This gives a procedure for learning equivariant representations without knowing the possibly nonlinear action of the group in the input space. Rather than enforcing such hard invariance constraints on the latent space, we show how to use invariants for "symmetry regularization" of the latent, while guaranteeing equivariance through other means. We also show the feasibility of learning disentangled representations using this approach and provide favorable qualitative and quantitative results on downstream tasks, including world modeling and reinforcement learning.

## 1 Introduction

Sample efficient representation learning is a critical open challenge in deep learning for AI. When we have prior information about transformations that are relevant to a particular domain, building representations that are aware of these transformations can lead to better sample efficiency and generalization. One way to use such symmetry priors is to make the network invariant to the given transformations. A generalization of this idea is called equivariance, where transforming the input transforms the output in a specific way. An equivariant network that makes good predictions for a particular input also generalizes to all input transformations, making symmetry a useful prior.

While recent years have witnessed a range of exciting equivariant deep models, there are several limitations. First, most equivariant networks constrain the network architecture, which often requires specialized implementations. Moreover, transformations considered by the existing methods are often assumed to be linear in both input and representation space. This is the case for architectures designed for finite permutation groups and continuous Lie groups. Approaches that go beyond linear transformations in the input space often assume access to group information – *i.e.*, the group member that transforms one input to another is known. This paper introduces a simple approach that addresses all of these limitations.

Our approach uses the invariants of a given linear representation of a transformation group. Previously invariants were used to connect different geometries and group theory in Klein's Erlangen

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program [33]. According to this view, geometries are concerned with invariant quantities under certain transformations. For example, Euclidean geometry is concerned with the length, angle, and parallelism of lines, among others, because Euclidean transformations preserve these. However, moving to the more general and less structured Affine geometry, notions of distance and angle are no longer relevant, while parallelism remains an invariant of the geometry. The corresponding symmetry groups are examples of Lie groups that have a subgroup relation, $E(n) < Aff(n)$, thereby enabling the groups to characterize a hierarchy (or lattice) of different geometries.

From this geometric perspective, our proposal in this work is to induce a geometry on the embedding and make it equivariant to a given group by enforcing the invariants of their defining action. For example, distance is the invariant for Euclidean geometry, which means all distance-preserving transformations are Euclidean. Therefore, to enforce equivariance to the Euclidean group, it is sufficient to ensure that the embedding of any two data points has the same distance before and after the same transformation of the inputs; see Figure 1. While this approach uses the *defining action* of different groups in the embedding space, the same group can have a non-linear and unknown action on the input space. In the pendulum example of Figure 1, the group $E(3)$ acts on the value of each pixel of the input image using an unknown and non-linear action. Moreover, this approach does not require the pairing of group members with transformations, a piece of information that is often unavailable.

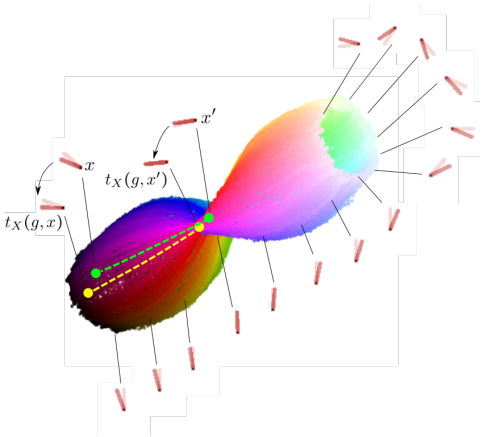


Figure 1: **$\mathbf{E(3)}$-equivariant embedding for the pendulum.** The input $x$ consists of a pair of images that identify both the angle and the angular velocity of a pendulum. The equivariant embedding learns to encode both: the true angle is shown by a change of color and angular velocity using a change of brightness. The two circular ends (black and white) correspond to states of maximum angular velocity in opposite directions. The SymReg objective for the *Euclidean group* learns this embedding by preserving the pairwise distance between the codes before $(f(x), f(x'))$ and after $(f(t_X(g, x)), f(t_X(g, x)))$ transformations of the input by $t_X$. Therefore dashed lines have equal lengths. For the pendulum, the transformations are in the form of applying positive or negative torque in some range.

In the rest of the paper, we arrive at the idea above from a different path: after reviewing related works in Section 2 and providing a background in Section 3, Section 4 observes that equivariance, in its general form, can be a weak inductive bias. This is because having an injective code is sufficient for equivariance to "any" transformation group. However, in this manifestation of equivariance, the group action on the embedding can be highly non-linear. Since the simplicity of the action on the embedding seems essential for equivariance to become a useful learning bias, Section 5 proposes to *regularize* the group action on the code to make it "simple". This *symmetry regularization* (SymReg) objective is group-dependent and the essence of our approach. Enforcing geometric invariants in the latent space is proposed as a symmetry regularization. While we focus on equivariant representation learning through self-supervision, in principle, supervised tasks can also benefit from the proposed SymReg. An important benefit of a symmetry-based representation is its ability to produce disentangled representations through group decomposition [28]. Section 6 studies disentanglement using SymReg. Section 7 presents a range of experiments to understand its behavior and puts it in the context of comparable baselines.

## 2 Related Works

Finding effective priors and objectives for deep representation learning is an integral part of the quest for AI [3]. Among these priors, learning equivariant deep representations has been the subject of many works over the past decade. Many recent efforts in this direction have focused on the design of equivariant maps [59, 13, 49, 35, 15, 24, 56, 19, 6] where the "linear" action of the group on the data is known. A particularly relevant example here is Villar et al. [56], which uses group invariants to construct equivariant maps where the group acts using its linear defining action in the input

space. Due to this constraint, the application of these models has been focused on fixed geometric data such as images [37], sets [62, 46], graphs [40, 34], spherical data and the (special) orthogonal group [14, 1, 52, 23], the Euclidean group [54, 57, 25] or other physically motivated groups such as the Lorentz [4] or Poincare group [56], among others.

In the present work, the group action is unknown and possibly non-linear. Our setup is closer to the body of work on generative representation learning [7, 11, 41], in which the (linear) transformation is applied to the latent space [47, 60, 36, 38, 16, 21]. Among these generative coding methods, transforming autoencoder [30] is a closely related early work, which in addition to equivariance, seeks to represent the part-whole hierarchy in the data. What additionally contrasts our work with the follow-up works on capsule networks [50, 39] is that SymReg is agnostic to the choice of architecture and training. We only rely on our objective function to enforce equivariance.

Since we consider learning equivariant representations through self-supervision, exciting recent progress in this area is also quite relevant [26, 43, 9, 55, 27, 63, 20, 42]. While the use of transformations is prominent in these works, in many settings, the objective encourages *invariance* to certain transformations, making such models useful for invariant downstream tasks such as classification. Similar to many of these methods, we also use transformed pairs to learn a representation, with the distinction of learning an *equivariant* representation. An exception is the recent work of Dangovski et al. [17], which learns an equivariant representation by separating the invariant embedding from the pose, where the relative pose is learned through supervision. Therefore, in that work, in contrast to ours, one needs to know the transformation that maps one input to another. When considering the Euclidean group, SymReg preserves distances in the embedding space under non-linear transformations of the input. This embedding should not be confused with isometric embedding [53], where the objective is to maintain the pairwise distances between points in the input and the embedding space.

## 3 Background on Symmetry Transformations

We can think of transformations as a set of bijective maps on a domain $X$. Since these maps are composable, we can identify their compositional structure using an abstract group $G$. For this reason, such transformations are called group actions. To formally define transformation groups, we first define an abstract group. A *group* $G$ is a set equipped with a binary operation, such that the set is closed under the operation $gg' \in G \ \forall g, g' \in G$, every $g \in G$ has a unique inverse such that $gg^{-1} = e$, where $e$ is the identity element of the group, and the group operations are associative $(gg')g'' = g(g'g'')$.

A $G$*-action* on a set $X$ is defined by a function $t : G \times X \to X$, which can be thought of as a bijective transformation parameterized by $g \in G$. In order to maintain the group structure, the action should satisfy the following two properties: (1) the action of the identity is the identity transformation $t(e, x) = x$; (2) the composition of two actions is equal to the action of the composition of group elements $t(g, t(g'x)) = t(gg', x)$. The action $t$ is *faithful* to $G$ if transformations of $X$ using each $g \in G$ are unique – *i.e.*, $\forall g, g' \ \exists x \in X$ s.t. $t(g, x) \neq t(g', x)$. If a $G$-action is defined on a set $X$, we call $X$ a $G$-set. Many groups are defined using their defining action; for example, $SO(3)$ is the group of rotations in 3D space. While this defining action is a linear transformation, the same group can act non-linearly on $\mathbb{R}^n$ using the action $t : SO(3) \times \mathbb{R}^n \to \mathbb{R}^n$.

## 4 Equivariance is Cheap, Actions Matter

A symmetry-based representation or embedding is a function $f : X \to Z$ such that both $X$ and $Z$ are $G$-sets, and furthermore, $f$ "knows about" $G$-actions, in the sense that transformations of the input using $t_X$ have the same effect as transformations of the output using some action $t_Z$:

$$f(t_X(g, x)) = t_Z(g, f(x)) \quad \forall g, x \in G \times X \tag{1}$$

The following claim shows that despite many efforts in designing equivariant networks, simply asking for the representation to be equivariant is not a strong inductive bias, and we argue that the action matters. Put another way, the strong performance of existing equivariant networks should be attributed to the fact that the group action on the embedding space is simple (linear).

**Proposition 4.1.** *Given a transformation group $t_X : G \times X \to X$, the function $f : X \to Z$ is an equivariant representation if $\forall g \in G, x, x' \in X$*

$$f(x) = f(x') \Leftrightarrow f(t_X(g, x)) = f(t_X(g, x')). \tag{2}$$

*That is, two embeddings are identical iff they are identical for all transformations.*

The proof is in the appendix. The condition above is satisfied by all injective functions, indicating that many functions are equivariant to any group.

**Corollary 4.2.** *Any* injective *function $f : X \to Z$ is equivariant to any transformation group $t_X : G \times X \to X$, if we define $G$ action on the embedding space as*

$$t_Z(g, z) \doteq f(t_X(g, f^{-1}(z))) \quad \forall g, z \in G \times Z \tag{3}$$

The ramification of the results above in what follows is two-fold:
**1.** While injectivity ensures equivariance, the group action on the embedding, as shown in Equation (3), can become highly non-linear. Intuitively, this action recovers $x = f^{-1}(z)$, applies the group action $x' = t_X(x)$ in the input domains and maps back to the embedding space $f(x')$ to ensure equivariance. In the following, we push $t_Z$ towards a simple linear $G$-action through optimization of $f$. This objective can be interpreted as a *symmetry regularization or a symmetry prior* (SymReg).

**2.** Although Corollary 4.2 uses injectivity of $f$ for the entire $X$, we only need this for the data manifold. In practice, one could enforce injectivity on the training dataset $D$ using a decoder, architectural choices such as momentum encoder [27], or loss functions defined on the training data, such as a hinge loss [26] $L_{\text{hinge}}(f, D) = \sum_{x,x' \neq x \in D} \max(\epsilon - \|f(x) - f(x')\|, 0)$ or other losses that monotonically decrease with distance, such as $\frac{1}{\|f(x)-f(x')\|}$, or its logarithm $-\log(\|f(x) - f(x')\|)$. In experiments, we use the logarithmic barrier function.

## 5 Symmetry Regularization Objectives

In learning equivariant representations, we often do not know the abstract group $G$ and how it transforms our data, $t_X$. We assume that one can pick a reasonable abstract group $G$ that "contains" the ground truth abstract group acting on the data – *i.e.*, $G$ action on the input may not be faithful. Our goal is to learn an $f : X \to Z$ that is equivariant w.r.t. the actions $t_X, t_Z$, where $t_X : G \times X \to X$ is unknown and $t_Z$ is some (simple) $G$-action on $Z$ of our choosing.

**More Informed but Less Practical Setting.** In the most informed case, the dataset also contains information about which group member $g \in G$ can be used to transform $x$ to $x'$ – that is, the dataset consists of triples $(x, g, x_t = t_X(g, x))$. By having access to this information, we can regularize the embedding using the following loss function: $L_G^{\text{informed}}(f, D) = \sum_{(x,g,x_t) \in D} \ell\big(f(x_t) - t_Z(g, f(x))\big)$, where $\ell$ is an appropriate loss function, such as the square loss. At its minimum, we have $f(x_t) = t_Z(g, f(x))$ or $f(t_X(g, x)) = t_Z(g, f(x))$, enforcing equivariance condition of Equation (1). However, even if the optimal value is not reached, due to its injectivity, $f$ is still $G$-equivariant, and the the objective above is regularizing the $G$ action on the code. This informed setup is used in equivariant contrastive learning of [17]. The assumption of having access to $g$ is realistic when we know the action $t_X$, so that we can generate $(x, g, x_t)$ triplets. Fortunately, using group invariants, we may still learn an equivariant embedding, even if we do not have the group information tied to the dataset.

Here, we introduce our method for several well-known groups first and then elaborate on the more general treatment.

*Example* 1 (**Euclidean Group**). The defining action of the Euclidean group $E(n)$ is the set of transformations that preserve the Euclidean distance between any two points in $\mathbb{R}^n$, a.k.a. isometries. These transformations are compositions of translations, rotations, and reflections. Since, for the real domain, all Euclidean isometries are linear and belong to $E(n)$, we can enforce the group structure on the embedding by ensuring that distances between the embeddings before and after any transformation match. For this, we need the dataset $D$ to be a set of pairs of pairs $\big((x, x_t = t_X(g, x)), (x', x'_t = t_X(g, x'))\big)$, where $x, x'$ are transformed using the same *unknown* group member $g$. Distance-preservation loss

below combined with injection loss are sufficient to produce an $E(n)$-regularized embedding:

$$L_{E(n)}(f, D) = \sum_{\big((x,x_t),(x',x'_t)\big)\in D} \ell\Big( \overbrace{\|f(x) - f(x')\|}^{\text{distance before the transformation}} - \overbrace{\|f(x_t) - f(x'_t)\|}^{\text{distance after the transformation}} \Big) \tag{4}$$

For example, in the standard RL setup, where we have access to triplets $(s, a, s')$, we can easily form $D$ by unrolling an episode and collecting two different state transitions corresponding to a particular action. In practice, with a finite number of actions, we can efficiently generate this dataset by keeping a separate buffer for each action where we store state transitions for that action and sample from that buffer to train the embedding function $f$. We provide the algorithm in Appendix C.

*Example* 2 (**Orthogonal and Unitary Groups**). The defining action of the orthogonal group $O(n)$ preserves the inner product between two vectors. The analogous group in the complex domain is the unitary group, which preserves the complex inner product. Our symmetry-regularization objective enforces this invariant: $L_{O(n)}(f, D) = \sum_{\big((x,x_t),(x',x'_t)\big)\in D} \ell\big( f(x)^\top f(x') - f(x_t)^\top f(x'_t) \big)$.

For the unitary group, one additionally needs to embed to complex domain $Z = \mathbb{C}^n$, where the only difference is in the definition of the inner product.

*Example* 3 (**Conformal Group**). The invariant of conformal geometry is the angle. In a Euclidean embedding, conformal transformations include a combination of translation, rotation, dilation, and inversion with respect to an $n-1$-sphere. To enforce this group structure, we need triplets of inputs before and after a transformation $\big((x, x_t), (x', x'_t), (x'', x''_t)\big)$, so that we can calculate the angle in the embedding. Conformal SymReg objective, which preserves angles, imposes a weaker constraint on the embedding than the distance preservation of the Euclidean group – since the latter implies the former. Moreover, it has an additional benefit that, compared to $L_{E(n)}$, the loss cannot be minimized by simply shrinking the embedding. Therefore in practice, the injection enforcing losses of Section 4 is not as crucial when using conformal symmetry regularization.

### 5.1 General Setting

Given a group $G$ acting linearly on a vector space $Z$, *invariant polynomials* associated with this action are those polynomials satisfying $P(t_Z(z, g)) = P(z)\ \forall g \in G$. These polynomials form an *algebra* studied in the field of *invariant theory* [58, 45]. In particular, a relevant problem is the question of whether there exists a finite set of bases for invariant polynomials for a given group representation. This question was one of Hilbert's 23 problems, and it was answered affirmatively by Hilbert himself for linear *reductive groups*, which includes classical Lie groups [29]. *Our proposal, in its most general form is to ensure invariance of polynomial bases within the orbits of the latent space before-after transformation of the input.*

Some examples of classical Lie groups and their invariants are: volume and orientation preservation by the Special Linear group, where the corresponding invariant polynomial is the determinant; Lorentz and Poincare groups are the analogs of the Orthogonal and Euclidean groups in the Minkowski space respectively, therefore equipped with similar invariants; the Symplectic group preserves another bilinear form. Finite groups also possess invariants. We show this use of invariants for SymReg through the important example of the symmetric group.

*Example* 4 (**Symmetric Group**). Symmetric polynomials $P(z_1, \ldots, z_n)$ that are invariant under all permutations of variables have a finite set of elementary bases:

$$e_1(z) = \sum_{1\le j\le n} z_j, \quad e_2(z) = \sum_{1\le j<k\le n} z_j z_k, \quad \ldots, \quad e_n(z) = z_1 z_2 \ldots z_n.$$

Assuming an n-dimensional embedding (*i.e.*, $z_j \in \mathbb{R}$), the corresponding SymReg objective penalizes change in these elementary basis before-after a transformation. At its minimum value, this penalty ensures that transformations of the input lead to permutations of the latent dimensions – however, with SymReg, this loss is used only to *regularize* the embedding. An alternative approach to SymReg for finite groups and, in particular, the Symmetric group is discussed in Appendix B.

**Choice of Lie group** Deciding on a Lie group for each application and in particular working with the corresponding invariants can be cumbersome. A simple alternative is to use an $E(n)$-equivariant

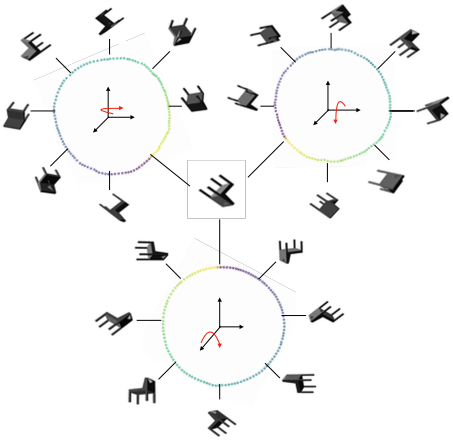

Figure 2: **Visualization of SymReg's latent projection for the rotating Chair dataset.** The chair is rotated in three orthogonal axes from 0 to $2\pi$. The latent embedding for each chair pose is projected from a 16D embedding space to a 2D space for visualization. The colors of the representations are mapped to the chair's angle of rotation. We notice that the mapping function $f$ learned is continuous with respect to the transformations of the object, and it maps the rotations along an axis to a circular manifold. This is true for each orthogonal axis of rotation. We observe a similar result for any other initial pose for the chair. A qualitative comparison with an existing method [26] is provided in Appendix D.1.

embedding for sufficiently large $n$. This is because Lie groups have isometric Euclidean embedding for sufficiently large $n$. We demonstrate this in the experiments with $SO(3)$ group in Section 7.1.

## 6 Decomposing the Representation

Higgins et al. [28] suggested a notion of disentangled representation based on decomposition of the abstract group into a direct product form $G = G_1 \times \ldots \times G_k$. There are two approaches to learning such decomposed representation using SymReg, depending on whether or not we can perform certain types of transformations in isolation. For example, an RL agent may transform its environment through actions like moving a single limb that can be performed in isolation. In this case, we call the decomposition *active* to contrast it with the *passive* case, where the action of different subgroups is always mixed in our dataset.

**Active Decomposition.** Let $G = \{(g_1, \ldots, g_k) \in G_1 \times \ldots \times G_k\}$ be a product group, where $G_i \cong \{(e, \ldots, e, g_i, e, \ldots, e) \in G\}$ can be identified with a normal subgroup of $G$. In active decomposition, sub-groups can act in isolation, and therefore we have $k$ types of tuples in our dataset $D_1, \ldots, D_k \subset D$. Each subset $D_i$ is associated with actions of a subgroup $G_i$ using $t_X((e, \ldots, e, g_i, e \ldots, e), \cdot))$, $g_i \in G_i$. In this setting, the representation $f : X \to Z = Z_1 \times \ldots \times Z_k$ can be thought of as $k$ separate functions where $f_i : X \to Z_i$ is equivariant to $G_i$-action and invariant to all $G_j, j \neq i$ actions. This gives the following objective

$$L_G^{\text{active}}(f, D) = \sum_{i=1}^{k} \underbrace{L_{G_i}(f_i, D_i)}_{\text{equivariance to } G_i} + \underbrace{L_{G/G_i}^{inv.}(f_i, D \backslash D_i)}_{\text{invariance to } G_j \text{ for } j \neq i}, \tag{5}$$

where $L_G^{inv.}(f, D)$ enforces invariance of $f$ to $G$-transformations in $D$ – e.g., by penalizing $\|f(x) - f(t_X(g, x))\|$.

**Passive Decomposition.** When we have no control over transformations, and we are simply given the data, it is still possible to use an abstract group that has a product form. Here again, $f : X \to Z = Z_1 \times \ldots \times Z_k$, but the loss function is simply enforced on each block separately – *i.e.*, $L_G^{\text{passive}}(f, D) = \sum_{i=1}^{k} L_{G_i}(f_i, D)$, where $L_{G_i}(f_i, D)$ is a SymReg objective from Section 5.

## 7 Experiments

We conducted many experiments to qualitatively study the representation learned by SymReg and its ability to produce a disentangled representation, and quantitatively compare it against simple baselines in representation learning and downstream RL tasks. For details on architecture and training, see Appendix G.

### 7.1 Qualitative Analysis

In this section, we visualize the representation learned for two examples from the Gym environment [5], including the pendulum and the mountain car (see Appendix D.2), followed by an experiment involving a rotating object where we know the ideal embedding is the $SO(3)$ manifold. Finally, Figure 3 visualizes a conformal embedding for double-bump world. In most cases, we also visualize a Variational AutoEncoder (VAE) [31] embedding for comparison. Our objective here is to visually demonstrate the behavior of SymReg and its remarkable ability to learn an embedding informed by the non-linear transformation of the input.

**The Pendulum.** For this experiment, the input $x$ is two consecutive frames of the pendulum that have been grayscaled and downsampled to $32 \times 32$ pixels. The action space is a range of torques that can be applied to the base of the pendulum. We use the action to transform the data. We use the objective of Equation (4) to learn an $E(3)$-equivariant representation. To efficiently estimate $L_{E(n)}$, we use a mini-batch that consists of 64 randomly sampled observations from the environment and their transformations via three randomly sampled actions ($4 \times 64$ samples in total). The model learns to parameterize the embedding using the angle and the angular momentum of the pendulum from the input data; see Figure 1. In order to compare, we visualize the learned latent of VAE and run similar experiments on the Mountain Car Environment in Appendix D.2).

**Rotating Chair.** We consider a 3D chair from ModelNet40 [61] and transform it through the action of the group $SO(3)$. The group action on the input is the 2D projection into a $48 \times 48$ image after the 3D rotation of the chair. While the group of interest is $SO(3)$, we use SymReg loss of Equation (4) following Section 5.1. We embed the chair in $\mathbb{R}^{16}$ using SymReg and visualize the latent by rotating the chair along three orthogonal axes and projecting the latent codes into a 2D space. Figure 2 shows three circular latent traversals of SymReg embedding corresponding to rotation around each axis, which is consistent with the structure of the $SO(3)$ manifold. The process of learning the $SO(3)$ manifold is a challenging task (see Appendix D.1), and previous works assumed that the group member corresponding to each transformation is given [47, 2]. In contrast, we only use the observations corresponding to similar actions during training and not the group members themselves. As we see later, this is critical in settings such as RL, where group information is unavailable.

**Conformal Embedding for Double-Bump World.** Double-bump world consists of a rectangular bump signal and a triangular bump signal, both cyclically shifted and superimposed. These transformations are given by a pair $(\Delta_1, \Delta_2)$ which cyclically shifts the rectangular bump by $\Delta_1$ and the triangular bump by $\Delta_2$. In our experiments, the signal length is 64, and the length of the bump is 16. SymReg embeds to a 4-dimensional conformal space in this example. Figure 3(right) shows the random projection of the embedding, where the colors change as the triangle bump moves. The figure suggests that SymReg can learn to represent a data point using the location of two bumps. For comparison, we show the embedding found by VAE and SimCLR [9].

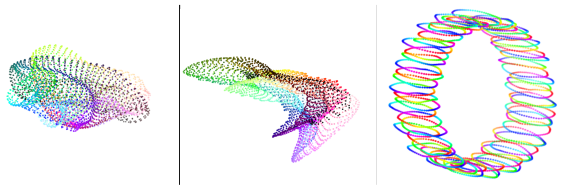

Figure 3: (left) SimCLR (middle) VAE (right) SymReg embeddings of double-bump world.

### 7.2 Experiments on Active and Passive Decomposition

In this section, we first contrast active and passive decomposition in their ability to disentangle the two bumps in the double bump world. We observe that while both can decompose the embedding into a product form $SO(2) \times SO(2)$, only active decomposition leads to disentanglement. Finally, we apply active decomposition to the more complex setting of ego-motion, where SymReg can decompose the representation of the agent's state into location and orientation.

**Decomposition of the Double-Bump World.** Here, we compare the active and passive decomposition for the same double-bump world. While the ground truth is $SO(2) \times SO(2)$, SymReg uses the larger group $E(2) \times E(2)$. In the active case, each subgroup moves one of the bumps, and the

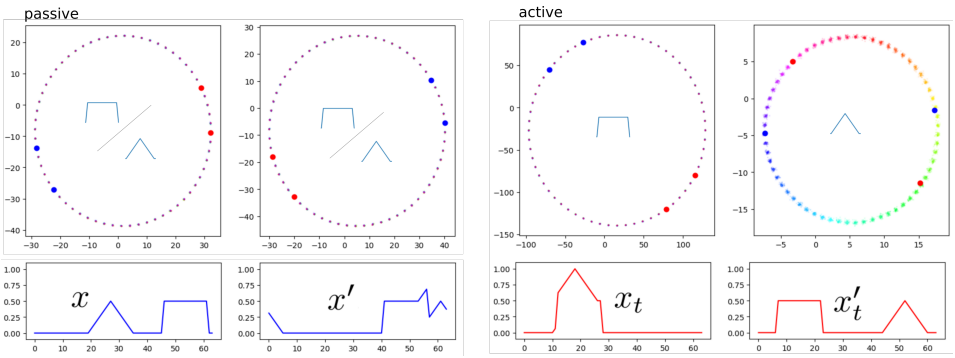


Figure 4: **Active versus passive decomposition for the double-bump world.** Four images at the bottom show a pair of inputs $(x, x')$ and their transformations $(x_t, x'_t)$. The figure shows the embedding for two inputs before, $(x, x')$ in blue, and after, $(x_t, x'_t)$ in red, the *same* transformation. This transformation cyclically shifts both the triangle and the square to the left, but the amount of translation is larger for the square. In both passive and active decomposition, the Euclidean distance is preserved by the transformation – the red points have the same distance from each other as the blue points on every manifold. In the active decomposition (right), one of the manifolds encodes the circular translation of the triangle bump, while the second one represents the location of the square bump. Various colors indicate the location of the triangle. In the case of passive decomposition (left), since the transformation of individual shapes does not guide the decomposition, the manifolds jointly encode the location of each bump type.

loss of Equation (5) is used to learn an embedding for each subgroup. In the passive case, both bumps move randomly. Figure 4 compares the decomposed embedding found in each case. While in both cases, the $SO(2) \times SO(2)$ torus is decomposed into a product of circles, only the active case successfully disentangles the two bumps. Note that the color of each point is based on the location of the triangle bump. Our results agree with Caselles-Dupré et al. [8], who claim that learning a disentangled representation requires interaction with the environment; see also [44, 41]. However, we note that while the disentangling of the bump movements does not happen in the passive case, we can still successfully "decompose" the embedding.

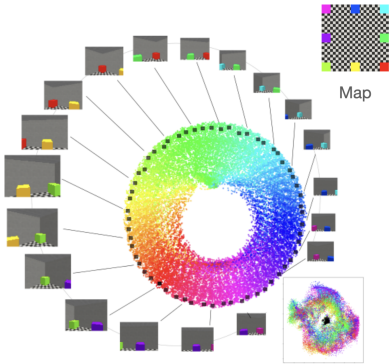


Figure 5: **Decomposition of the ego-motion.** The dataset contains a first-person view of a room. Transformations include right-left rotation and forward-backward movement. The equivariant embedding is produced by active decomposition using these two transformations, where the ring-structured manifold corresponds to the rotation action, and the smaller manifold corresponds to translations. Color coding shows the ground truth angle of the image. The black square markers show the traversal of the embedding as the agent rotates while standing in the middle of the room. Note that black squares are concentrated in the center of the second manifold.

**Active Decomposition for Ego-Motion.** We used a modified version of the single-room environment of MiniWorld [12] for this experiment. The agent is standing in a 3D room containing eight differently colored boxes around the walls. A map of the room can be seen in Figure 5. An observation consists of a first-person view of the room, downsampled to $32 \times 32$ pixels. The agent can rotate left/right or move forward/backward. We learn an $E(2) \times E(2)$ equivariant embedding using the active decomposition objective of Equation (5). Each mini-batch consists of 64 random observations and the result of applying all four actions in those states ($4 \times 64$ samples in total).

Figure 5 visualizes the embedding of the input in two sub-figures, where the more prominent figure shows the embedding corresponding to the rotation action, and the more petite figure (bottom right) shows the embedding corresponding to forward-backward movement. The first figure also shows the first-person view when the agent rotates while standing in the middle of the room. The corresponding markers collapse around the center of the second embedding, demonstrating an intuitive embedding

parameterized by rotation angle and location. Walking straight across the room also produces the expected behavior of traversing the second manifold while the rotation angle, for the most part, remains fixed (not shown).

### 7.3 Quantitative Evaluation in Downstream Tasks

#### 7.3.1 World Modelling

We select the Atari games Pong and Space Invaders as our environments for the world modeling experiments. These environments were previously used by Kipf et al. [32] to evaluate the Contrastive Structured World Model (C-SWM). We train the encoder using Euclidean SymReg of Equation (4), freeze it, and then learn a Multi-Layer Perceptron (MLP) based transition function in the latent space. Following Kipf et al. [32], we report Hits at Rank 1 (H@1) and Mean Reciprocal Rank (MRR), which are invariant to the embedding scale. These evaluation metrics measure the relative closeness of the following state's representation predicted by the transition model and the representation of the observed next state. We use a set of reference state representations to measure the relative closeness (embedding random observations from the experience buffer). Table 1 reports these measures and shows that a simple transition model learned on top of our embedding outperforms C-SWM in both games. Other reported baselines use an AutoEncodcer (AE) and a Variational AutoEncoder (VAE) to learn embeddings.

#### 7.3.2 Reinforcement Learning

Next, we consider three Mujoco environments: InvertedPendulum, Reacher, and Swimmer from OpenAI Gym [5] and learn directly from the image observations. We compare our model with Auto-Encoder (AE) and Self-supervised Learning (SSL) based baselines. While AE learns to reconstruct the image observations of the states, SSL learns to inject invariance (IN-SSL) or equivariance (EQ-SSL) to agent actions. Given a triplet $(s, a, s)$, IN-SSL maximizes the likelihood of $f(s)$ and $f(s')$ being similar (SimCLR [10]). EQ-SSL of Dangovski et al. [18], in this context, additionally predicts the action that leads to the state transition. We introduce two variations of each model. In the first variation, the low-dimensional embedding is used as a substitute for the high-dimensional input data without further adjustment (-decoupled). The second variation allows for fine-tuning during the reinforcement learning stage (-fine-tuned). We use random policy to collect trajectories for the pre-training and use Proximal Policy Optimization (PPO) [51] algorithm for the downstream RL task. To evaluate the data efficiency of these models, we report the average reward collected over 10 episodes in the first 100,000 steps for Reacher and Swimmer and 30,000 steps for Inverted Pendulum in Table 2 (since Inverted Pendulum generally learns faster, we took a fewer number of steps.)

Table 1: Hits at Rank 1 (H@1) and Mean Reciprocal Rank (MRR) of different methods.

| ENVIRONMENT | METHOD | H@1 | MRR |
|---|---|---|---|
| ATARI PONG | WORLD MODEL(AE) | $23.8_{\pm 3.3}$ | $44.7_{\pm 2.4}$ |
| | WORLD MODEL(VAE) | $1.0_{\pm 0.0}$ | $5.1_{\pm 0.1}$ |
| | C-SWM | $36.5_{\pm 5.6}$ | $56.2_{\pm 6.2}$ |
| | **OURS** | $\mathbf{45.2}_{\pm \mathbf{3.4}}$ | $\mathbf{60.2}_{\pm \mathbf{3.9}}$ |
| SPACE INVADERS | WORLD MODEL(AE) | $40.2_{\pm 3.3}$ | $59.6_{\pm 3.5}$ |
| | WORLD MODEL(VAE) | $1.0_{\pm 5.3}$ | $5.3_{\pm 0.1}$ |
| | C-SWM | $48.5_{\pm 7.0}$ | $66.1_{\pm 6.6}$ |
| | **OURS** | $\mathbf{54.2}_{\pm \mathbf{6.3}}$ | $\mathbf{68.7}_{\pm \mathbf{5.1}}$ |

We see that out of all the representation learning methods, learned representations of SymReg most adequately capture the structure of the environment in Inverted Pendulum since the RL agent just trained on the fixed representation (SymReg-decoupled) outperforms all of them, including vanilla PPO. In Reacher, SymReg, along with other non-generative models, performs poorly compared to the AE. We believe that this is because the representation is focused on transformations caused by the agent's actions while details that can be valuable from the reward's perspective — in this case, the small object that the Reacher should reach - are ignored. This observation points to a limitation of all non-generative approaches that fine-tuning can resolve. To further verify this, we combined SymReg with a Decoder (Dec-SymReg) and noticed a significant improvement in the performance of the decoupled variation. In Swimmer, again, we see that learning the agent's transformations is not enough to get all the reward information as the background movement decides how far the agent has swum. Indeed, allowing the encoder to fine-tune allows the representations to reflect the reward information and improve performance.

Table 2: Average reward collected over 10 episodes for various models in Inverted Pendulum, Reacher and Swimmer. We provide the standard errors using 5 random seeds.

| METHODS | INVERTED PENDULUM | REACHER | SWIMMER |
|---|---|---|---|
| VANILLA | $500_{\pm 150}$ | $-11_{\pm 2.5}$ | $25.6_{\pm 3.4}$ |
| AE-DECOUPLED | $30_{\pm 15}$ | $-13_{\pm 3.0}$ | $16_{\pm 3.9}$ |
| AE-FINETUNED | $580_{\pm 130}$ | $-11.5_{\pm 3.2}$ | $26_{\pm 4.3}$ |
| IN-SSL-DECOUPLED | $100_{\pm 17}$ | $-15_{\pm 2.6}$ | $12_{\pm 2.5}$ |
| IN-SSL-FINETUNED | $550_{\pm 21}$ | $-12_{\pm 4.1}$ | $25.9_{\pm 4.8}$ |
| EQ-SSL-DECOUPLED | $456_{\pm 190}$ | $-14.8_{\pm 3.1}$ | $18_{\pm 4.5}$ |
| EQ-SSL-FINETUNED | $710_{\pm 120}$ | $\mathbf{-10}_{\pm \mathbf{2.6}}$ | $27_{\pm 3.5}$ |
| SYMREG-DECOUPLED | $800_{\pm 180}$ | $-14.5_{\pm 3.1}$ | $21_{\pm 4.1}$ |
| DEC-SYMREG-DECOUPLED | $600_{\pm 200}$ | $-12.8_{\pm 2.7}$ | $19_{\pm 5.6}$ |
| SYMREG-FINETUNED | $\mathbf{950}_{\pm \mathbf{50}}$ | $-10_{\pm 3.4}$ | $\mathbf{31.5}_{\pm \mathbf{3.9}}$ |

## Conclusion

We proposed to learn equivariant representations by learning an injective embedding that is regularized towards a simple linear action using group invariants. We demonstrate this to be a simple, intuitive, and yet effective approach for representation learning. In the future, we would like to understand data characteristics that motivate the choice of one Lie group over others. We would also like to explore SymReg for the symmetric group and its combination with Euclidean groups as a way to represent various objects in the Euclidean space. We would also like to investigate further the best choice of objectives or mechanism for preventing collapse in conjunction with SymReg.

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1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See the Conclusion section (Section 7.3.2).
   (c) Did you discuss any potential negative societal impacts of your work? [No] We do not foresee any negative societal impact by our work.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] See the appendix.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See the corresponding section for each experiment.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] All of the quantitative experiments contain confidence intervals.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] This paper does not contain any large scale experiments and the specific resources that were used is not relevant to the work.
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] We have cited the environments that we used in the experiments.
   (b) Did you mention the license of the assets? [No] The license is available on each environments GitHub page.
   (c) Did you include any new assets either in the supplemental material or as a URL? [No] We did not produce any new assets.
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [No] The environments that we used are open-source.
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5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
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   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

## A Proof of Claims

*Proof.* of Claim 1
Let $\sim_f$ be an equivalence relation on $X$, such that two points are "equivalent" if they have the same embedding $x \sim x' \Leftrightarrow f(x) = f(x')$. We use $[x]_\sim$ to denote the equivalence class of $x$. To get an intuition for this result, first consider an injective $f$, where the equivalence classes are trivial $[x]_\sim = x$. In this case for any $G$-set $X$, $f$ is $G$-equivariant with $G$-action on $Z$ defined by

$$t_Z(g, y) \doteq f(t_X(g, f^{-1}(y))) \quad \forall g, y \in G \times Z \tag{6}$$

Now to see why $f$ is equivariant to any action $t_X$ and the corresponding $t_Z$ defined above, simply replace the definition of $t_Z$ into definition of equivariance Equation (1)

$$t_Z(g, f(x)) = f(t_X(g, f^{-1}(f(x)))) = f(t_X(g, x)) \tag{7}$$

For general functions, note that $f^{-1}(f(x)) = [x]_\sim$. The equation above makes sense iff $t_X(g, x') = t_X(g, x'') \forall x', x'' \in [x]_\sim$, which is basically the assumption of Equation (2). This means $[t_X(g, x)]_\sim \doteq [t_X(g, x)]_\sim$, and using the $t_Z$ of Equation (6) in the definition of equivariance, we see that its condition is satisfied

$$t_Z(g, f(x)) = f(t_X(g, f^{-1}(f(x)))) = f(t_X(g, [x]_\sim)) = f([t_X(g, x)]_\sim) = f(t_X(g, x)) \tag{8}$$

□

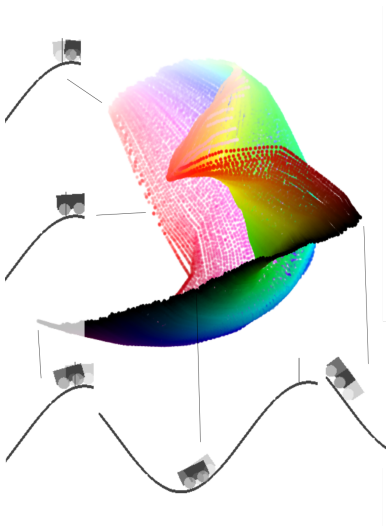


Figure 6: The $E(3)$ equivariant representation of the mountain-car. Each state $x \in X$ is a concatenation of two consecutive frames so as to inform about both position and velocity of the car. The colors encode the true position of the car, and the brightness of the colors shows the positive-negative velocity. The transformation $t_X(g, x)$ changes the velocity through positive/negative acceleration. By preserving the distance between pairs of instances in which the same acceleration is applied, transformation coding is able to recover a manifold that is parameters by velocity and location.

## B Finite Groups

Given two instances $x, x' \in D$, we may "optimize" for the choice of $g \in G$, such that $t_Z(f(x), g) \approx f(x')$:

$$L_G(f, D) = \sum_{x, x' \in D} \min_{g \in G} \ell\big(f(x) - t_Z\big(g, f(x')\big)\big). \tag{9}$$

This approach may be suitable for finite groups, where one can enumerate $g \in G$ to perform the minimization inside the outer optimization. Since any finite group has a permutation representation, we can always define $t_Z$ to permute blocks of the latent representation. In this case, the loss function above involves a search to find the best permutation within a group that matches the embeddings before and after the transformation. For some permutation groups, this search can be performed more efficiently. For example, in the case of the symmetric group, the loss function of Equation (9) is also sometimes known as earth mover's distance [22], and the Hungarian algorithm can be used for the optimization. One can also use the Chamfer distance to approximate this loss function. [†]

[†] In contrast to the existing permutation equivariant networks, here the permutation action on the input is not limited to the permutation of a known group of variables – that is, the representation could be equivariant to shuffling of a priori unknown entities/objects in the input.

## C Algorithm

Although SymReg is using simple loss functions for equivariant representations, for the benefit of clarity, here we give the algorithm for $E(n)$ equivariance. Algorithm 1 gives a generic training step of SymReg for $E(n)$, and Algorithm 2 provides its integration in RL environments for encoder pre-training. The policy used to collect data in the RL setting can be a behavior policy for offline RL and a random policy for online RL. Note that there are many ways SymReg can be exploited in RL but in this work we limit ourselves to preliminary representation learning experiments.

**Algorithm 1:** SymReg for $E(n)$ - Training Steps

```
Denote the parameters of the encoder as θ
Denote the encoder as e_θ
Denote the batch size by B
Given a batch of samples {(x_1^i, x_2^i), (x̂_1^i, x̂_2^i)}_{i=1}^B ;  // pairs with same transformation
l_eqv, l_barrier = 0, 0
z_all = {}
for i in range(1, B) do
    z_1^i, z_2^i, ẑ_1^i, ẑ_2^i = e_θ(x_1^i), e_θ(x_2^i), e_θ(x̂_1^i), e_θ(x̂_2^i) ;        // encode the date
    l_eqv = l_eqv + (‖z_1^i − z_2^i‖ − ‖ẑ_1^i − ẑ_2^i‖)^2;                  // compute SymReg loss
    z_all = z_all ∪ {z_1^i, z_2^i, ẑ_1^i, ẑ_2^i}
end
for z_i in z_all do
    for z_j in z_all do
        if z_i != z_j then
            l_barrier = l_barrier − log(‖z_i − z_j‖) ;                  // compute barrier loss
        end
    end
end
l_eqv, l_barrier = l_eqv/B, l_barrier/(4B − 1)^2;                  // normalize the losses
l_total = l_eqv + l_barrier;                                        // compute the total loss
θ ← optmize((θ), l_total);                                          // update the encoder params
```

**Algorithm 2:** SymReg for $E(n)$ in RL environments

```
Denote the parameters of the encoder θ
Denote the encoder as e_θ
Denote the action space by A
if A is continuous then
    Discretize A into k actions {a_1, ..., a_k}
end
Initialize k replay buffers {B_1, ..., B_k} for each action
for step in max_pretraining_steps do
    Collect {s, a, s'} using a given policy and add to the buffer B_i of a
    if step > warm_up_steps then
        Sample a batch B of buffers with repetition
        Sample two state transitions from each sampled buffer
        Create a batch of samples {(s_1^i, s_2^i), (ŝ_1^i, ŝ_2^i)}_{i=1}^B
        Perform SymReg training step with e_θ and {(s_1^i, s_2^i), (ŝ_1^i, ŝ_2^i)}_{i=1}^B
    end
end
if finetune then
    Use pre-trained e_θ for downstream RL algorithm
else
    Use frozen e_θ as feature extractor for downstream RL algorithm
end
```

# D Additional Experiments

## D.1 Rotating Chair

We compare the learnt representations of our method with the existing manifold learning method DrLIM [26] for the rotating Chair example. DrLIM objective computes the Euclidean distance between the representations and uses that to contrast positive samples with negative ones. This is achieved by minimizing the distance between the positive pairs and maximizing the hinged distances between the negative pairs. In the rotating chair, positive samples are in close proximity in the $SO(3)$ manifold, that is samples that are different from each other by a small rotation. For this experiment, we use a higher dimensional latent space because it is easier to embed the SO(3) manifold in that. For a fair comparison, we keep most of the settings similar to SymReg including the amount of data used to train.

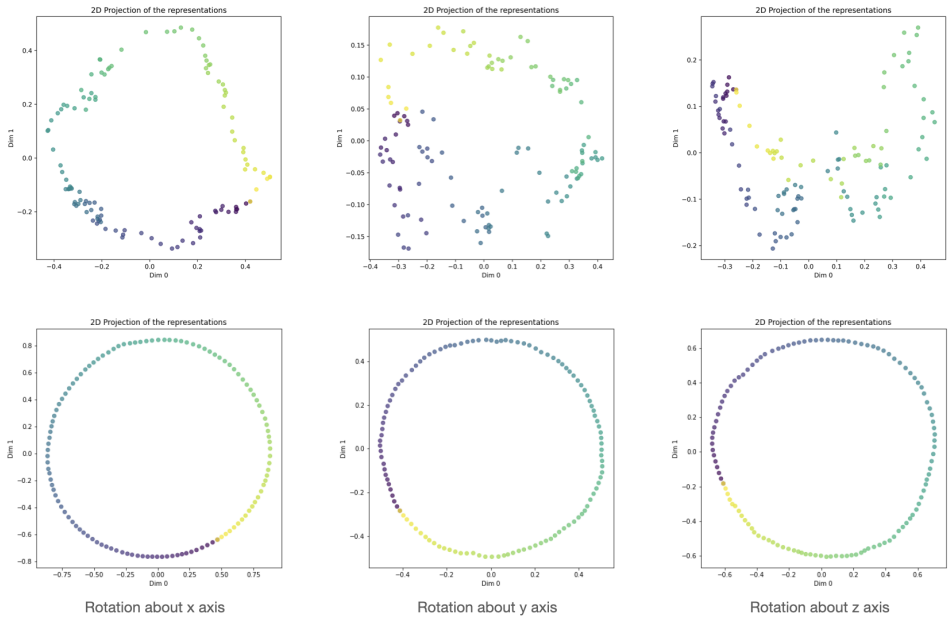


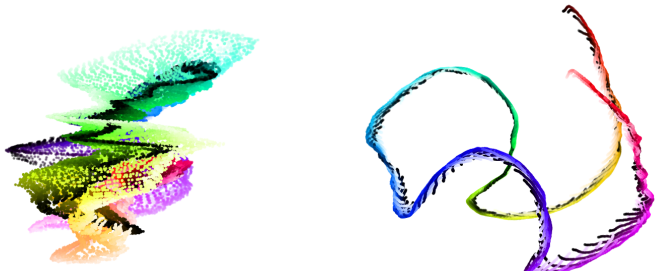

Figure 7: **Comparison of the latent projections of SymReg with DrLIM[26]** for the rotating Chair dataset. The chair is rotated in three orthogonal axes from 0 to $2\pi$. The latent embedding for each chair pose for both the methods is projected from 16D embedding space to a 2D space for visualization. The colors of the representations are mapped to the chair's angle of rotation. We see that while DrLIM struggles to learn the correct manifold SymReg learns it easily.

## D.2 The Mountain Car

In the Mountain Car environment, an observation $x$ consists of two consecutive frames that have been grayscaled and downsampled to $32 \times 32$ pixels. The action-space of the environment is a range of accelerations that can be applied to the car. Figure 6 shows the learned embedding in $\mathbb{R}^3$. Figure 6 shows the learned embedding in $\mathbb{R}^3$. Colors show the change in the location and the brightness shows the velocity. The learned representaion is quite intuitive, and the model learns to parameterize the manifold using the location and velocity of the car. We also provide visualizations of the learnt embeddings of VAE for the pendulum and mountain car environment in Figure 8.

# E Addition of Invariant Features

While we focused on equivariant codes, one may also consider an invariant component in the code which can account for variations in the data that are not due to transformations – that is, we have $f : X \to Z \times Y$, where $Y$ is the invariant part of the code that identifies distinct *orbits*. Let



((a)) The Pendulum ((b)) The Mountain Car 2

Figure 8: Visualization of the representations learnt using a VAE.

$f^{\text{inv.}} : X \to Y$ denote the invariant component of $f$. To learn $f^{\text{inv.}}$, one could use a loss of the form $\ell(f^{\text{inv.}}(x) - f^{\text{inv.}}(t_X(g, x)))$ which enforces invariance for points on the same orbit. At the same time, an injection loss, similar to those of Section 4, pushes apart the points that are not in the same orbit. This invariant component is therefore very similar to what is used in contrastive coding.

## F Effect of Transformations on the Embedding Manifold

When using transformation coding, the transformation can have a complex relationship with the ideal parameterizations of the manifold. For example, the location and velocity of the mountain-car or the pendulum (as ideal parameters for the manifold), are non-trivially related to the action that accelerates the movement. A natural question here is about the effect of the choice of transformation on the manifold. In practice, we observe that, in low-dimensional embedding, the geometry of the manifold is quite sensitive to the choice of transformation. As an example, Figure 9 presents an alternative embedding for the pendulum of Figure 1 obtained by simply decreasing the amount of time between taking an action, and observing its outcome from $\delta t = .05 \to \delta t = .01$. Whether or not this (potential) sensitivity is a bug or feature may depend on the application setting.

## G Implementation Details

**Pendulum and Mountain Car** We use the same setting for both of these environments. The neural network first applies three convolutional layers with $3 \times 3$ kernels, 24 output channels, and "same" padding, each followed by $2 \times 2$ max pooling and ReLU activation. Finally, a Multi-Layer Perceptron (MLP) with a single hidden layer of size 128 and ReLU activation is applied to embed the representation into $\mathbb{R}^3$. The model was trained for 5000 steps with the Adam optimizer set to a learning rate of $10^{-3}$. The log-barrier coefficient was set to 1, and we used a weight decay coefficient of $10^{-7}$.

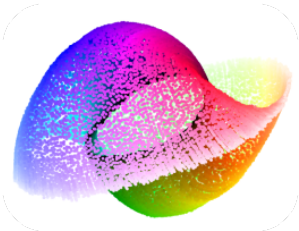


Figure 9: Alternative embedding for the pendulum.

**Bump World** The neural network used for conformal coding experiments in Section 7.1 is a 4-layer MLP with hidden layers of size 128 and ReLU activation functions. The embedding space is $\mathbb{R}^4$ which is then randomly projected to $\mathbb{R}^3$ for visualization. The log-barrier coefficient and weight decay coefficient were both set to $10^{-7}$. The mini-batches used for training consist of 64 randomly sampled observations from the environment and their transformations via 15 randomly sampled transformations ($16 \times 64$ samples in total). The model was trained for $10,000$ steps with the Adam optimizer set to a learning rate of $10^{-3}$.

The neural network used in active and passive decomposition experiments in Section 7.2 is a 3-layer MLP with hidden layers of size 128 and ELU activation functions. The embedding space is $\mathbb{R}^4$ which we interpret as two $\mathbb{R}^2$s. We optimize the model for 5000 steps using the Adam optimizer with an initial learning rate of $10^{-2}$ which is halved every 1000. We use a barrier coefficient of 1 and a

weight decay coefficient of $10^{-5}$. The mini-batches used for training consist of 64 randomly sampled observations from the environment and their transformations via 7 randomly sampled transformations ($8 \times 64$ samples in total).

**Gym Mini-world** The neural network architecture and training settings are similar to those of the pendulum and mountain car experiments.

**Rotating chair** The neural network used for the Chair dataset first applies three convolutional layers with $3 \times 3$ kernels, increasing number of output channels from 16 to 32 to 64, 1 padding and stride of 2. It is followed by a ReLU activation. Finally, a Multi-Layer Perceptron (MLP) with a single hidden layer of size 128 and ReLU activation is applied to embed the representation into $\mathbb{R}^{16}$. The model was trained with 10000 unique rotations repeated with multiple initial points with the Adam optimizer set to a learning rate of $10^{-3}$. We stick to small rotation angles to make the setup similar to dynamical environments. The log-barrier loss coefficient was set to 1, and we used a weight decay coefficient of $10^{-7}$.

**Pong and Space Invaders** The neural network used for the Chair dataset first applies three convolutional layers with $7 \times 7$ , $5 \times 5$ and $3 \times 3$ kernels. We again increase the number of output channels from 16 to 32 to 64 and use a padding of 1 and stride of 2 for all of them. The convolutions are followed by ReLU activations. Finally, a linear layer is applied to embed the representation into $\mathbb{R}^{32}$. We train the encoder with samples collected from around 100k environment steps. The log-barrier loss coefficient was again set to 1.

**Pendulum, Reacher and Swimmer** The encoder and pre-training setting used for this part is similar to the one used for Pong and space invaders. We use an MLP to learn the policy from the extracted features. We use stable baselines 3 for our PPO implementation. [48].