Provable Generalization of Overparameterized Meta-learning Trained with SGD

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Abstract

Despite the superior empirical success of deep meta-learning, theoretical understanding of overparameterized meta-learning is still limited. This paper studies the generalization of a widely used meta-learning approach, Model-Agnostic Meta-Learning (MAML), which aims to find a good initialization for fast adaptation to new tasks. Under a mixed linear regression model, we analyze the generalization properties of MAML trained with SGD in the overparameterized regime. We provide both upper and lower bounds for the excess risk of MAML, which captures how SGD dynamics affect these generalization bounds. With such sharp characterizations, we further explore how various learning parameters impact the generalization capability of overparameterized MAML, including explicitly identifying typical data and task distributions that can achieve diminishing generalization error with overparameterization, and characterizing the impact of adaptation learning rate on both excess risk and the early stopping time. Our theoretical findings are further validated by experiments.

1 Introduction

Meta-learning [27] is a learning paradigm which aims to design algorithms that are capable of gaining knowledge from many previous tasks and then using it to improve the performance on future tasks efficiently. It has exhibited great power in various machine learning applications spanning over few-shot image classification [39, 40], reinforcement learning [25] and intelligent medicine [24].

One prominent type of meta-learning approaches is an optimization-based method, Model-Agnostic Meta-Learning (MAML) [20], which achieves impressive results in different tasks [36, 4, 2]. The idea of MAML is to learn a good initialization $\boldsymbol{\omega}^*$, such that for a new task we can adapt quickly to a good task parameter starting from $\boldsymbol{\omega}^*$. MAML takes a bi-level implementation: the inner-level initializes at the meta parameter and takes task-specific updates using a few steps of gradient descent (GD), and the outer-level optimizes the meta parameter across all tasks.

With the superior empirical success, theoretical justifications have been provided for MAML and its variants over the past few years from both optimization [22, 47, 18, 30] and generalization perspectives [1, 14, 19, 11]. However, most existing analyses did not take overparameterization

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into consideration, which we deem as crucial to demystify the remarkable generalization ability of deep meta-learning [50, 27]. More recently, [45] studied the MAML with overparameterized deep neural nets and derived a complexity-based bound to quantify the difference between the empirical and population loss functions at their optimal solutions. However, complexity-based generalization bounds tend to be weak in the high dimensional, especially in the overparameterized regime. Recent works [6, 53] developed more precise bounds for overparameterized setting under a mixed linear regression model, and identified the effect of adaptation learning rate on the generalization. Yet, they considered only the simple isotropic covariance for data and tasks, and did not explicitly capture how the generalization performance of MAML depends on the data and task distributions. Therefore, the following important problem still remains largely open:

*Can **overparameterized** MAML generalize well to a new task, under general data and task distributions?*

In this work, we utilize the mixed linear regression, which is widely adopted in theoretical studies for meta-learning [32, 6, 15, 3], as a proxy to address the above question. In particular, we assume that each task τ is a noisy linear regression and the associated weight vector is sampled from a common distribution. Under this model, we consider one-step MAML meta-trained with stochastic gradient descent (SGD), where we minimize the loss evaluated at single GD step further ahead for each task. Such settings correspond to real-world implementations of MAML [21, 33, 27] and are extensively considered in theoretical analysis [18, 10, 19]. The focus of this work is the overparameterized regime, i.e., the data dimension d is far larger than the meta-training iterations T ($d \gg T$).

1.1 Our Contributions

Our goal is to characterize the generalization behaviours of the MAML output in the overparameterized regime, and to explore how different problem parameters, such as data and task distributions, the adaptation learning rate β^{tr}, affect the test error. The main contributions are highlighted below.

- Our first contribution is a sharp characterization (both upper and lower bounds) of the excess risk of MAML trained by SGD. The results are presented in a general manner, which depend on a new notion of effective meta weight, data spectrum, task covariance matrix, and other hyperparameters such as training and test learning rates. In particular, the **effective meta weight** captures an essential property of MAML, where the inner-loop gradient updates have distinctive effects on different dimensions of data eigenspace, i.e., the importance of "leading" space will be magnified whereas the "tail" space will be suppressed.
- We investigate the influence of data and task distributions on the excess risk of MAML. For $\log$-decay data spectrum, our upper and lower bounds establish a sharp phase transition of the generalization. Namely, the excess risk vanishes for large T (where benign fitting occurs) if the data spectrum decay rate is faster than the task diversity rate, and non-vanishing risk occurs otherwise. In contrast, for polynomial or exponential data spectrum decays, excess risk always vanishes for large T irrespective of the task diversity spectrum.
- We showcase the important role the adaptation learning rate β^{tr} plays in the excess risk and the early stopping time of MAML. We provably identify a novel tradeoff between the different impacts of β^{tr} on the "leading" and "tail" data spectrum spaces as the main reason behind the phenomena that the excess risk will first increase then decrease as β^{tr} changes from negative to positive values under general data settings. This complements the explanation based only on the "leading" data spectrum space given in [6] for the isotropic case. We further theoretically illustrate that β^{tr} plays a similar role in determining the early stopping time, i.e., the iteration at which MAML achieves steady generalization error.

Notations. We will use bold lowercase and capital letters for vectors and matrices respectively. $\mathcal{N}\left(0, \sigma^2\right)$ denotes the Gaussian distribution with mean 0 and variance σ^2. We use $f(x) \lesssim g(x)$ to denote the case $f(x) \leq cg(x)$ for some constant $c > 0$. We use the standard big-O notation and its variants: $\mathcal{O}(\cdot), \Omega(\cdot)$, where T is the problem parameter that becomes large. Occasionally, we use the symbol $\widetilde{\mathcal{O}}(\cdot)$ to hide $\text{polylog}(T)$ factors. $\mathbf{1}_{(\cdot)}$ denotes the indicator function. Let $x^+ = \max\{x, 0\}$.

2 Related Work

Optimization theory for MAML-type approaches Theoretical guarantee of MAML was initially provided in [21] by proving a universal approximation property under certain conditions. One line of theoretical works have focused on the optimization perspective. [18] established the convergence guarantee of one-step MAML for general nonconvex functions, and [30] extended such results to the multi-step setting. [22] analyzed the regret bound for online MAML. [47, 45] studied the global optimality of MAML with sufficiently wide deep neural nets (DNN). Recently, [12] studied MAML from a representation point of view, and showed that MAML can provably recover the ground-truth subspace.

Statistical theory for MAML-type approaches. One line of theoretical analyses lie in the statistical aspect. [19] studied the generalization of MAML on recurring and unseen tasks. Information theory-type generalization bounds for MAML were developed in [31, 11]. [10] characterized the gap of generalization error between MAML and Bayes MAML. [45] provided the statistical error bound for MAML with overparameterized DNN. Our work falls into this category, where the overparameterization has been rarely considered in previous works. Note that [45] only derived the generalization bound from the complexity-based perspective to study the difference between the empirical and population losses for the obtained optimization solutions. Such complexity bound is typically related to the data dimension [35] and may yield vacuous bound in the high dimensional regime. However, our work show that the generalization error of MAML can be small even the data dimension is sufficiently large.

Overparamterized meta-learning. [17, 42] studied overparameterized meta-learning from a representation learning perspective. The most relevant papers to our work are [53, 6], where they derived the population risk in overparameterized settings to show the effect of the adaptation learning rate for MAML. Our analysis differs from these works from two essential perspectives: i). we analyze the excess risk of MAML based on the optimization trajectory of SGD in non-asymptotic regime, highlighting the dependence of iterations T, while they directly solved the MAML objective asymptotically; ii). [53, 6] mainly focused on the simple isotropic case for data and task covariance, while we explicitly explore the role of data and task distributions under general settings.

Overparameterized linear model. There has been several recent progress in theoretical understanding of overparameterized linear model under different scenarios, where the main goal is to provide non-asymptotic generalization guarantees, such as studies of linear regression [5], ridge regression [43], constant-stepsize SGD [51], decaying-stepsize SGD [48], GD [49], Gaussian Mixture models [46]. This paper aims to derive the non-asymptotic excess risk bound for MAML under mixed linear model, which can be independent of data dimension d and still converge as the iteration T enlarges.

3 Preliminary

3.1 Meta Learning Formulation

In this work, we consider a standard meta-learning setting [19], where a number of tasks share some similarities, and the learner aims to find a good model prior by leveraging task similarities, so that the learner can quickly find a desirable model for a new task by adapting from such an initial prior.

Learning a proper initialization. Suppose we are given a collection of tasks $\{\tau_t\}_{t=1}^T$ sampled from some distribution $\mathcal{T}$. For each task τ_t, we observe N samples $\mathcal{D}_t \triangleq (\mathbf{X}_t, \mathbf{y}_t) = \left\{(\mathbf{x}_{t,j}, y_{t,j}) \in \mathbb{R}^d \times \mathbb{R}\right\}_{j\in[N]} \overset{i.i.d.}{\sim} \mathbb{P}_{\phi_t}(y|\mathbf{x})\mathbb{P}(\mathbf{x})$, where ϕ_t is the model parameter for the t-th task. The collection of $\{\mathcal{D}_t\}_{t=1}^T$ is denoted as $\mathcal{D}$. Suppose that $\mathcal{D}_t$ is randomly split into training and validation sets, denoted respectively as $\mathcal{D}_t^{\text{in}} \triangleq (\mathbf{X}_t^{\text{in}}, \mathbf{y}_t^{\text{in}})$ and $\mathcal{D}_t^{\text{out}} \triangleq (\mathbf{X}_t^{\text{out}}, \mathbf{y}_t^{\text{out}})$, correspondingly containing n_1 and n_2 samples (i.e., $N = n_1 + n_2$). We let $\boldsymbol{\omega} \in \mathbb{R}^d$ denote the initialization variable. Each task τ_t applies an inner algorithm $\mathcal{A}$ with such an initial and obtains an output $\mathcal{A}(\boldsymbol{\omega}; \mathcal{D}_t^{\text{in}})$. Thus, the adaptation performance of $\boldsymbol{\omega}$ for task τ_t can be measured by the mean squared loss over the validation set given by $\ell(\mathcal{A}(\boldsymbol{\omega}; \mathcal{D}_t^{\text{in}}); \mathcal{D}_t^{\text{out}}) := \frac{1}{2n_2}\sum_{j=1}^{n_2}\left(\langle \mathbf{x}_{t,j}^{\text{out}}, \mathcal{A}(\boldsymbol{\omega}; \mathcal{D}_t^{\text{in}})\rangle - y_{t,j}^{\text{out}}\right)^2$. The goal of meta-learning is to find an optimal initialization $\widehat{\boldsymbol{\omega}}^* \in \mathbb{R}^d$ by minimizing the following empirical

meta-training loss:

$$\min_{\boldsymbol{\omega}\in\mathbb{R}^d} \widehat{\mathcal{L}}(\mathcal{A},\boldsymbol{\omega};\mathcal{D}) \quad \text{where} \quad \widehat{\mathcal{L}}(\mathcal{A},\boldsymbol{\omega};\mathcal{D}) = \frac{1}{T}\sum_{t=1}^{T} \ell(\mathcal{A}(\boldsymbol{\omega};\mathcal{D}_t^{\text{in}});\mathcal{D}_t^{\text{out}}). \tag{1}$$

In the testing process, suppose a new task τ sampled from $\mathcal{T}$ is given, which is associated with the dataset $\mathcal{Z}$ consisting of m points with the task. We apply the learned initial $\widehat{\boldsymbol{\omega}}^*$ as well as the inner algorithm $\mathcal{A}$ on $\mathcal{Z}$ to produce a task predictor. Then the test performance can be evaluated via the following population loss:

$$\mathcal{L}(\mathcal{A},\boldsymbol{\omega}) = \mathbb{E}_{\tau\sim\mathcal{T}}\mathbb{E}_{\mathcal{Z},(\mathbf{x},y)\sim\mathbb{P}_\phi(y|\mathbf{x})\mathbb{P}(\mathbf{x})}\left[\ell\left(\mathcal{A}\left(\boldsymbol{\omega};\mathcal{Z}\right);(\mathbf{x},y)\right)\right]. \tag{2}$$

Inner Loop with one-step GD. Our focus of this paper is the popular meta-learning algorithm MAML [20], where inner stage takes a few steps of GD update initialized from $\boldsymbol{\omega}$. We consider one step for simplicity, which is commonly adopted in the previous studies [6, 12, 23]. Formally, for any $\boldsymbol{\omega}\in\mathbb{R}^d$, and any dataset $(\mathbf{X},\mathbf{y})$ with n samples, the inner loop algorithm for MAML with a learning rate β is given by

$$\mathcal{A}(\boldsymbol{\omega};(\mathbf{X},\mathbf{y})) := \boldsymbol{\omega} - \beta\nabla_{\boldsymbol{\omega}}\ell\left(\boldsymbol{\omega};(\mathbf{X},\mathbf{y})\right) = (\mathbf{I}-\frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\boldsymbol{\omega} + \frac{\beta}{n}\mathbf{X}^\top\mathbf{y}. \tag{3}$$

We allow the learning rate to differ at the meta-training and testing stages, denoted as β^{tr} and β^{te} respectively. Moreover, in subsequent analysis, we will include the dependence on the learning rate to the inner loop algorithm and loss functions as $\mathcal{A}(\boldsymbol{\omega},\beta;(\mathbf{X},\mathbf{y}))$, $\widehat{\mathcal{L}}(\mathcal{A},\boldsymbol{\omega},\beta;\mathcal{D})$ and $\mathcal{L}(\mathcal{A},\boldsymbol{\omega},\beta)$.

Outer Loop with SGD. We adopt SGD to iteratively update the meta initialization variable $\boldsymbol{\omega}$ based on the empirical meta-training loss eq. (1), which is how MAML is implemented in practice [21]. Specifically, we use the constant stepsize SGD with iterative averaging [19, 15, 14], and the algorithm is summarized in Algorithm 1. Note that at each iteration, we use one task for updating the meta parameter, which can be easily generalized to the case with a mini-batch tasks for each iteration.

Algorithm 1 MAML with SGD

Input: Stepsize $\alpha > 0$, meta learning rate $\beta^{\text{tr}} > 0$
Initialization: $\boldsymbol{\omega}_0$
  for $t = 1$ to T **do**
    Receive task τ_t with data $\mathcal{D}_t$
    Randomly divided into training and validation set: $\mathcal{D}_t^{in} = (\mathbf{X}_t^{in},\mathbf{y}_t^{in})$, $\mathcal{D}_t^{out} = (\mathbf{X}_t^{out},\mathbf{y}_t^{out})$
    Update $\boldsymbol{\omega}_{t+1} = \boldsymbol{\omega}_t - \alpha\nabla\ell(\mathcal{A}(\boldsymbol{\omega},\beta^{\text{tr}};\mathcal{D}_t^{\text{in}});\mathcal{D}_t^{\text{out}})$
  end for
  return $\overline{\boldsymbol{\omega}}_T = \frac{1}{T}\sum_{t=0}^{T-1}\boldsymbol{\omega}_t$

Meta Excess Risk of SGD. Let $\boldsymbol{\omega}^*$ denote the optimal solution to the population meta-test error eq. (2). We define the following excess risk for the output $\overline{\boldsymbol{\omega}}_T$ of SGD:

$$R(\overline{\boldsymbol{\omega}}_T,\beta^{\text{te}}) \triangleq \mathbb{E}\left[\mathcal{L}(\mathcal{A},\overline{\boldsymbol{\omega}}_T,\beta^{\text{te}})\right] - \mathcal{L}(\mathcal{A},\boldsymbol{\omega}^*,\beta^{\text{te}}) \tag{4}$$

which identifies the difference between adapting from the SGD output $\overline{\boldsymbol{\omega}}_T$ and from the optimal initialization $\boldsymbol{\omega}^*$. Assuming that each task contains a fixed constant number of samples, the total number of samples over all tasks is $\mathcal{O}(T)$. Hence, the overparameterized regime can be identified as $d \gg T$, which is the focus of this paper, and is in contrast to the well studied underparameterized setting with finite dimension d ($d \ll T$). The goal of this work is to characterize the impact of SGD dynamics, demonstrating how the iteration T affects the excess risk, which has not been considered in the previous overparameterized MAML analysis [6, 53].

3.2 Task and Data Distributions

To gain more explicit knowledge of MAML, we specify the task and data distributions in this section.

Mixed Linear Regression. We consider a canonical case in which the tasks are linear regressions. This setting has been commonly adopted recently in [6, 3, 32]. Given a task τ, its model parameter ϕ is determined by $\boldsymbol{\theta} \in \mathbb{R}^d$, and the output response is generated as follows:

$$y = \boldsymbol{\theta}^\top \mathbf{x} + z, \quad \mathbf{x} \sim \mathcal{P}_{\mathbf{x}}, \quad z \sim \mathcal{P}_z \tag{5}$$

where $\mathbf{x}$ is the input feature, which follows the same distribution $\mathcal{P}_{\mathbf{x}}$ across different tasks, and z is the i.i.d. Gaussian noise sampled from $\mathcal{N}(0, \sigma^2)$. The task signal $\boldsymbol{\theta}$ has the mean $\boldsymbol{\theta}^*$ and the covariance $\Sigma_{\boldsymbol{\theta}} \triangleq \mathbb{E}[\boldsymbol{\theta}\boldsymbol{\theta}^\top]$. Denote the distribution of $\boldsymbol{\theta}$ as $\mathcal{P}_{\boldsymbol{\theta}}$. We do not make any additional assumptions on $\mathcal{P}_{\boldsymbol{\theta}}$, whereas recent studies on MAML [6, 53] assume it to be Gaussian and isotropic.

Data distribution. For the data distribution $\mathcal{P}_{\mathbf{x}}$, we first introduce some mild regularity conditions:

1. $\mathbf{x} \in \mathbb{R}^d$ is mean zero with covariance operator $\boldsymbol{\Sigma} = \mathbb{E}[\mathbf{x}\mathbf{x}^\top]$;
2. The spectral decomposition of $\boldsymbol{\Sigma}$ is $\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top = \sum_{i>0} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$, with decreasing eigenvalues $\lambda_1 \geq \cdots \geq \lambda_d > 0$, and suppose $\sum_{i>0} \lambda_i < \infty$.
3. $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{x}$ is $\sigma_{\mathbf{x}}$-subGaussian.

To analyze the stochastic approximation method SGD, we take the following standard fourth moment condition [51, 29, 7].

Assumption 1 (Fourth moment condition)**.** *There exist positive constants $c_1, b_1 > 0$, such that for any positive semidefinite (PSD) matrix $\mathbf{A}$, it holds that*

$$b_1 \operatorname{tr}(\boldsymbol{\Sigma}\mathbf{A})\Sigma + \Sigma\mathbf{A}\boldsymbol{\Sigma} \preceq \mathbb{E}_{\mathbf{x}\sim\mathcal{P}_{\mathbf{x}}}\left[\mathbf{x}\mathbf{x}^\top\mathbf{A}\mathbf{x}\mathbf{x}^\top\right] \preceq c_1 \operatorname{tr}(\boldsymbol{\Sigma}\mathbf{A})\Sigma$$

For the Gaussian distribution, it suffices to take $c_1 = 3, b_1 = 2$.

3.3 Connection to a Meta Least Square Problem.

After instantiating our study on the task and data distributions in the last section, note that $\nabla\ell(\mathcal{A}(\boldsymbol{\omega}, \beta^{\text{tr}}; \mathcal{D}_t^{\text{in}}); \mathcal{D}_t^{\text{out}})$ is linear with respect to $\boldsymbol{\omega}$. Hence, we can reformulate the problem eq. (1) as a least square (LS) problem with transformed meta inputs and output responses.

Proposition 1 (Meta LS Problem)**.** *Under the mixed linear regression model, the expectation of the meta-training loss eq. (1) taken over task and data distributions can be rewritten as:*

$$\mathbb{E}\left[\widehat{\mathcal{L}}(\mathcal{A}, \boldsymbol{\omega}, \beta^{tr}; \mathcal{D})\right] = \mathcal{L}(\mathcal{A}, \boldsymbol{\omega}, \beta^{tr}) = \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\|\mathbf{B}\boldsymbol{\omega} - \boldsymbol{\gamma}\|^2\right]. \tag{6}$$

The meta data are given by

$$\begin{aligned}
\mathbf{B} &= \frac{1}{\sqrt{n_2}}\mathbf{X}^{out}\left(\mathbf{I} - \frac{\beta^{tr}}{n_1}\mathbf{X}^{in^T}\mathbf{X}^{in}\right) \\
\boldsymbol{\gamma} &= \frac{1}{\sqrt{n_2}}\left(\mathbf{X}^{out}\left(\mathbf{I} - \frac{\beta^{tr}}{n_1}\mathbf{X}^{in^T}\mathbf{X}^{in}\right)\boldsymbol{\theta} + \mathbf{z}^{out} - \frac{\beta^{tr}}{n_1}\mathbf{X}^{out}\mathbf{X}^{in^\top}\mathbf{z}^{in}\right)
\end{aligned} \tag{7}$$

where $\mathbf{X}^{in} \in \mathbb{R}^{n_1 \times d}$,$\mathbf{z}^{in} \in \mathbb{R}^{n_1}$,$\mathbf{X}^{out} \in \mathbb{R}^{n_2 \times d}$ and $\mathbf{z}^{out} \in \mathbb{R}^{n_2}$ denote the inputs and noise for training and validation. Furthermore,we have

$$\boldsymbol{\gamma} = \mathbf{B}\boldsymbol{\theta}^* + \boldsymbol{\xi} \quad \textit{with meta noise } \mathbb{E}[\boldsymbol{\xi} \mid \mathbf{B}] = 0. \tag{8}$$

Therefore, the meta-training objective is equivalent to searching for a $\boldsymbol{\omega}$, which is close to the task mean $\boldsymbol{\theta}^*$. Moreover, with the specified data and task model, the optimal solution for meta-test loss eq. (2) can be directly calculated [23], and we obtain $\boldsymbol{\omega}^* = \mathbb{E}[\boldsymbol{\theta}] = \boldsymbol{\theta}^*$. Hence, the meta excess risk eq. (4) is identical to the standard excess risk [5] for the linear model eq. (8), i.e., $R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) = \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\|\mathbf{B}\overline{\boldsymbol{\omega}}_T - \boldsymbol{\gamma}\|^2 - \|\mathbf{B}\boldsymbol{\theta}^* - \boldsymbol{\gamma}\|^2\right]$, but with more complicated input and output data expressions. The following analysis will focus on this transformed linear model.

Furthermore, we can calculate the statistical properties of the reformed input $\mathbf{B}$, and obtain the meta-covariance:

$$\mathbb{E}[\mathbf{B}^\top\mathbf{B}] = (\mathbf{I} - \beta^{\text{tr}}\boldsymbol{\Sigma})^2\boldsymbol{\Sigma} + \frac{\beta^{\text{tr}2}}{n_1}(\mathbf{F} - \boldsymbol{\Sigma}^3)$$

where $\mathbf{F} = \mathbb{E}[\mathbf{x}\mathbf{x}^\top \Sigma \mathbf{x}\mathbf{x}^\top]$. Let $\mathbf{X} \in \mathbb{R}^{n\times d}$ denote the collection of n i.i.d. samples from $\mathcal{P}_{\mathbf{x}}$, and denote

$$\mathbf{H}_{n,\beta} = \mathbb{E}[(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\Sigma(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})] = (\mathbf{I} - \beta\mathbf{\Sigma})^2\mathbf{\Sigma} + \frac{\beta^2}{n}(\mathbf{F} - \mathbf{\Sigma}^3).$$

We can then write $\mathbb{E}[\mathbf{B}^\top\mathbf{B}] = \mathbf{H}_{n_1,\beta^{\text{tr}}}$. Regarding the form of $\mathbf{B}$ and $\mathbf{H}_{n_1,\beta^{\text{tr}}}$, we need some further conditions on the higher order moments of the data distribution.

Assumption 2 (Commutity)**.** $\mathbf{F} = \mathbb{E}[\mathbf{x}\mathbf{x}^\top\mathbf{\Sigma}\mathbf{x}\mathbf{x}^\top]$ *commutes with the data covariance* $\mathbf{\Sigma}$.

Assumption 2 holds for Gaussian data. Such commutity of $\mathbf{\Sigma}$ has also been considered in [51].

Assumption 3 (Higher order moment condition)**.** *Given* $|\beta| < \frac{1}{\lambda_1}$ *and* $\mathbf{\Sigma}$*, there exists a constant* $C(\beta, \mathbf{\Sigma}) > 0$*, for large* $n > 0$*, s.t. for any unit vector* $\mathbf{v} \in \mathbb{R}^d$*, we have:*

$$\mathbb{E}[\|\mathbf{v}^\top\mathbf{H}_{n,\beta}^{-\frac{1}{2}}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n,\beta}^{-\frac{1}{2}}\mathbf{v}\|^2] < C(\beta, \mathbf{\Sigma}). \tag{9}$$

In Assumption 3, the analytical form of $C(\beta, \Sigma)$ can be derived if $\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{x}$ is Gaussian. Moreover, if $\beta = 0$, then we obtain $C(\beta, \mathbf{\Sigma}) = 1$. Further technical discussions are presented in Appendix.

4 Main Results

In this section, we present our analyses on generalization properties of MAML optimized by average SGD and derive insights on the effect of various parameters. Specifically, our results consist of three parts. First, we characterize the meta excess risk of MAML trained with SGD. Then, we establish the generalization error bound for various types of data and task distributions, to reveal which kind of overparameterization regarding data and task is essential for diminishing meta excess risk. Finally, we explore how the adaptation learning rate β^{tr} affects the excess risk and the training dynamics.

4.1 Performance Bounds

Before starting our results, we first introduce relevant notations and concepts. We define the following rates of interest (See Remark 3 for further discussions)

$$\begin{aligned} c(\beta, \mathbf{\Sigma}) &:= c_1(1 + 8|\beta|\lambda_1\sqrt{C(\beta, \mathbf{\Sigma})}\sigma_x^2 + 64\sqrt{C(\beta, \mathbf{\Sigma})}\sigma_x^4\beta^2\operatorname{tr}(\mathbf{\Sigma}^2)) \\ f(\beta, n, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}}) &:= c(\beta, \mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}_{\boldsymbol{\theta}}\mathbf{\Sigma}) + 4c_1\sigma^2\sigma_x^2\beta^2\sqrt{C(\beta, \mathbf{\Sigma})}\operatorname{tr}(\mathbf{\Sigma}^2) + \sigma^2/n \\ g(\beta, n, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}}) &:= \sigma^2 + b_1\operatorname{tr}(\mathbf{\Sigma}_{\boldsymbol{\theta}}\mathbf{H}_{n,\beta}) + \beta^2\mathbf{1}_{\beta\le 0}b_1\operatorname{tr}(\mathbf{\Sigma}^2)/n. \end{aligned}$$

Moreover, for a positive semi-definite matrix $\mathbf{H}$, s.t. $\mathbf{H}$ and $\mathbf{\Sigma}$ can be diagonalized simultaneously, let $\mu_i(\mathbf{H})$ denote its corresponding eigenvalues for $\mathbf{v}_i$, i.e. $\mathbf{H} = \sum_i \mu_i(\mathbf{H})\mathbf{v}_i\mathbf{v}_i^\top$ (Recall $\mathbf{v}_i$ is the i-th eigenvector of $\mathbf{\Sigma}$).

We next introduce the following new notion of the *effective meta weight*, which will serve as an important quantity for capturing the generalization of MAML.

Definition 1 (Effective Meta Weights)**.** For $|\beta^{\text{tr}}|, |\beta^{\text{te}}| < 1/\lambda_1$, given step size α and iteration T, define

$$\Xi_i(\mathbf{\Sigma}, \alpha, T) = \begin{cases} \mu_i(\mathbf{H}_{m,\beta^{\text{te}}})/\left(T\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\right) & \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) \ge \frac{1}{\alpha T}; \\ T\alpha^2\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mu_i(\mathbf{H}_{m,\beta^{\text{te}}}) & \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \frac{1}{\alpha T}. \end{cases} \tag{10}$$

We call $\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})/\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})$ and $\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})$ the **meta ratio** (See Remark 2).

We omit the arguments of the effective meta weight Ξ_i for simplicity in the following analysis.

Our first results characterize matching upper and lower bounds on the meta excess risk of MAML in terms of the effective meta weight.

Theorem 1 (Upper Bound)**.** *Let* $\omega_i = \langle \boldsymbol{\omega}_0 - \boldsymbol{\theta}^*, \mathbf{v}_i\rangle$*. If* $|\beta^{tr}|, |\beta^{te}| < 1/\lambda_1$*,* n_1 *is large ensuring that* $\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) > 0$*,* $\forall i$ *and* $\alpha < 1/\left(c(\beta^{tr}, \mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma})\right)$*, then the meta excess risk* $R(\overline{\boldsymbol{\omega}}_T, \beta^{te})$ *is bounded above as follows*

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) \le \textit{Bias} + \textit{Var}$$

where

$$\textit{Bias} = \frac{2}{\alpha^2 T}\sum_i \Xi_i \frac{\omega_i^2}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})}$$

$$\textit{Var} = \frac{2}{(1-\alpha c(\beta^{tr},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))}\left(\sum_i \Xi_i\right)$$

$$\times[\underbrace{f(\beta^{tr},n_2,\sigma,\mathbf{\Sigma}_{\boldsymbol{\theta}},\mathbf{\Sigma})}_{V_1} + \underbrace{2c(\beta^{tr},\mathbf{\Sigma})\sum_i\left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\geq\frac{1}{\alpha T}}}{T\alpha\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})<\frac{1}{\alpha T}}\right)\lambda_i\omega_i^2}_{V_2}]$$

Remark 1. The primary error source of the upper bound are two folds. The bias term corresponds to the error if we directly implement GD updates towards the meta objective eq. (6). The variance error is composed of the disturbance of meta noise $\boldsymbol{\xi}$ (the V_1 term), and the randomness of SGD itself (the V_2 term). Regardless of data or task distributions, for proper stepsize α, we can easily derive that the bias term is $\mathcal{O}(\frac{1}{T})$, and the V_2 term is also $\mathcal{O}(\frac{1}{T})$, which is dominated by V_1 term ($\Omega(1)$). Hence, to achieve the vanishing risk, we need to understand the roles of Ξ_i and $f(\cdot)$

Remark 2 (Effective Meta Weights). By Definition 1, we separate the data eigenspace into "**leading**" ($\geq \frac{1}{\alpha T}$) and "**tail**" ($< \frac{1}{\alpha T}$) spectrum spaces with different meta weights. The meta ratios indicate the impact of one-step gradient update. For large n, $\mu_i(\mathbf{H}_{n,\beta}) \approx (1-\beta\lambda_i)^2\lambda_i$, and hence a larger β^{tr} in training will increase the weight for "leading" space and decrease the weight for "tail" space, while a larger β^{te} always decreases the weight.

Remark 3 (Role of $f(\cdot)$). $f(\cdot)$ in variance term consists of various sources of meta noise $\boldsymbol{\xi}$, including inner gradient updates (β), task diversity ($\mathbf{\Sigma}_{\boldsymbol{\theta}}$) and noise from regression tasks (σ). As mentioned in Remark 1, understanding $f(\cdot)$ is critical in our analysis. Yet, due to the multiple randomness origins, techniques for classic linear regression [51, 29] cannot be directly applied here. Our analysis overcomes such non-trivial challenges. $g(\cdot)$ in Theorem 2 plays a similar role to $f(\cdot)$.

Therefore, Theorem 1 implies that overparameterization is crucial for diminishing risk under the following conditions:

- For $f(\cdot)$: $\operatorname{tr}(\mathbf{\Sigma}\mathbf{\Sigma}_{\boldsymbol{\theta}})$ and $\operatorname{tr}(\mathbf{\Sigma}^2)$ is small compared to T;
- For Ξ_i: the dimension of "leading" space is $o(T)$, and the summation of meta ratio over "tail" space is $o(\frac{1}{T})$.

We next provide a lower bound on the meta excess risk, which matches the upper bound in order.

Theorem 2 (Lower Bound). *Following the similar notations in Theorem 1, Then*

$$R(\overline{\boldsymbol{\omega}}_T,\beta^{te}) \geq \frac{1}{100\alpha^2 T}\sum_i \Xi_i\frac{\omega_i^2}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \frac{1}{n_2}\cdot\frac{1}{(1-\alpha c(\beta^{tr},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))}\sum_i \Xi_i$$

$$\times[\frac{1}{100}g(\beta^{tr},n_1,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}}) + \frac{b_1}{1000}\sum_i\left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\geq\frac{1}{\alpha T}}}{T\alpha\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})<\frac{1}{\alpha T}}\right)\lambda_i\omega_i^2].$$

Our lower bound can also be decomposed into bias and variance terms as the upper bound. The bias term well matches the upper bound up to absolute constants. The variance term differs from the upper bound only by $\frac{1}{n_2}$, where n_2 is the batch size of each task, and is treated as a constant (i.e., does not scale with T) [28, 41] in practice. Hence, in the overparameterized regime where $d \gg T$ and T tends to be sufficiently large, the variance term also matches that in the upper bound w.r.t. T.

4.2 The Effects of Task Diversity

From Theorem 1 and Theorem 2, we observe that the task diversity $\mathbf{\Sigma}_{\boldsymbol{\theta}}$ in $f(\cdot)$ and $g(\cdot)$ plays a crucial role in the performance guarantees for MAML. In this section, we explore several types of data distributions to further characterize the effects of the task diversity.

We take the single task setting as a comparison with meta-learning, where the task diversity diminishes (tentatively say $\mathbf{\Sigma}_{\boldsymbol{\theta}} \to \mathbf{0}$), i.e., each task parameter $\boldsymbol{\theta} = \boldsymbol{\theta}^*$. In such a case, it is unnecessary to do one-step gradient in the inner loop and we set $\beta^{\text{tr}} = 0$, which is equivalent to directly running

SGD. Formally, the **single task setting** can be described as outputting $\overline{\boldsymbol{\omega}}_T^{\text{sin}}$ with iterative SGD that minimizes $\widehat{\mathcal{L}}(\mathcal{A}, \boldsymbol{\omega}, 0; \mathcal{D})$ with meta linear model as $\boldsymbol{\gamma} = \frac{1}{\sqrt{n_2}}(\mathbf{X}^{out}\boldsymbol{\theta}^* + \mathbf{z}^{\text{out}})$.

Theorem 1 implies that the data spectrum should decay fast, which leads to a small dimension of "leading" space and small meta ratio summation over "tail" space. Let us first consider a relatively slow decaying case: $\lambda_k = k^{-1}\log^{-p}(k+1)$ for some $p > 1$. Applying Theorem 1, we immediately derive the theoretical guarantees for single task:

Lemma 1 (Single Task)**.** *If $|\beta^{te}| < \frac{1}{\lambda_1}$ and if the spectrum of $\boldsymbol{\Sigma}$ satisfies $\lambda_k = k^{-1}\log^{-p}(k+1)$, then $R(\overline{\boldsymbol{\omega}}_T^{sin}, \beta^{te}) = \mathcal{O}(\frac{1}{\log^p(T)})$*

At the test stage, if we set $\beta^{\text{te}} = 0$, then the meta excess risk for the single task setting, i.e., $R(\overline{\boldsymbol{\omega}}_T^{\text{sin}}, 0)$, is exactly the excess risk in classical linear regression [51]. Lemma 1 can be regarded as a generalized version of Corollary 2.3 in [51], where they provide the upper bound for $R(\overline{\boldsymbol{\omega}}_T^{\text{sin}}, 0)$, while we allow a one-step fine-tuning for testing.

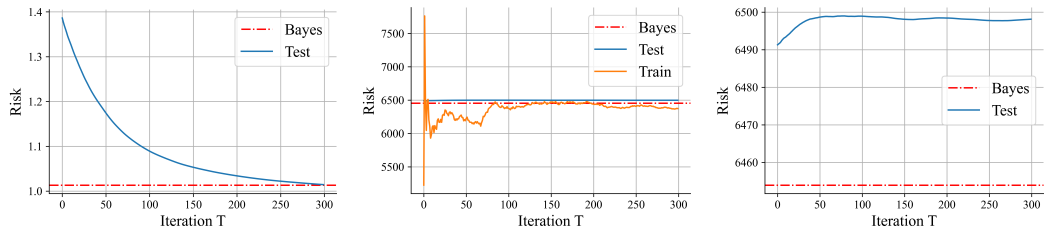
Lemma 1 suggests that the log-decay is sufficient to assure that $R(\overline{\boldsymbol{\omega}}_T^{\text{sin}}, 0)$ is diminishing when $d \gg T$. However, in meta-learning with multi-tasks, the task diversity captured by the task spectral distribution can highly affect the meta excess risk. In the following, our Theorem 1 and Theorem 2 (i.e., upper and lower bounds) establish a sharp phase transition of the generalization for MAML for the same data spectrum considered in Lemma 1, which is in contrast to the single task setting (see Lemma 1), where log-decay data spectrum always yields vanishing excess risk.

Proposition 2 (MAML, log-Decay Data Spectrum)**.** *Given $|\beta^{tr}|, |\beta^{te}| < \frac{1}{\lambda_1}$, under the same data distribution as in Lemma 1, and the spectrum of $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$, denoted as ν_i, satisfies $\nu_k = \log^r(k+1)$ for some $r > 0$, then*

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) = \begin{cases} \Omega(\log^{r-2p+1}(T)) & r \geq 2p-1 \\ \mathcal{O}(\frac{1}{\log^{p-(r-p+1)^+}(T)}) & r < 2p-1 \end{cases}$$

Proposition 2 implies that under log-decay data spectrum parameterized by p, the meta excess risk of MAML experiences a phase transition determined by the spectrum parameter r. Since large r implies large eigenvalues and high variations for task vectors, we adopt r to measure the diversity of task distributions, and call r as the task diversity in the sequel. While slower task diversity rate $r < 2p-1$ guarantees vanishing excess risk, faster task diversity rate $r \geq 2p-1$ necessarily results in non-vanishing excess risk. Proposition 2 and Lemma 1 together indicate that while log-decay data spectrum always yields benign fitting (vanishing risk) in the single task setting, it can yield non-vanishing risk in meta learning due to fast task diversity rate.

We further validate our theoretical results in Proposition 2 by experiments. We consider the case $p = 2$. As shown in Figure 1a, when $r < 2p-1$, the test error quickly converges to the Bayes error. When $r > 2p-1$, Figure 1b illustrates that MAML already converges on the training samples, but the test error (which is further zoomed in Figure 1c) levels off and does not vanish, showing MAML generalizes poorly when $r > 2p-1$.



(a) $\nu_i = 0.25\log^{1.5}(i+1)$ (b) $\nu_i = 0.25\log^{8}(i+1)$ (c) $\nu_i = 0.25\log^{8}(i+1)$

Figure 1: The effects of task diversity. $d = 500$, $T = 300$, $\lambda_i = \frac{1}{i\log(i+1)^2}$, $\beta^{\text{tr}} = 0.02$, $\beta^{\text{te}} = 0.2$.

Furthermore, we show that the above phase transition that occurs for log-decay data distributions no longer exists for data distributions with faster decaying spectrum.

Proposition 3 (MAML, Fast-Decay Data Spectrum)**.** *Under the same task distribution as in Proposition 2, i.e., the spectrum of $\mathbf{\Sigma_\theta}$, denoted as ν_i, satisfies $\nu_k = \log^r(k+1) = \widetilde{O}(1)$ for some $r > 0$, and the data distribution satisfies:*

1. *$\lambda_k = k^{-q}$ for some $q > 1$, $R(\overline{\boldsymbol{\omega}}_T^{sin}, \beta^{te}) = \mathcal{O}\left(\frac{1}{T^{\frac{q-1}{q}}}\right)$ and $R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) = \widetilde{\mathcal{O}}\left(\frac{1}{T^{\frac{q-1}{q}}}\right)$;*
2. *$\lambda_k = e^{-k}$, $R(\overline{\boldsymbol{\omega}}_T^{sin}, \beta^{te}) = \widetilde{\mathcal{O}}(\frac{1}{T})$ and $R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) = \widetilde{\mathcal{O}}(\frac{1}{T})$.*

4.3 On the Role of Adaptation Learning Rate

The analysis in [6] suggests a surprising observation that a negative learning rate (i.e., when β^{tr} takes a negative value) optimizes the generalization for MAML under mixed linear regression models. Their results indicate that the testing risk initially increases and then decreases as β^{tr} varies from negative to positive values around zero for Gaussian isotropic input data and tasks. Our following proposition supports such a trend, but with a novel tradeoff in SGD dynamics as a new reason for the trend, under more general data distributions. Denote $\overline{\boldsymbol{\omega}}_T^{\beta}$ as the average SGD solution of MAML after T iterations that uses β as the inner loop learning rate.

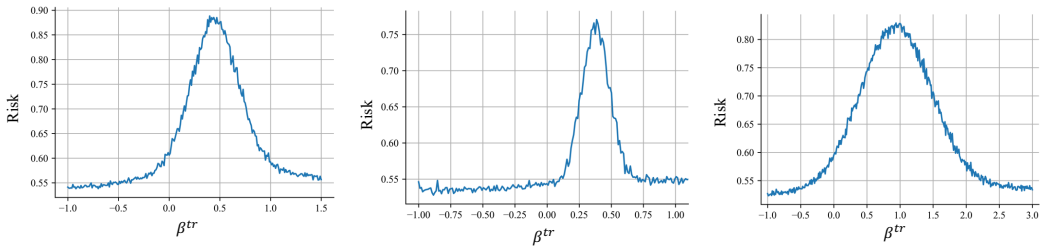
Proposition 4. *Let $s = T\log^{-p}(T)$ and $d = T\log^q(T)$, where $p, q > 0$. Suppose $\mathcal{P}_{\mathbf{x}}$ is Gaussian and the spectrum of $\mathbf{\Sigma}$ satisfies*

$$\lambda_k = \begin{cases} 1/s, & k \le s \\ 1/(d-s), & s+1 \le k \le d. \end{cases}$$

Suppose the spectral parameter ν_i of $\mathbf{\Sigma_\theta}$ is $O(1)$, and let the step size $\alpha = \frac{1}{2c(\beta^{tr}, \mathbf{\Sigma})\,\mathrm{tr}(\mathbf{\Sigma})}$. Then for large n_1, $|\beta^{tr}|, |\beta^{te}| < \frac{1}{\lambda_1}$, we have

$$R(\overline{\boldsymbol{\omega}}_T^{\beta^{tr}}, \beta^{te}) \lesssim \mathcal{O}\Big(\frac{1}{\log^p(T)}\Big)\frac{1}{(1-\beta^{tr}\lambda_1)^2} + \mathcal{O}\Big(\frac{1}{\log^q(T)}\Big)\Big(1-\beta^{tr}\lambda_d\Big)^2 + \widetilde{\mathcal{O}}(\frac{1}{T}). \tag{11}$$

The first two terms in the bound of eq. (11) correspond to the impact of effective meta weights Ξ_i on the "leading" and "tail" spaces, respectively, as we discuss in Remark 2. Clearly, the learning rate β^{tr} plays a tradeoff role in these two terms, particularly when p is close to q. This explains the fact that the test error first increases and then decreases as β^{tr} varies from negative to positive values around zero. Such a tradeoff also serves as the reason for the first-increase-then-decrease trend of the test error under more general data distributions as we demonstrate in Figure 2. This complements the reason suggested in [6], which captures only the quadratic form $\frac{1}{(1-\beta^{\text{tr}}\lambda_1)^2}$ of β^{tr} for isotropic $\mathbf{\Sigma}$, where there exists only the "leading" space without "tail" space.



(a) $\lambda_i = \frac{1}{i\log(i+1)^2}$ (b) $\lambda_i = \frac{1}{i\log(i+1)^3}$ (c) $\lambda_i = \frac{1}{i^2}$

Figure 2: $R(\overline{\boldsymbol{\omega}}_T^{\beta^{\text{tr}}}, \beta^{\text{te}})$ as a function of β^{tr}. $d = 200$, $T = 100$, $\mathbf{\Sigma_\theta} = \frac{0.8^2}{d}\mathbf{I}$, $\beta^{\text{te}} = 0.2$.

Based on the above results, incorporating with our dynamics analysis, we surprisingly find that β^{tr} not only affects the final risk, but also plays a pivot role towards the early iteration that the testing error tends to be steady. To formally study such a property, we define the stopping time as follows.

Definition 2 (Stopping time)**.** Given $\beta^{\text{tr}}, \beta^{\text{te}}$, for any $\epsilon > 0$, the corresponding stopping time $t_\epsilon(\beta^{\text{tr}}, \beta^{\text{te}})$ is defined as:

$$t_\epsilon(\beta^{\text{tr}}, \beta^{\text{te}}) = \min t \quad \text{s.t.} \quad R(\overline{\boldsymbol{\omega}}_t^{\beta^{\text{tr}}}; \beta^{\text{te}}) < \epsilon.$$

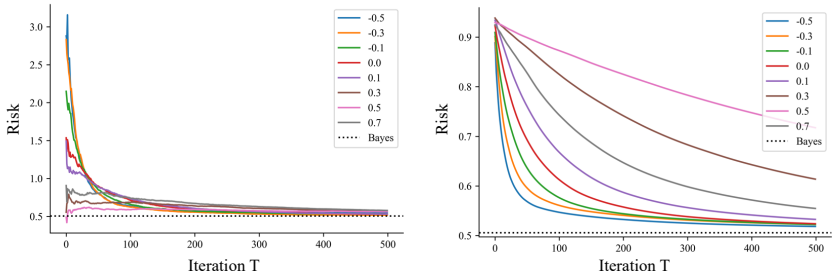
In the sequel, we may omit the arguments in t_ϵ for simplicity. We consider the similar data distribution in Proposition 4 but parameterized by K, i.e., $s = K\log^{-p}(K)$ and $d = K\log^{q}(K)$, where $p, q > 0$. Then we can derive the following characterization for t_ϵ.

Corollary 1. *If the assumptions in Proposition 4 hold and $p = q$. Further, let $\mathbf{\Sigma_\theta} = \eta^2\mathbf{I}$, and $|\beta^{tr}| < \frac{1}{\lambda_1}$. Then for $t_\epsilon(\beta^{tr}, \beta^{te}) \in (s, K]$, we have:*

$$\exp\left(\epsilon^{-\frac{1}{p}}\left[\frac{L_l}{(1-\beta^{tr}\lambda_1)^2} + L_t(1-\beta^{tr}\lambda_d)^2\right]^{\frac{1}{p}}\right) \le t_\epsilon \le \exp\left(\epsilon^{-\frac{1}{p}}\left[\frac{U_l}{(1-\beta^{tr}\lambda_1)^2} + U_t(1-\beta^{tr}\lambda_d)^2\right]^{\frac{1}{p}}\right) \tag{12}$$

where L_l, L_t, U_l, $U_t > 0$ are factors for "leading" and "tail" spaces that are independent of K[2].

Equation (12) suggests that the early stopping time t_ϵ is also controlled by the tradeoff role that β^{tr} plays in the "leading" (U_l, L_l) and "tail" spaces (U_t, L_t), which takes a similar form as the bound in Proposition 4. Therefore, the trend for t_ϵ in terms of β^{tr} will exhibit similar behaviours as the final excess risk, and hence the optimal β^{tr} for the final excess risk will lead to an earliest stopping time. We plot the training and test errors for different β^{tr} in Figure 3, under the same data distributions as Figure 2a to validate our theoretical findings. As shown in Figure 3a, β^{tr} does not make much difference in the training stage (the process converges for all β^{tr} when T is larger than 100). However, in Figure 3b at test stage, β^{tr} significantly affects the iteration when the test error starts to become relatively flat. Such an early stopping time first increases then decreases as β^{tr} varies from -0.5 to 0.7, which resembles the change of final excess risk in Figure 2a.



(a) Training Risk (b) Test Error

Figure 3: Training and test curves for different β^{tr}. $d = 500$, $\lambda_i = \frac{1}{i\log^2(i+1)}$,$\mathbf{\Sigma_\theta} = \frac{0.8^2}{d}\mathbf{I}$, $\beta^{\text{te}} = 0.2$.

5 Conclusions

In this work, we give the theoretical treatment towards the generalization property of MAML based on their optimization trajectory in non-asymptotic and overparameterized regime. We provide both upper and lower bounds on the excess risk of MAML trained with average SGD. Furthermore, we explore which type of data and task distributions are crucial for diminishing error with overparameterization, and discover the influence of adaption learning rate both on the generalization error and the dynamics, which brings novel insights towards the distinct effects of MAML's one-step gradient updates on "leading" and "tail" parts of data eigenspace.

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[2]Such terms have been suppressed for clarity. Details are presented in the appendix.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

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A Proof of Proposition 1

We first show how to connect the loss function associated with MAML to a Meta Least Square Problem.

Proposition 5 (Proposition 1 Restated)**.** *Under the mixed linear regression model, the expectation of the meta-training loss taken over task and data distributions can be rewritten as:*

$$\mathbb{E}\left[\widehat{\mathcal{L}}(\mathcal{A}, \boldsymbol{\omega}, \beta^{tr}; \mathcal{D})\right] = \mathcal{L}(\mathcal{A}, \boldsymbol{\omega}, \beta^{tr}) = \mathbb{E}_{\mathbf{B},\gamma}\frac{1}{2}\left[\|\mathbf{B}\boldsymbol{\omega} - \boldsymbol{\gamma}\|^2\right]. \tag{13}$$

The meta data are given by

$$\mathbf{B} = \frac{1}{\sqrt{n_2}}\mathbf{X}^{out}\Big(\mathbf{I} - \frac{\beta^{tr}}{n_1}\mathbf{X}^{in^T}\mathbf{X}^{in}\Big) \tag{14}$$

$$\boldsymbol{\gamma} = \frac{1}{\sqrt{n_2}}\Big(\mathbf{X}^{out}\Big(\mathbf{I} - \frac{\beta^{tr}}{n_1}\mathbf{X}^{in^T}\mathbf{X}^{in}\Big)\boldsymbol{\theta} + \mathbf{z}^{out} - \frac{\beta^{tr}}{n_1}\mathbf{X}^{out}\mathbf{X}^{in^\top}\mathbf{z}^{in}\Big) \tag{15}$$

where $\mathbf{X}^{in} \in \mathbb{R}^{n_1\times d}$,$\mathbf{z}^{in} \in \mathbb{R}^{n_1}$,$\mathbf{X}^{out} \in \mathbb{R}^{n_2\times d}$ *and* $\mathbf{z}^{out} \in \mathbb{R}^{n_2}$ *denote the inputs and noise for training and validation. Furthermore,we have*

$$\boldsymbol{\gamma} = \mathbf{B}\boldsymbol{\theta}^* + \boldsymbol{\xi} \quad \textit{with meta noise } \mathbb{E}[\boldsymbol{\xi} \mid \mathbf{B}] = 0. \tag{16}$$

Proof. We first rewrite $\mathcal{L}(\mathcal{A}, \boldsymbol{\omega}, \beta^{\text{tr}})$ as follows:

$$\begin{aligned}
\mathcal{L}(\mathcal{A}, \boldsymbol{\omega}, \beta^{\text{tr}}) &= \mathbb{E}\left[\ell(\mathcal{A}(\boldsymbol{\omega}, \beta^{\text{tr}}; \mathcal{D}^{\text{in}}); \mathcal{D}^{\text{out}})\right] \\
&= \mathbb{E}\left[\frac{1}{2n_2}\sum_{j=1}^{n_2}\left(\langle \mathbf{x}_j^{\text{out}}, \mathcal{A}(\boldsymbol{\omega}, \beta^{\text{tr}}; \mathcal{D}^{\text{in}})\rangle - y_j^{\text{out}}\right)^2\right] \\
&= \mathbb{E}\left[\frac{1}{2n_2}\|\mathbf{X}^{\text{out}}\Big(\mathbf{I} - \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{in}^T}\mathbf{X}^{\text{in}}\Big)\boldsymbol{\omega} + \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{in}^T}\mathbf{y}^{\text{in}} - \mathbf{y}^{\text{out}}\|^2\right].
\end{aligned}$$

Using the mixed linear model:

$$\mathbf{y}^{\text{in}} = \mathbf{X}^{\text{in}}\boldsymbol{\theta} + \mathbf{z}^{\text{in}}, \quad \mathbf{y}^{\text{out}} = \mathbf{X}^{\text{out}}\boldsymbol{\theta} + \mathbf{z}^{\text{out}}, \tag{17}$$

we have

$$\begin{aligned}
&\mathcal{L}(\mathcal{A}, \boldsymbol{\omega}, \beta^{\text{tr}}) \\
&= \mathbb{E}\left[\frac{1}{2n_2}\|\mathbf{X}^{\text{out}}\Big(\mathbf{I} - \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{in}^T}\mathbf{X}^{\text{in}}\Big)\boldsymbol{\omega}\right. \\
&\left. - \Big(\mathbf{X}^{\text{out}}\Big(\mathbf{I} - \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{in}^T}\mathbf{X}^{\text{in}}\Big)\boldsymbol{\theta} + \mathbf{z}^{out} - \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{out}}\mathbf{X}^{\text{in}^\top}\mathbf{z}^{\text{in}}\Big)\|^2\right] \\
&= \mathbb{E}_{\mathbf{B},\gamma}\frac{1}{2}\left[\|\mathbf{B}\boldsymbol{\omega} - \boldsymbol{\gamma}\|^2\right].
\end{aligned}$$

Moreover, note that $\boldsymbol{\theta} - \boldsymbol{\theta}^*$ has mean zero and is independent of data and noise, and define

$$\boldsymbol{\xi} = \frac{1}{\sqrt{n_2}}\left(\mathbf{X}^{\text{out}}\left(\mathbf{I} - \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{in}^T}\mathbf{X}^{\text{in}}\right)(\boldsymbol{\theta} - \boldsymbol{\theta}^*) + \mathbf{z}^{\text{out}} - \frac{\beta^{\text{tr}}}{n_1}\mathbf{X}^{\text{out}}\mathbf{X}^{\text{in}^\top}\mathbf{z}^{in}\right). \tag{18}$$

We call $\boldsymbol{\xi}$ as meta noise, and then we have

$$\boldsymbol{\gamma} = \mathbf{B}\boldsymbol{\theta}^* + \boldsymbol{\xi} \quad \text{and} \quad \mathbb{E}[\boldsymbol{\xi} \mid \mathbf{B}] = 0.$$

□

Lemma 2 (Meta Excess Risk)**.** *Under the mixed linear regression model, the meta excess risk can be rewritten as follows:*

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) = \frac{1}{2}\mathbb{E}\|\overline{\boldsymbol{\omega}}_T - \boldsymbol{\theta}^*\|^2_{\mathbf{H}_{m,\beta^{te}}}$$

where $\|\mathbf{a}\|^2_{\mathbf{A}} = \mathbf{a}^T\mathbf{A}\mathbf{a}$*. Moreover, the Bayes error is given by*

$$\mathcal{L}(\mathcal{A}, \boldsymbol{\omega}^*, \beta^{te}) = \frac{1}{2}\operatorname{tr}(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{H}_{m,\beta^{te}}) + \frac{\sigma^2\beta^{te2}}{2m} + \frac{\sigma^2}{2}.$$

Proof. Recall that

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) \triangleq \mathbb{E}\left[\mathcal{L}(\mathcal{A}, \overline{\boldsymbol{\omega}}_T, \beta^{\text{te}})\right] - \mathcal{L}(\mathcal{A}, \boldsymbol{\omega}^*, \beta^{\text{te}})$$

where $\boldsymbol{\omega}^*$ denotes the optimal solution to the population meta-test error. Under the mixed linear model, such a solution can be directly calculated [23], and we obtain $\boldsymbol{\omega}^* = \mathbb{E}[\boldsymbol{\theta}] = \boldsymbol{\theta}^*$. Hence,

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) = \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\|\mathbf{B}\overline{\boldsymbol{\omega}}_T - \boldsymbol{\gamma}\|^2 - \|\mathbf{B}\boldsymbol{\theta}^* - \boldsymbol{\gamma}\|^2\right],$$

where

$$\begin{aligned}
\mathbf{B} =& \mathbf{x}^{\text{out}\top}\left(\mathbf{I} - \frac{\beta^{\text{te}}}{m}\mathbf{X}^{\text{in}T}\mathbf{X}^{\text{in}}\right) \\
\boldsymbol{\gamma} =& \mathbf{x}^{\text{out}\top}\left(\mathbf{I} - \frac{\beta^{\text{te}}}{m}\mathbf{X}^{\text{in}T}\mathbf{X}^{\text{in}}\right)\boldsymbol{\theta} + \mathbf{z}^{\text{out}} - \frac{\beta^{\text{te}}}{m}\mathbf{x}^{\text{out}\top}\mathbf{X}^{\text{in}\top}\mathbf{z}^{\text{in}},
\end{aligned} \tag{19}$$

and $\mathbf{x}^{\text{out}} \in \mathbb{R}^d$, $\mathbf{z}^{\text{out}} \in \mathbb{R}^d$, $\mathbf{X}^{\text{in}} \in \mathbb{R}^{m\times d}$ and $\mathbf{z}^{\text{in}} \in \mathbb{R}^m$. The forms of $\mathbf{B}$ and $\boldsymbol{\gamma}$ are slightly different since we allow a new adaptation rate β^{te} and the inner loop has m samples at test stage. Similarly

$$\xi = \left(\underbrace{\mathbf{x}^{\text{out}\top}\left(\mathbf{I} - \frac{\beta^{\text{te}}}{m}\mathbf{X}^{\text{in}T}\mathbf{X}^{\text{in}}\right)(\boldsymbol{\theta} - \boldsymbol{\theta}^*)}_{\xi_1} + \underbrace{\mathbf{z}^{\text{out}}}_{\xi_2} \underbrace{- \frac{\beta^{\text{tr}}}{m}\mathbf{x}^{\text{out}\top}\mathbf{X}^{\text{in}\top}\mathbf{z}^{in}}_{\xi_3}\right). \tag{20}$$

Then we have

$$\begin{aligned}
R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) &= \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\|\mathbf{B}\overline{\boldsymbol{\omega}}_T - \boldsymbol{\gamma}\|^2 - \|\mathbf{B}\boldsymbol{\theta}^* - \boldsymbol{\gamma}\|^2\right] \\
&= \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\|\mathbf{B}(\overline{\boldsymbol{\omega}}_T - \boldsymbol{\theta}^*)\|^2\right] \\
&= \frac{1}{2}\mathbb{E}\|\overline{\boldsymbol{\omega}}_T - \boldsymbol{\theta}^*\|^2_{\mathbf{H}_{m,\beta^{\text{te}}}}
\end{aligned}$$

where the last equality follows because $\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\right] = \mathbf{H}_{m,\beta^{\text{te}}}$ at the test stage.

The Bayes error can be calculated as follows:

$$\begin{aligned}
\mathcal{L}(\mathcal{A}, \boldsymbol{\omega}^*, \beta^{\text{te}}) &= \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\|\mathbf{B}\boldsymbol{\theta}^* - \boldsymbol{\gamma}\|^2\right] = \mathbb{E}_{\mathbf{B},\boldsymbol{\gamma}} \frac{1}{2}\left[\xi^2\right] \\
&\overset{(a)}{=} \frac{1}{2}\left(\mathbb{E}\left[\xi_1^2\right] + \mathbb{E}\left[\xi_2^2\right] + \mathbb{E}\left[\xi_3^2\right]\right) \\
&= \frac{1}{2}(\operatorname{tr}(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{H}_{m,\beta^{\text{te}}}) + \frac{{\beta^{\text{te}}}^2\sigma^2}{m} + \sigma^2)
\end{aligned}$$

where (a) follows because ξ_1, ξ_2, ξ_3 are independent and have zero mean conditioned on $\mathbf{X}^{\text{in}}$ and $\mathbf{x}^{\text{out}}$. □

B Analysis for Upper Bound (Theorem 1)

B.1 Preliminaries

We first introduce some additional notations.

Definition 3 (Inner product of matrices)**.** For any two matrices $\mathbf{C}, \mathbf{D}$, the inner product of them is defined as

$$\langle \mathbf{C}, \mathbf{D}\rangle = \operatorname{tr}(\mathbf{C}^\top\mathbf{D}).$$

We will use the following property about the inner product of matrices throughout our proof.

Property 3. *If $\mathbf{C} \succeq 0$ and $\mathbf{D} \succeq \mathbf{D}'$, then we have $\langle \mathbf{C}, \mathbf{D}\rangle \geq \langle \mathbf{C}, \mathbf{D}'\rangle$.*

Definition 4 (Linear operator)**.** Let $\otimes$ denote the tensor product. Define the following linear operators on symmetric matrices:

$$\mathcal{M} = \mathbb{E}\left[\mathbf{B}^\top \otimes \mathbf{B}^\top \otimes \mathbf{B} \otimes \mathbf{B}\right] \quad \widetilde{\mathcal{M}} := \mathbf{H}_{n_1,\beta^{\text{tr}}} \otimes \mathbf{H}_{n_1,\beta^{\text{tr}}} \quad \mathcal{I} := \mathbf{I} \otimes \mathbf{I}$$

$$\mathcal{T} := \mathbf{H}_{n_1,\beta^{\text{tr}}} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{H}_{n_1,\beta^{\text{tr}}} - \alpha\mathcal{M}, \quad \widetilde{\mathcal{T}} = \mathbf{H}_{n_1,\beta^{\text{tr}}} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{H}_{n_1,\beta^{\text{tr}}} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}} \otimes \mathbf{H}_{n_1,\beta^{\text{tr}}}.$$

We next define the operation of the above linear operators on a symmetric matrix $\mathbf{A}$ as follows.

$$\mathcal{M} \circ \mathbf{A} = \mathbb{E}\left[\mathbf{B}^\top \mathbf{B}\mathbf{A}\mathbf{B}^\top \mathbf{B}\right], \quad \widetilde{\mathcal{M}} \circ \mathbf{A} = \mathbf{H}_{n_1,\beta^{\text{tr}}}\mathbf{A}\mathbf{H}_{n_1,\beta^{\text{tr}}}, \quad \mathcal{I} \circ \mathbf{A} = \mathbf{A},$$
$$\mathcal{T} \circ \mathbf{A} = \mathbf{H}_{n_1,\beta^{\text{tr}}}\mathbf{A} + \mathbf{A}\mathbf{H}_{n_1,\beta^{\text{tr}}} - \alpha\mathbb{E}\left[\mathbf{B}^\top \mathbf{B}\mathbf{A}\mathbf{B}^\top \mathbf{B}\right]$$
$$\widetilde{\mathcal{T}} \circ \mathbf{A} = \mathbf{H}_{n_1,\beta^{\text{tr}}}\mathbf{A} + \mathbf{A}\mathbf{H}_{n_1,\beta^{\text{tr}}} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}\mathbf{A}\mathbf{H}_{n_1,\beta^{\text{tr}}}.$$

Based on the above definitions, we have the following equations hold.

$$(\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{A} = \mathbb{E}\left[\left(\mathbf{I} - \alpha\mathbf{B}^\top\mathbf{B}\right)\mathbf{A}\left(\mathbf{I} - \alpha\mathbf{B}^\top\mathbf{B}\right)\right]$$
$$(\mathcal{I} - \alpha\widetilde{\mathcal{T}}) \circ \mathbf{A} = (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{A}(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}).$$

For the linear operators, we have the following technical lemma.

Lemma 3. *We call the linear operator $\mathcal{O}$ a PSD mapping, if for every symmetric PSD matrix $\mathbf{A}$, $\mathcal{O} \circ \mathbf{A}$ is also PSD matrix. Then we have:*

(i) $\mathcal{M}$, $\widetilde{\mathcal{M}}$ and $(\mathcal{M} - \widetilde{\mathcal{M}})$ are all PSD mappings.

(ii) $\widetilde{\mathcal{T}} - \mathcal{T}$, $\mathcal{I} - \alpha\mathcal{T}$ and $\mathcal{I} - \alpha\widetilde{\mathcal{T}}$ are all PSD mappings.

(iii) If $0 < \alpha < \frac{1}{\max_i\{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\}}$, then $\widetilde{\mathcal{T}}^{-1}$ exists, and is a PSD mapping.

(iv) If $0 < \alpha < \frac{1}{\max_i\{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\}}$, $\widetilde{\mathcal{T}}^{-1} \circ \mathbf{H}_{n_1,\beta^{tr}} \preceq \mathbf{I}$.

(v) If $0 < \alpha < \frac{1}{c(\beta^{tr},\mathbf{\Sigma})\,\text{tr}(\mathbf{\Sigma})}$, then $\mathcal{T}^{-1} \circ \mathbf{A}$ exists for PSD matrix $\mathbf{A}$, and $\mathcal{T}^{-1}$ is a PSD mapping.

Proof. Items (i) and (iii) directly follow from the proofs in [29, 51]. For (iv), by the existence of $\widetilde{\mathcal{T}}^{-1}$, we have

$$\begin{aligned}
\widetilde{\mathcal{T}}^{-1} \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} &= \sum_{t=0}^{\infty} \alpha(\mathcal{I} - \alpha\widetilde{\mathcal{T}})^t \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&= \sum_{t=0}^{\infty} \alpha(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^t \mathbf{H}_{n_1,\beta^{\text{tr}}}(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^t \\
&\preceq \sum_{t=0}^{\infty} \alpha(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^t \mathbf{H}_{n_1,\beta^{\text{tr}}} = \mathbf{I}.
\end{aligned}$$

For (v), for any PSD matrix $\mathbf{A}$, consider

$$\mathcal{T}^{-1} \circ \mathbf{A} = \alpha\sum_{k=0}^{\infty}(\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{A}.$$

We first show that $\sum_{k=0}^{\infty}(\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{A}$ is finite, and then it suffices to show that the trace is finite, i.e.,

$$\sum_{k=0}^{\infty} \text{tr}\left((\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{A}\right) < \infty. \tag{21}$$

Let $\mathbf{A}_k = (\mathcal{I} - \gamma\mathcal{T})^k \circ \mathbf{A}$. Combining with the definition of $\mathcal{T}$, we obtain

$$\text{tr}\left(\mathbf{A}_k\right) = \text{tr}\left(\mathbf{A}_{k-1}\right) - 2\alpha\,\text{tr}\left(\mathbf{H}_{n_1,\beta^{\text{tr}}}\mathbf{A}_{k-1}\right) + \alpha^2\,\text{tr}\left(\mathbf{A}\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{B}^\top\mathbf{B}\right]\right).$$

Letting $\mathbf{A} = \mathbf{I}$ in Proposition 6, we have $\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{B}^\top\mathbf{B}\right] \preceq c(\beta^{\text{tr}},\mathbf{\Sigma})\,\text{tr}(\mathbf{\Sigma})\mathbf{H}_{n_1,\beta^{\text{tr}}}$. Hence

$$\begin{aligned}
\text{tr}\left(\mathbf{A}_k\right) &\le \text{tr}\left(\mathbf{A}_{k-1}\right) - \left(2\alpha - \alpha^2 c(\beta^{\text{tr}},\mathbf{\Sigma})\,\text{tr}(\mathbf{\Sigma})\right)\text{tr}\left(\mathbf{H}_{n_1,\beta^{\text{tr}}}\mathbf{A}_{k-1}\right) \\
&\le \text{tr}\left((\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{A}_{k-1}\right) \qquad \text{by } \alpha < \frac{1}{c(\beta^{\text{tr}},\mathbf{\Sigma})\,\text{tr}(\mathbf{\Sigma})} \\
&\le \left(1 - \alpha\min_i\{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}\right)\text{tr}\left(\mathbf{A}_{k-1}\right).
\end{aligned}$$

If $\alpha < \frac{1}{\min_i\{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}}$, then we substitute it into eq. (21) and obtain

$$\sum_{k=0}^{\infty} \operatorname{tr}\left((\mathcal{I}-\alpha\mathcal{T})^k \circ \mathbf{A}\right) = \sum_{k=0}^{\infty} \operatorname{tr}\left(\mathbf{A}_k\right) \leq \frac{\operatorname{tr}(\mathbf{A})}{\alpha \min_i\{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}} < \infty$$

which guarantees the existence of $\mathcal{T}^{-1}$. Moreover, $\mathbf{A}_k$ is a PSD matrix for every k since $\mathcal{I}-\alpha\mathcal{T}$ is a PSD mapping. The $\mathcal{T}^{-1} \circ \mathbf{A} = \alpha \sum_{k=0}^{\infty} \mathbf{A}_k$ must be a PSD matrix, which implies that $\mathcal{T}^{-1}$ is PSD mapping. □

Property 4 (Commutity)**.** *Suppose Assumption 2 holds, then for all $n > 0$, $|\beta| < 1/\lambda_1$, $\mathbf{H}_{n,\beta}$ with different n and β commute with each other.*

B.2 Fourth Moment Upper Bound for Meta Data

In this section, we provide a technical result for the fourth moment of meta data $\mathbf{B}$, which is essential throughout the proof of our upper bound.

Proposition 6. *Suppose Assumptions 1-3 hold. Given $|\beta| < \frac{1}{\lambda_1}$, for any PSD matrix $\mathbf{A}$, we have*

$$\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{A}\mathbf{B}^\top\mathbf{B}\right] \preceq c(\beta^{tr},\mathbf{\Sigma})\mathbb{E}\left[\operatorname{tr}(\mathbf{A}\mathbf{\Sigma})\right]\mathbf{H}_{n_1,\beta^{tr}}$$

where $c(\beta,\mathbf{\Sigma}) := c_1\left(1+8|\beta|\lambda_1\sqrt{C(\beta,\mathbf{\Sigma})}\sigma_x^2 + 64\sqrt{C(\beta,\mathbf{\Sigma})}\sigma_x^4\beta^2\operatorname{tr}(\mathbf{\Sigma}^2)\right)$.

Proof. Recall that $\mathbf{B} = \frac{1}{\sqrt{n_2}}\mathbf{X}^{\text{out}}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}})$. With a slight abuse of notations, we write β^{tr} as β, $\mathbf{X}^{\text{in}}$ as $\mathbf{X}$ in this proof. First consider the case $\beta \geq 0$. By the definition of $\mathbf{B}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{A}\mathbf{B}^\top\mathbf{B}\right]\\
&= \mathbb{E}\left[(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\frac{1}{n_2}\mathbf{X}^{\text{out}\top}\mathbf{X}^{\text{out}}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{A}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\frac{1}{n_2}\mathbf{X}^{\text{out}\top}\mathbf{X}^{\text{out}}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}\mathbf{X})\right]\\
&\preceq c_1\mathbb{E}\left[\operatorname{tr}\left((\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{A}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\
&\preceq c_1\mathbb{E}\left[\operatorname{tr}\left(\mathbf{A}(\mathbf{\Sigma}+\frac{\beta^2}{n_1^2}\mathbf{X}^\top\mathbf{X}\mathbf{\Sigma}\mathbf{X}^\top\mathbf{X})\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]
\end{aligned}$$

where the second inequality follows from Assumption 1. Let $\mathbf{x}_i$ denote the i-th row of $\mathbf{X}$. Note that $\mathbf{x}_i = \Sigma^{\frac{1}{2}}\mathbf{z}_i$, where $\mathbf{z}_i$ is independent σ_x-sub-gaussian vector. For any $\mathbf{x}_{i_1},\mathbf{x}_{i_2},\mathbf{x}_{i_3},\mathbf{x}_{i_4}$, where $1 \leq i_1,i_2,i_3,i_4 \leq n_1$, we have:

$$\begin{aligned}
&\mathbb{E}\left[\operatorname{tr}(\mathbf{A}\mathbf{x}_{i_1}\mathbf{x}_{i_2}^\top\mathbf{\Sigma}\mathbf{x}_{i_3}\mathbf{x}_{i_4}^\top)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\
&= \mathbb{E}\left[\operatorname{tr}(\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{A}\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{z}_{i_1}\mathbf{z}_{i_2}^\top\mathbf{\Sigma}^2\mathbf{z}_{i_3}\mathbf{z}_{i_4}^\top)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\
&= \sum_{k,j}\mu_k\lambda_j^2\mathbb{E}\left[(\mathbf{z}_{i_4}^\top\mathbf{u}_k)(\mathbf{z}_{i_1}^\top\mathbf{u}_k)(\mathbf{z}_{i_4}^\top\mathbf{v}_j)(\mathbf{z}_{i_1}^\top\mathbf{v}_j)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\Sigma(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]
\end{aligned}$$

where the SVD of $\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{A}\mathbf{\Sigma}^{\frac{1}{2}}$ is $\sum_j \mu_j\mathbf{u}_j\mathbf{u}_j^\top$, the SVD of $\mathbf{\Sigma}$ is $\sum_j \lambda_j\mathbf{v}_j\mathbf{v}_j^\top$. For any unit vector $\mathbf{w} \in \mathbb{R}^d$, we have:

$$\begin{aligned}
&\mathbf{w}^\top\mathbb{E}\left[\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\operatorname{tr}(\mathbf{A}\mathbf{x}_{i_1}\mathbf{x}_{i_2}^\top\mathbf{\Sigma}\mathbf{x}_{i_3}\mathbf{x}_{i_4}^\top)(\mathbf{I}-\frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I}-\frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n,\beta}^{-\frac{1}{2}}\right]\mathbf{w}\\
&\leq \sum_{k,j}\mu_k\lambda_j^2\sqrt{\mathbb{E}\left[((\mathbf{z}_{i_4}^\top\mathbf{u}_k)(\mathbf{z}_{i_1}^\top\mathbf{u}_k)(\mathbf{z}_{i_4}^\top\mathbf{v}_j)(\mathbf{z}_{i_1}^\top\mathbf{v}_j)^2)\right]}\\
&\quad\times\sqrt{\mathbb{E}\left[\|\mathbf{w}^\top\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\Sigma(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\mathbf{w}\|^2\right]}\\
&\leq 64\sqrt{C(\beta,\mathbf{\Sigma})}\sigma_x^4\operatorname{tr}(A\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}^2)
\end{aligned}$$

where the first inequality follows from the Cauchy Schwarz inequality; the last inequality is due to Assumption 3 and the property of sub-Gaussian distributions [44]. Therefore,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\operatorname{tr}(\mathbf{A}\mathbf{x}_{i_1}\mathbf{x}_{i_2}^\top\boldsymbol{\Sigma}\mathbf{x}_{i_3}\mathbf{x}_{i_4}^\top)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\right]\\&\preceq 64\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^4\operatorname{tr}(A\boldsymbol{\Sigma}^2)\mathbf{I}\end{aligned}$$

which implies

$$\mathbb{E}\left[\operatorname{tr}(\mathbf{A}\mathbf{x}_{i_1}\mathbf{x}_{i_2}^\top\boldsymbol{\Sigma}\mathbf{x}_{i_3}\mathbf{x}_{i_4}^\top)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\preceq 64\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^4\operatorname{tr}(A\boldsymbol{\Sigma}^2)\mathbf{H}_{n_1,\beta}.$$

Hence,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{A}\mathbf{B}^\top\mathbf{B}\right]\\&\preceq c_1\mathbb{E}\left[\operatorname{tr}\left(\mathbf{A}(\boldsymbol{\Sigma}+64\sqrt{C}\sigma_x^4\beta^2\boldsymbol{\Sigma}\operatorname{tr}(\boldsymbol{\Sigma}^2))\right)(\mathbf{I}-\frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\right]\\&\preceq c_1(1+64\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^4\beta^2\operatorname{tr}(\boldsymbol{\Sigma}^2))\mathbb{E}\left[\operatorname{tr}(\mathbf{A}\boldsymbol{\Sigma})\right]\mathbf{H}_{n_1,\beta}.\end{aligned}$$

Now we turn to $\beta<0$, and derive

$$\begin{aligned}&\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{A}\mathbf{B}^\top\mathbf{B}\right]\\&\preceq c_1\mathbb{E}\left[\operatorname{tr}\left((\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{A}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\&=c_1\mathbb{E}\left[\operatorname{tr}\left(\mathbf{A}(\boldsymbol{\Sigma}-\underbrace{\frac{\beta}{n_1}(\mathbf{X}^\top\mathbf{X}\boldsymbol{\Sigma}+\boldsymbol{\Sigma}\mathbf{X}^\top\mathbf{X})}_{\mathbf{J}_1}+\frac{\beta^2}{n_1^2}\mathbf{X}^\top\mathbf{X}\boldsymbol{\Sigma}\mathbf{X}^\top\mathbf{X})\right)\right.\\&\left.\cdot(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right].\end{aligned}$$

We can bound the extra term $\mathbf{J}_1$ in the similar way as $\beta>0$. For any $\mathbf{x}_i$, $1\le i\le n_1$, we have

$$\begin{aligned}&\mathbb{E}\left[\operatorname{tr}\left(\mathbf{A}\mathbf{x}_i\mathbf{x}_i^\top\boldsymbol{\Sigma}\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\&=\mathbb{E}\left[\operatorname{tr}\left(\mathbf{z}_i^\top\boldsymbol{\Sigma}^{\frac{3}{2}}\mathbf{A}\boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{z}_i\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\&=\sum_k\iota_k\mathbb{E}\left[(\mathbf{z}_i^\top\boldsymbol{\kappa}_k)^2(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\end{aligned}$$

where the SVD of $\boldsymbol{\Sigma}^{\frac{3}{2}}\mathbf{A}\boldsymbol{\Sigma}^{\frac{1}{2}}$ is $\sum_k\iota_k\boldsymbol{\kappa}_k\boldsymbol{\kappa}_k^\top$. Similarly, for any unit vector $\mathbf{w}\in\mathbb{R}^d$, we can obtain

$$\begin{aligned}&\mathbf{w}^\top\mathbb{E}\left[\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\operatorname{tr}\left(\mathbf{A}\mathbf{x}_i\mathbf{x}_i^\top\boldsymbol{\Sigma}\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\right]\mathbf{w}\\&\le\sum_k\iota_k\sqrt{\mathbb{E}[(\mathbf{z}_i^\top\boldsymbol{\kappa}_k)^4]}\sqrt{\mathbb{E}[\|\mathbf{w}^\top\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n_1,\beta}^{-\frac{1}{2}}\mathbf{w}\|^2]}\\&\le 4\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^2\operatorname{tr}(A\boldsymbol{\Sigma}^2)\end{aligned}$$

which implies:

$$\mathbb{E}\left[\operatorname{tr}\left(\mathbf{A}\mathbf{x}_i\mathbf{x}_i^\top\boldsymbol{\Sigma}\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\preceq 4\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^2\operatorname{tr}(\mathbf{A}\boldsymbol{\Sigma}^2)\mathbf{H}_{n_1,\beta}.$$

Hence,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{B}^\top\mathbf{B}\mathbf{A}\mathbf{B}^\top\mathbf{B}\right]\\&\preceq c_1\mathbb{E}\left[\operatorname{tr}\left(\mathbf{A}(\boldsymbol{\Sigma}-8\beta\sqrt{C}\sigma_x^2\boldsymbol{\Sigma}^2+64\sqrt{C}\sigma_x^4\beta^2\boldsymbol{\Sigma}\operatorname{tr}(\boldsymbol{\Sigma}^2))\right)(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I}-\frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right]\\&\preceq c_1\left(1-8\beta\lambda_1\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^2+64\sqrt{C(\beta,\boldsymbol{\Sigma})}\sigma_x^4\beta^2\operatorname{tr}(\boldsymbol{\Sigma}^2)\right)\mathbb{E}\left[\operatorname{tr}(\mathbf{A}\boldsymbol{\Sigma})\right]\mathbf{H}_{n_1,\beta}.\end{aligned}$$

Together with the discussions for $\beta > 0$, we have

$$c(\beta, \mathbf{\Sigma}) = c_1(1 + 8|\beta|\lambda_1\sqrt{C(\beta, \mathbf{\Sigma})}\sigma_x^2 + 64\sqrt{C(\beta, \mathbf{\Sigma})}\sigma_x^4\beta^2 \operatorname{tr}(\mathbf{\Sigma}^2)),$$

which completes the proof. □

B.3 Bias-Variance Decomposition

We will use the bias-variance decomposition similar to theoretical studies of classic linear regression [29, 16, 51]. Consider the error at each iteration: $\boldsymbol{\varrho}_t = \boldsymbol{\omega}_t - \boldsymbol{\theta}^*$, where $\boldsymbol{\omega}_t$ is the SGD output at each iteration t. Then the update rule can be written as:

$$\boldsymbol{\varrho}_t := (\mathbf{I} - \alpha\mathbf{B}_t^\top\mathbf{B}_t)\boldsymbol{\varrho}_{t-1} + \alpha\mathbf{B}_t^\top\boldsymbol{\xi}_t$$

where $\mathbf{B}_t, \boldsymbol{\xi}_t$ are the meta data and noise at iteration t (see eqs. (14) and (18)). It is helpful to consider $\boldsymbol{\varrho}_t$ as the sum of the following two random processes:

- If there is no meta noise, the error comes from the bias:

$$\boldsymbol{\varrho}_t^{\text{bias}} := (\mathbf{I} - \alpha\mathbf{B}_t^\top\mathbf{B}_t)\boldsymbol{\varrho}_{t-1}^{\text{bias}} \quad \boldsymbol{\varrho}_t^{\text{bias}} = \boldsymbol{\varrho}_0.$$

- If the SGD trajectory starts from $\boldsymbol{\theta}^*$, the error originates from the variance:

$$\boldsymbol{\varrho}_t^{\text{var}} := (\mathbf{I} - \alpha\mathbf{B}_t^\top\mathbf{B}_t)\boldsymbol{\varrho}_{t-1}^{\text{var}} + \alpha\mathbf{B}_t^\top\boldsymbol{\xi}_t \quad \boldsymbol{\varrho}^{\text{var}} = \mathbf{0}$$

  and $\mathbb{E}[\boldsymbol{\varrho}_t^{\text{var}}] = 0$.

With slightly abused notations, we have:

$$\boldsymbol{\varrho}_t = \boldsymbol{\varrho}_t^{\text{bias}} + \boldsymbol{\varrho}_t^{\text{var}}.$$

Define the averaged output of $\boldsymbol{\varrho}_t^{\text{bias}}$, $\boldsymbol{\varrho}_t^{\text{var}}$ and $\boldsymbol{\varrho}_t$ after T iterations as:

$$\overline{\boldsymbol{\varrho}}_T^{\text{bias}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\varrho}_t^{\text{bias}}, \quad \overline{\boldsymbol{\varrho}}_T^{\text{var}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\varrho}_t^{\text{var}}, \quad \overline{\boldsymbol{\varrho}}_T = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\varrho}_t. \tag{22}$$

Similarly, we have

$$\overline{\boldsymbol{\varrho}}_T = \overline{\boldsymbol{\varrho}}_T^{\text{bias}} + \overline{\boldsymbol{\varrho}}_T^{\text{var}}.$$

Now we are ready to introduce the bias-variance decomposition for the excess risk.

Lemma 4 (Bias-variance decomposition)**.** *Following the notations in eq. (22), then the excess risk can be decomposed as*

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) \leq 2\mathcal{E}_{bias} + 2\mathcal{E}_{var}$$

where

$$\mathcal{E}_{bias} = \frac{1}{2}\langle\mathbf{H}_{m,\beta^{te}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{bias} \otimes \overline{\boldsymbol{\varrho}}_T^{bias}]\rangle, \quad \mathcal{E}_{var} = \frac{1}{2}\langle\mathbf{H}_{m,\beta^{te}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{var} \otimes \overline{\boldsymbol{\varrho}}_T^{var}]\rangle. \tag{23}$$

Proof. By Lemma 2, we have

$$\begin{aligned}
R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) &= \frac{1}{2}\langle\mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T \otimes \overline{\boldsymbol{\varrho}}_T]\rangle \\
&= \frac{1}{2}\langle\mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[(\overline{\boldsymbol{\varrho}}_T^{\text{bias}} + \overline{\boldsymbol{\varrho}}_T^{\text{var}}) \otimes (\overline{\boldsymbol{\varrho}}_T^{\text{bias}} + \overline{\boldsymbol{\varrho}}_T^{\text{var}})]\rangle \\
&\leq 2\left(\frac{1}{2}\langle\mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{bias}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{bias}}]\rangle + \frac{1}{2}\langle\mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}]\rangle\right)
\end{aligned}$$

where the last inequality follows because for vector-valued random variables $\mathbf{u}$ and $\mathbf{v}$, $\mathbb{E}\|\mathbf{u}+\mathbf{v}\|_H^2 \leq \left(\sqrt{\mathbb{E}\|\mathbf{u}\|_H^2} + \sqrt{\mathbb{E}\|\mathbf{v}\|_H^2}\right)^2$ and from Cauchy-Schwarz inequality. □

For $t = 0, 1, \cdots, T-1$, consider the following bias and variance iterates:

$$\begin{aligned} \mathbf{D}_t &= (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{D}_{t-1} \quad \text{and} \quad \mathbf{D}_0 = (\boldsymbol{\omega}_t - \boldsymbol{\theta}^*)(\boldsymbol{\omega}_t - \boldsymbol{\theta}^*)^\top \\ \mathbf{V}_t &= (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{V}_{t-1} + \alpha^2\Pi \quad \text{and} \quad \mathbf{V}_0 = \mathbf{0} \end{aligned} \tag{24}$$

where $\Pi = \mathbb{E}[\mathbf{B}^\top \boldsymbol{\xi}\boldsymbol{\xi}^\top \mathbf{B}]$. One can verify that

$$\mathbf{D}_t = \mathbb{E}\left[\boldsymbol{\varrho}_t^{\text{bias}} \otimes \boldsymbol{\varrho}_t^{\text{bias}}\right], \quad \mathbf{V}_t = \mathbb{E}\left[\boldsymbol{\varrho}_t^{\text{var}} \otimes \boldsymbol{\varrho}_t^{\text{var}}\right].$$

With such notations, we can further bound the bias and variance terms.

Lemma 5. *Following the notations in eq. (24), we have*

$$\mathcal{E}_{bias} \leq \frac{1}{\alpha T^2}\left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{tr}})^T\right)\mathbf{H}_{n_1,\beta^{tr}}^{-1}\mathbf{H}_{m,\beta^{te}}, \sum_{t=0}^{T-1}\mathbf{D}_t \right\rangle, \tag{25}$$

$$\mathcal{E}_{var} \leq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\left\langle (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{tr}})^{k-t}\mathbf{H}_{m,\beta^{te}}, \mathbf{V}_t \right\rangle. \tag{26}$$

Proof. Similar calculations have appeared in the prior works [29, 51]. However, our meta linear model contains additional terms, and hence we provide a proof here for completeness. We first have

$$\begin{aligned} \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}] &= \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=0}^{T-1}\mathbb{E}[\boldsymbol{\varrho}_t^{\text{var}} \otimes \boldsymbol{\varrho}_k^{\text{var}}] \\ &\preceq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\mathbb{E}[\boldsymbol{\varrho}_t^{\text{var}} \otimes \boldsymbol{\varrho}_k^{\text{var}}] + \mathbb{E}[\boldsymbol{\varrho}_k^{\text{var}} \otimes \boldsymbol{\varrho}_t^{\text{var}}] \end{aligned}$$

where the last inequality follows because we double count the diagonal terms $t = k$.

For $t \leq k$, $\mathbb{E}[\boldsymbol{\varrho}_k^{\text{var}} | \boldsymbol{\varrho}_t^{\text{var}}] = (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}\boldsymbol{\varrho}_t^{\text{var}}$, since $\mathbb{E}[\mathbf{B}_t^\top \boldsymbol{\xi}_t | \boldsymbol{\varrho}_{t-1}] = \mathbf{0}$. From this, we have

$$\mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}] \preceq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\mathbf{V}_t(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t} + \mathbf{V}_t(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}.$$

Substituting the above inequality into $\frac{1}{2}\langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}]\rangle$, we obtain:

$$\begin{aligned} \mathcal{E}_{\text{var}} &= \frac{1}{2}\langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}]\rangle \\ &\leq \frac{1}{2T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}\rangle + \langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}\rangle \\ &= \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\langle (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}\mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t\rangle \end{aligned}$$

where the last inequality follows from Assumption 2 that F and $\boldsymbol{\Sigma}$ commute, and hence $\mathbf{H}_{m,\beta^{\text{te}}}$ and $\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}$ commute.

For the bias term, similarly we have:

$$\mathcal{E}_{\text{bias}} \leq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\langle (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}\mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{D}_t\rangle \tag{27}$$

$$= \frac{1}{\alpha T^2}\sum_{t=0}^{T-1}\langle \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t}\right)\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1}\mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{D}_t\rangle \tag{28}$$

$$\leq \frac{1}{\alpha T^2}\langle \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T}\right)\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1}\mathbf{H}_{m,\beta^{\text{te}}}, \sum_{t=0}^{T-1}\mathbf{D}_t\rangle \tag{29}$$

which completes the proof. □

B.4 Bounding the Bias

Now we start to bound the bias term. By Lemma 5, we focus on bounding the summation of $\mathbf{D}_t$, i.e. $\sum_{t=0}^{T-1} \mathbf{D}_t$. Consider $\mathbf{S}_t := \sum_{k=0}^{t-1} \mathbf{D}_k$, and the following lemma shows the properties of $\mathbf{S}_t$

Lemma 6. *$\mathbf{S}_t$ satisfies the recursion form:*

$$\mathbf{S}_t = (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{S}_{t-1} + \mathbf{D}_0.$$

Moreover, if $\alpha < \frac{1}{c(\beta^{tr}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})}$, then we have:

$$\mathbf{D}_0 = \mathbf{S}_0 \preceq \mathbf{S}_1 \preceq \cdots \preceq \mathbf{S}_\infty$$

where $\mathbf{S}_\infty := \sum_{k=0}^{\infty} (\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{D}_0 = \alpha^{-1}\mathcal{T}^{-1} \circ \mathbf{D}_0$.

Proof. By eq. (24), we have

$$\begin{aligned}
\mathbf{S}_t &= \sum_{k=0}^{t-1} \mathbf{D}_k = \sum_{k=0}^{t-1} (\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{D}_0 \\
&= \mathbf{D}_0 + (\mathcal{I} - \alpha\mathcal{T}) \circ \left(\sum_{k=0}^{t-2} (\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{D}_0 \right) \\
&= \mathbf{D}_0 + (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{S}_{t-1}.
\end{aligned}$$

By Lemma 3, $(\mathcal{I} - \alpha\mathcal{T})$ is PSD mapping, and hence $\mathbf{D}_t = (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{D}_{t-1}$ is a PSD matirx for every t, which implies $\mathbf{S}_{t-1} \preceq \mathbf{S}_{t-1} + \mathbf{D}_t = \mathbf{S}_t$. The form of $\mathbf{S}_\infty$ can be directly obtained by Lemma 3. □

Then we can decompose $\mathbf{S}_t$ as follows:

$$\begin{aligned}
\mathbf{S}_t &= \mathbf{D}_0 + (\mathcal{I} - \alpha\widetilde{\mathcal{T}}) \circ \mathbf{S}_{t-1} + \alpha(\widetilde{\mathcal{T}} - \mathcal{T}) \circ \mathbf{S}_{t-1} \\
&= \mathbf{D}_0 + (\mathcal{I} - \alpha\widetilde{\mathcal{T}}) \circ \mathbf{S}_{t-1} + \alpha^2(\mathcal{M} - \widetilde{\mathcal{M}}) \circ \mathbf{S}_{t-1} \\
&\preceq \mathbf{D}_0 + (\mathcal{I} - \alpha\widetilde{\mathcal{T}}) \circ \mathbf{S}_{t-1} + \alpha^2\mathcal{M} \circ \mathbf{S}_T \\
&= \sum_{k=0}^{t-1} (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^k \circ (\mathbf{D}_0 + \alpha^2\mathcal{M} \circ \mathbf{S}_T)
\end{aligned} \tag{30}$$

where the inequality follows because $\mathbf{S}_t \preceq \mathbf{S}_T$ for any $t \leq T$. Therefore, it is crucial to understand $\mathcal{M} \circ \mathbf{S}_T$.

Lemma 7. *For any symmetric matrix $\mathbf{A}$, if $\alpha < \frac{1}{c(\beta^{tr}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})}$, it holds that*

$$\mathcal{M} \circ \mathcal{T}^{-1} \circ \mathbf{A} \preceq \frac{c(\beta^{tr}, \mathbf{\Sigma}) \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{tr}}^{-1}\mathbf{A}\right)}{1 - \alpha c(\beta^{tr}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \cdot \mathbf{H}_{n_1,\beta^{tr}}.$$

Proof. Denote $\mathbf{C} = \mathcal{T}^{-1} \circ \mathbf{A}$. Recalling $\widetilde{\mathcal{T}} = \mathcal{T} + \alpha\mathcal{M} - \alpha\widetilde{\mathcal{M}}$, we have

$$\begin{aligned}
\widetilde{\mathcal{T}} \circ \mathbf{C} &= \mathcal{T} \circ \mathbf{C} + \alpha\mathcal{M} \circ \mathbf{C} - \alpha\widetilde{\mathcal{M}} \circ \mathbf{C} \\
&\preceq \mathbf{A} + \alpha\mathcal{M} \circ \mathbf{C}.
\end{aligned}$$

Recalling that $\widetilde{\mathcal{T}}^{-1}$ exists and is a PSD mapping, we then have

$$\begin{aligned}
\mathcal{M} \circ \mathbf{C} &\preceq \alpha\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1} \circ \mathcal{M} \circ \mathbf{C} + \mathcal{M} \circ \widetilde{\mathcal{T}}^{-1} \circ \mathbf{A} \\
&\preceq \sum_{k=0}^{\infty} (\alpha\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1})^k \circ (\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1} \circ \mathbf{A}).
\end{aligned} \tag{31}$$

By Proposition 6, we have $\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1} \circ \mathbf{A} \preceq \underbrace{c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}\widetilde{\mathcal{T}}^{-1} \circ \mathbf{A})}_{J_2} \mathbf{H}_{n_1,\beta^{\text{tr}}}$. Substituting back into eq. (31), we obtain:

$$\begin{aligned}
\sum_{k=0}^{\infty} (\alpha\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1})^k \circ (\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1} \circ \mathbf{A}) &\preceq \sum_{k=0}^{\infty} (\alpha\mathcal{M} \circ \widetilde{\mathcal{T}}^{-1})^k \circ (J_2 \mathbf{H}_{n_1,\beta^{\text{tr}}}) \\
&\preceq J_2 \sum_{k=0}^{\infty} (\alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}))^k \mathbf{H}_{n_1,\beta^{\text{tr}}} \preceq \frac{J_2}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \mathbf{H}_{n_1,\beta^{\text{tr}}}
\end{aligned}$$

where the second inequality follows since $\widetilde{\mathcal{T}}^{-1} \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} \preceq \mathbf{I}$ (Lemma 3) and $\mathcal{M} \circ \mathbf{I} \preceq c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}) \mathbf{H}_{n_1,\beta^{\text{tr}}}$ (Proposition 6).

Finally, we bound J_2 as follows:

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{\Sigma}\widetilde{\mathcal{T}}^{-1} \circ \mathbf{A}\right) &= \alpha \operatorname{tr}\left(\sum_{k=0}^{\infty} \mathbf{\Sigma}(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k \mathbf{A} (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k\right) \\
&= \alpha \operatorname{tr}\left(\sum_{k=0}^{\infty} \mathbf{\Sigma}(\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{2k} \mathbf{A}\right) \\
&= \operatorname{tr}\left(\mathbf{\Sigma}\left(2\mathbf{H}_{n_1,\beta^{\text{tr}}} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}^2\right)^{-1} \mathbf{A}\right) \\
&\leq \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{A}\right)
\end{aligned}$$

where the second equality follows because $\mathbf{\Sigma}$ and $\mathbf{H}_{n_1,\beta^{\text{tr}}}$ commute, and the last inequality holds since $\alpha < \frac{1}{\max_i\{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}}$. Putting all these results together completes the proof. □

Lemma 8 (Bounding $\mathcal{M} \circ \mathbf{S}_T$)**.**

$$\mathcal{M} \circ \mathbf{S}_T \preceq \frac{c(\beta^{tr}, \mathbf{\Sigma}) \cdot \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{tr}}^{-1}\left[\mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T\right] \circ \mathbf{D}_0\right)}{\alpha(1 - c(\beta^{tr}, \mathbf{\Sigma})\alpha \operatorname{tr}(\mathbf{\Sigma}))} \cdot \mathbf{H}_{n_1,\beta^{tr}}.$$

Proof. $\mathbf{S}_T$ can be further derived as follows:

$$\mathbf{S}_T = \sum_{k=0}^{T-1} (\mathcal{I} - \alpha\mathcal{T})^k \circ \mathbf{D}_0 = \alpha^{-1}\mathcal{T}^{-1} \circ \left[\mathcal{I} - (\mathcal{I} - \alpha\mathcal{T})^T\right] \circ \mathbf{D}_0.$$

Since $\widetilde{\mathcal{T}} - \mathcal{T}$ is a PSD mapping by Lemma 3, we have $\mathcal{I} - \alpha\widetilde{\mathcal{T}} \leq \mathcal{I} - \alpha\mathcal{T}$. Hence $\mathcal{I} - (\mathcal{I} - \alpha\mathcal{T})^T \preceq \mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T$. Combining with the fact that $\mathcal{T}^{-1}$ is also a PSD mapping, we have:

$$\mathbf{S}_T \preceq \alpha^{-1}\mathcal{T}^{-1} \circ \left[\mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T\right] \circ \mathbf{D}_0.$$

Letting $\mathbf{A} = \left[\mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T\right] \circ \mathbf{D}_0$ in Lemma 7, we obtain:

$$\begin{aligned}
\mathcal{M} \circ \mathbf{S}_T &\preceq \alpha^{-1}\mathcal{M} \circ \mathcal{T}^{-1} \circ \left[\mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T\right] \circ \mathbf{D}_0 \\
&\preceq \frac{c(\beta^{\text{tr}}, \mathbf{\Sigma}) \cdot \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1}\left[\mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T\right] \circ \mathbf{D}_0\right)}{\alpha(1 - c(\beta^{\text{tr}}, \mathbf{\Sigma})\alpha \operatorname{tr}(\mathbf{\Sigma}))} \cdot \mathbf{H}_{n_1,\beta^{\text{tr}}}.
\end{aligned}$$

□

Now we are ready to derive the upper bound on the bias term.

Lemma 9 (Bounding the bias). *If $\alpha < \frac{1}{c(\beta^{tr},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma})}$, for sufficiently large n_1, s.t. $\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) > 0$, $\forall i$, then we have*

$$\mathcal{E}_{bias} \leq \sum_i \left(\frac{1}{\alpha^2 T^2} \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) \geq \frac{1}{\alpha T}} + \mu_i^2(\mathbf{H}_{n_1,\beta^{tr}}) \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) < \frac{1}{\alpha T}} \right) \frac{\omega_i^2 \mu_i(\mathbf{H}_{m,\beta^{te}})}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})^2}$$
$$+ \frac{2c(\beta^{tr},\mathbf{\Sigma})}{T\alpha\left(1 - c(\beta^{tr},\mathbf{\Sigma})\alpha \operatorname{tr}(\mathbf{\Sigma})\right)} \sum_i \left(\frac{1}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) \geq \frac{1}{\alpha T}} + T\alpha \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) < \frac{1}{\alpha T}} \right) \cdot \lambda_i \omega_i^2$$
$$\times \sum_i \left(\frac{1}{T} \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) \geq \frac{1}{\alpha T}} + T\alpha^2 \mu_i(\mathbf{H}_{n_1,\beta^{tr}})^2 \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) < \frac{1}{\alpha T}} \right) \cdot \frac{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})}{\mu_i(\mathbf{H}_{m,\beta^{te}})}.$$

Proof. Applying Lemma 8 to eq. (30), we can obtain:

$$\mathbf{S}_t \preceq \sum_{k=0}^{t-1} (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^k \circ \left(\frac{\alpha c(\beta^{\text{tr}},\mathbf{\Sigma}) \cdot \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \left[\mathcal{I} - (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^T \right] \circ \mathbf{D}_0 \right)}{1 - c(\beta,\Sigma)\alpha \operatorname{tr}(\Sigma)} \cdot \mathbf{H}_{n_1,\beta^{\text{tr}}} + \mathbf{D}_0 \right)$$
$$= \sum_{k=0}^{t-1} (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k \cdot$$
$$\left(\underbrace{\frac{\alpha c(\beta^{\text{tr}},\mathbf{\Sigma}) \cdot \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} (\mathbf{D}_0 - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^T \mathbf{D}_0 (\mathbf{I} - \alpha\mathbf{H}_n)^T) \right)}{1 - c(\beta^{\text{tr}},\mathbf{\Sigma})\alpha \operatorname{tr}(\mathbf{\Sigma})} \cdot \mathbf{H}_{n_1,\beta^{\text{tr}}}}_{\mathbf{G}_1} + \underbrace{\mathbf{D}_0}_{\mathbf{G}_2} \right)$$
$$\cdot (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k.$$

Letting $t = T$, and substituting the upper bound of $\mathbf{S}_T$ into the bias term in Lemma 5, we obtain:

$$\mathcal{E}_{\text{bias}} \leq \frac{1}{\alpha T^2} \sum_{k=0}^{T-1} \left\langle ((\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{2k} - (\mathbf{I} - \alpha\mathbf{H}_{n,\beta})^{T+2k}) \mathbf{H}_{n,\beta}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{G}_1 + \mathbf{G}_2 \right\rangle$$
$$\leq \frac{1}{\alpha T^2} \sum_{k=0}^{T-1} \left\langle ((\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T+k}) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{G}_1 + \mathbf{G}_2 \right\rangle.$$

We first consider

$$d_1 = \frac{1}{\alpha T^2} \sum_{k=0}^{T-1} \left\langle ((\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T+k}) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{G}_1 \right\rangle.$$

Since $\mathbf{H}_{n_1,\beta^{\text{tr}}}$, $\mathbf{H}_{m,\beta^{\text{te}}}$ and $\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}$ commute, we have

$$d_1 = \frac{c(\beta^{\text{tr}},\mathbf{\Sigma}) \cdot \operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} (\mathbf{D}_0 - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^T \mathbf{D}_0 (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^T) \right)}{(1 - c(\beta^{\text{tr}},\mathbf{\Sigma})\alpha \operatorname{tr}(\mathbf{\Sigma}))T^2}$$
$$\times \sum_{k=0}^{T-1} \left\langle \left((\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T+k} \right), \mathbf{H}_{m,\beta^{\text{te}}} \right\rangle.$$

For the first term, since $\mathbf{\Sigma}$, $\mathbf{H}_{n_1,\beta^{\text{tr}}}$ and $\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}$ can be diagonalized simultaneously, considering the eigen-decompositions under the basis of $\mathbf{\Sigma}$ and recalling $\mathbf{\Sigma} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$, we have:

$$\operatorname{tr}\left(\mathbf{\Sigma}\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} [\mathbf{D}_0 - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^T \mathbf{D}_0 (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^T] \right)$$
$$= \sum_i \left(1 - (1 - \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{2T} \right) \cdot (\langle \mathbf{w}_0 - \mathbf{w}^*, \mathbf{v}_i \rangle)^2 \frac{\lambda_i}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}$$
$$\leq 2 \sum_i \left(\mathbf{1}_{\lambda_i(\mathbf{H}_{n,\beta}) \geq \frac{1}{\alpha T}} + T\alpha\mu_i(\mathbf{H}_{n,\beta}) \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \frac{1}{\alpha T}} \right) \cdot (\langle \mathbf{w}_0 - \mathbf{w}^*, \mathbf{v}_i \rangle)^2 \frac{\lambda_i}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}$$

where the last inequality holds since $1-(1-\alpha x)^{2T} \le \min\{2, 2T\alpha x\}$.

For the second term, similarly, $\mathbf{H}_{m,\beta^{\text{te}}}$ and $\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}}$ can be diagonalized simultaneously. We then have

$$\begin{aligned}
&\sum_{k=0}^{T-1}\left\langle\left((\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k-(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T+k}\right),\mathbf{H}_{m,\beta^{\text{te}}}\right\rangle\\
&\le\sum_{k=0}^{T-1}\sum_i[(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^k-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T+k}]\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})\\
&=\frac{1}{\alpha}\sum_i[1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^T]^2\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\
&\le\frac{1}{\alpha}\sum_i\left(\mathbf{1}_{\lambda_i(\mathbf{H}_{n,\beta})\ge\frac{1}{\alpha T}}+T^2\alpha^2\lambda_i(\mathbf{H}_{n,\beta})\mathbf{1}_{\lambda_i(\mathbf{H}_{n,\beta})<\frac{1}{\alpha T}}\right)\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}.
\end{aligned}$$

Now we turn to:

$$d_2=\frac{1}{\alpha T^2}\sum_{k=0}^{T-1}\left\langle((\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k-(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T+k})\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1}\mathbf{H}_{m,\beta^{\text{te}}},\mathbf{G}_2\right\rangle.$$

Considering the orthogonal decompositions of $\mathbf{H}_{m,\beta^{\text{te}}}$ and $\mathbf{H}_{n_1,\beta^{\text{tr}}}$ under $\mathbf{V}$, $\mathbf{H}_{n_1,\beta^{\text{tr}}}=\mathbf{V}\mathbf{\Lambda}_1\mathbf{V}^\top$, $\mathbf{H}_{m,\beta^{\text{te}}}=\mathbf{V}\mathbf{\Lambda}_2\mathbf{V}^\top$, where the diagonal entries of $\mathbf{\Lambda}_1$ are $\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})$ (and $\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})$ for $\mathbf{\Lambda}_2$). Then we have:

$$\begin{aligned}
d_2&=\frac{1}{\alpha T^2}\sum_{k=0}^{T-1}\left\langle\underbrace{\left((\mathbf{I}-\alpha\mathbf{\Lambda}_1)^k-(\mathbf{I}-\alpha\mathbf{\Lambda}_1)^{T+k}\right)\mathbf{\Lambda}_1^{-1}\mathbf{\Lambda}_2}_{\mathbf{J}_3},\mathbf{V}^\top\mathbf{D}_0\mathbf{V}\right\rangle\\
&=\frac{1}{\alpha T^2}\sum_{k=0}^{T-1}\sum_i\left[(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^k-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T+k}\right]\frac{\omega_i^2\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\
&=\frac{1}{\alpha^2T^2}\sum_i\left[1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^T\right]^2\frac{\omega_i^2\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\
&\le\frac{1}{\alpha^2T^2}\sum_i\left(\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\ge\frac{1}{\alpha T}}+\alpha^2T^2\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\frac{\omega_i^2\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})}
\end{aligned}$$

where $\omega_i=\langle\boldsymbol{\omega}_0-\boldsymbol{\theta}^*,\mathbf{v}_i\rangle$ is the diagonal entry of $\mathbf{V}^\top\mathbf{D}_0\mathbf{V}$ and the second equality holds since $\mathbf{J}_3$ is a diagonal matrix. □

B.5 Bounding the Variance

Note that the noisy part $\Pi=\mathbb{E}[\mathbf{B}^\top\boldsymbol{\xi}\boldsymbol{\xi}^\top\mathbf{B}]$ in eq. (24) is important in the variance iterates. In order to analyze the variance term, we first understand the role of Π by the following lemma.

Lemma 10 (Bounding the noise)**.**

$$\Pi=\mathbb{E}[\mathbf{B}^\top\boldsymbol{\xi}\boldsymbol{\xi}^\top\mathbf{B}]\preceq f(\beta^{tr},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})\mathbf{H}_{n_1,\beta^{tr}}$$

where $f(\beta,n,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})=[c(\beta,\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}_{\boldsymbol{\theta}}\mathbf{\Sigma})+4c_1\sigma^2\sigma_x^2\beta^2\sqrt{C(\beta,\mathbf{\Sigma})}\operatorname{tr}(\mathbf{\Sigma}^2)+\sigma^2/n]$.

Proof. With a slight abuse of notations, we write β^{tr} as β in this proof. By definition of meta data and noise, we have

$$\begin{aligned}
\Pi&=\mathbb{E}[\mathbf{B}^\top\boldsymbol{\xi}\boldsymbol{\xi}^\top\mathbf{B}]\\
&=\frac{\sigma^2}{n_2}\mathbf{H}_{n_1,\beta}+\mathbb{E}[\mathbf{B}^\top\mathbf{B}\Sigma_{\boldsymbol{\theta}}\mathbf{B}^\top\mathbf{B}]+\sigma^2\cdot\frac{\beta^2}{n_2n_1^2}\mathbb{E}[\mathbf{B}^\top\mathbf{X}^{\text{out}}\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}}\mathbf{X}^{\text{out}\top}\mathbf{B}].
\end{aligned}$$

The second term can be directly bounded by Proposition 6:

$$\mathbb{E}[\mathbf{B}^\top\mathbf{B}\Sigma_{\boldsymbol{\theta}}\mathbf{B}^\top\mathbf{B}]\preceq c(\beta,\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}_{\boldsymbol{\theta}}\mathbf{\Sigma})\mathbf{H}_{n_1,\beta}.$$

For the third term, we utilize the technique similar to Proposition 6, and by Assumption 1, we have:

$$\begin{aligned}
&\sigma^2 \cdot \frac{\beta^2}{n_2 n_1^2} \mathbb{E}\left[\mathbf{B}^\top \mathbf{X}^{\text{out}} \mathbf{X}^{\text{in}\top} \mathbf{X}^{\text{in}} \mathbf{X}^{\text{out}\top} \mathbf{B}\right] \\
&\preceq \sigma^2 c_1 \cdot \frac{\beta^2}{n_1^2} \mathbb{E}\left[\operatorname{tr}(\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}}\Sigma)(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}})\mathbf{\Sigma}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}})\right].
\end{aligned}$$

Following the analysis for $\mathbf{J}_1$ in the proof of Proposition 6, and letting $\mathbf{A} = \mathbf{I}$, we obtain:

$$\frac{1}{n_1^2}\mathbb{E}\left[\operatorname{tr}(\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}}\Sigma)(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}})\mathbf{\Sigma}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^{\text{in}\top}\mathbf{X}^{\text{in}})\right] \preceq 4\sqrt{C(\beta, \mathbf{\Sigma})}\sigma_x^2 \operatorname{tr}(\mathbf{\Sigma}^2)\mathbf{H}_{n_1,\beta}.$$

Putting all these results together completes the proof. □

Lemma 11 (Property of $\mathbf{V}_t$)**.** *If the stepsize satisfies $\alpha < \frac{1}{c(\beta^{tr}, \mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma})}$, it holds that*

$$\mathbf{0} = \mathbf{V}_0 \preceq \mathbf{V}_1 \preceq \cdots \preceq \mathbf{V}_\infty \preceq \frac{\alpha f(\beta^{tr}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{1 - \alpha c(\beta^{tr}, \mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma})}\mathbf{I}.$$

Proof. Similar calculations has appeared in prior works [29, 51]. However, our analysis of the meta linear model needs to handle the complicated meta noise, and hence we provide a proof here for completeness.

We first show that $\mathbf{V}_{t-1} \preceq \mathbf{V}_t$. By recursion:

$$\begin{aligned}
\mathbf{V}_t &= (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{V}_{t-1} + \alpha^2 \Pi \\
&\overset{(a)}{=} \alpha^2 \sum_{k=0}^{t-1} (\mathcal{I} - \alpha\mathcal{T})^k \circ \Pi \\
&= \mathbf{V}_{t-1} + \alpha^2 (\mathcal{I} - \alpha\mathcal{T})^{t-1} \circ \Pi \\
&\overset{(b)}{\succeq} \mathbf{V}_{t-1}
\end{aligned}$$

where (a) holds by solving the recursion and (b) follows because $\mathcal{I} - \alpha\mathcal{T}$ is a PSD mapping.

The existence of $\mathbf{V}_\infty$ can be shown in the way similar to the proof of Lemma 3. We first have

$$\mathbf{V}_t = \alpha^2 \sum_{k=0}^{t-1} (\mathcal{I} - \alpha\mathcal{T})^k \circ \Pi \preceq \alpha^2 \sum_{k=0}^{\infty} \underbrace{(\mathcal{I} - \alpha\mathcal{T})^k \circ \Pi}_{\mathbf{A}_k}.$$

By previous analysis in Lemma 3 , if $\alpha < \frac{1}{c(\beta^{\text{tr}}, \mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma})}$, we have

$$\operatorname{tr}(\mathbf{A}_k) \leq \left(1 - \alpha \min_i \{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}\right) \operatorname{tr}(\mathbf{A}_{t-1}).$$

Therefore,

$$\operatorname{tr}(\mathbf{V}_t) \leq \alpha^2 \sum_{k=0}^{\infty} \operatorname{tr}(\mathbf{A}_k) \leq \frac{\alpha \operatorname{tr}(\Pi)}{\min_i \{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}} < \infty.$$

The trace of $\mathbf{V}_t$ is uniformly bounded from above, which indicates that $\mathbf{V}_\infty$ exists.

Finally, we bound $\mathbf{V}_\infty$. Note that $\mathbf{V}_\infty$ is the solution to:

$$\mathbf{V}_\infty = (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{V}_\infty + \alpha^2 \Pi.$$

Then we can write $\mathbf{V}_\infty$ as $\mathbf{V}_\infty = \mathcal{T}^{-1} \circ \alpha\Pi$. Following the analysis in the proof of Lemma 7, we have:

$$\begin{aligned}
\widetilde{\mathcal{T}} \circ \mathbf{V}_\infty &= \widetilde{\mathcal{T}} \circ \mathcal{T}^{-1} \circ \alpha\Pi \\
&\preceq \alpha\Pi + \alpha\mathcal{M} \circ \mathbf{V}_\infty \\
&\preceq \alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})\mathbf{H}_{n_1,\beta^{\text{tr}}} + \alpha\mathcal{M} \circ \mathbf{V}_\infty
\end{aligned}$$

where the last inequality follows from Lemma 10. Applying $\widetilde{\mathcal{T}}^{-1}$, which exists and is a PSD mapping, to the both sides, we have

$$\begin{aligned}
\mathbf{V}_\infty &\preceq \alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}}) \cdot \widetilde{\mathcal{T}}^{-1} \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} + \alpha \widetilde{\mathcal{T}}^{-1} \circ \mathcal{M} \circ \mathbf{V}_\infty \\
&\overset{(a)}{\preceq} \alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}}) \cdot \sum_{t=0}^{\infty} \left(\alpha \widetilde{\mathcal{T}}^{-1} \circ \mathcal{M}\right)^t \circ \widetilde{\mathcal{T}}^{-1} \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\overset{(b)}{\preceq} \alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}}) \sum_{t=0}^{\infty} (\alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}))^t \mathbf{I} \\
&= \frac{\alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \mathbf{I}
\end{aligned}$$

where (a) holds by directly solving the recursion; (b) follows from the fact that $\widetilde{\mathcal{T}}^{-1} \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} \preceq \mathbf{I}$ from Lemma 3 and $\mathcal{M} \circ \mathbf{I} \preceq c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}) \mathbf{H}_{n_1,\beta^{\text{tr}}}$ by letting $\mathbf{A} = \mathbf{I}$ in Proposition 6. □

Now we are ready to provide the upper bound on the variance term.

Lemma 12 (Bounding the Variance)**.** *If $\alpha < \frac{1}{c(\beta^{tr}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})}$, for sufficiently large n_1, s.t. $\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) > 0$, $\forall i$, then we have*

$$\begin{aligned}
\mathcal{E}_{var} \leq & \frac{f(\beta^{tr}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1 - \alpha c(\beta^{tr}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}))} \\
& \times \sum_i \left(\frac{1}{T} \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) \geq \frac{1}{\alpha T}} + T\alpha^2 \mu_i^2(\mathbf{H}_{n_1,\beta^{tr}}) \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) < \frac{1}{\alpha T}} \right) \frac{\mu_i(\mathbf{H}_{m,\beta^{te}})}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})}.
\end{aligned}$$

Proof. Recall

$$\begin{aligned}
\mathbf{V}_t &= (\mathcal{I} - \alpha \mathcal{T}) \circ \mathbf{V}_{t-1} + \alpha^2 \Pi \\
&= (\mathcal{I} - \alpha \widetilde{\mathcal{T}}) \circ \mathbf{V}_{t-1} + \alpha^2 (\mathcal{M} - \widetilde{\mathcal{M}}) \circ \mathbf{V}_{t-1} + \alpha^2 \Pi \\
&\preceq (\mathcal{I} - \alpha \widetilde{\mathcal{T}}) \circ \mathbf{V}_{t-1} + \alpha^2 \mathcal{M} \circ \mathbf{V}_{t-1} + \alpha^2 \Pi. \qquad (32)
\end{aligned}$$

By the uniform bound on $\mathbf{V}_t$ and $\mathcal{M}$ is a PSD mapping, we have:

$$\begin{aligned}
\mathcal{M} \circ \mathbf{V}_t &\preceq \mathcal{M} \circ \mathbf{V}_\infty \\
&\overset{(a)}{\preceq} \mathcal{M} \circ \frac{\alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \mathbf{I} \\
&\overset{(b)}{\preceq} \frac{\alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}}) c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \cdot \mathbf{H}_{n_1,\beta^{\text{tr}}}
\end{aligned}$$

where (a) directly follows from Lemma 11; (b) holds because $\mathcal{M} \circ \mathbf{I} \preceq c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma}) \mathbf{H}_{n_1,\beta^{\text{tr}}}$ (letting $\mathbf{A} = \mathbf{I}$ in Proposition 6). Substituting it back into eq. (32), we have:

$$\begin{aligned}
\mathbf{V}_t &\preceq (\mathcal{I} - \alpha \widetilde{\mathcal{T}}) \circ \mathbf{V}_{t-1} + \alpha^2 \frac{\alpha f c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \cdot \mathbf{H}_{n_1,\beta^{\text{tr}}} + \alpha^2 f \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&= (\mathcal{I} - \alpha \widetilde{\mathcal{T}}) \circ \mathbf{V}_{t-1} + \frac{\alpha^2 f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\overset{(a)}{=} \frac{\alpha^2 f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} \sum_{k=0}^{t-1} (\mathbf{I} - \alpha \widetilde{\mathcal{T}})^k \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\overset{(b)}{\preceq} \frac{\alpha f(\beta^{\text{tr}}, n_2, \sigma, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{1 - \alpha c(\beta^{\text{tr}}, \mathbf{\Sigma}) \operatorname{tr}(\mathbf{\Sigma})} (\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n,\beta})^t)
\end{aligned}$$

where (a) holds by solving the recursion and (b) is due to the fact that

$$\begin{aligned}\sum_{k=0}^{t-1}(\mathbf{I}-\alpha\widetilde{\mathcal{T}})^k\circ\mathbf{H}_{n_1,\beta^{\text{tr}}} &= \sum_{k=0}^{t-1}(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k\mathbf{H}_{n_1,\beta^{\text{tr}}}(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k\\ &\preceq \sum_{k=0}^{t-1}(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k\mathbf{H}_{n_1,\beta^{\text{tr}}}\\ &= \frac{1}{\alpha}[\mathbf{I}-(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^t].\end{aligned}$$

Substituting the bound for $\mathbf{V}_t$ back into the variance term in Lemma 5, we have

$$\begin{aligned}\mathcal{E}_{\text{var}} &\leq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{k=t}^{T-1}\left\langle(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}\mathbf{H}_{m,\beta^{\text{te}}},\mathbf{V}_t\right\rangle\\ &= \frac{1}{\alpha T^2}\sum_{t=0}^{T-1}\left\langle(\mathbf{I}-(\mathbf{I}-\alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t})\mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1}\mathbf{H}_{m,\beta^{\text{te}}},\mathbf{V}_t\right\rangle\\ &\leq \frac{f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))T^2}\sum_{t=0}^{T-1}\left\langle\mathbf{I}-(\mathbf{I}-\alpha\mathbf{H}_{n,\beta})^{T-t},\left(\mathbf{I}-(\mathbf{I}-\alpha\mathbf{H}_{n,\beta})^t\right)\mathbf{H}_{n,\beta}^{-1}\mathbf{H}_{m,\eta}\right\rangle.\end{aligned}$$

Simultaneously diagonalizing $\mathbf{H}_{n_1,\beta^{\text{tr}}}$ and $\mathbf{H}_{m,\beta^{\text{te}}}$ as the analysis in Lemma 9, we have

$$\begin{aligned}\mathcal{E}_{\text{var}} \leq& \frac{f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))T^2}\\ &\cdot\sum_i\sum_{t=0}^{T-1}\left(1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T-t}\right)\left(1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^t\right)\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\ \leq& \frac{f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))T^2}\\ &\cdot\sum_i\sum_{t=0}^{T-1}\left(1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T}\right)\left(1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T}\right)\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\ =& \frac{f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))T}\sum_i\left(1-(1-\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T}\right)^2\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\ \leq& \frac{f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))T}\sum_i\left(\min\{1,\alpha T\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\}\right)^2\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\\ \leq& \frac{f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))}\\ &\cdot\sum_i\left(\frac{1}{T}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}}+T\alpha^2\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{1}_{\mu i(\mathbf{H}_{n,\beta})<\frac{1}{\alpha T}}\right)\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})},\end{aligned}$$

which completes the proof. □

B.6 Proof of Theorem 1

Theorem 5 (Theorem 1 Restated)**.** *Let $\omega_i = \langle\boldsymbol{\omega}_0-\boldsymbol{\theta}^*,\mathbf{v}_i\rangle$. If $|\beta^{tr}|,|\beta^{te}| < 1/\lambda_1$, n_1 is large ensuring that $\mu_i(\mathbf{H}_{n_1,\beta^{tr}})>0$, $\forall i$ and $\alpha < 1/\left(c(\beta^{tr},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma})\right)$, then the meta excess risk $R(\overline{\boldsymbol{\omega}}_T,\beta^{te})$ is bounded above as follows*

$$R(\overline{\boldsymbol{\omega}}_T,\beta^{te}) \leq \textit{Bias} + \textit{Var}$$

where

$$
\begin{aligned}
\textit{Bias} &= \frac{2}{\alpha^2 T}\sum_i \Xi_i \frac{\omega_i^2}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} \\
\textit{Var} &= \frac{2}{(1-\alpha c(\beta^{tr},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))}\left(\sum_i \Xi_i\right) \\
&\times[f(\beta^{tr},n_2,\sigma,\mathbf{\Sigma}_{\boldsymbol{\theta}},\mathbf{\Sigma}) + 2c(\beta^{tr},\mathbf{\Sigma})\textstyle\sum_i\left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\geq\frac{1}{\alpha T}}}{T\alpha\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})<\frac{1}{\alpha T}}\right)\lambda_i\omega_i^2].
\end{aligned}
$$

Proof. By Lemma 4, we have

$$
R(\overline{\boldsymbol{\omega}}_T,\beta^{\text{te}}) \leq 2\mathcal{E}_{\text{bias}} + 2\mathcal{E}_{\text{var}}.
$$

Using Lemma 9 to bound $\mathcal{E}_{\text{bias}}$, and Lemma 12 to bound $\mathcal{E}_{\text{var}}$, we have

$$
\begin{aligned}
&R(\overline{\boldsymbol{\omega}}_T,\beta^{\text{te}}) \\
&\leq \frac{2f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma},\mathbf{\Sigma}_{\boldsymbol{\theta}})}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))} \\
&\times\sum_i\left(\frac{1}{T}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}} + T\alpha^2\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \\
&+\frac{4c(\beta^{\text{tr}},\mathbf{\Sigma})}{T\alpha(1-c(\beta^{\text{tr}},\mathbf{\Sigma})\alpha\operatorname{tr}(\mathbf{\Sigma}))}\sum_i\left(\frac{1}{T}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}} + T\alpha^2\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^2\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right) \\
&\times\sum_i\left(\frac{1}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}} + T\alpha\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\cdot\lambda_i\left(\langle\boldsymbol{\omega}_0-\boldsymbol{\theta}^*,\mathbf{v}_i\rangle\right)^2 \\
&+2\sum_i\left(\frac{1}{\alpha^2T^2}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}} + \mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\frac{\omega_i^2\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^2}.
\end{aligned}
$$

Incorporating with the definition of effective meta weight

$$
\Xi_i(\mathbf{\Sigma},\alpha,T) = \begin{cases}\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})/\left(T\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\right) & \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) \geq \frac{1}{\alpha T};\\ T\alpha^2\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mu_i(\mathbf{H}_{m,\beta^{\text{te}}}) & \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \frac{1}{\alpha T},\end{cases} \tag{33}
$$

we obtain

$$
\left(\frac{1}{T}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}} + T\alpha^2\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} = \Xi_i(\mathbf{\Sigma},\alpha,T).
$$

Therefore,

$$
R(\overline{\boldsymbol{\omega}}_T,\beta^{\text{te}}) \leq \text{Bias} + \text{Var}
$$

where

$$
\begin{aligned}
\text{Bias} &= \frac{2}{\alpha^2 T}\sum_i \Xi_i \frac{\omega_i^2}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \\
\text{Var} &= \frac{2}{(1-\alpha c(\beta^{\text{tr}},\mathbf{\Sigma})\operatorname{tr}(\mathbf{\Sigma}))}\left(\sum_i \Xi_i\right) \\
&\times[f(\beta^{\text{tr}},n_2,\sigma,\mathbf{\Sigma}_{\boldsymbol{\theta}},\mathbf{\Sigma}) + \underbrace{2c(\beta^{\text{tr}},\mathbf{\Sigma})\sum_i\left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}}}{T\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\lambda_i\omega_i^2}_{V_2}].
\end{aligned}
$$

Note that the term V_2 is obtained by our analysis for $\mathcal{E}_{\text{bias}}$. However, it originates from the stochasticity of SGD, and hence we treat this term as the variance in our final results. □

C Analysis for Lower Bound (Theorem 2)

C.1 Fourth Moment Lower Bound for Meta Nosie

Similarly to upper bound, we need some technical results for the fourth moment of meta data $\mathbf{B}$ and noise $\boldsymbol{\xi}$ to proceed the lower bound analysis.

Lemma 13. *Suppose Assumption 1-3 hold. Given $|\beta^{tr}| < \frac{1}{\lambda_1}$, for any PSD matrix $\mathbf{A}$, we have*

$$\mathbb{E}[\mathbf{B}^\top \mathbf{B}\mathbf{A}\mathbf{B}^\top \mathbf{B}] \succeq \mathbf{H}_{n_1,\beta^{tr}}\mathbf{A}\mathbf{H}_{n_1,\beta^{tr}} + \frac{b_1}{n_2}\operatorname{tr}(\mathbf{H}_{n_1,\beta^{tr}}\mathbf{A})\mathbf{H}_{n_1,\beta^{tr}} \tag{34}$$

$$\Pi \succeq \frac{1}{n_2} g(\beta^{tr}, n_1, \sigma, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma})\mathbf{H}_{n_1,\beta^{tr}} \tag{35}$$

where $g(\beta, n, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) := \sigma^2 + b_1 \operatorname{tr}(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{H}_{n,\beta}) + \beta^2 \mathbf{1}_{\beta\le 0} b_1 \operatorname{tr}(\boldsymbol{\Sigma}^2)/n$.

Proof. With a slight abuse of notations, we write β^{tr} as β, $\mathbf{X}^{\text{in}}$ as $\mathbf{X}$ in this proof. Note that $\mathbf{x} \in \mathbb{R}^d \sim \mathcal{P}_{\mathbf{x}}$ is independent of $\mathbf{X}^{\text{in}}$. We first derive

$$\begin{aligned}
&\mathbb{E}[\mathbf{B}^\top \mathbf{B}\mathbf{A}\mathbf{B}^\top \mathbf{B}] \\
&= \frac{1}{n_2}\mathbb{E}\left[(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{x}\mathbf{x}^\top(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{A}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{x}\mathbf{x}^\top(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right] \\
&+ \frac{n_2-1}{n_2}\mathbb{E}\left[(\mathbf{I} - \frac{\beta}{n_2}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{A}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right] \\
&\overset{(a)}{\succeq} \frac{b_1}{n_2}\mathbb{E}\left[\operatorname{tr}(\mathbf{A}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\Sigma(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X}))(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\Sigma(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\right] \\
&+ \mathbf{H}_{n_1,\beta}\mathbf{A}\mathbf{H}_{n_1,\beta} \\
&\succeq \frac{b_1}{n_2}\operatorname{tr}(\mathbf{H}_{n_1,\beta}\mathbf{A})\mathbf{H}_{n_1,\beta} + \mathbf{H}_{n_1,\beta}\mathbf{A}\mathbf{H}_{n_1,\beta}
\end{aligned}$$

where (a) is implied by Assumption 1.

Recall that Π takes the following form:

$$\Pi = \frac{\sigma^2}{n_2}\mathbf{H}_{n_1,\beta} + \mathbb{E}[\mathbf{B}^\top\mathbf{B}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{B}^\top\mathbf{B}] + \sigma^2 \cdot \frac{\beta^2}{n_2 n_1^2}\mathbb{E}[\mathbf{B}^\top\mathbf{X}^{\text{out}}\mathbf{X}^\top\mathbf{X}\mathbf{X}^{\text{out}\top}\mathbf{B}].$$

The second term can be directly bounded by letting $\mathbf{A} = \boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ in eq. (34), and we have:

$$\mathbb{E}[\mathbf{B}^\top\mathbf{B}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{B}^\top\mathbf{B}] \succeq \frac{b_1}{n_2}\operatorname{tr}(\mathbf{H}_{n_1,\beta}\boldsymbol{\Sigma}_{\boldsymbol{\theta}})\mathbf{H}_{n_1,\beta}.$$

For the third term:

$$\begin{aligned}
&\frac{1}{n_2}\mathbb{E}[\mathbf{B}^\top\mathbf{X}^{\text{out}}\mathbf{X}^\top\mathbf{X}\mathbf{X}^{\text{out}\top}\mathbf{B}] \\
&= \frac{1}{n_2}\mathbb{E}\left[(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\mathbf{x}\mathbf{x}^\top\mathbf{X}^\top\mathbf{X}\mathbf{x}\mathbf{x}^\top(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right] \\
&+ \frac{n_2-1}{n_2}\mathbb{E}\left[(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}\mathbf{X}^\top\mathbf{X}\boldsymbol{\Sigma}(\mathbf{I} - \frac{\beta}{n_1}\mathbf{X}^\top\mathbf{X})\right] \\
&\succeq \frac{n_1 b_1 \operatorname{tr}(\boldsymbol{\Sigma}^2)}{n_2}\mathbf{H}_{n_1,\beta}\mathbf{1}_{\beta\le 0}
\end{aligned}$$

Putting these results together completes the proof. □

C.2 Bias-Variance Decomposition

For the lower bound analysis, we also decompose the excess risk into bias and variance terms.

Lemma 14 (Bias-variance decomposition, lower bound)**.** *Following the notations in eq. (24), the excess risk can be decomposed as follows:*

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) \geq \underline{\mathcal{E}_{bias}} + \underline{\mathcal{E}_{var}}$$

where

$$\underline{\mathcal{E}_{bias}} = \frac{1}{2T^2} \cdot \sum_{t=0}^{T-1} \sum_{k=t}^{T-1} \left\langle (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^{k-t} \mathbf{H}_{m,\beta^{te}}, \mathbf{D}_t \right\rangle,$$

$$\underline{\mathcal{E}_{var}} = \frac{1}{2T^2} \cdot \sum_{t=0}^{T-1} \sum_{k=t}^{T-1} \left\langle (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^{k-t} \mathbf{H}_{m,\beta^{te}}, \mathbf{V}_t \right\rangle.$$

Proof. The proof is similar to that for Lemma 5, and the inequality sign is reversed since we only calculate the half of summation. In particular,

$$\begin{aligned}
\mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}] &= \frac{1}{T^2} \sum_{1 \leq t < k \leq T-1} \mathbb{E}[\boldsymbol{\varrho}_t^{\text{var}} \otimes \boldsymbol{\varrho}_k^{\text{var}}] + \frac{1}{T^2} \sum_{1 \leq k < t \leq T-1} \mathbb{E}[\boldsymbol{\varrho}_t^{\text{var}} \otimes \boldsymbol{\varrho}_k^{\text{var}}] \\
&\succeq \frac{1}{T^2} \sum_{1 \leq t < k \leq T-1} \mathbb{E}[\boldsymbol{\varrho}_t^{\text{var}} \otimes \boldsymbol{\varrho}_k^{\text{var}}].
\end{aligned}$$

For $t \leq k$, $\mathbb{E}[\boldsymbol{\varrho}_k^{\text{var}} | \boldsymbol{\varrho}_t^{\text{var}}] = (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t} \boldsymbol{\varrho}_t^{\text{var}}$, since $\mathbb{E}[\mathbf{B}_t^\top \boldsymbol{\xi}_t | \boldsymbol{\varrho}_{t-1}] = \mathbf{0}$. From this

$$\mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}] \succeq \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{k=t}^{T-1} \mathbf{V}_t (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t}.$$

Plugging this into $\frac{1}{2}\langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}] \rangle$, we obtain:

$$\begin{aligned}
&\frac{1}{2}\langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbb{E}[\overline{\boldsymbol{\varrho}}_T^{\text{var}} \otimes \overline{\boldsymbol{\varrho}}_T^{\text{var}}] \rangle \\
&\geq \frac{1}{2T^2} \sum_{t=0}^{T-1} \sum_{k=t+1}^{T-1} \langle \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t} \rangle \\
&= \frac{1}{2T^2} \sum_{t=0}^{T-1} \sum_{k=t}^{T-1} \langle (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t \rangle \\
&= \underline{\mathcal{E}_{\text{var}}}.
\end{aligned}$$

The proof is the same for the term $\underline{\mathcal{E}_{\text{bias}}}$. □

C.3 Bounding the Bias

We first bound the summation of $\mathbf{D}_t$, i.e. $\mathbf{S}_k = \sum_{t=0}^{k-1} \mathbf{D}_t$.

Lemma 15 (Bounding $\mathbf{S}_t$)**.** *If the stepsize satisfies $\alpha < 1/(2 \max_i \{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\})$, then for any $k \geq 2$, it holds that*

$$\begin{aligned}
\mathbf{S}_k \succeq \frac{b_1}{4n_2} \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^{k/2} \right) \mathbf{D}_0 \right) \cdot \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^{k/2} \right) \\
+ \sum_{t=0}^{k-1} (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^t \cdot \mathbf{D}_0 \cdot (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^t.
\end{aligned}$$

Proof. By eq. (30), since $\widetilde{\mathcal{M}} - \mathcal{M}$ is a PSD mapping, we have

$$\begin{aligned}
\mathbf{S}_k &= \mathbf{D}_0 + (\mathcal{I} - \alpha \widetilde{\mathcal{T}}) \circ \mathbf{S}_{k-1} + \alpha^2 (\mathcal{M} - \widetilde{\mathcal{M}}) \circ \mathbf{S}_{k-1} \qquad (36) \\
&\succeq \sum_{t=0}^{k-1} (\mathcal{I} - \alpha \widetilde{\mathcal{T}})^t \circ \mathbf{D}_0 \\
&= \sum_{t=0}^{k-1} (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \cdot \mathbf{D}_0 \cdot (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t.
\end{aligned}$$

Note that for PSD $\mathbf{A}$,

$$(\mathcal{M} - \widetilde{\mathcal{M}}) \circ \mathbf{A} = \mathbb{E}[\mathbf{B}^\top \mathbf{B}\mathbf{A}\mathbf{B}^\top \mathbf{B}] - \mathbf{H}_{n_1,\beta^{\text{tr}}} \mathbf{A} \mathbf{H}_{n_1,\beta^{\text{tr}}}$$

By Lemma 13, we have

$$\begin{aligned}
(\mathcal{M} - \widetilde{\mathcal{M}}) \circ \mathbf{S}_k &\succeq \frac{b_1}{n_2} \operatorname{tr}\left(\mathbf{H}_{n_1,\beta^{\text{tr}}} \mathbf{S}_k\right) \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\succeq \frac{b_1}{n_2} \operatorname{tr}\left(\sum_{t=0}^{k-1} (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{2t} \mathbf{H}_{n_1,\beta^{\text{tr}}} \cdot \mathbf{D}_0\right) \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\succeq \frac{b_1}{n_2} \operatorname{tr}\left(\sum_{t=0}^{k-1} (\mathbf{I} - 2\alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{t} \mathbf{H}_{n_1,\beta^{\text{tr}}} \cdot \mathbf{D}_0\right) \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\succeq \frac{b_1}{2n_2\alpha} \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^k\right) \mathbf{D}_0\right) \mathbf{H}_{n_1,\beta^{\text{tr}}}. \qquad (37)
\end{aligned}$$

Substituting eq. (37) back into eq. (36), and solving the recursion, we obtain

$$\begin{aligned}
\mathbf{S}_k \succeq& \sum_{t=0}^{k-1} (\mathcal{I} - \alpha \widetilde{\mathcal{T}})^t \circ \left\{ \frac{b_1 \alpha}{2n_2} \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-1-t}\right) \mathbf{D}_0\right) \mathbf{H} + \mathbf{D}_0 \right\} \\
=& \frac{b_1 \alpha}{2n_2} \underbrace{\sum_{t=0}^{k-1} \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-1-t}\right) \mathbf{D}_0\right) \cdot (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{2t} \mathbf{H}_{n_1,\beta^{\text{tr}}}}_{\mathbf{J}_4} \\
&+ \sum_{t=0}^{k-1} (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \cdot \mathbf{D}_0 \cdot (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t.
\end{aligned}$$

The term $\mathbf{J}_4$ can be further bounded by the following:

$$\begin{aligned}
\mathbf{J}_4 &\succeq \sum_{t=0}^{k-1} \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-1-t}\right) \mathbf{D}_0\right) \cdot (\mathbf{I} - 2\alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\succeq \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k/2}\right) \mathbf{D}_0\right) \cdot \sum_{t=0}^{k/2-1} (\mathbf{I} - 2\alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\succeq \frac{1}{2\alpha} \operatorname{tr}\left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k/2}\right) \mathbf{D}_0\right) \cdot \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k/2}\right)
\end{aligned}$$

which completes the proof. □

Then we can bound the bias term.

Lemma 16 (Bounding the bias)**.** *Let $\omega_i = \langle \boldsymbol{\omega}_0 - \boldsymbol{\theta}^*, \mathbf{v}_i \rangle$. If $\alpha < \frac{1}{c(\beta^{tr}, \boldsymbol{\Sigma}) \operatorname{tr}(\boldsymbol{\Sigma})}$, for sufficiently large n_1, s.t. $\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) > 0$, $\forall i$, then we have*

$$\begin{aligned}
\underline{\mathcal{E}_{bias}} \geq& \frac{1}{100\alpha^2 T} \sum_i \Xi_i \frac{\omega_i^2}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \frac{b_1}{1000 n_2 (1 - \alpha c(\beta^{tr}, \boldsymbol{\Sigma}) \operatorname{tr}(\boldsymbol{\Sigma}))} \sum_i \Xi_i \\
&\times \sum_i \left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) \geq \frac{1}{\alpha T}}}{T\alpha \mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) < \frac{1}{\alpha T}} \right) \lambda_i \omega_i^2.
\end{aligned}$$

Proof. From Lemma 14, we have

$$\begin{aligned}
\underline{\mathcal{E}_{bias}} &= \frac{1}{2T^2} \cdot \sum_{t=0}^{T-1} \sum_{k=t}^{T-1} \left\langle (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{D}_t \right\rangle \\
&= \frac{1}{2\alpha T^2} \cdot \sum_{t=0}^{T-1} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t} \right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{D}_t \right\rangle \\
&\geq \frac{1}{2\alpha T^2} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/2} \right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \sum_{t=0}^{T/2} \mathbf{D}_t \right\rangle \\
&\geq \frac{1}{2\alpha T^2} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/2} \right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{S}_{\frac{T}{2}} \right\rangle .
\end{aligned}$$

Applying Lemma 15 to $\mathbf{S}_{\frac{T}{2}}$, we obtain:

$$\underline{\mathcal{E}_{bias}} \geq \underline{d_1} + \underline{d_2}$$

where

$$\begin{aligned}
\underline{d_1} &= \frac{b_1}{8\alpha n_2 T^2} \operatorname{tr} \left(\left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/4} \right) \mathbf{D}_0 \right) \\
&\quad \times \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/2} \right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \ \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/4} \right) \right\rangle \\
\underline{d_2} &= \frac{1}{2\alpha T^2} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/2} \right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \right. \\
&\quad \left. \sum_{t=0}^{T/2-1} (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \cdot \mathbf{D}_0 \cdot (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \right\rangle .
\end{aligned}$$

Moreover,

$$\begin{aligned}
\underline{d_2} &\geq \frac{1}{2\alpha T^2} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/2} \right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \sum_{t=0}^{T/2-1} (\mathbf{I} - 2\alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^t \mathbf{D}_0 \right\rangle ; \\
&\geq \frac{1}{4\alpha^2 T^2} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{\text{tr}}})^{T/2} \right)^2 \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-2} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{D}_0 \right\rangle .
\end{aligned}$$

Using the diagonalizing technique similar to the proof for Lemma 9, we have

$$\underline{d_1} \geq \frac{b_1}{8\alpha n_2 T^2} \left(\sum_i \left(1 - (1 - \alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T/4} \right) \omega_i^2 \right) \tag{38}$$

$$\times \left(\sum_i \left(1 - (1 - \alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T/4} \right)^2 \frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \right), \tag{39}$$

$$\underline{d_2} \geq \frac{1}{4\alpha^2 T^2} \sum_i \left(1 - (1 - \alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T/4} \right)^2 \frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \omega_i^2 . \tag{40}$$

We use the following fact to bound the polynomial term. For $h_1(x) = 1 - (1-x)^{\frac{T}{4}}$, we have

$$h_1(x) \geq \begin{cases} \frac{1}{5} & x \geq 1/T \\ \frac{T}{5} x & x < 1/T \end{cases}$$

i.e., $1 - (1 - \alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}))^{T/4} \geq \left(\frac{1}{5} \mathbf{1}_{\alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) \geq \frac{1}{T}} + \frac{\alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}{5} \mathbf{1}_{\alpha \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \frac{1}{T}} \right)$. Substituting this back into eqs. (39) and (40), and using the definition of effective meta weight Ξ_i complete the proof. □

C.4 Bounding the Variance

We first bound the term $\mathbf{V}_t$.

Lemma 17 (Bounding $\mathbf{V}_t$)**.** *If the stepsize satisfies $\alpha < 1/(\max_i\{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})\})$, it holds that*

$$\mathbf{V}_t \succeq \frac{\alpha g(\beta^{tr}, n_1, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{2} \cdot \left(\mathbf{I} - (\mathbf{I} - \alpha \mathbf{H}_{n_1,\beta^{tr}})^{2t}\right).$$

Proof. With a slight abuse of notations, we write $g(\beta^{\text{tr}}, n_1, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})$ as g. By definition,

$$\begin{aligned}
\mathbf{V}_t &= (\mathcal{I} - \alpha\mathcal{T}) \circ \mathbf{V}_{t-1} + \alpha^2 \Pi \\
&= (\mathcal{I} - \alpha\widetilde{\mathcal{T}}) \circ \mathbf{V}_{t-1} + (\mathcal{M} - \widetilde{\mathcal{M}}) \circ \mathbf{V}_{t-1} + \alpha^2 \Pi \\
&\overset{(a)}{\succeq} (\mathcal{I} - \alpha\widetilde{\mathcal{T}}) \circ \mathbf{V}_{t-1} + \alpha^2 g \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&\overset{(b)}{=} \alpha^2 g \cdot \sum_{k=0}^{t-1} (\mathcal{I} - \alpha\widetilde{\mathcal{T}})^k \circ \mathbf{H}_{n_1,\beta^{\text{tr}}} \\
&= \alpha^2 g \cdot \sum_{k=0}^{t-1} (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k \mathbf{H}_{n_1,\beta^{\text{tr}}} (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^k \quad \text{(by the definition of } \mathcal{I} - \alpha\widetilde{\mathcal{T}}) \\
&= \alpha g \cdot \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{2t}\right) \cdot (2\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{-1} \\
&\overset{(c)}{\succeq} \frac{\alpha g}{2} \cdot \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{2t}\right)
\end{aligned}$$

where (a) follows from the Lemma 13, (b) follows by solving the recursion and (c) holds since we directly replace $(2\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{-1}$ by $(2\mathbf{I})^{-1}$. □

Lemma 18 (Bounding the variance)**.** *Let $\omega_i = \langle \boldsymbol{\omega}_0 - \boldsymbol{\theta}^*, \mathbf{v}_i \rangle$. If $\alpha < \frac{1}{c(\beta^{tr}, \mathbf{\Sigma})\,\mathrm{tr}(\mathbf{\Sigma})}$, for sufficiently large n_1, s.t. $\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) > 0$, $\forall i$, for $T > 10$, then we have*

$$\underline{\mathcal{E}_{var}} \geq \frac{g(\beta^{tr}, n_1, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})}{100 n_2 (1 - \alpha c(\beta^{tr}, \mathbf{\Sigma})\,\mathrm{tr}(\mathbf{\Sigma}))} \sum_i \Xi_i.$$

Proof. From Lemma 14, we have

$$\begin{aligned}
\underline{\mathcal{E}_{var}} &= \frac{1}{2T^2} \cdot \sum_{t=0}^{T-1} \sum_{k=t}^{T-1} \left\langle (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{k-t} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t \right\rangle \\
&= \frac{1}{2\alpha T^2} \cdot \sum_{t=0}^{T-1} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t}\right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \mathbf{V}_t \right\rangle.
\end{aligned}$$

Then applying Lemma 17, and writting $g(\beta^{\text{tr}}, n_1, \mathbf{\Sigma}, \mathbf{\Sigma}_{\boldsymbol{\theta}})$ as g, we obtain

$$\begin{aligned}
\underline{\mathcal{E}_{var}} &\geq \frac{g}{4T^2} \cdot \sum_{t=0}^{T-1} \left\langle \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t}\right) \mathbf{H}_{n_1,\beta^{\text{tr}}}^{-1} \mathbf{H}_{m,\beta^{\text{te}}}, \left(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H}_{n_1,\beta^{\text{tr}}})^{2t}\right) \right\rangle \\
&= \frac{g}{4T^2} \sum_i \frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \sum_{t=0}^{T-1} (1 - (1 - \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t}))(1 - (1 - \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^{2t})) \\
&\geq \frac{g}{4T^2} \sum_i \frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \sum_{t=0}^{T-1} (1 - (1 - \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^{T-t-1}))(1 - (1 - \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^{t})) \quad (41)
\end{aligned}$$

where the equality holds by applying the diagonalizing technique again. Following the trick similar to that in [51] to lower bound the function $h_2(x) := \sum_{t=0}^{T-1} \left(1 - (1-x)^{T-t-1}\right)(1 - (1-x)^t)$ defined on $x \in (0, 1)$, for $T > 10$, we have

$$f(x) \geq \begin{cases} \frac{T}{20}, & \frac{1}{T} \leq x < 1 \\ \frac{3T^3}{50} x^2, & 0 < x < \frac{1}{T} \end{cases}$$

Substituting this back into eq. (41), and using the definition of effective meta weight Ξ_i completes the proof. □

C.5 Proof of Theorem 2

Theorem 6 (Theorem 2 Restated)**.** *Let $\omega_i = \langle \boldsymbol{\omega}_0 - \boldsymbol{\theta}^*, \mathbf{v}_i \rangle$. If $|\beta^{tr}|, |\beta^{te}| < 1/\lambda_1$, n_1 is large ensuring that $\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) > 0$, $\forall i$ and $\alpha < 1/\left(c(\beta^{tr}, \boldsymbol{\Sigma}) \operatorname{tr}(\boldsymbol{\Sigma})\right)$. For $T > 10$, the meta excess risk $R(\overline{\boldsymbol{\omega}}_T, \beta^{te})$ is bounded below as follows*

$$
\begin{aligned}
R(\overline{\boldsymbol{\omega}}_T, \beta^{te}) \geq & \frac{1}{100\alpha^2 T} \sum_i \Xi_i \frac{\omega_i^2}{\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \frac{1}{n_2} \cdot \frac{1}{(1 - \alpha c(\beta^{tr}, \boldsymbol{\Sigma}) \operatorname{tr}(\boldsymbol{\Sigma}))} \sum_i \Xi_i \\
& \times [\frac{1}{100} g(\beta^{tr}, n_1, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) + \frac{b_1}{1000} \sum_i \left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) \geq \frac{1}{\alpha T}}}{T\alpha\mu_i(\mathbf{H}_{n_1,\beta^{tr}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{tr}}) < \frac{1}{\alpha T}} \right) \lambda_i \omega_i^2].
\end{aligned}
$$

Proof. The proof can be completed by combining Lemmas 16 and 18. □

D Proofs for Section 4.2

D.1 Proof of Lemma 1

Proof of Lemma 1. For the single task setting, we first simplify our notations in Theorem 1 as follows.

$$
c(0, \boldsymbol{\Sigma}) = c_1, \quad f(0, n_2, \sigma, \boldsymbol{\Sigma}, \mathbf{0}) = \sigma^2/n_2, \quad \mathbf{H}_{n_1,\beta^{\text{tr}}} = \boldsymbol{\Sigma}.
$$

By Theorem 1, we have

$$
\begin{aligned}
\text{Bias} &= \frac{2}{\alpha^2 T} \sum_i \left(\frac{1}{T} \mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}} + T\alpha^2 \lambda_i^2 \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right) \frac{\omega_i^2 \mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\lambda_i^2} \\
&\leq \frac{2}{\alpha^2 T} \sum_i (\alpha\lambda_i \mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}} + \alpha\lambda_i \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}}) \frac{\omega_i^2 \mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\lambda_i^2}.
\end{aligned}
$$

For large m, we have $\mu_i(\mathbf{H}_{m,\beta^{\text{te}}}) = (1 - \beta^{\text{te}}\lambda_i)^2 \lambda_i + o(1)$. Therefore,

$$
\text{Bias} \leq \frac{2(1 - \beta^{\text{te}}\lambda_d)^2}{\alpha^2 T} \sum_i \omega_i^2 \leq \mathcal{O}(\frac{1}{T}).
$$

For the variance term,

$$
\begin{aligned}
\text{Var} = & \frac{2}{(1 - \alpha c_1 \operatorname{tr}(\boldsymbol{\Sigma}))} \underbrace{\sum_i \left(\frac{1}{T} \mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}} + T\alpha^2 \lambda_i^2 \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right) \frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\lambda_i}}_{J_5} \\
& \times [\frac{\sigma^2}{n_2} + 2c_1 \sum_i \left(\frac{\mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}}}{T\alpha\lambda_i} + \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right) \lambda_i \omega_i^2].
\end{aligned}
$$

It is easy to check that

$$
\sum_i \left(\frac{\mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}}}{T\alpha\lambda_i} + \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right) \lambda_i \omega_i^2 \leq \sum_i \left(\frac{\mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}}}{T\alpha} + \frac{1}{\alpha T} \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right) \omega_i^2 \leq \mathcal{O}(1/T).
$$

Moreover,

$$
J_5 \leq (1 - \beta^{\text{te}}\lambda_d)^2 \sum_i \left(\frac{1}{T} \mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}} + T\alpha^2 \lambda_i^2 \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right).
$$

The term $\sum_i \left(\frac{1}{T} \mathbf{1}_{\lambda_i \geq \frac{1}{\alpha T}} + T\alpha^2 \lambda_i^2 \mathbf{1}_{\lambda_i < \frac{1}{\alpha T}} \right)$ has the form similar to Corollary 2.3 in [51] and we directly have $J_5 = \mathcal{O}\left(\log^{-p}(T)\right)$, which implies

$$
\text{Var} = \mathcal{O}\left(\log^{-p}(T)\right).
$$

Thus we complete the proof. □

D.2 Proof of Proposition 2

Proof of Proposition 2. We first consider the bias term in Theorems 1 and 2 (up to absolute constants):

$$\text{Bias} = \frac{2}{\alpha^2 T}\sum_i \left(\frac{1}{T}\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq \frac{1}{\alpha T}} + \alpha^2 T\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\frac{\omega_i^2\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})^2}.$$

If $\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) \geq \frac{1}{\alpha T}$, $\frac{1}{T} \leq \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})$; and if $\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \frac{1}{\alpha T}$, then $\alpha^2 T\mu_i^2(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})$. Hence

$$\text{Bias} \leq \frac{1}{\alpha^2 T}\sum_i \frac{\omega_i^2\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} \leq \frac{2}{\alpha^2 T}\cdot\max_i \frac{\mu_i(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\|\boldsymbol{\omega}_0-\boldsymbol{\theta}^*\|^2 = \mathcal{O}(\frac{1}{T}).$$

Moreover,

$$\begin{aligned}
&\sum_i \left(\frac{\mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})\geq\frac{1}{\alpha T}}}{T\alpha\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})} + \mathbf{1}_{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})<\frac{1}{\alpha T}}\right)\lambda_i\omega_i^2 \\
&\overset{(a)}{\leq} \frac{1}{\alpha T}\sum_i \frac{\lambda_i}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\omega_i^2 \\
&\leq \frac{1}{\alpha T}\max_i \frac{\lambda_i}{\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})}\|\boldsymbol{\omega}_0-\boldsymbol{\theta}^*\|^2 = \mathcal{O}(\frac{1}{T})
\end{aligned}$$

where (a) holds since we directly upper bound $\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}})$ by $\frac{1}{\alpha T}$ when $\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) < \frac{1}{\alpha T}$. Therefore, it is essential to analyze $f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})\,(\sum_i \Xi_i)$ and $g(\beta^{\text{tr}}, n_1, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})\,(\sum_i \Xi_i)$ from variance term in the upper and lower bounds respectively.

Then we calculate some rates of interesting in Theorems 1 and 2 under the specific data and task distributions in Proposition 2.

If the spectrum of $\boldsymbol{\Sigma}$ satisfies $\lambda_k = k^{-1}\log^{-p}(k+1)$, then it is easily verified that $\text{tr}(\boldsymbol{\Sigma}^s) = O(1)$ for $s = 1, \cdots, 4$. By discussions on Assumption 3 in Appendix F, we have $C(\beta, \boldsymbol{\Sigma}) = \Theta(1)$ for given β. Hence,

$$\begin{aligned}
c(\beta^{\text{tr}}, \boldsymbol{\Sigma}) &= \Theta(1) \\
f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) &= c(\beta^{\text{tr}}, \boldsymbol{\Sigma})\,\text{tr}(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\boldsymbol{\Sigma}) + \Theta(1) \\
g(\beta^{\text{tr}}, n_1, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) &= b_1\,\text{tr}(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{H}_{n_1,\beta^{\text{tr}}}) + \Theta(1).
\end{aligned}$$

If $r \geq 2p-1$, then we have $g(\beta^{\text{tr}}, n_1, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \geq \Omega\left(\log^{r-p+1}(d)\right) \geq \Omega\left(\log^{r-p+1}(T)\right)$.

Let $k^\dagger := \text{card}\{i : \mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) \geq 1/\alpha T\}$. For large n_1, we have $\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) = (1-\beta^{\text{tr}}\lambda_i)^2\lambda_i + o(1)$. If $k^\dagger = \mathcal{O}\left(T/\log^p(T+1)\right)$, then

$$\min_{1\leq i\leq k^\dagger+1}\mu_i(\mathbf{H}_{n_1,\beta^{\text{tr}}}) = \omega\left(\frac{\log^p(T)}{T[\log(T)-p\log(\log(T))]^p}\right) = \omega\left(\frac{1}{T}\right)$$

which contradicts the definition of $k^\dagger$. Hence $k^\dagger = \Omega\left(T/\log^p(T+1)\right)$. Then

$$\sum_i \Xi_i \geq \Omega\left(k^\dagger\cdot\frac{1}{T}\right) = \Omega\left(\frac{1}{\log^p(T)}\right).$$

Therefore, by Theorem 2, $R(\overline{\boldsymbol{\omega}}, \beta^{\text{te}}) = \Omega\left(\log^{r-2p+1}(T)\right)$.

For $r < 2p-1$, if $d = T^l$, where l can be sufficiently large ($d \gg T$) but still finite, then

- If $p-1 < r < 2p-1$, $f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \leq \mathcal{O}(\log^{r-p+1} T)$;
- If $r \leq p-1$, $f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \leq \mathcal{O}\Big(\log\left(\log(T)\right)\Big)$.

Following the analysis similar to that for Corollary 2.3 in [51], we have $\sum_i \Xi_i = \mathcal{O}(\frac{1}{\log^p(T)})$. Then by Theorem 1

$$R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) = \mathcal{O}\left(\frac{1}{\log^{p-(r-p+1)^+}(T)}\right).$$

□

D.3 Proof of Proposition 3

Proof of Proposition 3. Following the analysis in Appendix D.2, it is essential to analyze $f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})\,(\sum_i \Xi_i)$. If $d = T^l$, where l can be sufficiently large but still finite, then

$$f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) = \widetilde{\Theta}(1)$$

for $\lambda_k = k^q$ $(q > 1)$ or $\lambda_k = e^{-k}$.

Following the analysis similar to that for Corollary 2.3 in [51], we have

- If $\lambda_k = k^q$ $(q > 1)$, then $\sum_i \Xi_i = \mathcal{O}\left(\frac{1}{T^{\frac{q-1}{q}}}\right)$;
- If $\lambda_k = e^{-k}$, then $\sum_i \Xi_i = \mathcal{O}\left(\frac{\log(T)}{T}\right)$.

Substituting these results back into Theorem 1, we obtain

- If $\lambda_k = k^q$ $(q > 1)$, then $R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) = \widetilde{\mathcal{O}}\left(\frac{1}{T^{\frac{q-1}{q}}}\right)$;
- If $\lambda_k = e^{-k}$, then $R(\overline{\boldsymbol{\omega}}_T, \beta^{\text{te}}) = \widetilde{\mathcal{O}}\left(\frac{1}{T}\right)$.

□

E Proofs for Section 4.3

E.1 Proof of Proposition 4

Proof of Proposition 4. Following the analysis in Appendix D.2, it is crucial to analyze $f(\beta^{\text{tr}}, n_2, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})\,(\sum_i \Xi_i)$.

Then we calculate the rate of interest in Theorems 1 and 2 under some specific data and task distributions in Proposition 4. We have $\text{tr}(\boldsymbol{\Sigma}^2) = \frac{1}{s} + \frac{1}{d-s} = \Theta(\frac{\log^p(T)}{T})$. Moreover, by discussions on Assumption 3 in Appendix F, $C(\beta, \boldsymbol{\Sigma}) = \Theta(1)$. Hence

$$c(\beta, \boldsymbol{\Sigma}) := c_1 + \widetilde{\mathcal{O}}(\frac{1}{T});$$

$$f(\beta, n, \sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) := 2c_1\mathcal{O}(1) + \frac{\sigma^2}{n} + \widetilde{\mathcal{O}}\left(\frac{1}{T}\right).$$

By the definition of Ξ_i, we have

$$\begin{aligned}
\sum_i \Xi_i &= \mathcal{O}\left(s \cdot \frac{\mu_1(\mathbf{H}_{m,\beta^{\text{te}}})}{T\mu_1(\mathbf{H}_{n_1,\beta^{\text{tr}}})} + \frac{1}{d-s} \cdot T\frac{\mu_d(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mu_d(\mathbf{H}_{m,\beta^{\text{te}}})}{\lambda_d^2}\right) \\
&= \mathcal{O}\left(\frac{1}{\log^p(T)}\right)\frac{\mu_1(\mathbf{H}_{m,\beta^{\text{te}}})}{\mu_1(\mathbf{H}_{n_1,\beta^{\text{tr}}})} + \mathcal{O}\Big(\frac{1}{\log^q(T)}\Big)\frac{\mu_d(\mathbf{H}_{n_1,\beta^{\text{tr}}})\mu_d(\mathbf{H}_{m,\beta^{\text{te}}})}{\lambda_d^2} \\
&= \mathcal{O}\left(\frac{1}{\log^p(T)}\right)\frac{(1-\beta^{\text{te}}\lambda_1)^2}{(1-\beta^{\text{tr}}\lambda_1)^2} + \mathcal{O}\Big(\frac{1}{\log^q(T)}\Big)(1-\beta^{\text{te}}\lambda_d)^2(1-\beta^{\text{tr}}\lambda_d)^2
\end{aligned}$$

where the last equality follows from the fact that for large n, we have $\mu_i(\mathbf{H}_{n,\beta}) = (1-\beta\lambda_i)^2\lambda_i + o(1)$. Combining with the bias term which is $\mathcal{O}(\frac{1}{T})$, and applying Theorem 1 completes the proof. □

E.2 Proof of Corollary 1

Proof of Corollary 1. For $t \in (s, K]$, by Theorem 2, one can verify that $t = \widetilde{\Theta}(K)$ for diminishing risk. Let $t = K \log^{-l}(K)$, where $p > l > 0$. Following the analysis in Appendix E.1, we have

$$R(\overline{\boldsymbol{\omega}}_t^{\beta^{\text{tr}}}, \beta^{\text{te}}) \lesssim \widetilde{\mathcal{O}}(\frac{1}{K}) + (2c_1\nu^2 + \frac{\sigma^2}{n_2}) \tag{42}$$

$$\times \left[\mathcal{O}\Big(\frac{1}{\log^{p-l}(K)}\Big)\frac{(1-\beta^{\text{te}}\lambda_1)^2}{(1-\beta^{\text{tr}}\lambda_1)^2} + \mathcal{O}\Big(\frac{1}{\log^{p+l}(K)}\Big)\Big(1-\beta^{\text{tr}}\lambda_d\Big)^2\Big(1-\beta^{\text{te}}\lambda_d\Big)^2\right]. \tag{43}$$

To clearly illustrate the trade-off in the stopping time, we let $l = 0$ for convenience. If $R(\overline{\boldsymbol{\omega}}_t^{\beta^{\text{tr}}}, \beta^{\text{te}}) < \epsilon$, we have

$$t_\epsilon \leq \exp\left(\epsilon^{-\frac{1}{p}}\Big[\frac{U_l}{(1-\beta^{\text{tr}}\lambda_1)^2} + U_t(1-\beta^{\text{tr}}\lambda_d)^2\Big]^{\frac{1}{p}}\right)$$

where

$$U_l = \mathcal{O}\Big((2c_1\nu^2 + \frac{\sigma^2}{n_2})(1-\beta^{\text{te}}\lambda_1)^2\Big) \quad \textit{and} \quad U_l = \mathcal{O}\Big((2c_1\nu^2 + \frac{\sigma^2}{n_2})(1-\beta^{\text{te}}\lambda_d)^2\Big).$$

The arguments are similar for the lower bound, and we can obtain:

$$L_l = \mathcal{O}\Big((2\frac{b_1\nu^2}{n_2} + \frac{\sigma^2}{n_2})(1-\beta^{\text{te}}\lambda_1)^2\Big) \quad \textit{and} \quad L_l = \mathcal{O}\Big((2\frac{b_1\nu^2}{n_2} + \frac{\sigma^2}{n_2})(1-\beta^{\text{te}}\lambda_d)^2\Big).$$

□

F Discussions on Assumptions

Discussions on Assumption 2 If $\mathcal{P}_{\mathbf{x}}$ is Gaussian distribution, then we have

$$\mathbf{F} = \mathbb{E}[\mathbf{x}\mathbf{x}^\top\boldsymbol{\Sigma}\mathbf{x}\mathbf{x}^\top] = 2\boldsymbol{\Sigma}^3 + \boldsymbol{\Sigma}\,\text{tr}(\boldsymbol{\Sigma}^2).$$

This implies that $\mathbf{F}$ and $\boldsymbol{\Sigma}$ commute because $\boldsymbol{\Sigma}^3$ and $\boldsymbol{\Sigma}$ commute. Moreover, in this case

$$\frac{\beta^2}{n}(\mathbf{F} - \boldsymbol{\Sigma}^3) = \frac{\beta^2}{n}(\boldsymbol{\Sigma}^3 + \boldsymbol{\Sigma}\,\text{tr}(\boldsymbol{\Sigma}^2)).$$

Therefore, if $n \gg \lambda_1(\lambda_1^2 + \text{tr}(\boldsymbol{\Sigma}^2))$, then the eigen-space of $\mathbf{H}_{n,\beta}$ will be dominated by $(\mathbf{I} - \beta\boldsymbol{\Sigma})^2\boldsymbol{\Sigma}$.

Discussions on Assumption 3 Assumption 3 is an eighth moment condition for $\mathbf{x} := \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{z}$, where $\mathbf{z}$ is a σ_x sub-Gaussian vector. Given β, for sufficiently large n s.t. $\mu_i(\mathbf{H}_{n,b}) > 0, \forall i$, and if $\text{tr}(\boldsymbol{\Sigma}^k)$ are all $O(1)$ for $k = 1, \cdots, 4$, then by the quadratic form and the sub-Gaussian property, which has finite higher order moments, we can conclude that $C(\beta, \Sigma) = \Theta(1)$.

The following lemma further shows that if $\mathcal{P}_{\mathbf{x}}$ is a Gaussian distribution, we can derive the analytical form for $C(\beta, \boldsymbol{\Sigma})$.

Lemma 19. *Given $|\beta| < \frac{1}{\lambda_1}$, for sufficiently large n s.t. $\mu_i(\mathbf{H}_{n,b}) > 0, \forall i$, and if $\mathcal{P}_{\mathbf{x}}$ is a Gaussian distribution, assuming $\boldsymbol{\Sigma}$ is diagonal, we have:*

$$C(\beta, \boldsymbol{\Sigma}) = 210(1 + \frac{\beta^4\,\text{tr}(\boldsymbol{\Sigma}^2)^2}{(1-\beta\lambda_1)^4}).$$

Proof. Let $\mathbf{e}_i \in \mathbb{R}^d$ denote the vector that the i-th coordinate is 1, and all other coordinates equal 0. For $\mathbf{x} \sim \mathcal{P}_{\mathbf{x}}$, denote $\mathbf{x}\mathbf{x}^\top = [x_{ij}]_{1\leq i,j\leq d}$. Then we have:

$$\begin{aligned}
&\mathbb{E}[\|\mathbf{e}_i^\top\mathbf{H}_{n,\beta}^{-\frac{1}{2}}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\boldsymbol{\Sigma}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n,\beta}^{-\frac{1}{2}}\mathbf{e}_i\|^2] \\
&\leq \mathbb{E}[\|\mathbf{e}_i^\top\mathbf{H}_{n,\beta}^{-\frac{1}{2}}(\mathbf{I} - \beta\mathbf{x}\mathbf{x}^\top)\boldsymbol{\Sigma}(\mathbf{I} - \beta\mathbf{x}\mathbf{x}^\top)\mathbf{H}_{n,\beta}^{-\frac{1}{2}}\mathbf{e}_i\|^2] \\
&= \mathbb{E}\left[(\mathbf{e}_i^\top\mathbf{H}_{n,\beta}^{-1}\mathbf{e}_i)^2\left(\sum_{j\neq i}\beta^2\lambda_j x_{ij}^2 + \lambda_i(1-\beta x_{ii})^2\right)^2\right]
\end{aligned}$$

For Gaussian distributions, we have

$$\mathbb{E}[x_{ij}^2 x_{ik}^2] = \begin{cases} 9\lambda_i^2\lambda_j^2 & j = k \text{ and } \neq i \\ 105\lambda_i^4 & i = j = k \\ 3\lambda_i^2\lambda_j\lambda_k & i \neq j \neq k \end{cases}$$

We can further obtain:

$$\begin{aligned}
&\mathbb{E}[\|\mathbf{e}_i^\top \mathbf{H}_{n,\beta}^{-\frac{1}{2}}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n,\beta}^{-\frac{1}{2}}\mathbf{e}_i\|^2] \\
&\leq 105(\mathbf{e}_i^\top \mathbf{H}_{n,\beta}^{-1}\mathbf{e}_i)^2 \left(\sum_{j\neq i} \beta^2\lambda_j^2 + (1-\beta\lambda_i)^2\right)^2 \\
&\overset{(a)}{\leq} 210(\mathbf{e}_i^\top \mathbf{H}_{n,\beta}^{-1}\mathbf{e}_i)^2[\beta^4 \operatorname{tr}(\mathbf{\Sigma}^2)^2 + (1-\beta\lambda_i)^4] \\
&\overset{(b)}{\leq} 210[(\mathbf{e}_i^\top \mathbf{H}_{n,\beta}^{-1}\mathbf{e}_i)^2\beta^4 \operatorname{tr}(\mathbf{\Sigma}^2)^2 + 1]
\end{aligned}$$

where (a) follows from the Cauchy-Schwarz inequality, and (b) follows the fact that $(\mathbf{e}_i^\top \mathbf{H}_{n,\beta}^{-1}\mathbf{e}_i)^2 = \frac{1}{[(1-\beta\lambda_i)\lambda_i^2 + \frac{\beta^2}{n}(\lambda_i^2 + \operatorname{tr}(\mathbf{\Sigma}^2)\lambda_i)]^2} \leq 1/(1-\beta\lambda_i)^4$.

Therefore, for any unit $\mathbf{v} \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\mathbb{E}[\|\mathbf{v}^\top \mathbf{H}_{n,\beta}^{-\frac{1}{2}}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{\Sigma}(\mathbf{I} - \frac{\beta}{n}\mathbf{X}^\top\mathbf{X})\mathbf{H}_{n,\beta}^{-\frac{1}{2}}\mathbf{v}\|^2] \\
&\leq \max_{\mathbf{v}} 210[(\mathbf{v}^\top \mathbf{H}_{n,\beta}^{-1}\mathbf{v})^2\beta^4 \operatorname{tr}(\mathbf{\Sigma}^2)^2 + 1] \leq 210\left(1 + \frac{\beta^4 \operatorname{tr}(\mathbf{\Sigma}^2)^2}{(1-\beta\lambda_1)^4}\right).
\end{aligned}$$

□

G Further Related Work

G.1 Underparameterized Setting

Provable guarantees of meta-learning have been extensively studied in the underparameterized regime, i.e. the number of tasks or the data size is much larger than the data dimension. Here we highlight some existing related studies and discuss their differences from ours.

[9] provides the generalization error bounds for S/Q meta-learners. Their generalization error bound is $O(\beta\sqrt{n}) + O(\frac{M}{\sqrt{n}})$, where the β is the uniform stability parameter, n is the number of task, and M is the uniform bound on the loss function. Note that with the square loss as considered in our setting, M can be as large as the dimension d of the input. For the overparameterized regime, where $d \gg n$, the bound becomes asymptotically large and not useful.

[19] shows the generalization guarantees of MAML on recurring and unseen tasks respectively, where the excess risk bound is roughly $\widetilde{O}(\frac{G^2}{T}) + O(\frac{G^2}{mn})$, depending on the total iterations T, the number of tasks m, the available data size for each task n, and the uniform bound for gradient norm G. Notice that the gradient norm typically scales polynomially with the input dimension d. Therefore, in the overparameterized regime ($d \gg mn$), the bound again becomes vacuous.

[13] aims to theoretically characterize the performance between MAML and the standard Empirical Risk Minimization (ERM). Firstly, they show that the empirical training solutions will converge to their population-optimal values with concentration bounds, which have terms $O(\frac{\sqrt{d}}{\sqrt{n}})$ or $O(\frac{\sqrt{d}}{\sqrt{\tau}})$ where n is the sample size and τ is the number of training episodes per task. Such bounds will be crude in the overparameterized regime($d \gg n, \tau$). Then from the generalization perspective, they only give excess risks for the population-optimal solutions, whereas we analyze the generalization property of empirical training solutions based on their optimization trajectory. Moreover, the comparison between the excess risk of MAML and ERM for the population-optimal solutions requires $m = \Omega(d)$, where m is the number of task. Therefore, such comparison does not apply to the overparameterized regime ($d \gg m$).

From what has been discussed above, all of these bounds, are useful (i.e., yield small or vanishing error) only for underparameterized regime, where sample size n is much larger than the dimension d of input (i.e., $n \gg d$), but become vacuous (i.e., yield large error bound) for overparameterized regime, where $n \ll d$. Thus, they fail to explain why the overparameterized neural network can still generalize well.

In contrast, our work provides a much more refined data-dependent upper bound by incorporating the data and task roles. Our bound can be written in a concise form as $O(\frac{h(\Sigma,\Sigma_\theta,\alpha,T)}{T})$ (see Theorem 1 in Section 4 for exact form of the bound), where the function $h(\cdot)$ is determined by data and task covariances Σ and Σ_θ, step-size α and the total number T of SGD iterations (here T scales linearly with the sample size). Note that the dimension d is implicitly captured in Σ, Σ_θ. In the overparameterized regime with $d \gg T$, the function $h(\cdot)$ explicitly captures the effect of data and task conditions to guarantee the value of $h(\cdot)$ to be small compared to T, and thus the excess risk can diminish under overparameterization. In Propositions 2 and 3, we further give specific examples about data and task covariances Σ, Σ_θ that yield small excess risk and good generalization.

Besides the above key difference, our paper also provides the following important results that were not studied in [9, 19, 13]. (a) We provide the lower bound to justify the tightness of our results in the overparameterized regime, whereas [9, 19, 13] do not have such a result. (b) Our results capture how the task diversity affects the excess risk in the overparameterized regime. In particular, we give an example (in Proposition 2), for which the excess risk exhibits a phase transition with respect to task diversity. Such an interesting behavior is not captured in [9, 19, 13].

G.2 Overparameterized Setting

Despite the overparameterization is crucial to demystify the remarkable generalization ability of deep meta-learning [50, 27], there are only a few theoretical analysis being developed to study the generalization of MAML in the overparameterized regime. [45] studied the MAML with overparameterized deep neural nets with a generalization gap quantifying the difference between the empirical and population loss functions at their optimal solutions. However, their bound is derived by conventional complexity-based techniques and does not consider the data or task-dependency similarly as [9, 19, 13] discussed in Appendix G.1, which tends to be weak in the high dimensional, especially in the overparameterized regime. Most related to our work are recent studies [6, 53], where they develop more precise generalization bounds for overparameterized setting under a mixed linear regression model. Yet, their empirical training solutions are directly calculated by taking the closed-form of training objective's minimum, which are not obtained by trained with SGD as ours. More crucially, they consider only the simple isotropic covariance for data and tasks, which are directly scaled by d, i.e. $\Sigma = \frac{1}{d}\mathbf{I}$. Thus, they do not explicitly capture how the generalization performance of MAML depends on the data and task distributions.

H Future Directions

This work takes a step towards understanding the benefits of overparameterization for MAML from the generalization aspect. There are many important future work directions, and we elaborate some interesting directions in this section.

H.1 Generalizing to Other Learning Methods

Our result can directly extend to the random feature(RF) model, adopting the similar analysis for nonlinear model in [26] : the data $\mathbf{x}$ is generated by $\sigma(\mathbf{W}\mathbf{z})$, where $\mathbf{W}$ is the random feature matrix and $\sigma(\cdot)$ is the nonlinear (activation) function. Hence, the RF model can be regarded as training a two-layer neural network where the weights in the first layer are chosen randomly and then fixed and only the output layer is optimized.

Beyond the fixed feature approach, understanding the overparameterization in neural networks is much more challenging. Recently, [8] made important progress towards this aspect. They studied the benign overfitting phenomenon in training a two-layer convolutional neural network, and provided new analysis to tackle the neural network learning process. One possible direction is to generalize our analysis to neural networks by further advancing the techniques in [8].

H.2 Meta-batch Setting

We can consider to incorporate a practical meta-batch setting in our framework, i.e. given a set of tasks in advance, and at each iteration, the task is sampled from the set with replacement uniformly at random and will be visited multiple times during optimization. This is broadly referred as multi-pass SGD [34], while we have studied the single-pass setting, that each task is used only once. For multi-pass SGD, the iterate analysis in Appendix B.3 will be more challenging since we have to consider the dependence on history that leads to a complicated calculation of the expected error matrix. To handle such complication, we may adopt some analysis techniques recently developed for multi-pass SGD [52, 37, 34], and further advance them to analyze meta-batch MAML in meta learning.

Intuitively, we expect in such a setting, meta-batch size (of tasks) will play an important role in determining the excess risk. Specifically, how meta-batch size compares with the effect of data covariance and task diversity (coupled with the number of SGD iterations) will determine whether benign fitting in the overparameterized regime can occur.

H.3 Longer Inner Loops

Another important direction is to study the MAML beyond the one-step gradient update in the inner loop. Two possible cases are discussed as follows.

If the inner loop continually takes gradient updates towards the per-task loss function until converges (additionally regularized by the distance between task-specific parameter and the model parameter), then the algorithm is equivalent to another well-known meta method, iMAML [38]. Under the mixed linear regression model, similarly, we can reformulate the problem as a meta least square problem with modified meta inputs and output responses [15, 3]. Therefore, our analysis can directly generalize to such a setting with modified meta least square problem.

If the inner loop only takes a few steps of gradient updates, the analysis will be more challenging, because the update of meta parameter in the outer loop will involve Hessian, and the analysis of such an algorithm will need to handle the complicated statistical correlation between the gradient and Hessian estimators [30], where new techniques are required. Recently, [12] proposed that one can treat the Hessian as identity operator in theoretical analysis, where the corresponding algorithm is called FO-MAML, and FO-MAML typically achieves a performance similar to MAML in practice [21]. Such relaxation may make the problem more tractable. Then the remaining problem is to extend our technique for overparameterized MAML to FO-MAML, which is an interesting research direction for future study.