# DECENTRALIZED SGD WITH ASYNCHRONOUS, LOCAL, AND QUANTIZED UPDATES

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## ABSTRACT

The ability to scale distributed optimization to large node counts has been one of the main enablers of recent progress in machine learning. To this end, several techniques have been explored, such as asynchronous, decentralized, or quantized communication–which significantly reduce the impact of synchronization, and the ability for nodes to perform several local model updates before communicating–which reduces the frequency of synchronization.

In this paper, we show that these techniques, which have so far been considered independently, can be jointly leveraged to minimize distribution cost for training neural network models via stochastic gradient descent (SGD). We consider a setting with minimal coordination: we have a large number of nodes on a communication graph, each with a local subset of data, performing independent SGD updates onto their local models. After some number of local updates, each node chooses an interaction partner uniformly at random from its neighbors, and averages a possibly quantized version of its local model with the neighbor's model. Our first contribution is in proving that, even under such a relaxed setting, SGD can still be guaranteed to converge to local minima under standard assumptions. The proof is based on a new connection with parallel load-balancing processes, and improves existing techniques by jointly handling decentralization, asynchrony, quantization, and local updates, and by bounding their impact. On the practical side, we implement variants of our algorithm and deploy them onto distributed environments, and show that they can successfully converge and scale for large-scale image classification and translation tasks, matching or even slightly improving the accuracy of previous methods.


## 1 INTRODUCTION

Distributed machine learning has become commonplace, and it is not unusual to encounter systems which distribute model training among tens or even hundreds of nodes. To facilitate distributed training, several independent techniques have been explored, such as communication-reduction, asynchronous updates, and decentralized execution. For some intuition, consider the classical *data-parallel* distribution strategy for SGD (Bottou, 2010). We are in the classical empirical risk minimization setting, where we have a set of samples $S$ from a distribution, and wish to minimize the function $f : \mathbb{R}^d \to \mathbb{R}$, which is the average of losses over samples from $S$ by finding $x^\star = \text{argmin}_x \sum_{s \in S} f_s(x)/|S|$. Assume that we have $P$ compute nodes which can process samples in parallel. Data-parallel SGD consists of parallel iterations, in which each node computes the gradient for one sample, followed by a gradient exchange. Globally, this leads to the iteration:

$$x_{t+1} = x_t - \eta_t \sum_{i=1}^{P} \tilde{g}_t^i(x_t),$$

where $\eta_t$ is the learning rate, $x_t$ is the value of the global parameter, initially $0^d$, and $\tilde{g}_t^i(x_t)$ is the stochastic gradient with respect to the parameter obtained by node $i$ at time $t$.

When extending this strategy to high node counts, two major bottlenecks are *communication* and *synchronization*. In particular, to maintain a consistent view of the parameter $x_t$, the nodes would need to broadcast and receive all gradients, and would need to synchronize with all other nodes, at the end of every iteration. Significant work has been dedicated to removing these two barriers. In particular, there has been progress on *communication-reduced* variants of SGD (Seide et al., 2014; Strom, 2015; Alistarh et al., 2017; Wen et al., 2017; Aji and Heafield, 2017; Dryden et al., 2016; Grubic et al., 2018; Davies et al., 2020), *asynchronous* variants (Recht et al., 2011; Sa et al., 2015;

Duchi et al., 2015)), as well as *large-batch* or *periodic model averaging* methods, which aim to reduce the *frequency* of communication (Goyal et al., 2017; You et al., 2017) and (Chen and Huo, 2016; Stich, 2018)), or even *decentralized synchronous* variants (Lian et al., 2017; Tang et al., 2018; Koloskova et al., 2019). Using such techniques, it is possible to scale SGD to tens or even hundreds of nodes, even for complex objectives such as the training of deep neural networks. However, for modern large-scale models, the communication and synchronization requirements of these parallel SGD can still become infeasible.

**Contribution.** In this paper, we investigate removing these scalability barriers by considering a drastically decoupled setting for parallel SGD: we are given a population of $n$ compute agents, located at vertices of a connected graph, each of which can execute sequential SGD steps on its own local model, based on a fraction of the data. Periodically, after performing some number of local optimization steps, a node can initiate a pairwise interaction with a neighbor that is chosen *uniformly at random*. While decentralized SGD variants (Lian et al., 2017; Tang et al., 2018; Assran et al., 2018; Wang and Joshi, 2018; Koloskova et al., 2019; 2020), and synchronous local SGD variants with global periodic averaging (Chen and Huo, 2016; Stich, 2018; Lin et al., 2018) have been considered in previous work, we are the first to consider decentralization, local updates, asynchrony, and quantization in conjunction. The resulting algorithm allows the asynchrony afforded by the pairwise interactions to be combined with the reduced communication frequency given by multiple local SGD steps, possibly adding communication-compression, thus minimizing the parallelization overheads. The key question is whether such an extremely decentralized SGD variant can still possibly converge. In brief, we show that SGD can still converge in this setting; however, we do observe trade-offs between the convergence rate and the degree of synchronization relaxation, in that the resulting algorithms may need additional gradient steps in order to attain a good solution.

More precisely, in our algorithm, each node $i$ is assigned a set of samples $S^i$, and maintains its own parameter estimate $x^i$. Each node $i$ performs local SGD steps on its model $x^i$ based on its local data, and then picks a neighbor uniformly at random to share information with, via *averaging* of the two models. (To streamline the exposition, we ignore quantization unless otherwise specified.) Effectively, if node $i$ interacts with node $j$, node $i$'s updated model becomes

$$x^i_{T+1} \leftarrow \frac{x^i_{T,H_i} + x^j_{T,H_j}}{2}, \tag{1}$$

where $j$ is the interaction partner, and the input models $x^i_{T,H}$ and $x^j_{T,H}$ have been obtained by iterating the SGD $H_i$ and $H_j$ times, respectively, locally from the previous interaction of either node. The update for node $j$ is symmetric, so that the two parameters match after the averaging step. In this paper, we analyze the above protocol, which we call SwarmSGD.

We show that, perhaps surprisingly, this simple decentralized SGD averaging dynamic with local updates provides strong convergence guarantees for non-convex objectives, under standard assumptions. Specifically, assuming that nodes interact every constant number of local SGD steps on average, SwarmSGD has a $\Theta(\sqrt{n})$ speedup in the non-convex case, matching results from previous work which considered decentralized dynamics but which synchronize upon every SGD step, (Lian et al., 2017; 2018). Our analysis also extends to regular graph topologies and non-blocking (delayed) averaging of iterates, and generally shows that the impact of decentralization, asynchrony, quantization, and local updates can be asymptotically negligible in some parameter regimes.

On the practical side, we show that this algorithm can be mapped to a distributed system setting, where agents correspond to compute nodes, connected by a dense communication topology. Specifically, we apply SwarmSGD to train deep neural networks on image classification and machine translation (NMT) tasks, deployed on the Piz Daint supercomputer (Piz). Experiments confirm the intuition that the average synchronization cost of SwarmSGD per iteration is low: it stays around $10\%$ or less of the batch computation time, and is constant as we increase the number of nodes. For example, using SwarmSGD deployed on 16 nodes, we are able to train a Transformer-XL Vaswani et al. (2017) model on WMT17 (En-Ge) $1.5\times$ faster than a highly-optimized large-batch SGD baseline, and to *slightly higher accuracy*, without additional hyper-parameter tuning. At the same time, our method appears to be faster and more accurate than the previous practical decentralized methods, e.g. (Chen and Huo, 2016; Lian et al., 2017; 2018; Assran et al., 2018), in the same setting.

Importantly, we also note a negative result: in less overparametrized settings such as training ResNet (He et al., 2016) models on ImageNet (Russakovsky et al., 2015), nodes do need to perform more iterations over the dataset relative to the baseline in order to recover full accuracy. This

is predicted by our analysis, and confirms similar findings in previous work (Assran et al., 2018). Overall, however, we believe our family of methods should be well-suited to training very large modern models, or to very large-scale settings, where global synchronization among all nodes is prohibitively expensive.

**Related Work.** The study of decentralized optimization algorithms dates back to Tsitsiklis (1984), and is related to the study of *gossip* algorithms for information dissemination (Kempe et al., 2003; Xiao and Boyd, 2004; Boyd et al., 2006). Gossip is usually studied in one of two models (Boyd et al., 2006): *synchronous*, structured in global rounds, where each node interacts with a randomly chosen neighbor, and *asynchronous*, where each node wakes up at times given by a local Poisson clock, and picks a random neighbor to interact with. The model we consider can be seen as equivalent to the asynchronous gossip model. The key differences between our work and averaging in the gossip model, e.g. Boyd et al. (2006), are that that 1) we consider local SGD steps, which would not make sense in the case of averaging fixed initial values; and 2) the gossip input model is *static* (node inputs are fixed, and node estimates must converge to the true mean), whereas we study a *dynamic* setting, where models are continually updated via SGD. Several optimization algorithms have been analyzed in this setting (Nedic and Ozdaglar, 2009; Johansson et al., 2009; Shamir and Srebro, 2014). Tang et al. (2018); Koloskova et al. (2019) analyze quantization in the synchronous gossip model.

Lian et al. (2017; 2018); Assran et al. (2018) consider SGD-type algorithms in gossip-like models. Specifically, they analyze the SGD averaging dynamic in the non-convex setting *but do not allow nodes to perform local updates or quantize*. In particular, nodes perform joint averaging upon every SGD step. Table 2 in the Appendix summarizes their assumptions, results, and rates. Their results are phrased in the synchronous gossip model, in which nodes interact in a sequence of perfect matchings, for which they provide $O(1/\sqrt{Tn})$ convergence rates under analytical assumptions. Lian et al. (2018) extends these results to a variant of the gossip model where updates can be performed based on stale information, similarly to our non-blocking extension.

Upon careful examination, one can find that their results can be extended to the asynchronous gossip setting we consider, *as long as nodes are not allowed to perform local SGD updates to their models (corresponding to $H = 1$)* or to quantize communication. Extending the analysis of distributed SGD to allow for local steps is challenging even in *centralized* models, see for instance Stich (2018). If we assume $H = 1$, our technique yields similar or better bounds relative to previous work in the decentralized model, as our potential analysis is specifically-tailored to this dynamic interaction model. For instance, for Assran et al. (2018), the speedup with respect to the number of nodes depends on a parameter $C$, which in turn, depends on 1) the dimension of the objective function domain, 2) the number iterations for the graph given by edge sets of all matrices used in averaging to be connected, and the 3) diameter of the aforementioned connected graph. In the dynamic interaction model we consider, the parameter $C$ will be at least *linear* in the number of nodes, which will eliminate any speedup from such an upper bound. We present a systematic comparison in terms of assumptions in Appendix B.

In sum, relative to prior work on decentralized algorithms, our contributions are as follows. We are the first to consider the impact of local updates, asynchrony, and quantization in conjunction with decentralized SGD. We show that the cost for the linear reduction in communication in $H$ given by local steps is at worst a squared convergence decrease in the parameter $H$. Our analysis technique relies on a fine-grained analysis of individual interactions, which is different than that of previous work, and can yield improved bounds even in the case where $H = 1$. By leveraging the lattice-based quantization scheme of Davies et al. (2020), we also allow for communication-compression. From the implementation perspective, the performance of our algorithm is superior to that of previous methods, notably D-PSGD (Lian et al., 2017), AD-PSGD (Lian et al., 2018) and SGP (Assran et al., 2018), mainly due to the ability to take local steps.

References Wang and Joshi (2018); Koloskova et al. (2020) provide analysis frameworks for the synchronous version of decentralized SGD with local updates, and possibly changing topologies. We emphasize that this is a different setting from ours, since it requires agents to take an equal number of gradient steps, and therefore do not allow for agents to progress at different speeds (asynchrony). In addition, we support quantization, and validate our analysis at scale.

Other references which consider dynamic interaction models are Nedic et al. (2017), who present a gradient tracking algorithm in a different dynamic graph model, and Hendrikx et al. (2018), who achieve exponential convergence rates in a gossip model where transmissions are synchronized across *edges*. The algorithm they consider is a more complex instance of accelerated coordinate

descent, and is therefore quite different from the simple dynamics we consider. Neither reference considers local updates, nor real-world scalability.

## 2 PRELIMINARIES

**The Distributed System Model.** We consider a model which consists of a set of $n \geq 2$ anonymous agents, or nodes, each of which is able to perform local computation. We assume that communication network of nodes is a $r$-regular graph $G$ with a spectral gap $\lambda_2$(second smallest eigenvalue of the Laplacian of $G$). The execution proceeds in discrete *steps*, where in each step we sample an edge of the graph $G$ uniformly at random and we allow the agents corresponding to the edge endpoints interact. Each of the two chosen agents updates its state according to a state update function, specified by the algorithm. The basic unit of time is a single pairwise interaction between two nodes. Notice however that in a real system $\Theta(n)$ of these interactions could occur in parallel. Thus, a standard global measure is *parallel time*, defined as the total number of interactions divided by $n$, the number of nodes. Parallel time intuitively corresponds to the *average* number of interactions per node to convergence. We note that our model is virtually identical to the population model of distributed computing (Angluin et al., 2006), or to asynchronous gossip models (Xiao and Boyd, 2004).

**Stochastic Optimization.** We assume that the agents wish to minimize a $d$-dimensional, differentiable function $f : \mathbb{R}^d \to \mathbb{R}$. Specifically, we will assume the empirical risk minimization setting, in which agents are given access to a set of data samples $S = \{s_1, \ldots, s_m\}$ coming from some underlying distribution $\mathcal{D}$, to a function $\ell_i : \mathbb{R}^d \to \mathbb{R}$ which encodes the loss of the argument at the sample $s_i$. The goal of the agents is to converge on a model $x^*$ which minimizes the empirical loss, that is

$$x^* = \text{argmin}_x f(x) = \text{argmin}_x (1/m) \sum_{i=1}^{m} \ell_i(x). \tag{2}$$

In this paper, we assume that the agents employ these samples to run a decentralized variant of SGD, described in detail in the next section. For this, we will assume that each agent $i$ has access to *stochastic gradients* $\widetilde{g}_i$ of the function $f$, which are functions such that

$$\mathbb{E}[\widetilde{g}_i(x)] = \nabla f(x). \tag{3}$$

Stochastic gradients can be computed by each agent by sampling i.i.d. the distribution $\mathcal{D}$, and computing the gradient of $f$ at $\theta$ with respect to that sample. (Our analysis can be extended to the case where each agent is sampling from its own partition of data, see Section I in the Appendix.) We will assume a subset of the following conditions about the objective function:

1. **Smooth Gradients**: The gradient $\nabla f(x)$ is $L$-Lipschitz continuous for some $L > 0$, i.e. for all $x, y \in \mathbb{R}^d$:
$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|. \tag{4}$$
2. **Bounded Variance**: The variance of the stochastic gradients is bounded i.e. for all $x \in \mathbb{R}^d$ and agent $i$:
$$\mathbb{E}\left\|\widetilde{g}_i(x) - \nabla f(x)\right\|^2 \leq \sigma^2. \tag{5}$$
3. **Bounded Second Moment**: The second moment of the stochastic gradients is bounded by some $M^2 > 0$, i.e. for all $x \in \mathbb{R}^d$ and agent $i$:
$$\mathbb{E}\left\|\widetilde{g}_i(x)\right\|^2 \leq M^2. \tag{6}$$

Note that throughout this paper for any random variable $X$, by $\mathbb{E}\|X\|^2$ we mean $\mathbb{E}[\|X\|^2]$.

## 3 THE SWARMSGD ALGORITHM

**Algorithm Description.** We now describe a decentralized variant of SGD, designed to be executed by a population of $n$ nodes, interacting over the edges of $r$-regular graph $G$. We assume that each node $i$ has access to local stochastic gradients $\widetilde{g}^i$, and maintains a model estimate $X^i$. For simplicity, we will assume that this initial estimate is $0^d$ at each agent, although its value may be arbitrary. We assume that each agent performs SGD steps on its local estimate $X^i$. At random times given by a clock of Poisson rate, we pick two neighboring agents $i$ and $j$ uniformly at random from $G$, and have them average their estimates. More precisely, the interaction can be described as follows:

Note that, for simplicity, the pseudocode is expressed sequentially, although in practice nodes perform their local SGD steps in parallel. Also, we have assumed a constant learning rate; we will detail the precise update values of the learning rate in the next section.

**Algorithm 1** Sequential SwarmSGD pseudocode for each interaction between nodes $i$ and $j$.

% Let $G$ be $r$-regular graph.
% Sample an edge $(i, j)$ of $G$ uniformly at random.
**Require:** agents $i$ and $j$ chosen for interaction
% choose $H_i$ and $H_j$
% agent $i$ performs $H_i$ local SGD steps
**for** $q = 1$ **to** $H_i$ **do**
  $X^i \leftarrow X^i - \eta \widetilde{g}^i(X^i)$
**end for**
% agent $j$ performs $H_j$ local SGD steps
**for** $q = 1$ **to** $H_j$ **do**
  $X^j \leftarrow X^j - \eta \widetilde{g}^j(X^j)$
**end for**
% agents *average* their estimates coordinate-wise
$avg \leftarrow (X^i + X^j)/2$
$X^i \leftarrow avg$
$X^j \leftarrow avg$

The main intuition behind the algorithm is that the independent SGD steps will allow nodes to explore local improvements to the objective function on their subset of the data, while the averaging steps provide a decentralized way for the models to remain in sync. We will show that, as long as the maximum number of local steps is bounded, this procedure still converges, in the sense that gradients calculated at average value of models are vanishing as we increase the number of interactions.

## 4 THE CONVERGENCE OF SWARMSGD

We begin by analyzing the convergence of the baseline SwarmSGD algorithm. Fix an integer $H \geq 1$. First, we will consider a version of the algorithm where $H_i$ and $H_j$ are independent geometrically distributed random variables with mean $H$. This corresponds to interaction times being chosen by a Poisson clock of constant rate. To handle the fact that the number of local steps upon an interaction is a random variable, in this first case we will require stochastic gradients to satisfy the *bounded second moment assumption*, specified above. Intuitively, this is required since otherwise the "distance travelled" by a node could be virtually unbounded. In this setting, we prove the following:

**Theorem 4.1.** *Let $f$ be an non-convex, $L$-smooth function, whose stochastic gradients satisfy the bounded second moment assumption above. Let the number of local stochastic gradient steps performed by each agent upon interaction be a geometric random variable with mean $H$. Let the learning rate we use be $\eta = n/\sqrt{T}$. Define $\mu_t = \sum_{i=1}^{n} X_t^i/n$, where $X_t^i$ is a value of model $i$ after $t$ interactions, be the average of the local parameters. Then, for learning rate $\eta = n/\sqrt{T}$ and any number of interactions $T \geq n^4$:*

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{4(f(\mu_0) - f(x^*))}{\sqrt{T}H} + \frac{2304H^2 \max(1, L^2)M^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2} + 1).$$

**Discussion.** We briefly pause to comment on this bound. Notice that each of the upper bound terms has an intuitive interpretation: the first represents the reduction in loss relative to the initialization, and gets *divided* by the number of local steps $H$, since progress is made in this term in every local step; the second represents the influence of the variance of each individual local step multiplied by a term which bounds the impact of the graph topology on the convergence. In particular, this term negatively impacts convergence fo large values of $H$, $L$, and $M$, but gets dampened if the graph is well-connected (i.e. large $\lambda_2$, for example in the case of the complete graph $\lambda_2 = n$).

Second, importantly, notice that time $T$ in this bound counts the *total* number of interactions. However, in practice $\Theta(n)$ pairwise interactions will occur in parallel. Therefore, it is natural to replace $T$ by $nT$ to obtain speedup with respect to wall-clock time. At the same time, notice that this speedup is dampened in the second term by the non-trivial additional variance due to noisy local gradient steps, a fact which we will revisit in the experimental section.

**Proof Overview.** At a high level, the argument rests on two technical ideas. The first idea is to show that, due to the pairwise averaging process, and in spite of the local steps, the nodes' parameters will have to remain concentrated around their mean $\mu_t$. The second is to show that, even though

stochastic gradients are taken at *perturbed, noisy* estimates of this mean, the impact of this noise on convergence can be bounded. We note that neither idea is straightforward to implement.

In particular, the main technical difficulty in the proof is to correctly "encode" the fact that parameters are well concentrated around the mean. For this, we define the potential $\Gamma_t$, which denotes the variance of models after $t$ interactions. More formally,

$$\Gamma_t = \sum_{i=1}^{n} \|X_t^i - \mu_t\|^2, \tag{7}$$

where $\mu_t = \sum_{i=1}^{n} X_t^i/n$. We bound the expected evolution of $\Gamma_t$ in terms of $r$, the degree of nodes in interaction the graph $G$, and $\lambda_2$, the second smallest eigenvalue of the Laplacian of $G$. For both algorithm variants we consider, our bound depends on the learning rate, number of local steps, and the bound provided by the assumption on the stochastic gradients (the bound $M^2$ and $\sigma^2$). The critical point is that the upper bound on the expectation of $\Gamma_t$ *does not depend* on the number of interactions $t$. Our approach is roughly similar to the analysis of static load balancing schemes, e.g. Berenbrink et al. (2009). Two key elements of novelty in our case are that (1) for us the load balancing process is *dynamic*, in the sense that loads (gradients) get continually added; (2) the load-balancing process we consider is multi-dimensional, whereas usually the literature considers simple scalar weights. The complete argument is presented in the Appendix.

Our technique is quite powerful, as it allows for a number of non-trivial extensions:

**Extension 1: Removing the second-moment bound.** In the first extension of the algorithm, we assume that the number of local steps performed by each agent is fixed and is equal to $H$. In this case we are able to remove the bounded second moment assumption, and are able to prove convergence by just assuming that stochastic gradients have bounded *variance*:

**Theorem 4.2.** *Let $f$ be an non-convex, $L$-smooth, function satisfying condition 5, whose minimum $x^\star$ we are trying to find via the SwarmSGD procedure given in Algorithm 1. Let $H$ be the number of local stochastic gradient steps performed by each agent upon interaction. Define $\mu_t = \sum_{i=1}^{n} X_t^i/n$, where $X_t^i$ is a value of model $i$ after $t$ interactions. For learning rate $\eta = \frac{n}{\sqrt{T}}$ and $T = O\left(n^4 H^2 \max(1, L^2)(\frac{r^2}{\lambda_2^2} + 1)^2\right)$ we have that:*

$$\frac{\sum_{i=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2}{T} \leq \frac{1}{\sqrt{T}H}\mathbb{E}[f(\mu_0) - f(\mu_t)] + \frac{388H^2 \max(1, L^2)\sigma^2}{\sqrt{T}}\left(\frac{r^2}{\lambda_2^2} + 1\right).$$

**Discussion.** Notice that, relative to Theorem 4.1, we have the same quadratic dependency on the number of local steps $H$ and on $L$, but now the second moment bound is replaced by the variance term. Further, this our analysis can be extended to the case where each agent is sampling from its own partition of data, matching the setting assumed by Lian et al. (2017), and our implementation. Please see Section I in the Appendix for details.

**Extension 2: Non-blocking averaging.** Algorithm 1 is *blocking*, in that it requires both nodes to complete their local iterations at the same time before they can interact. In practice, nodes can average their local updates without synchronizing, as follows. Each node $i$ keeps two copies of the model: the *live copy* $X_i$ (on which local SGD iterations are applied) and the *communication copy* $Y_i$, which can be accessed asynchronously by communicating nodes. When completing its local steps, a node $i$ first checks if some other node averaged against its *communication copy* $Y_i$ since its last communication step. If the answer is yes, it simply applies its locally-generated update to its communication model $Y_i$, updates the live copy so that $X_i = Y_i$, and proceeds with the next iteration of local computation. If no other node has averaged against its communication copy, then the node actively seeks a random communication partner $j$, and averages its live copy against its model $Y_j$, updating both to $(X_i + Y_j)/2$. The node then proceeds with the next iteration of local computation. Please see Appendix F for the precise definition of the algorithm, and for the formal convergence guarantee for this variant.

**Extension 3: Quantization.** For large models, the cost of the averaging step can become significant, due to bandwidth constraints. To remove the bandwidth bottleneck, we allow the averaging step to be performed with respect to *quantized* versions of the two models. While communication-compression has been considered in a decentralized context before, e.g. Tang et al. (2018); Lu and Sa (2020), our proposal approaches it differently. Instead of modifying the algorithm or maintaining neighbor information at each node, we instantiate the lattice-based quantization scheme recently introduced

| Model / Dataset | SGD Top-1 | LB SGD Top-1 | SwarmSGD | Parameters |
|---|---|---|---|---|
| ResNet20 / CIFAR-10 | 91.7% (200 epochs) | 91.5% (200 epochs) | 91.79% (280 epochs) | 4 local steps |
| ResNet18 / ImageNet | 69.76 % (90 epochs) | 69.17% (90 epochs) | 69.79% (240 epochs) | 3 local steps |
| ResNet50 / ImageNet | 76.14% (90 epochs) | 75.43% (90 epochs) | 75.68% (240 epochs) | 2 local steps |

Table 1: Parameters for full Top-1 validation accuracy on CIFAR-10 and ImageNet running on 32 nodes. Swarm step count represents *local SGD steps per model* between two averaging steps, epochs are counted in terms of total passes over the data.

in Davies et al. (2020), which has the useful property that its error does not depend on the *norm of the inputs*, but only *on their variance*, i.e. on the distance between them. In Appendix G, we prove that a variant scheme can be used in the context of our algorithm, ensuring the same notion of convergence (up to constants due to quantization) as the basic algorithm.

## 5 Experimental Results

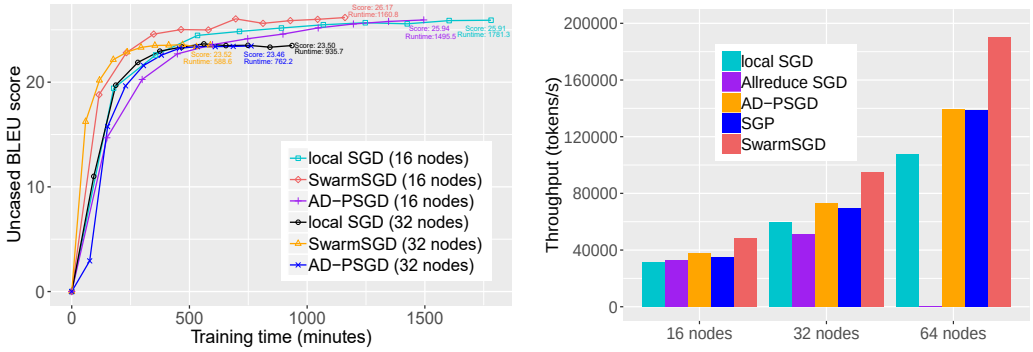
In this section, we validate our analysis, by applying the algorithm to training deep neural networks for image classification and speech recognition. We map the algorithm onto a multi-node supercomputing setting, in which we have a large number of compute nodes, connected by fast communication links. The key overhead in this setting is *synchronization*: at large node counts, the cost of synchronizing all nodes so they execute in lock-step can be very high, see e.g. Li et al. (2019) for numerical results on different workloads. SwarmSGD mitigates this overhead, since nodes synchronize only sporadically and in pairs. Harnessing the computational power of this large-scale distributed setting is still an underexplored area (Ben-Nun and Hoefler, 2019).

**Target System and Implementation.** We run SwarmSGD on the CSCS Piz Daint supercomputer, which is composed of Cray XC50 nodes, each with a Xeon E5-2690v3 CPU and an NVIDIA Tesla P100 GPU, using a state-of-the-art Aries interconnect. Please see (Piz) for hardware details. We implemented SwarmSGD in Pytorch and TensorFlow using NCCL and MPI-based primitives. Both variants implement the version with non-blocking averaging. The Pytorch implementation is on top of SGP framework of SGP (Assran et al., 2018), and uses SwarmSGD to train ResNets on the CIFAR-10/100 (Krizhevsky et al., 2014) and ImageNet (Russakovsky et al., 2015) datasets, while we use the TensorFlow implementation to train a much larger Transformer-XL model (Vaswani et al., 2017) on the WMT17 (En-Ge) dataset. We note that all algorithms used the same communication topology (fully-connected with random pairings), and that SGP was run with overlap factor 1, as suggested in Assran et al. (2018).

**Training Process.** Our training methodology follows data-parallel training, with some differences due to decentralization, and is identical to previous work on decentralized and local SGD, e.g. Lian et al. (2017); Assran et al. (2018); Lin et al. (2018). Training proceeds in *epochs*, each of which corresponds to processes collectively performing a full pass over the dataset. At the beginning of each epoch, we re-shuffle the dataset and partition it among processes (Lin et al., 2018).

As noted in previous work (Lian et al., 2017; 2018; Assran et al., 2018) variants of decentralized SGD are not always able to recover sequential SGD accuracy within the same number of epochs as this baseline. This is justified by Theorems 4.1 and 4.2, which predict that the slower mixing (and higher local model variance) can affect convergence. Thus, in some experiments, we will allow the decentralized schemes to execute for more epochs, by a constant *multiplier* factor between 1 and 3. Once we have fixed the number of epochs, *we do not alter the other training hyperparameters*: in particular, the learning rate schedule, momentum and weight decay terms are identical to sequential SGD, for each individual model.

**Accuracy Recovery and Speed.** We first examined whether SwarmSGD can in fact recover full accuracy versus the sequential or large-batch SGD baselines. In Table 1 we provide an overview of parameter values to recover or exceed large-batch SGD accuracy (following (Goyal et al., 2017)) using SwarmSGD, on the ResNet/ImageNet/CIFAR tasks. We execute for 32 nodes on ImageNet, and 8 nodes on CIFAR-10. (Local batch sizes are 256 for ResNet20 and ResNet18, and 128 for ResNet50. Quantization is not applied.) The results show that Swarm can recover or slightly exceed the accuracy of the large-batch baselines, but may require significant additional passes over the data (up to $2.7\times$) to do so, which would negate its performance benefits in this setting. This somewhat negative finding is in line with previous work on decentralized methods (Assran et al., 2018).




(a) Convergence versus Time.

(b) Throughput vs previous work. Higher is better.

Figure 1: Convergence and Scalability on the Transformer/WMT Task with multiplier = 1.

Next, we examine accuracy for the WMT17 task, given in Figure 1(a), in accuracy-vs-time format, for 16 and 32 nodes, executing for 10 global epochs. Here, the large-batch SGD (LB-SGD) baseline (BLEU score 26.1 at 16 nodes) is a poor alternative at high node counts: its throughput is very low, due to the size of the model (see Figure 1(b)). At 16 nodes, Swarm *slightly exceeds* the baseline accuracy at 26.17 BLEU, for an end-to-end speedup of $\sim 1.5\times$. In the same setting, Swarm outperforms all other decentralized methods (the fastest previous method, AD-PSGD, is $30\%$ slower, and less accurate), both in terms of BLEU score, and in terms of end-to-end time. At 32 nodes, all decentralized methods reach lower scores ($\sim 23.5$) after 10 epochs. However, we observed experimentally that running Swarm for an additional $5$ epochs at 32 nodes recovered a BLEU score of $\sim 25.9$, $30\%$ faster than the 16-node version in terms of end-to-end time (omitted for readability).

In addition, we investigated 1) the accuracy of the *real average* of all models throughout training: it is usually more accurate than an arbitrary model, but not significantly, corroborating the claim that individual models tend to stay close to the mean; 2) the influence of the number of local steps on accuracy: perhaps surprisingly, we were able to recover baseline accuracy on ResNet18/ImageNet for up to $4$ local steps; 3) the impact of quantization on convergence, where we were able to recover accuracy when applying 8-bit model quantization to Swarm. We encourage the reader to examine the experimental report in the Appendix, which contains data on these experiments.

**Discussion.** Generally, the performance and accuracy of SwarmSGD are superior to previous decentralized methods (see Figure 1 for an illustration, and the Appendix for a performance breakdown). In particular, a closer examination of execution times shows that time per node per batch (including communication and computation) is largely *constant* with respect to the number of nodes, which should give our method close-to-ideal scaling behaviour. This advantage relative to previous schemes, notably AD-PSGD, comes mainly from the *reduction in communication frequency*: Swarm communicates less often, and therefore incurs lower average communication cost.

The main disadvantage of Swarm is that, similar to previous decentralized methods, it may need additional data passes in order to fully recover accuracy at high node counts. However, we also note that our method does not benefit or require the same level of hyperparameter tuning as large-batch SGD, e.g. (Goyal et al., 2017). We find it interesting that this accuracy issue is less prevalent in the context of large, over-parametrized models, such as the Transformer, where Swarm could be a practically-viable alternative to large-batch SGD within the same number of epochs.

## 6 Conclusion and Future Work

We have analyzed the convergence of SGD in an extremely decoupled model of distributed computing, in which nodes mostly perform independent SGD updates, interspersed with intermittent pairwise averaging steps. We have shown that SGD is able to still converge in this restrictive setting, and moreover can even achieve speedup in the number of nodes in terms of iteration time. The empirical results in a supercomputing environment complement and confirm our analysis, showing that this method can be competitive with standard solutions. SwarmSGD provides good analytic and empirical properties, even though each model only directly observes a small fraction of the data. Our work opens several avenues for extensions. One natural but non-trivial direction is to study compressed communication; another is to further generalize the bounds in terms of their the assumptions on the objective and on the system setting considered.

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# A SUMMARY OF THE APPENDIX SECTIONS

Appendix contains the following sections:

- In **Section B** we compare SwarmSGD with some of the existing algorithms. We list convergence bounds and the assumptions needed to achieve them.
- In **Section C** we provide crucial properties for the load balancing on the graph.
- In **Section D** we provide definitions for the local steps we use in the later sections.
- In **Section E** we provide the sketch of proof of the Theorem 4.1, which shows the convergence of SwarmSGD assuming the second moment bound on the gradients. Recall that the number of local steps in this case is a geometric random variable with mean $H$.
- In **Section F** we provide the of proof for the non-blocking version of the swarm SGD algorithm. We again assume the second moment bound on the gradients and the number of local steps is a geometric random variable with mean $H$.
- In **Section G** we provide the of proof for the quantized version of the swarm SGD algorithm. We again assume the second moment bound on the gradients and the number of local steps is a geometric random variable with mean $H$.
- In **Section H** we prove the Theorem 4.2. In this case we do not assume the second moment bound and the number of the local steps performed by each agent is a fixed number $H$.
- In **Section I** we show that SwarmSGD analysis can be extended to the case where each agent is sampling from its own partition of data.
- In **Section J** we provide additional experimental results for SwarmSGD.

# B COMPARISON OF RESULTS

In this section we compare convergence rates of existing algorithms, while specifying the bounds they require for convergence. In the tables $T$-corresponds to the parallel time and $n$ is a number of processes. We use the following notations for needed bounds(or assumptions):

1. $\boldsymbol{\sigma^2}$ - bound on the variance of gradient.
2. $\mathbf{M^2}$ - bound on the second moment of gradient.
3. $\mathbf{d}$ - bounded dimension.
4. $\boldsymbol{\lambda_2}$ - bounded spectral gap of the averaging matrix(interaction graph in case of SwarmSGD).
5. $\boldsymbol{\tau}$ - bounded message delay.
6. $\mathbf{r}$ - interaction graph is $r$-regular.
7. $\boldsymbol{\Delta}$ - bounded diameter of interaction graph.

| Algorithm | Assumptions | Convergence Rate |
|---|---|---|
| SwarmSGD | $\sigma^2, \lambda_2, r$ | $O(1/\sqrt{Tn})$ |
| SwarmSGD | $M^2, \lambda_2, r$ | $O(1/\sqrt{Tn})$ |
| AD-PSGD Lian et al. (2018) | $\sigma^2, \lambda_2, \tau$ | $O(1/\sqrt{Tn})$ |
| SGP Assran et al. (2018) | $\sigma^2, d, \Delta, \tau$ | $O(1/\sqrt{Tn})$ |

Table 2: **Comparison of theoretical results in the non-convex case.**

**Discussion.** We compare in more detail against Lian et al. (2018) and Assran et al. (2018), since these are the only other papers which do not require explicit global synchronization in the form of rounds. (By contrast, e.g. (Wang and Joshi, 2018; Koloskova et al., 2020) require that nodes synchronize in rounds, so that at every point in time each node has taken the same number of steps.)

In Assran et al. (2018), all nodes perform gradient steps at each iteration, but averaging steps can be delayed by $\tau$ iterations. Unfortunately, in this case the mixing time depends on the dimension - $d$

(more precisely, it contains $\sqrt{d}$ factor!), on the delay bound $\tau$, and on $\Delta$, defined as the number of iterations over which the interaction graph is well connected. Additionally, the analysis is not suitable for the random interactions. On the other hand, Lian et al. (2018) consider random interaction matrices and do not require the agents to perform the same number of gradient steps. Unlike our model, in their case more than two nodes can interact during the averaging step.

To circumvent the global synchronization issue, Lian et al. (2018) allow agents to have outdated views during the averaging step. Yet, we would like to emphasize that they require blocking during the averaging steps, while we allow some amount of non-blocking property. By some amount we means that algorithm needs blocking only in the case when some node takes more than two consecutive iterations to complete it's local gradient steps. This means that for each node $i$ to complete $H_i$ local steps should not more take more then $O(n)$ global steps (since each node interacts with probability $2/n$ at each step), this assumption also holds for Lian et al. (2018).

In summary, our algorithm reduces the synchronization required by averaging steps, by considering pairwise interactions and by introducing local steps and providing non-blocking version of the algorithm as well (In SGP and AD-PSGD, agents perform one local step and one averaging step per iteration). We would like to point out that we also allow a random number of local steps between interactions in the case when we have second moment bound on the stochastic gradient, which reduces synchronization costs even further. Finally, our algorithm requires $T \geq O(n^4)$ number of iterations to achieve the convergence rate of $O(1/\sqrt{Tn})$ in the case of blocking algorithm and $T = \Omega(n^6)$ in general. (By contrast, Lian et al. (2018) requires $T = \Omega(n^6)$.)

## C Properties of the Load Balancing

In this section provide the useful lemmas which will help as in the later sections.

We are given a simple undirected graph $G$, with $n$ nodes(for convenience we number them from 1 to $n$) and edge set $E$. Each node is adjacent to exactly $r$ nodes.

Each node $i$ of graph $G$ keeps local vector model $X_t^i \in \mathbb{R}^d$ ($t$ is the number of interactions or steps), let $X_t = (X_t^1, X_t^2, ..., X_t^n)$ be a vector of local models at step $t$.

Interaction(step) is defined as follows: we pick edge $e = (u, v)$ of $G$ uniformly at random and update the vector models correspondingly.

Let $\mu_t = \sum_{i=1}^n X_t^i/n$ be the average of models at step $t$ and let $\Gamma_t = \sum_{i=1}^n \|X_t^i - \mu_t\|^2$ be a potential at time step $t$.

Let $\mathcal{L}$ be the Laplacian matrix of $G$ and let let $\lambda_2$ be a second smallest eigenvalue of $\mathcal{L}$.

First we state the following lemma from Ghosh and Muthukrishnan (1996)

**Lemma C.1.**

$$\lambda_2 = \min_{v=(v_1,v_2,...,v_n)} \Big\{ \frac{v^T \mathcal{L} v}{v^T v} | \sum_{i=1}^n v_i = 0 \Big\} \tag{8}$$

Now, we show that Lemma C.1 can be used to lower bound $\sum_{(i,j)\in E} \|X_t^i - X_t^j\|^2$:

**Lemma C.2.**

$$\sum_{(i,j)\in E} \|X_t^i - X_t^j\|^2 \geq \lambda_2 \sum_{i=1}^n \|X_t^i - \mu_t\|^2 = \lambda_2 \Gamma_t. \tag{9}$$

*Proof.* Observe that

$$\sum_{(i,j)\in E} \|X_t^i - X_t^j\|^2 = \sum_{(i,j)\in E} \|(X_t^i - \mu_t) - (X_t^j - \mu_t)\|^2. \tag{10}$$

Also, notice that Lemma C.1 means that for every vector $v = (v_!, v_2, ..., v_n)$ such that $\sum_{i=1}^n v_i = 0$, we have :

$$\sum_{(i,j)\in E} (v_i - v_j)^2 \geq \lambda_2 \sum_{i=1}^n v_i^2. \tag{11}$$

Since $\sum_{i=1}^{n}(X_t^i - \mu_t)$ is a 0 vector, we can apply the above inequality to the each of $d$ components of the vectors
$X_t^1 - \mu_t, X_t^2 - \mu_t, ..., X_t^n - \mu_t$ separately and by properties of two norm we get the proof of the lemma. □

## D Definitions for the Local Steps

In this section we provide the formal definition of the local steps performed by our algorithms. Recall that $X_t^i$ is a local model of node $i$ at step $t$. Let $H_t^i$ be a number of local steps node $i$ performs in the case when it is chosen for interaction at step $t+1$. $H_t^i$ can be fixed throughout the whole algorithm, that is : for each time step $t$ and node $i$, $H_t^i = H$. The choices of $H_t^i$ depend on the existence of the second moment bound (6). Let:

$$\widetilde{h}_i^0(X_t^i) = 0.$$

and for $1 \leq q \leq H_t^i$ let:

$$\widetilde{h}_i^q(X_t^i) = \widetilde{g}_i(X_t^i - \sum_{s=0}^{q-1} \eta \widetilde{h}_i^s(X_t^i)).$$

Also , for $1 \leq q \leq H_t^i$, let

$$h_i^q(X_t^i) = \mathbb{E}[\widetilde{g}_i(X_t^i - \sum_{s=0}^{q-1} \eta \widetilde{h}_i^s(X_t^i))] = \nabla f(X_t^i - \sum_{s=0}^{q-1} \eta \widetilde{h}_i^s(X_t^i)).$$

be the expected value of $\widetilde{h}_i^q(X_t^i)$ taken over the randomness of the stochastic gradient $\widetilde{g}_i$. Let $\widetilde{h}_i(X_t^i)$ be the sum of $H_t^i$ local stochastic gradients we computed:

$$\widetilde{h}_i(X_t^i) = \sum_{q=1}^{H_t^i} \widetilde{h}_i^q(X_t^i).$$

For the simplicity we avoid using index $t$ in the left side of the above definition, since it is clear that if the local steps are applied to model $X_t^i$ we compute them in the case when node $i$ interacts at step $t+1$. The update step in Swarm SGD (Algorithm 1) is (before averaging)

$$X_{t+1}^i = X_t^i - \eta \widetilde{h}_i(X_t^i) = X_t^i - \eta \sum_{q=1}^{H_t^i} \widetilde{h}_i^q(X_t^i) = X_t^i - \eta \sum_{q=1}^{H_t^i} \widetilde{g}_i(X_t^i - \sum_{s=0}^{q-1} \eta \widetilde{h}_i^s(X_t^i))$$

Notice that

$$\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 = \mathbb{E}\|\widetilde{g}_i(X_t^i - \sum_{s=0}^{q-1} \eta \widetilde{h}_i^s(X_t^i))\|^2 \overset{\text{Assumption 6}}{\leq} M^2. \tag{12}$$

## E Analysis under second moment bound and random number of local steps

In this section we consider Algorithm 1, where for each node $i$, $H_i$ is a geometric random variable with mean $H$. We also assume that (6) holds. We provide only the sketch of the proof since the proof for the non-blocking version of algorithm in Section F is more general. If nodes $i$ and $j$ interact at step $t+1$ and their local models have values of $X_t^i$ and $X_t^j$ after step $t$. Their new model values become :

$$X_{t+1}^i = X_{t+1}^j = (X_t^i + X_t^j - \eta \widetilde{h}_i(X_t^i) - -\eta \widetilde{h}_j(X_t^j))/2.$$

Recall that $\mu_t$ is average of the values of models after time step $t$ and $\Gamma_t = \sum_{i=1}^{n} \|X_t^i - \mu_t\|^2$ First of all we can prove that (See Lemma F.1, the result can be achieved even though algorithms differ):

$$\mathbb{E}[\Gamma_{t+1}] \leq \mathbb{E}[\Gamma_t](1 - \frac{\lambda_2}{2rn}) + (2 + \frac{4r}{\lambda_2})\eta^2 \sum_{i=1}^{n} \Big(4\mathbb{E}\|\widetilde{h}_i(X_t^i)\|^2\Big). \tag{13}$$

Next we have that Lemma F.2 holds and this means that Lemmas F.3 and F.4 hold as well, we get:

$$\mathbb{E}[\Gamma_t] \leq (\frac{40r}{\lambda_2} + \frac{80r^2}{\lambda_2^2})n\eta^2 H^2 M^2. \tag{14}$$

and

$$\sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -\widetilde{h}_i(X_t^i)\rangle \leq 2HL^2\mathbb{E}[\Gamma_t] - \frac{3Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2 + 12H^3 nL^2M^2\eta^2. \tag{15}$$

Next in the similar fashion as in the proof of theorem F.8 we can show that :

$$\mathbb{E}[f(\mu_{t+1})] \leq \mathbb{E}[f(\mu_t)] + \frac{2\eta}{n^2}\sum_{i=1}^{n}\mathbb{E}\Big\langle \nabla f(\mu_t), -\widetilde{h}_i(X_t^i)\Big\rangle + \frac{20L\eta^2H^2M^2}{n^2}. \tag{16}$$

This allows to show that:

**Theorem 4.1.** *Let $f$ be an non-convex, $L$-smooth function, whose stochastic gradients satisfy the bounded second moment assumption above. Let the number of local stochastic gradient steps performed by each agent upon interaction be a geometric random variable with mean $H$. Let the learning rate we use be $\eta = n/\sqrt{T}$. Define $\mu_t = \sum_{i=1}^{n} X_t^i/n$, where $X_t^i$ is a value of model $i$ after $t$ interactions, be the average of the local parameters. Then, for learning rate $\eta = n/\sqrt{T}$ and any number of interactions $T \geq n^4$:*

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{4(f(\mu_0) - f(x^*))}{\sqrt{T}H} + \frac{2304H^2\max(1, L^2)M^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2} + 1).$$

*Proof.* We again skip the calculations and follow steps from the proof of theorem F (Note that constants can be improved, but for the simplicity we keep them the same). Which after applying (14) and (15) results in:

$$\mathbb{E}[f(\mu_{t+1})] - \mathbb{E}[f(\mu_t)] \leq (\frac{160r}{\lambda_2} + \frac{320r^2}{\lambda_2^2})\frac{\eta^3H^3M^2L^2n}{n^2} - \frac{Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2 + \frac{76H^3L^2M^2\eta^3}{n} + \frac{20L\eta^2H^2M^2}{n^2}.$$

once we sum up the above inequality $t = 0$ to $t = T - 1$ and massage terms we get (Additionally recall that $\mathbb{E}[f(\mu_T)] \geq f(x^*)$):

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{4n(f(\mu_0) - f(x^*))}{TH\eta} + (\frac{640r}{\lambda_2} + \frac{1280r^2}{\lambda_2^2})\eta^2H^2M^2L^2T + \frac{80TL\eta HM^2}{n} + 304TH^2L^2M^2\eta^2.$$

Finally since $\eta = n/\sqrt{T} \leq \frac{1}{n}$(because $T \geq n^4$) we get the proof of the lemma. Note that the difference between this theorem and theorem F.8 is that we have lower bound of $n^4$ for $T$ here, instead of $n^4(n+1)^2$, the reason is that we are not required to use Lemmas F.7 and F.5 since our algorithm allows blocking and this means that interacting agents do not have incomplete values for the models. □

# F Analysis of the Nonblocking Variant, with Second Moment Bound and random number of local steps

First we define how non-blocking property changes our interactions. Let $i$ and $j$ be nodes which interact at step $t+1$, we set

$$X_{t+1/2}^i = \frac{X_t^i}{2} + \frac{X_t^{j'}}{2},$$
$$X_{t+1/2}^j = \frac{X_t^j}{2} + \frac{X_t^{i'}}{2},$$

and

$$X^i_{t+1} = X^i_{t+1/2} - \eta \widetilde{h}_i(X^i_t),$$
$$X^j_{t+1} = X^j_{t+1/2} - \eta \widetilde{h}_j(X^j_t),$$

where for each node $k$, if $p^k_t + 1$ is the last time interacting before and including step $t$:

$$X^{k\prime}_t = X^i_{p^k_t+1/2} = X_{p^k_t+1} + \eta \widetilde{h}_k(X^k_{p^k_t}) = X^k_t + \eta \widetilde{h}_k(X^k_{p^k_t}). \tag{17}$$

Intuitively the last definition means that node $k$ has computed $X^i_{p^k_t+1/2}$ but has not finished computing $X_{p^k_t+1}$, hence when some other node tries to read $X_{p^k_t+1}$, it reads the value which is missing local gradient update step, but it does not have to wait for node $k$ to finish computing. Since $p^k_t + 1$ is the last step node $k$ interacted we have that $X^k_t = X^k_{p^k_t+1}$. More formally: Notice the differences

**Algorithm 2** Sequential non-blocking SwarmSGD pseudocode for each interaction between nodes $i$ and $j$.

% Let $G$ be $r$-regular graph.
% Sample an edge $(i, j)$ of $G$ uniformly at random.
**Require:** agents $i$ and $j$ chosen for interaction, $i$ is initiator
% choose $H_i$ and $H_j$
% agent $i$ performs $H_i$ local SGD steps
$S^i \leftarrow X_i$
**for** $q = 1$ **to** $H_i$ **do**
  $X^i \leftarrow X^i - \eta \widetilde{g}^i(X^i)$
**end for**
% agent $j$ performs $H_j$ local SGD steps
$S^j \leftarrow X_j$
**for** $q = 1$ **to** $H_j$ **do**
  $X^j \leftarrow X^j - \eta \widetilde{g}^j(X^j)$
**end for**
% agents *average* their estimates coordinate-wise
$avg \leftarrow (S^i + S^j)/2$
$X^i \leftarrow (S^i + X^{j\prime})/2 + (X^i - S^i)$
$X^j \leftarrow (S^j + X^{i\prime})/2 + (X^j - S^j)$

between the main algorithm and non-blocking one, first local gradient steps is applied only after the averaging steps (this corresponds to term $X^i - S^i$ for node $i$ and second nodes get access to the model of interacting partner which might not be complete for the reasons described above (for example, node $i$ is forced to use $X'_j$ instead of $X_j$ in it's averaging step). If node $i$ is initiator of the interaction and chosen interaction partner $j$ is still computing the local gradients from it's previous interaction this algorithm allows node $i$ not to wait for $j$ to finish computation. In this case, I will just leave it's value $X^{i\prime}$. Notice that since $i$ is finished computation it does not need to pass pass it's outdated model to $j$, but we assume the worst case.

We proceed by proving the following lemma which upper bounds the expected change in potential:

**Lemma F.1.** *For any time step $t$ we have:*

$$\mathbb{E}[\Gamma_{t+1}] \leq \mathbb{E}[\Gamma_t](1 - \frac{\lambda_2}{2rn}) + (2 + \frac{4r}{\lambda_2})\eta^2 \sum_{i=1}^{n} \left(4\mathbb{E}\|\widetilde{h}_i(X^i_t)\|^2 + \mathbb{E}\|\widetilde{h}_i\left(X^i_{p^i_t}\right)\|^2\right).$$

*Proof.* First we bound change in potential $\Delta_t = \Gamma_{t+1} - \Gamma_t$ for some fixed time step $t > 0$.

For this, let $\Delta^{i,j}_t$ be a change in potential when we choose agents $(i, j) \in E$ for interaction (While calculating $\Delta^{i,j}_t$ we assume that $X_t$ is fixed). Let $R^i_t = -\eta \widetilde{h}_i(X^i_t) + \frac{\eta \widetilde{h}_j\left(X^j_{p^j_t}\right)}{2}$ and

$R_t^j = -\eta \widetilde{h}_j(X_t^j) + \frac{\eta \widetilde{h}_i\left(\frac{X^i}{p_t^i}\right)}{2}$. We have that:

$$X_{t+1}^i = \frac{X_t^i + X_t^j}{2} + R_t^i.$$
$$X_{t+1}^j = \frac{X_t^i + X_t^j}{2} + R_t^j.$$
$$\mu_{t+1} = \mu_t + \frac{R_t^i + R_t^j}{n}.$$

This gives us that:

$$X_{t+1}^i - \mu_{t+1} = \frac{X_t^i + X_t^j}{2} + \frac{n-1}{n} R_t^i - \frac{1}{n} R_t^j - \mu_t.$$
$$X_{t+1}^i - \mu_{t+1} = \frac{X_t^i + X_t^j}{2} + \frac{n-1}{n} R_t^j - \frac{1}{n} R_t^i - \mu_t.$$

For $k \neq i, j$ we get that

$$X_{t+1}^k - \mu_{t+1} = X_t^k - \frac{1}{n}(R_t^i + R_t^j) - \mu_t.$$

Hence:

$$\begin{aligned}
\Delta_t^{i,j} &= \left\| \frac{X_t^i + X_t^j}{2} + \frac{n-1}{n} R_t^i - \frac{1}{n} R_t^j - \mu_t \right\|^2 - \left\| X_t^i - \mu_t \right\|^2 \\
&\quad + \left\| \frac{X_t^i + X_t^j}{2} + \frac{n-1}{n} R_t^j - \frac{1}{n} R_t^i - \mu_t \right\|^2 - \left\| X_t^j - \mu_t \right\|^2 \\
&\quad + \sum_{k \neq i,j} \left( \left\| X_t^k - \frac{1}{n}(R_t^i + R_t^j) - \mu_t \right\|^2 - \left\| X_t^k - \mu_t \right\|^2 \right) \\
&= 2 \left\| \frac{X_t^i - \mu_t}{2} + \frac{X_t^j - \mu_t}{2} \right\|^2 - \left\| X_t^i - \mu_t \right\|^2 - \left\| X_t^j - \mu_t \right\|^2 \\
&\quad + \left\langle X_t^i - \mu_t + X_t^j - \mu_t, \frac{n-2}{n} R_t^i + \frac{n-2}{n} R_t^j \right\rangle \\
&\quad + \left\| \frac{n-1}{n} R_t^i - \frac{1}{n} R_t^j \right\|^2 + \left\| \frac{n-1}{n} R_t^j - \frac{1}{n} R_t^i \right\|^2 \\
&\quad + \sum_{k \neq i,j} 2 \left\langle X_t^k - \mu_t, -\frac{1}{n}(R_t^i + R_t^j) \right\rangle \\
&\quad + \sum_{k \neq i,j} (\frac{1}{n})^2 \|R_t^i + R_t^j\|^2.
\end{aligned}$$

Observe that:

$$\sum_{k=1}^n \left\langle X_t^k - \mu_t, -\frac{1}{n}(R_t^i + R_t^j) \right\rangle = 0.$$

After combining the above two equations, we get that:

$$\begin{aligned}
\Delta_t^{i,j} &= -\frac{\|X_t^i - X_t^j\|^2}{2} + \left\langle X_t^i - \mu_t + X_t^j - \mu_t, R_t^i + R_t^j \right\rangle \\
&\quad + \frac{n-2}{n^2} \left\| R_t^i + R_t^j \right\|^2 + \left\| \frac{n-1}{n} R_t^i - \frac{1}{n} R_t^j \right\|^2 + \left\| \frac{n-1}{n} R_t^j - \frac{1}{n} R_t^i \right\|^2 \\
&\overset{\text{Cauchy-Schwarz}}{\leq} -\frac{\|X_t^i - X_t^j\|^2}{2} + \left\langle X_t^i - \mu_t + X_t^j - \mu_t, R_t^i + R_t^j \right\rangle \\
&\quad + 2\left( \frac{n-2}{n^2} + \frac{1}{n^2} + \frac{(n-1)^2}{n^2} \right) \left( \|R_t^i\|^2 + \|R_t^j\|^2 \right).
\end{aligned} \tag{18}$$

Recall that $R_t^i = -\eta \widetilde{h}_i(X_t^i) + \frac{\eta \widetilde{h}_j\left(X^j_{p_t^j}\right)}{2}$ and $R_t^j = -\eta \widetilde{h}_j(X_t^j) + \frac{\eta \widetilde{h}_i\left(X^i_{p_t^i}\right)}{2}$, Using Cauchy-Schwarz inequality we get that

$$\|R_t^i\|^2 \leq 2\eta^2 \|\widetilde{h}_i(X_t^i)\|^2 + \frac{\eta^2}{2}\|\widetilde{h}_j\left(X^j_{p_t^j}\right)\|^2.$$
$$\|R_t^j\|^2 \leq 2\eta^2 \|\widetilde{h}_j(X_t^j)\|^2 + \frac{\eta^2}{2}\|\widetilde{h}_i\left(X^i_{p_t^i}\right)\|^2.$$

Denote $2\eta^2 \|\widetilde{h}_i(X_t^i)\|^2 + \frac{\eta^2}{2}\|\widetilde{h}_i\left(X^i_{p_t^i}\right)\|^2$ by $S_t^i$ and $2\eta^2 \|\widetilde{h}_i(X_t^j)\|^2 + \frac{\eta^2}{2}\|\widetilde{h}_j\left(X^j_{p_t^j}\right)\|^2$ by $S_t^j$. Hence (18) can be rewritten as:

$$\begin{aligned}
\Delta_t^{i,j} &\leq -\frac{\|X_t^i - X_t^j\|^2}{2} + \left\langle X_t^i - \mu_t + X_t^j - \mu_t, R_t^i + R_t^j \right\rangle \\
&\quad + 2\Big(\frac{n-2}{n^2} + \frac{1}{n^2} + \frac{(n-1)^2}{n^2}\Big)\Big(S_t^i + S_t^j\Big) \\
&\leq -\frac{\|X_t^i - X_t^j\|^2}{2} + \left\langle X_t^i - \mu_t + X_t^j - \mu_t, R_t^i + R_t^j \right\rangle \\
&\quad + 2\Big(S_t^i + S_t^j\Big).
\end{aligned}$$

Further:

$$\begin{aligned}
\left\langle X_t^i - \mu_t + X_t^j - \mu_t, R_t^i + R_t^j \right\rangle &\overset{\text{Young}}{\leq} \frac{\lambda_2 \Big\| X_t^i - \mu_t + X_t^j - \mu_t \|^2}{8r} + \frac{2r\Big\|R_t^i + R_t^j\Big\|^2}{\lambda_2} \\
&\overset{\text{Cauchy-Schwarz}}{\leq} \frac{\lambda_2 \Big\| X_t^i - \mu_t \Big\|^2 + \lambda_2 \Big\| X_t^j - \mu_t \Big\|^2}{4r} + \frac{4r\Big\|R_t^i\Big\|^2 + 4r\Big\|R_t^j\Big\|^2}{\lambda_2} \\
&\leq \frac{\lambda_2 \Big\| X_t^i - \mu_t \Big\|^2 + \lambda_2 \Big\| X_t^j - \mu_t \Big\|^2}{4r} + \frac{4r(S_t^i + S_t^j)}{\lambda_2}.
\end{aligned}$$

This gives us:

$$\begin{aligned}
\sum_{(i,j)\in E} \Delta_t^{i,j} &\leq \sum_{(i,j)\in E} \Bigg( -\frac{\|X_t^i - X_t^j\|^2}{2} + \frac{\lambda_2 \Big\| X_t^i - \mu_t \Big\|^2 + \lambda_2 \Big\| X_t^j - \mu_t \Big\|^2}{4r} + \frac{4r(S_t^i + S_t^j)}{\lambda_2} \\
&\qquad + 2(S_t^i + S_t^j)\Bigg) \\
&\overset{\text{Lemma C.2}}{\leq} -\frac{\lambda_2 \Gamma_t}{2} + \sum_{i=1}^n (2r + \frac{4r^2}{\lambda_2}) S_t^i + \sum_{i=1}^n \frac{\lambda_2 \Big\| X_t^i - \mu_t \Big\|^2}{4} \\
&= -\frac{\lambda_2 \Gamma_t}{4} + \sum_{i=1}^n (2r + \frac{4r^2}{\lambda_2}) S_t^i.
\end{aligned}$$

Next, we use the above inequality to upper bound $\Delta_t$ in expectation:

$$\begin{aligned}
\mathbb{E}[\Delta_t | X_0, X_1, ..., X_t] &= \frac{1}{rn/2} \sum_{(i,j)\in E} \mathbb{E}[\Delta_t^{i,j} | X_0, X_1, ..., X_t] \\
&\leq \frac{1}{rn/2}\Big( -\frac{\lambda_2 \Gamma_t}{4} + \sum_{i=1}^n (2r + \frac{4r^2}{\lambda_2}) \mathbb{E}\Big[S_t^i | X_0, X_1, ..., X_t\Big]\Big) \\
&= -\frac{\lambda_2 \Gamma_t}{2rn} + \sum_{i=1}^n (4 + \frac{8r}{\lambda_2}) \frac{\mathbb{E}\Big[S_t^i | X_0, X_1, ..., X_t\Big]}{n}.
\end{aligned}$$

Finally, we remove the conditioning:

$$\mathbb{E}[\Delta_t] = \mathbb{E}[\mathbb{E}[\Delta_t | X_0, X_1, ..., X_t]] \leq -\frac{\lambda_2 \mathbb{E}[\Gamma_t]}{2rn} + (4 + \frac{8r}{\lambda_2}) \sum_{i=1}^{n} \frac{\mathbb{E}[S_t^i]}{n}.$$

By considering the definition of $\Delta_t$ and $S_t^i$, we get the proof of the lemma. □

Next, we upper bound the second moment of local updates , for any step $t$ and node $i$:

**Lemma F.2.**

$$\sum_{i=1}^{n} \mathbb{E}\|\eta \widetilde{h}_i(X_t^i)\|^2 \leq 2\eta^2 n H^2 M^2.$$

*Proof.*

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{E}\|\eta \widetilde{h}_i(X_t^i)\|^2 &= \eta^2 \sum_{u=1}^{\infty} Pr[H_t^i = u] \sum_{i=1}^{n} \mathbb{E}\| \sum_{q=1}^{u} \widetilde{h}_i^q(X_t^i)\|^2 \\
&\leq \eta^2 \sum_{u=1}^{\infty} Pr[H_t^i = u] \sum_{i=1}^{n} u \sum_{q=1}^{u} \mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \\
&\overset{(6)}{\leq} \eta^2 \sum_{u=1}^{\infty} Pr[H_t^i = u] u^2 \sum_{i=1}^{n} M^2 \leq 2n\eta^2 H^2 M^2.
\end{aligned}$$

Where in the last step we used

$$\sum_{u=1}^{\infty} Pr[H_t^i = u] u^2 = \mathbb{E}[(H_t^i)^2] = 2H^2 - H \leq 2H^2.$$

□

This allows us to upper bound the potential in expectation for any step $t$.

**Lemma F.3.**

$$\mathbb{E}[\Gamma_t] \leq (\frac{40r}{\lambda_2} + \frac{80r^2}{\lambda_2^2}) n\eta^2 H^2 M^2. \tag{19}$$

*Proof.* We prove by using induction. Base case $t = 0$ trivially holds. For an induction step step we assume that $\mathbb{E}[\Gamma_t] \leq (\frac{40r}{\lambda_2} + \frac{80r^2}{\lambda_2^2}) n\eta^2 H^2 M^2 r^2$. We get that :

$$\begin{aligned}
\mathbb{E}[\Gamma_{t+1}] &\leq \mathbb{E}[\Gamma_t](1 - \frac{\lambda_2}{2rn}) + (2 + \frac{4r}{\lambda_2})\eta^2 \sum_{i=1}^{n} \left(4\mathbb{E}\|\widetilde{h}_i(X_t^i)\|^2 + \mathbb{E}\|\widetilde{h}_i\left(X_{p_t^i}^i\right)\|^2\right) \\
&\overset{\text{Lemma (F.2)}}{\leq} (1 - \frac{\lambda_2}{2rn})\mathbb{E}[\Gamma_t] + (20 + \frac{40r}{\lambda_2}) H^2 M^2 \eta^2 \\
&\leq (1 - \frac{\lambda_2}{2rn})(\frac{40r}{\lambda_2} + \frac{80r^2}{\lambda_2^2}) n\eta^2 H^2 M^2 + (20 + \frac{40}{\lambda_2}) H^2 M^2 \eta^2 \\
&= (\frac{40r}{\lambda_2} + \frac{80r^2}{\lambda_2^2}) n\eta^2 H^2 M^2.
\end{aligned}$$

□

The next lemma allows us to upper bound $\sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -\widetilde{h}_i(X_t^i)\rangle$ which will be used later once we apply $L$-smoothness to upper bound $f(\mu_{t+1})$. The intuition is as follows, if for each $i$, $\widetilde{h}_i(X_t^i)$ was just a sum of single stochastic gradient($H_i = 1$) by unbiasedness property we would have to upper bound $\sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -\nabla f(X_t^i)\rangle = \sum_{i=1}^{n} \Big(\mathbb{E}\langle \nabla f(\mu_t), \nabla f(\mu_t) - \nabla f(X_t^i)\rangle - \mathbb{E}\|\nabla f(\mu_t)\|^2\Big)$, which can be done by using $L$-smoothness and then definition of $\Gamma_t$.

**Lemma F.4.** *For any time step t.*

$$\sum_{i=1}^{n}\mathbb{E}\langle\nabla f(\mu_t),-\widetilde{h}_i(X_t^i)\rangle \leq 2HL^2\mathbb{E}[\Gamma_t]-\frac{3Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2+12H^3nL^2M^2\eta^2. \tag{20}$$

*Proof.*

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{E}\langle\nabla f(\mu_t),-\widetilde{h}_i(X_t^i)\rangle &= \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\mathbb{E}\langle\nabla f(\mu_t),-\sum_{q=1}^{u}\widetilde{h}_i^q(X_t^i)\rangle\\
&= \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\sum_{q=1}^{u}\Big(\mathbb{E}\langle\nabla f(\mu_t),\nabla f(\mu_t)-h_i^q(X_t^i)\rangle-\mathbb{E}\|\nabla f(\mu_t)\|^2\Big)\\
&= \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\sum_{q=1}^{u}\Big(\mathbb{E}\langle\nabla f(\mu_t),\nabla f(\mu_t)-\nabla f(X_t^i-\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\rangle-\mathbb{E}\|\nabla f(\mu_t)\|^2\Big)
\end{aligned}$$

Using Young's inequality we can upper bound $\mathbb{E}\langle\nabla f(\mu_t),\nabla f(\mu_t)-\nabla f(X_t^i-\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\rangle$ by $\frac{\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}+\mathbb{E}\Big\|\nabla f(\mu_t)-\nabla f(X_t^i-\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\Big\|^2$. plugging this in the above inequality we get:

$$\begin{aligned}
&\sum_{i=1}^{n}\mathbb{E}\langle\nabla f(\mu_t),-\widetilde{h}_i(X_t^i)\rangle \leq\\
&\quad\leq \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\sum_{q=1}^{u}\Big(\mathbb{E}\|\nabla f(\mu_t)-\nabla f(X_t^i-\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2-\frac{3\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}\Big)\\
&\quad\overset{(4)}{\leq} \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\sum_{q=1}^{u}\Big(L^2\mathbb{E}\|\mu_t-X_t^i+\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2-\frac{3\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}\Big).
\end{aligned}$$

Next we use Cauchy-Schwarz inequality on $\mathbb{E}\|\mu_t-X_t^i+\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2$

$$\begin{aligned}
&\sum_{i=1}^{n}\mathbb{E}\langle\nabla f(\mu_t),-\widetilde{h}_i(X_t^i)\rangle \leq\\
&\quad\leq \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\sum_{q=1}^{u}\Big(2L^2\mathbb{E}\|\mu_t-X_t^i\|^2+2L^2\mathbb{E}\|\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2-\frac{3\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}\Big)
\end{aligned}$$

Term $\mathbb{E}\|\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2$ can be upper bounded by $q^2M^2$ using Cauchy-Schwarz and assumption (6). Hence:

$$\begin{aligned}
&\sum_{i=1}^{n}\mathbb{E}\langle\nabla f(\mu_t),-\widetilde{h}_i(X_t^i)\rangle \leq\\
&\quad\leq \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]\sum_{q=1}^{u}\Big(2L^2\mathbb{E}\|\mu_t-X_t^i\|^2+2L^2\eta^2q^2M^2-\frac{3\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}\Big)\\
&\quad= \sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]u\Big(2L^2\mathbb{E}\|\mu_t-X_t^i\|^2-\frac{3\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}\Big)\\
&\qquad\qquad+\sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]u(u+1)(2u+1)L^2M^2\eta^2/3
\end{aligned}\tag{21}$$

Note that:

$$\sum_{i=1}^{n}\sum_{u=1}^{\infty}Pr[H_t^i=u]u\Big(2L^2\mathbb{E}\|\mu_t-X_t^i\|^2-\frac{3\mathbb{E}\|\nabla f(\mu_t)\|^2}{4}\Big)=2HL^2\mathbb{E}[\Gamma_t]-\frac{3Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2. \tag{22}$$

Also:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{u=1}^{\infty} Pr[H_t^i = u]u(u+1)(2u+1)L^2M^2\eta^2/3 \\ \leq \sum_{i=1}^{n}\sum_{u=1}^{\infty} Pr[H_t^i = u]2u^3L^2M^2\eta^2 \\ \leq 12H^3nL^2M^2\eta^2. \end{aligned} \tag{23}$$

Where in the last step we used (Recall that $H_t^i$ is a geometric random variable with mean $H$):

$$\sum_{u=1}^{\infty} Pr[H_t^i = u]u^3 = \mathbb{E}[(H_t^i)^3] = 6H^3 - 6H^2 + H \leq 6H^3.$$

By plugging inequalities (23) and (22) into inequality (21) we get the proof of the lemma. □

Our next goal is to upper bound $\sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -\widetilde{h}_i(X^i_{p_t^i})\rangle$.

**Lemma F.5.**

$$\sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), \widetilde{h}_i(X^i_{p_t^i})\rangle \leq 2HL^2\sum_{i=1}^{n}\mathbb{E}\|\mu_t - X^i_{p_t^i}\|^2 + \frac{5Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2 + 12H^3nL^2M^2\eta^2. \tag{24}$$

*Proof.* The proof is very similar to the proof of lemma F.4, except that when we subtract and add term $\mathbb{E}|\nabla f(\mu_t)\|^2$ in the proof it will eventually end up with a positive sign (After using Young's inequality it will have factor of $\frac{1}{4}+1$ instead of factor of $\frac{1}{4}-1$) and we have $\sum_{i=1}^{n}\mathbb{E}\|\mu_t - X^i_{p_t^i}\|^2$ instead of $\Gamma_t = \sum_{i=1}^{n}\mathbb{E}\|\mu_t - X_t^i\|^2$. Thus, we omit the proof in this case. □

Next step is to upper bound $\sum_{i=1}^{n}\mathbb{E}\|\mu_t - X^i_{p_t^i}\|^2$, for this we will need the following lemma:

**Lemma F.6.** *For any node $i$ and time step $t$,*

$$\mathbb{E}\|\mu_t - \mu_{p_t^i}\|^2 \leq 10\eta^2H^2M^2.$$

*Proof.* Notice that

$$\mathbb{E}\|\mu_t - \mu_{p_t^i}\|^2 = \sum_{t'=0}^{t} Pr[p_t^i = t']\mathbb{E}\Big\|\sum_{s=t'}^{t-1}\mu_{s+1} - \mu_s\Big\|^2 \leq \sum_{t'=0}^{t}\sum_{s=t'}^{t-1} Pr[p_t^i = t'](t-t')\mathbb{E}\Big\|\mu_{s+1} - \mu_s\Big\|^2. \tag{25}$$

where we used Cauchy-Schwarz inequality in the lastt step. Fix step $s$. Let $u$ and $v$ be nodes which interact at step $s+1$. We have that

$$\begin{aligned}\mathbb{E}\|\mu_{s+1} - \mu_s\|^2 = \mathbb{E}\Big\| - \frac{\eta\widetilde{h}_u(X_s^u)}{n} + \frac{\eta\widetilde{h}_u\Big(X^u_{p_s^u}\Big)}{2n} - \frac{\eta\widetilde{h}_v(X_s^v)}{n} + \frac{\eta\widetilde{h}_v\Big(X^v_{p_s^v}\Big)}{2n}\Big\|^2 \\ \leq \frac{4\eta^2}{n^2}\mathbb{E}\|\widetilde{h}_u(X_s^u)\|^2 + \frac{4\eta^2}{n^2}\mathbb{E}\|\widetilde{h}_v(X_s^v)\|^2 + \\ + \frac{\eta^2}{n^2}\mathbb{E}\|\widetilde{h}_u(X^u_{p_s^u})\|^2 + \frac{\eta^2}{n^2}\mathbb{E}\|\widetilde{h}_v(X^v_{p_s^v})\|^2.\end{aligned}$$

Where we again used Cauchy-Schwarz, inequality since the expectation is taken only over the randomness of sampling and number of local steps We can use approach from lemma $F.2$ to upper bound $E\|\widetilde{h}_u(X_s^u)\|^2$, $\mathbb{E}\|\widetilde{h}_v(X_s^v)\|^2$, $\mathbb{E}\|\widetilde{h}_u(X^u_{p_s^u})\|^2$ and $\mathbb{E}\|\widetilde{h}_v(X^v_{p_s^v})\|^2$.

(in the lemma we upper bound sum of $n$ similar terms, but with $\eta^2$). Hence:

$$\begin{aligned}\mathbb{E}\|\mu_t - \mu_{p_t^i}\|^2 \leq \sum_{t'=0}^{t}\sum_{s=t'}^{t-1} Pr[p_t^i = t'](t-t')\mathbb{E}\Big\|\mu_{s+1} - \mu_s\Big\|^2 \\ \leq \sum_{t'=0}^{t}\sum_{s=t'}^{t-1} Pr[p_t^i = t'](t-t')^2\frac{20\eta^2H^2M^2}{n^2} = \frac{20\eta^2H^2M^2}{n^2}\mathbb{E}[(p_t^i - t)^2].\end{aligned}$$

$t - p_t^i$ is a geometric random variable with mean $n/2$ (because probability that node $i$ interacts is $2/n$ at every step). Thus, $\mathbb{E}[(p_t^i - t)^2] = 2(\mathbb{E}[t - p_t^i])^2 - \mathbb{E}[t - p_t^i] \leq \frac{n^2}{2}$. Thus, $\mathbb{E}\|\mu_t - \mu_{p_t^i}\|^2 \leq 10\eta^2 H^2 M^2$. □

Finally we can show that:

**Lemma F.7.** *For any node $i$ and time step $t$,*

$$\sum_{i=1}^{n} \mathbb{E}\|\mu_t - X_{p_t^i} i\|^2 \leq 20nH^2M^2\eta^2 + (\frac{80r}{\lambda_2} + \frac{160r^2}{\lambda_2^2})n^2\eta^2H^2M^2.$$

*Proof.* Using Cauchy-Schwarz inequality we get:

$$\sum_{i=1}^{n} \mathbb{E}\|\mu_t - X_{p_t^i}^i\|^2 \leq \sum_{i=1}^{n}(2\mathbb{E}\|\mu_t - \mu_{p_t^i}\|^2 + 2\mathbb{E}[\mu_{p_t^i} - X_{p_t^i}^i\|^2)$$
$$\leq \sum_{i=1}^{n}(2\mathbb{E}\|\mu_t - \mu_{p_t^i}\|^2 + 2\mathbb{E}[\Gamma_{p_t^i}]) \leq 20nH^2M^2\eta^2 + (\frac{80r}{\lambda_2} + \frac{160r^2}{\lambda_2^2})n^2\eta^2H^2M^2.$$

Where the last inequality comes from Lemmas F.3 and F.6. □

Now we are ready to prove the main theorem.

**Theorem F.8.** *Let $f$ be an non-convex, $L$-smooth, function satisfying assumption 6, whose minimum $x^\star$ we are trying to find via the non-blocking version of SwarmSGD procedure (See, algorithm 2). Let the number of local stochastic gradient steps performed by each agent upon interaction be a geometric random variable with mean $H$. Let the learning rate we use be $\eta = n/\sqrt{T}$. Define $\mu_t = \sum_{i=1}^{n} X_t^i/n$, where $X_t^i$ is a value of model $i$ after $t$ interactions. Then, for learning rate $\eta = n/\sqrt{T}$ and any $T \geq n^4(n+1)^2$:*

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{4(f(\mu_0) - f(x^*))}{\sqrt{T}H} + \frac{2304H^2\max(1, L^2)M^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2} + 1).$$

*Proof.* Let $\mathbb{E}_t$ be the expectation which is conditioned on $\{X_1^t, X_2^t, ..., X_n^t\}$. By $L$-smoothness we have that

$$\mathbb{E}_t[f(\mu_{t+1})] \leq f(\mu_t) + \mathbb{E}_t\langle \nabla f(\mu_t), \mu_{t+1} - \mu_t\rangle + \frac{L}{2}\mathbb{E}_t\|\mu_{t+1} - \mu_t\|^2.$$

After removing conditioning:

$$\mathbb{E}[f(\mu_{t+1})] = \mathbb{E}[\mathbb{E}_t[f(\mu_{t+1})]] \leq \mathbb{E}[f(\mu_t)] + \mathbb{E}\langle \nabla f(\mu_t), \mu_{t+1} - \mu_t\rangle + \frac{L}{2}\mathbb{E}\|\mu_{t+1} - \mu_t\|^2. \quad (26)$$

First we look at $\mathbb{E}[\mu_{t+1} - \mu_t]$. If agents $i$ and $j$ interact, (which happens with probability $\frac{1}{rn/2}$). We have that $\mu_{t+1} - \mu_t = -\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j) + \frac{\eta}{2n}\widetilde{h}_i(X_{p_t^i}^i) + \frac{\eta}{2n}\widetilde{h}_j(X_{p_t^j}^j)$. Hence we get that

$$\mathbb{E}_t[\mu_{t+1} - \mu_t] = \frac{1}{rn/2}\sum_{(i,j)\in E}\left(\mathbb{E}_t[-\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j)] + \frac{\eta}{2n}\widetilde{h}_i(X_{p_t^i}^i) + \frac{\eta}{2n}\widetilde{h}_j(X_{p_t^j}^j).\right)$$
$$= -\frac{2\eta}{n^2}\sum_{i=1}^{n}\mathbb{E}_t[\widetilde{h}_i(X_t^i)] + \frac{\eta}{n^2}\sum_{i=1}^{n}\widetilde{h}_i(X_{p_t^i}^i).$$

and

$$\mathbb{E}_[\mu_{t+1} - \mu_t] = \mathbb{E}[\mathbb{E}_t[\mu_{t+1} - \mu_t]] = -\frac{2\eta}{n^2}\sum_{i=1}^{n}\mathbb{E}[\widetilde{h}_i(X_t^i)] + \frac{\eta}{n^2}\sum_{i=1}^{n}\mathbb{E}[\widetilde{h}_i(X_{p_t^i}^i)].$$

Next we look at $\mathbb{E}\|\mu_{t+1}-\mu_t\|^2$. If agents $i$ and $j$ interact, (which happens with probability $\frac{1}{rn/2}$). We have that $\mu_{t+1}-\mu_t = -\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j) + \frac{\eta}{2n}\widetilde{h}_i(X_{p_t^i}^i) + \frac{\eta}{2n}\widetilde{h}_j(X_{p_t^j}^j)$. Hence we get that

$$\begin{aligned}
\mathbb{E}_t\|\mu_{t+1}-\mu_t\|^2 &= \frac{1}{rn/2}\sum_{(i,j)\in E}\mathbb{E}_t\Big\| -\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j) + \frac{\eta}{2n}\widetilde{h}_i(X_{p_t^i}^i) + \frac{\eta}{2n}\widetilde{h}_j(X_{p_t^j}^j)\Big\|^2 \\
&\overset{Cauchy-Schwarz}{\leq} \frac{1}{rn/2}\sum_{(i,j)\in E}\frac{\eta^2}{n^2}\Big(4\mathbb{E}_t\|\widetilde{h}_i(X_t^i)\|^2 + 4\mathbb{E}_t\|\frac{\eta}{n}\widetilde{h}_j(X_t^j)\|^2 \\
&\qquad + \|\widetilde{h}_i(X_{p_t^i}^i)\|^2 + \|\widetilde{h}_j(X_{p_t^j}^j)\|^2\Big) \\
&= \frac{2}{n}\sum_{i=1}^n \frac{4\eta^2}{n^2}\|\widetilde{h}_i(X_t^i)\|^2 + \frac{2}{n}\sum_{i=1}^n\frac{\eta^2}{n^2}\|\widetilde{h}_i(X_{p_t}^i)\|^2 \\
&\overset{\text{Lemma F.2}}{\leq} \frac{16\eta^2H^2M^2}{n^2} + \frac{2}{n}\sum_{i=1}^n\frac{\eta^2}{n^2}\|\widetilde{h}_i(X_{p_t}^i)\|^2.
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\|\mu_{t+1}-\mu_t\|^2 &= \mathbb{E}[[\mathbb{E}_t\|\mu_{t+1}-\mu_t\|^2]] \\
&\leq \frac{16\eta^2H^2M^2}{n^2} + \frac{2}{n}\sum_{i=1}^n\frac{\eta^2}{n^2}\mathbb{E}\|\widetilde{h}_i(X_{p_t}^i)\|^2 \overset{\text{Lemma F.2}}{\leq} \frac{20\eta^2H^2M^2}{n^2}.
\end{aligned}$$

Hence, we can rewrite (26) as:

$$\begin{aligned}
\mathbb{E}[f(\mu_{t+1})] \leq \mathbb{E}[f(\mu_t)] &+ \frac{2\eta}{n^2}\sum_{i=1}^n\mathbb{E}\Big\langle\nabla f(\mu_t), -\widetilde{h}_i(X_t^i)\Big\rangle + \frac{\eta}{n^2}\sum_{i=1}^n\mathbb{E}\Big\langle\nabla f(\mu_t), \widetilde{h}_i(X_{p_t^i}^i)\Big\rangle \\
&+ \frac{20L\eta^2H^2M^2}{n^2}.
\end{aligned}$$

Next, we use Lemmas F.4 and F.5:

$$\begin{aligned}
\mathbb{E}[f(\mu_{t+1})] &\leq \mathbb{E}[f(\mu_t)] + \frac{2\eta}{n^2}\Big(2HL^2\mathbb{E}[\Gamma_t] - \frac{3Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2 + 12H^3nL^2M^2\eta^2\Big) \\
&\quad + \frac{\eta}{n^2}\Big(2HL^2\sum_{i=1}^n\mathbb{E}\|\mu_t - X_{p_t^i}^i\|^2 + \frac{5Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2 + 12H^3nL^2M^2\eta^2\Big) \\
&\quad + \frac{4L\eta^2H^2M^2}{n^2} \\
&= \mathbb{E}[f(\mu_t)] + \frac{4HL^2\eta\mathbb{E}[\Gamma_t]}{n^2} + \frac{2HL^2\eta\Big(\sum_{i=1}^n\mathbb{E}\|\mu_t - X_{p_t^i}^i\|^2\Big)}{n^2} - \frac{H\eta}{4n}\mathbb{E}\|\nabla f(\mu_t)\|^2 \\
&\quad + \frac{36H^3L^2M^2\eta^3}{n} + \frac{20L\eta^2H^2M^2}{n^2}.
\end{aligned}$$

We use Lemmas F.3 and F.7 to upper bound $\mathbb{E}[\Gamma_t]$ and $\sum_{i=1}^n\mathbb{E}\|\mu_t - X_{p_t^i}^i\|^2$ respectively :

$$\begin{aligned}
\mathbb{E}[f(\mu_{t+1})] - \mathbb{E}[f(\mu_t)] &\leq (\frac{160r}{\lambda_2} + \frac{320r^2}{\lambda_2^2})\frac{\eta^3H^3M^2L^2(n^2+n)}{n^2} - \frac{Hn}{4}\mathbb{E}\|\nabla f(\mu_t)\|^2 \\
&\quad + \frac{76H^3L^2M^2\eta^3}{n} + \frac{20L\eta^2H^2M^2}{n^2}.
\end{aligned}$$

by summing the above inequality for $t=0$ to $t=T-1$, we get that

$$\begin{aligned}
\mathbb{E}[f(\mu_T)] - f(\mu_0) &\leq \sum_{t=0}^{T-1}\Big((\frac{160r}{\lambda_2} + \frac{320r^2}{\lambda_2^2})\frac{\eta^3H^3M^2L^2(n+1)}{n} - \frac{\eta H}{4n}\mathbb{E}\|\nabla f(\mu_t)\|^2 \\
&\quad + \frac{20L\eta^2H^2M^2}{n^2} + \frac{76H^3L^2M^2\eta^3}{n}\Big).
\end{aligned}$$

From this we get that :

$$\sum_{t=0}^{T-1}\frac{\eta H}{4n}\mathbb{E}\|\nabla f(\mu_t)\|^2 \leq f(\mu_0)-\mathbb{E}[f(\mu_T)]+(\frac{160r}{\lambda_2}+\frac{320r^2}{\lambda_2^2})\frac{\eta^3H^3M^2L^2T(n+1)}{n} + \frac{20TL\eta^2H^2M^2}{n^2}+\frac{76TH^3L^2M^2\eta^3}{n}.$$

Note that $\mathbb{E}[f(\mu_T)] \geq f(x^*)$, hence after multiplying the above inequality by $\frac{4n}{\eta HT}$ we get that

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{4n(f(\mu_0)-f(x^*))}{TH\eta}+(\frac{640r}{\lambda_2}+\frac{1280r^2}{\lambda_2^2})\eta^2H^2M^2L^2T(n+1) + \frac{80TL\eta HM^2}{n}+304TH^2L^2M^2\eta^2.$$

Observe that $\eta = n/\sqrt{T} \leq \frac{1}{n(n+1)}$, since $T \geq n^4(n+1)^2$. This allows us to finish the proof:

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2 &\leq \frac{4n(f(\mu_0)-f(x^*))}{TH\eta}+(\frac{640r}{\lambda_2}+\frac{1280r^2}{\lambda_2^2})\frac{L^2\eta M^2H^2}{n} \\ &\quad + \frac{80L\eta HM^2}{n}+\frac{304H^2L^2M^2\eta}{n} \\ &= \frac{(4f(\mu_0)-f(x^*))}{\sqrt{T}H}+(\frac{640r}{\lambda_2}+\frac{1280r^2}{\lambda_2^2})\frac{L^2M^2H^2}{\sqrt{T}} \\ &\quad + \frac{80LHM^2}{\sqrt{T}}+\frac{304H^2L^2M^2}{\sqrt{T}} \\ &\leq \frac{4(f(\mu_0)-f(x^*))}{\sqrt{T}H}+\frac{2304H^2\max(1,L^2)M^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2}+1).\end{aligned}$$

□

## G Analysis of Quantized Averaging, assuming second moment bound and random number of local steps

First we define how quantization of models changes our interactions. Both the algorithm and the analysis in this case are similar to those of Section F. Let $i$ and $j$ be nodes which interact at step $t+1$, we set

$$X_{t+1/2}^i = \frac{X_t^i}{2}+\frac{X_t^{j\prime}}{2},$$
$$X_{t+1/2}^j = \frac{X_t^j}{2}+\frac{X_t^{i\prime}}{2},$$

and

$$X_{t+1}^i = X_{t+1/2}^i - \eta\widetilde{h}_i(X_t^i),$$
$$X_{t+1}^j = X_{t+1/2}^j - \eta\widetilde{h}_j(X_t^j),$$

where for each node $k$, $X_t^{k\prime}$ is a quantized version of the model $X_t^k$. We use the quantization provided in Davies et al. (2020). The key property of this quantization scheme is summarized below:

**Lemma G.1.** *Let $q$ be a parameter we will fix later. If the inputs $x^u$, $x^v$ at nodes $u$ and $v$, respectively satisfy $\|x^u - x^v\| \leq q^{q^d}\epsilon$, then with probability at least $1-\log\log(\frac{1}{\epsilon}\|x^u-x^v\|)\cdot O(q^{-d})$, the quantization algorithm of Davies et al. (2020) provides node $v$ with an unbiased estimate $x^{u\prime}$ of $x^u$, with $\|x^{u\prime}-x^u\| \leq (q^2+7)\epsilon$, and uses $O\left(d\log(\frac{q}{\epsilon}\|x^u-x^v\|)\right)$ bits to do so.*

For our purposes, $x^u$ and $x^v$ are the local models of the nodes $u$ and $v$ and $d$ is their dimension (we omit the time step here). In the following , we refer to the above lemma as quantization lemma. Recall that in section E, for each node $k$, if $p_t^k + 1$ is the last time interacting before and including step $t$:

$$X_t^{k\prime} = X^i_{p_t^k+1/2} = X_t^k + \eta \widetilde{h}_k(X^k_{p_t^k}). \tag{27}$$

**Analysis Outline.** The crucial property we used in the analysis is that $\mathbb{E}[\widetilde{h}_k(X^k_{p_t^k})] \leq 2H^2M^2\eta^2$ (see Lemma F.2). We also used that for $\widetilde{h}_k(X^k_{p_t^k})$ , we can use Smoothness property (4) in Lemma F.5 and F.7. In our case we plan to use the quantization lemma above. For this first notice that the estimate is unbiased: this means that $\mathbb{E}[X_t^{k\prime}] = X_t^k$, eliminating the need to use Lemma F.5 and subsequently F.7, since $\sum_{i=1}^n \mathbb{E}\langle \nabla f(\mu_t), X_t^i - X_t^{i\prime}\rangle = 0$ in our case. Secondly if we set $(q^2+7)\epsilon = H\eta M$ we also satisfy Lemma F.2. This means that entire analysis can be replicated, and even further as in the case of section E we will only need $T \geq n^4$ (In one case we do not use lemma F.5 at all, and in the second case upper bound can be replaced by 0, which is the same as not using it). Now we concentrate on calculating the probability that $\|x^u - x^v\| \leq q^{q^d}\epsilon$ (which we call the distance criterion) required by the quantization lemma to hold over $T$ steps and each pair of nodes. We also need to calculate the probability with which we fail to decode.

Assume that Lemma F.3 holds for step $t$, as in the proof of this lemma we will use induction and F.1 (we omit the proof since it will be exactly the same given that the conditions discuss above hold). That is $\mathbb{E}[\Gamma_t] \leq 2H^2M^2\eta^2$, notice that for pair of nodes $u$ and $v$, $\|X_t^u - X_t^v\|^2 \leq 2\Gamma_t$ (Using Cauchy-Schwarz). Hence we need to calculate probability that $\Gamma_t \geq (q^{q^d}\epsilon)^2/2$. Using Markov's inequality probability of this happening is at most

$$\frac{2\mathbb{E}[\Gamma_t]}{(q^{q^d}\epsilon)^2} \leq \frac{4n(q^2+7)^2\epsilon^2}{(q^{q^d}\epsilon)^2} = \frac{n(q^2+7)^2}{(q^{q^d})^2}.$$

We set $q = 2 + T^{3/d}$, this means that $(q^{q^d})^2 \geq 2^{T^3}$. So given that $T \geq n^4$, we have that $Pr[\Gamma_t \geq (q^{q^d}\epsilon)^2/2] \leq O(1/T^4)$. With this probability the distance criterion is satisfied. Given that it is satisfied, we also have failure probability $\log\log(\frac{1}{\epsilon}\|X_t^u - X_t^v\|) \cdot O(q^{-d}) = O(\frac{\log\log T}{T^3})$. So, the total probability of failure, either due to contravening the distance criterion or by probabilistic failure, is at most $O(T^{-2})$. Hence with probability $O(1/T^2)$ we can use Lemma $F.1$ and prove that $\mathbb{E}[\Gamma_{t+1}] \leq 2H^2M^2\eta^2$. What is left is to union bound over $T$ steps (Note that we do not need to union bound over all pairs of $u$ and $v$, since can assume that $u$ and $v$ are the ones which interact at step $t+1$), and we get that with probability $O(1/T) = O(1/n^4)$ the quantization algorithm never fails and distance criterion is always satisfied. The total number of bits used per step is $O(d\log q)$. With this we can state the main theorem:

**Theorem G.2.** *Let $f$ be an non-convex, $L$-smooth function, whose stochastic gradients satisfy the bounded second moment assumption above. Consider quantized version of the algorithm 1. Let the number of local stochastic gradient steps performed by each agent upon interaction be a geometric random variable with mean $H$. Let the learning rate we use be $\eta = n/\sqrt{T}$. Define $\mu_t = \sum_{i=1}^n X_t^i/n$, where $X_t^i$ is a value of model $i$ after $t$ interactions, be the average of the local parameters. Then, for learning rate $\eta = n/\sqrt{T}$ and any number of interactions $T \geq n^4$, with probability at least $1 - O(1/n^4)$ we have that:*

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{4(f(\mu_0) - f(x^*))}{\sqrt{T}H} + \frac{2304H^2\max(1, L^2)M^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2} + 1).$$

*and additionally we use $O(d + \log(T))$ communication bits per step.*

## H Fixed number of local steps with Variance Bound

We again assume that the optimization function is non-convex with variance bound (5) and smoothness property (4). We use the constant learning rate $\eta$ and fixed local steps sizes $H_t^i = H$, for each node $i$ and step $t$.

We proceed by proving the following lemma which upper bounds the expected change in potential:

**Lemma H.1.** *For any time step $t$ , we have:*

$$\mathbb{E}[\Gamma_{t+1}] \leq (1 - \frac{\lambda_2}{2rn})\mathbb{E}[\Gamma_t] + (2 + \frac{8r}{\lambda_2})\eta^2 \sum_{i=1}^{n} \frac{\mathbb{E}\|\widetilde{h}_i(X_t^i)\|^2}{n}$$

*Proof.* First we bound change in potential $\Delta_t = \Gamma_{t+1} - \Gamma_t$ for some fixed time step $t > 0$. For this, let $\Delta_t^{i,j}$ be a change in potential when we choose agents $(i, j) \in E$ for interaction. We have that:

$$\begin{aligned}
X_{t+1}^i &= (X_t^i + X_t^j + \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j))/2. \\
X_{t+1}^j &= (X_t^i + X_t^j + \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j))/2. \\
\mu_{t+1} &= \mu_t + \eta\widetilde{h}_i(X_t^i)/n + \eta\widetilde{h}_j(X_t^j)/n.
\end{aligned}$$

This gives us that:

$$\begin{aligned}
X_{t+1}^i - \mu_{t+1} &= \frac{X_t^i + X_t^j}{2} + \frac{n-2}{2n}\eta\widetilde{h}_i(X_t^i) + \frac{n-2}{2n}\eta\widetilde{h}_j(X_t^j) - \mu_t. \\
X_{t+1}^j - \mu_{t+1} &= \frac{X_t^i + X_t^j}{2} + \frac{n-2}{2n}\eta\widetilde{h}_i(X_t^i) + \frac{n-2}{2n}\eta\widetilde{h}_j(X_t^j) - \mu_t.
\end{aligned}$$

For $k \neq i, j$ we get that

$$X_{t+1}^k - \mu_{t+1} = X_t^k - \frac{1}{n}(\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)) - \mu_t.$$

Hence:

$$\begin{aligned}
\Delta_t^{i,j} &= \left\|(X_t^i + X_t^j)/2 + \frac{n-2}{2n}(\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)) - \mu_t\right\|^2 - \left\|X_t^i - \mu_t\right\|^2 \\
&\quad + \left\|(X_t^i + X_t^j)/2 + \frac{n-2}{2n}(\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)) - \mu_t\right\|^2 - \left\|X_t^j - \mu_t\right\|^2 \\
&\quad + \sum_{k\neq i,j}\left(\left\|X_t^k - \frac{1}{n}(\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)) - \mu_t\right\|^2 - \left\|X_t^k - \mu_t\right\|^2\right) \\
&= 2\left\|\frac{X_t^i - \mu_t}{2} + \frac{X_t^j - \mu_t}{2}\right\|^2 - \left\|X_t^i - \mu_t\right\|^2 - \left\|X_t^j - \mu_t\right\|^2 \\
&\quad + 2\left\langle X_t^i - \mu_t + X_t^j - \mu_t, \frac{n-2}{2n}\eta\widetilde{h}_i(X_t^i) + \frac{n-2}{2n}\eta\widetilde{h}_j(X_t^j)\right\rangle \\
&\quad + 2(\frac{n-2}{2n})^2\|\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\|^2 \\
&\quad + \sum_{k\neq i,j} 2\left\langle X_t^k - \mu_t, -\frac{1}{n}(\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j))\right\rangle \\
&\quad + \sum_{k\neq i,j} (\frac{1}{n})^2\|\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\|^2.
\end{aligned}$$

Observe that:

$$\sum_{k=1}^{n}\left\langle X_t^k - \mu_t, -\frac{1}{n}(\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j))\right\rangle = 0.$$

After combining the above two equations, we get that:

$$\begin{aligned}
\Delta_t^{i,j} &= -\frac{\|X_t^i - X_t^j\|^2}{2} + \Big\langle X_t^i - \mu_t + X_t^j - \mu_t, \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\Big\rangle \\
&\quad + \Big(2(\frac{n-2}{2n})^2 + (n-2)(\frac{1}{n})^2\Big)\|\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\|^2 \\
&\overset{\text{Cauchy-Schwarz}}{\leq} -\frac{\|X_t^i - X_t^j\|^2}{2} + \Big\langle X_t^i - \mu_t + X_t^j - \mu_t, \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\Big\rangle \\
&\quad + 2\Big(2(\frac{n-2}{2n})^2 + (n-2)(\frac{1}{n})^2\Big)\Big(\|\eta\widetilde{h}_i(X_t^i)\|^2 + \|\eta\widetilde{h}_j(X_t^j)\|^2\Big) \\
&\leq -\frac{\|X_t^i - X_t^j\|^2}{2} + \Big\langle X_t^i - \mu_t + X_t^j - \mu_t, \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\Big\rangle \\
&\quad + \Big(\|\eta\widetilde{h}_i(X_t^i)\|^2 + \|\eta\widetilde{h}_j(X_t^j)\|^2\Big).
\end{aligned}$$

This gives us:

$$\begin{aligned}
\sum_{(i,j)\in E}\Delta_t^{i,j} &\leq \sum_{(i,j)\in E}\Bigg(-\frac{\|X_t^i - X_t^j\|^2}{2} + \Big\langle X_t^i - \mu_t + X_t^j - \mu_t, \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\Big\rangle \\
&\qquad + \|\eta\widetilde{h}_i(X_t^i)\|^2 + \|\eta\widetilde{h}_j(X_t^j)\|^2\Bigg) \\
&\overset{\text{Lemma C.2}}{\leq} -\frac{\lambda_2\Gamma_t}{2} + \sum_{i=1}^n r\|\eta\widetilde{h}_i(X_t^i)\|^2 \\
&\quad + \sum_{(i,j)\in E}\Big\langle X_t^i - \mu_t + X_t^j - \mu_t, \eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\Big\rangle \\
&\overset{\text{Young}}{\leq} -\frac{\lambda_2\Gamma_t}{2} + \sum_{i=1}^n r\|\eta\widetilde{h}_i(X_t^i)\|^2 \\
&\quad + \sum_{(i,j)\in E}\left(\frac{\lambda_2\Big\|X_t^i - \mu_t + X_t^j - \mu_t\|^2}{8r} + \frac{2r\Big\|\eta\widetilde{h}_i(X_t^i) + \eta\widetilde{h}_j(X_t^j)\Big\|^2}{\lambda_2}\right) \\
&\overset{\text{Cauchy-Schwarz}}{\leq} -\frac{\lambda_2\Gamma_t}{2} + \sum_{i=1}^n r\|\eta\widetilde{h}_i(X_t^i)\|^2 \\
&\quad + \sum_{(i,j)\in E}\Bigg(\frac{\lambda_2\Big\|X_t^i - \mu_t\Big\|^2 + \lambda_2\Big\|X_t^j - \mu_t\Big\|^2}{4r} \\
&\qquad + \frac{4r\Big\|\eta\widetilde{h}_i(X_t^i)\Big\|^2 + 4r\Big\|\eta\widetilde{h}_j(X_t^j)\Big\|^2}{\lambda_2}\Bigg) \\
&= -\frac{\lambda_2\Gamma_t}{2} + \sum_{i=1}^n r\|\eta\widetilde{h}_i(X_t^i)\|^2 + \sum_{i=1}^n \frac{\lambda_2\Big\|X_t^i - \mu_t\Big\|^2}{4} \\
&\quad + \sum_{i=1}^n \frac{4r^2\|\eta\widetilde{h}_i(X_t^i)\|^2}{\lambda_2}
\end{aligned}$$

Next, we use definition of $\Gamma_t$:

$$\sum_{(i,j)\in E}\Delta_t^{i,j} \leq -\frac{\lambda_2\Gamma_t}{4} + \sum_{i=1}^n (r + \frac{4r^2}{\lambda_2})\|\eta\widetilde{h}_i(X_t^i)\|^2. \tag{28}$$

Next, we use the above inequality to upper bound $\Delta_t$ in expectation:

$$\begin{aligned}\mathbb{E}[\Delta_t|X_0, X_1, ..., X_t] &= \frac{1}{rn/2}\sum_{(i,j)\in E}\mathbb{E}[\Delta_t^{i,j}|X_0, X_1, ..., X_t] \\ &\leq \frac{1}{rn/2}\Big(-\frac{\lambda_2\Gamma_t}{4}+\sum_{i=1}^{n}(r+\frac{4r^2}{\lambda_2})\mathbb{E}\Big[\|\eta\widetilde{h}_i(X_t^i)\|^2|X_0, X_1, ..., X_t\Big]\Big) \\ &= -\frac{\lambda_2\Gamma_t}{2rn}+\sum_{i=1}^{n}(2+\frac{8r}{\lambda_2})\eta^2\frac{\mathbb{E}\Big[\|\widetilde{h}_i(X_t^i)\|^2|X_0, X_1, ..., X_t\Big]}{n}.\end{aligned}$$

Finally, we remove the conditioning:

$$\mathbb{E}[\Delta_t] = \mathbb{E}[\mathbb{E}[\Delta_t|X_0, X_1, ..., X_t]] \leq -\frac{\lambda_2\mathbb{E}[\Gamma_t]}{2rn}+(2+\frac{8r}{\lambda_2})\eta^2\sum_{i=1}^{n}\frac{\mathbb{E}\|\widetilde{h}_i(X_t^i)\|^2}{n}.$$

By considering the definition of $\Delta_t$, we get the proof of the lemma. □

**Lemma H.2.** *For any $1 \leq q \leq H$ and step $T$, we have that*

$$\sum_{i=1}^{n}\mathbb{E}\|\nabla f(\mu_t)-h_i^q(X_t^i)\|^2 \leq 2L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}2L^2\eta^2\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2.$$

*Proof.*

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{E}\|\nabla f(\mu_t)-h_i^q(X_t^i)\|^2 &= \sum_{i=1}^{n}\mathbb{E}\|\nabla f(\mu_t)-\nabla f(X_t^i-\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2 \\ &\leq \sum_{i=1}^{n}L^2\mathbb{E}\|\mu_t-X_t^i+\eta\widetilde{h}_i^{q-1}(X_t^i))\|^2 \\ &\overset{Cauchy-Schwarz}{\leq} \sum_{i=1}^{n}2L^2\mathbb{E}\|X_t^i-\mu_t\|^2+\sum_{i=1}^{n}2L^2\eta^2\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2 \\ &= 2L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}2L^2\eta^2\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2.\end{aligned}$$

□

**Lemma H.3.** *For any $1 \leq q \leq H$ and step $T$, we have that*

$$\sum_{i=1}^{n}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \leq n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}12L^2\eta^2\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2.$$

*Proof.*

$$\sum_{i=1}^{n}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \leq \sum_{i=1}^{n}(\sigma^2+\mathbb{E}\|h_i^q(X_t^i)\|^2) \leq n\sigma^2+\sum_{i=1}^{n}\mathbb{E}\|\nabla f(X_t^i-\sum_{s=0}^{q-1}\eta\widetilde{h}_i^s(X_t^i))\|^2$$

$$\leq n\sigma^2+\sum_{i=1}^{n}\mathbb{E}\Bigg\|\nabla f(X_t^i-\eta\widetilde{h}_i^{q-1}(X_t^i))-\nabla f(\mu_t)+\nabla f(\mu_t)$$

$$-\sum_{j=1}^{n}\nabla f(X_t^j-\sum_{s=0}^{q-1}\eta\widetilde{h}_j^s(X_t^j))/n+\sum_{j=1}^{n}\nabla f(X_t^j-\sum_{s=0}^{q-1}\eta\widetilde{h}_j^s(X_t^j))/n\Bigg\|^2$$

$$\overset{Cauchy-Schwarz}{\leq} n\sigma^2+\sum_{i=1}^{n}3\mathbb{E}\|\nabla f(\mu_t)-h_i^q(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}(\nabla f(\mu_t)-h_i^q(X_t^i))/n\|^2$$

$$+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2$$

$$\overset{Cauchy-Schwarz}{\leq} n\sigma^2+\sum_{i=1}^{n}6\mathbb{E}\|\nabla f(\mu_t)-h_i^q(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2$$

$$\overset{\text{Lemma H.2}}{\leq} n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}12L^2\eta^2\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2.$$

□

Next we use the above lemma to show the upper bound for $\sum_{q=1}^{H}\sum_{i=1}^{n}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2$:

**Lemma H.4.** *For $\eta \leq \frac{1}{6LH}$, we have that :*

$$\sum_{q=1}^{H}\sum_{i=1}^{n}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \leq 2Hn\sigma^2+24HL^2\mathbb{E}[\Gamma_t]+6n\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2$$

*Proof.* Notice that if $\eta \leq \frac{1}{6LH}$ the Lemma H.3 gives us that :

$$\sum_{i=1}^{n}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \leq n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}\frac{1}{2H^2}\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2$$

$$\leq n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}\frac{q}{2H^2}\sum_{s=0}^{q-1}\mathbb{E}\|\widetilde{h}_i^s(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2$$

$$\leq n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+\sum_{i=1}^{n}\frac{1}{2H}\sum_{s=0}^{q-1}\mathbb{E}\|\widetilde{h}_i^s(X_t^i)\|^2+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2. \tag{29}$$

For $0 \leq q \leq H$, let

$$R_q=\sum_{i=1}^{n}\sum_{s=0}^{q}\mathbb{E}\|\widetilde{h}_i^s(X_t^i)\|^2.$$

Observe that the inequality 29 can be rewritten as:

$$R_q-R_{q-1} \leq \frac{1}{2H}R_{q-1}+n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2.$$

which is the same as

$$R_q \leq (1+\frac{1}{2H})R_{q-1}+n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+3n\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2.$$

By unrolling the recursion we get that

$$R_H \leq \sum_{q=0}^{H-1}(1+\frac{1}{2H})^q\Big(n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+3n\mathbb{E}\|\sum_{i=1}^n h_i^{H-q}(X_t^i)/n\|^2\Big)$$

Since,

$$(1+\frac{1}{2H})^H \leq (e^{\frac{1}{2H}})^H = e^{1/2} \leq 2$$

we have that

$$\begin{aligned}R_H &= \sum_{q=1}^H\sum_{i=1}^n \mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \leq 2\Bigg(\sum_{q=1}^H\Big(n\sigma^2+12L^2\mathbb{E}[\Gamma_t]+3n\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2\Big)\Bigg)\\ &= 2Hn\sigma^2+24HL^2\mathbb{E}[\Gamma_t]+6n\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2.\end{aligned}$$

□

Next we derive the upper bound for $\sum_{t=0}^T \mathbb{E}[\Gamma_t]$:

**Lemma H.5.** *For $\eta \leq \frac{1}{10HL\sqrt{2r/\lambda_2+8r^2/\lambda_2^2}}$, we have that :*

$$\sum_{t=0}^T\mathbb{E}[\Gamma_t] \leq \frac{8nr\eta^2\sigma^2H^2T}{\lambda_2}(2+\frac{8r}{\lambda_2})+\frac{24nr\eta^2H}{\lambda_2}(2+\frac{8r}{\lambda_2})\sum_{t=1}^T\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2.$$

*Proof.* By Lemma H.1 we get that:

$$\begin{aligned}\mathbb{E}[\Gamma_{t+1}] &\leq (1-\frac{\lambda_2}{2rn})\mathbb{E}[\Gamma_t]+\frac{\eta^2H}{n}(2+\frac{8r}{\lambda_2})\sum_{q=1}^H\sum_{i=1}^n\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2\\ &\overset{\text{Lemma H.4}}{\leq} (1-\frac{\lambda_2}{2rn})\mathbb{E}[\Gamma_t]\\ &\qquad+\frac{\eta^2H}{n}(2+\frac{8r}{\lambda_2})\Big(2Hn\sigma^2+24HL^2\mathbb{E}[\Gamma_t]+6n\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2\Big)\\ &= (1-\frac{\lambda_2}{2rn})\mathbb{E}[\Gamma_t]+2\eta^2\sigma^2H^2(2+\frac{8r}{\lambda_2})+\frac{24H^2L^2\eta^2}{n}(2+\frac{8r}{\lambda_2})\mathbb{E}[\Gamma_t]\\ &\qquad+6\eta^2H(2+\frac{8r}{\lambda_2})\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2\end{aligned}$$

Notice that for $\eta \leq \frac{1}{10HL\sqrt{2r/\lambda_2+8r^2/\lambda_2^2}}$ we can rewrite the above inequality as

$$\mathbb{E}[\Gamma_{t+1}] \leq (1-\frac{\lambda_2}{4nr})\mathbb{E}[\Gamma_t]+2\eta^2\sigma^2H^2(2+\frac{8r}{\lambda_2})+6\eta^2H(2+\frac{8r}{\lambda_2})\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2.$$

since $\sum_{i=0}^\infty(1-\frac{\lambda_2}{4nr})^i \leq \frac{1}{1-(1-\frac{\lambda_2}{4nr})} = \frac{4nr}{\lambda_2}$ we get that:

$$\begin{aligned}\sum_{t=0}^T\mathbb{E}[\Gamma_t] &\leq \frac{4nr}{\lambda_2}\Big(2\eta^2\sigma^2H^2(2+\frac{8r}{\lambda_2})+6\eta^2H(2+\frac{4r}{\lambda_2})\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2)\\ &= \frac{8nr\eta^2\sigma^2H^2T}{\lambda_2}(2+\frac{8r}{\lambda_2})+\frac{24nr\eta^2H}{\lambda_2}(2+\frac{8r}{\lambda_2})\sum_{t=1}^T\sum_{q=1}^H\mathbb{E}\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2.\end{aligned}$$

□

Now, we are ready to prove the following theorem

**Theorem 4.2.** *Let $f$ be an non-convex, $L$-smooth, function satisfying condition 5, whose minimum $x^\star$ we are trying to find via the SwarmSGD procedure given in Algorithm 1. Let $H$ be the number of local stochastic gradient steps performed by each agent upon interaction. Define $\mu_t = \sum_{i=1}^n X_t^i/n$, where $X_t^i$ is a value of model $i$ after $t$ interactions. For learning rate $\eta = \frac{n}{\sqrt{T}}$ and $T \geq 32400 n^4 H^2 \max(1, L^2)(\frac{r^2}{\lambda_2^2}+1)^2$ we have that:*

$$\frac{\sum_{i=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2}{T} \leq \frac{1}{\sqrt{T}H}\mathbb{E}[f(\mu_0) - f(\mu_t)] + \frac{388H^2\max(1,L^2)\sigma^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2}+1).$$

*Proof.* Let $\mathbb{E}_t$ be the expectation which is conditioned on $\{X_1^t, X_2^t, ..., X_n^t\}$. By $L$-smoothness we have that

$$\mathbb{E}_t[f(\mu_{t+1})] \leq f(\mu_t) + \mathbb{E}_t\langle \nabla f(\mu_t), \mu_{t+1} - \mu_t\rangle + \frac{L}{2}\mathbb{E}_t\|\mu_{t+1} - \mu_t\|^2. \tag{30}$$

First we look at $\mathbb{E}_t[\mu_{t+1} - \mu_t]$. If agents $i$ and $j$ interact, (which happens with probability $\frac{1}{rn/2}$). We have that $\mu_{t+1} - \mu_t = -\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j)$. Hence we get that

$$\mathbb{E}_t[\mu_{t+1} - \mu_t] = \frac{1}{rn/2}\sum_{(i,j)\in E}\mathbb{E}_t[-\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j)] = \frac{2}{n}\sum_{i=1}^n \mathbb{E}_t[-\frac{\eta}{n}\widetilde{h}_i(X_t^i)]$$
$$= -\frac{2\eta}{n^2}\sum_{i=1}^n \mathbb{E}_t[\widetilde{h}_i(X_t^i)] \overset{(3)}{=} -\frac{2\eta}{n^2}\sum_{i=1}^n\sum_{q=1}^H \mathbb{E}_t[h_i^q(X_t^i)].$$

using the above inequality we get:

$$\mathbb{E}_t\langle \nabla f(\mu_t), \mu_{t+1} - \mu_t\rangle = \langle \nabla f(\mu_t), \mathbb{E}_t[\mu_{t+1} - \mu_t]\rangle = \langle \nabla f(\mu_t), -\frac{2\eta}{n^2}\sum_{i=1}^n\sum_{q=1}^H \mathbb{E}_t[h_i^q(X_t^i)]\rangle$$
$$= \frac{\eta}{n}\sum_{q=1}^H\Big(\mathbb{E}_t\|\nabla f(\mu_t) - \sum_{i=1}^n h_i^q(X_t^i)/n\|^2 - \|\nabla f(\mu_t)\|^2 - \mathbb{E}_t\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2\Big)$$
$$= \frac{\eta}{n}\sum_{q=1}^H\Big(\mathbb{E}_t\|\sum_{i=1}^n(\nabla f(\mu_t) - h_i^q(X_t^i))/n\|^2 - \|\nabla f(\mu_t)\|^2 - \mathbb{E}_t\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2\Big)$$
$$\overset{Cauchy-Schwarz}{\leq} \frac{\eta}{n}\sum_{q=1}^H\Big(\frac{1}{n}\mathbb{E}_t\|\sum_{i=1}^n \nabla f(\mu_t) - h_i^q(X_t^i)\|^2 - \|\nabla f(\mu_t)\|^2 - \mathbb{E}_t\|\sum_{i=1}^n h_i^q(X_t^i)/n\|^2\Big)$$

Next we look at $\mathbb{E}_t\|\mu_{t+1} - \mu_t\|^2$. If agents $i$ and $j$ interact, (which happens with probability $\frac{1}{rn/2}$). We have that $\mu_{t+1} - \mu_t = -\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j)$. Hence we get that

$$\mathbb{E}_t\|\mu_{t+1} - \mu_t\|^2 = \frac{1}{rn/2}\sum_{(i,j)\in E}\mathbb{E}_t\Big\|-\frac{\eta}{n}\widetilde{h}_i(X_t^i) - \frac{\eta}{n}\widetilde{h}_j(X_t^j)\Big\|^2$$
$$\overset{Cauchy-Schwarz}{\leq} \frac{1}{rn/2}\sum_{(i,j)\in E}\frac{\eta^2}{n^2}\Big(2\mathbb{E}_t\|\widetilde{h}_i(X_t^i)\|^2 + 2\mathbb{E}_t\|\frac{\eta}{n}\widetilde{h}_j(X_t^j)\|^2\Big)$$
$$= \frac{2}{n}\sum_{i=1}^n\frac{2\eta^2}{n^2}\|\widetilde{h}_i(X_t^i)\| \overset{Cauchy-Schwarz}{\leq} \frac{4\eta^2 H}{n^3}\sum_{i=1}^n\sum_{q=1}^H \mathbb{E}_t\|\widetilde{h}_i^q(X_t^i)\|^2$$

Hence, we can rewrite (30) as:

$$\mathbb{E}_t[f(\mu_{t+1})] \leq f(\mu_t) + \frac{\eta}{n}\sum_{q=1}^{H}\Big(\frac{1}{n}\mathbb{E}_t\|\sum_{i=1}^{n}\nabla f(\mu_t) - h_i^q(X_t^i)\|^2 - \|\nabla f(\mu_t)\|^2 - \mathbb{E}_t\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2\Big) + \frac{2L\eta^2 H}{n^3}\sum_{i=1}^{n}\sum_{q=1}^{H}\mathbb{E}_t\|\widetilde{h}_i^q(X_t^i)\|^2$$

Next, we remove conditioning:

$$\begin{aligned}
\mathbb{E}[f(\mu_{t+1})] &= \mathbb{E}[\mathbb{E}_t[f(\mu_{t+1})]] \\
&\leq \mathbb{E}[f(\mu_t)] + \frac{\eta}{n}\sum_{q=1}^{H}\Big(\frac{1}{n}\mathbb{E}\|\sum_{i=1}^{n}\nabla f(\mu_t) - h_i^q(X_t^i)\|^2 - \mathbb{E}\|\nabla f(\mu_t)\|^2 - \mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2\Big) \\
&\quad + \frac{2L\eta^2 H}{n^3}\sum_{i=1}^{n}\sum_{q=1}^{H}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \\
&\overset{\text{Lemma H.2}}{\leq} \mathbb{E}[f(\mu_t)] + \frac{\eta}{n}\sum_{q=1}^{H}\Big(\frac{1}{n}(2L^2\mathbb{E}[\Gamma_t] + \sum_{i=1}^{n}2L^2\eta^2\mathbb{E}\|\sum_{s=0}^{q-1}\widetilde{h}_i^s(X_t^i)\|^2) - \mathbb{E}\|\nabla f(\mu_t)\|^2 - \mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2\Big) \\
&\quad + \frac{2L\eta^2 H}{n^3}\sum_{i=1}^{n}\sum_{q=1}^{H}\mathbb{E}\|\widetilde{h}_i^q(X_t^i)\|^2 \\
&\overset{\text{Lemma H.4}}{\leq} \mathbb{E}[f(\mu_t)] + \frac{2\eta L^2 H}{n^2}\mathbb{E}[\Gamma_t] - \frac{H\eta}{n}\mathbb{E}\|\nabla f(\mu_t)\|^2 \\
&\quad + \frac{2L^2\eta^3}{n^2}\Big(2Hn\sigma^2 + 24HL^2\mathbb{E}[\Gamma_t] + 6n\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2\Big) \\
&\quad + \frac{2LH\eta^2}{n^3}\Big(2Hn\sigma^2 + 24HL^2\mathbb{E}[\Gamma_t] + 6n\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2\Big) \\
&\quad - \frac{\eta}{n}\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2
\end{aligned}$$

Next we choose $\eta \leq \frac{1}{8L}$ and $\eta \leq \frac{n}{60LH}$, so that $\frac{12L^2\eta^3}{n} \leq \frac{\eta}{5n}$ and $\frac{12LH\eta^2}{n^2} \leq \frac{\eta}{5n}$. This together with the above inequalities allows us to derive the following upper bound for $\mathbb{E}[f(\mu_{t+1})]$(we eliminate terms with positive multiplicative factor $\mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2$):

$$\begin{aligned}
\mathbb{E}[f(\mu_{t+1})] &\leq \mathbb{E}[f(\mu_t)] + \frac{2\eta L^2 H}{n^2}\mathbb{E}[\Gamma_t] - \frac{H\eta}{n}\mathbb{E}\|\nabla f(\mu_t)\|^2 \\
&\quad + \frac{2L^2\eta^3}{n^2}\Big(2Hn\sigma^2 + 24HL^2\mathbb{E}[\Gamma_t]\Big) + \frac{2LH\eta^2}{n^3}\Big(2Hn\sigma^2 + 24HL^2\mathbb{E}[\Gamma_t]\Big) \\
&\quad - \frac{3\eta}{5n}\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2.
\end{aligned}$$

We proceed by summing up the above inequality for $0 \leq t \leq T-1$:

$$\sum_{t=0}^{T-1} \mathbb{E}[f(\mu_{t+1})] \leq \sum_{t=0}^{T-1} \mathbb{E}[f(\mu_t)] + \frac{4L^2 H\eta^3\sigma^2 T}{n} + \frac{4LH^2\eta^2\sigma^2 T}{n^2} - \sum_{i=0}^{T-1} \frac{\eta H}{n}\mathbb{E}\|\nabla f(\mu_t)\|^2 + \frac{2\eta L^2 H}{n^2}\sum_{t=0}^{T-1}\mathbb{E}[\Gamma_t] + \frac{48L^4 H\eta^3}{n^2}\sum_{t=0}^{T-1}\mathbb{E}[\Gamma_t] + \frac{48L^3H^2\eta^2}{n^3}\sum_{t=0}^{T-1}\mathbb{E}[\Gamma_t] - \sum_{t=0}^{T-1}\frac{3\eta}{5n}\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n} h_i^q(X_t^i)/n\|^2 \tag{31}$$

Next we use Lemma H.5:

$$\begin{aligned}
&\frac{2\eta L^2 H}{n^2}\sum_{t=0}^{T-1}\mathbb{E}[\Gamma_t] + \frac{48L^4H\eta^3}{n^2}\sum_{t=0}^{T-1}\mathbb{E}[\Gamma_t] + \frac{48L^3H^2\eta^2}{n^3}\sum_{t=0}^{T-1}\mathbb{E}[\Gamma_t] \\
&\leq \frac{2\eta L^2 H}{n^2}\Bigg(\frac{8nr\eta^2\sigma^2H^2T}{\lambda_2}(2+\frac{8r}{\lambda_2}) + \frac{24nr\eta^2 H}{\lambda_2}(2+\frac{8r}{\lambda_2})\sum_{t=1}^{T-1}\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2\Bigg) \\
&+ \frac{48L^4H\eta^3}{n^2}\Bigg(\frac{8nr\eta^2\sigma^2H^2T}{\lambda_2}(2+\frac{8r}{\lambda_2}) + \frac{24nr\eta^2 H}{\lambda_2}(2+\frac{8r}{\lambda_2})\sum_{t=1}^{T-1}\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2\Bigg) \\
&+ \frac{48L^3H^2\eta^2}{n^3}\Bigg(\frac{8nr\eta^2\sigma^2H^2T}{\lambda_2}(2+\frac{8r}{\lambda_2}) + \frac{24nr\eta^2 H}{\lambda_2}(2+\frac{8r}{\lambda_2})\sum_{t=1}^{T-1}\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2\Bigg)
\end{aligned}$$

By choosing $\eta \leq \frac{1}{16HL\sqrt{2r/\lambda_2+8r^2/\lambda_2^2}}$, $\eta \leq \frac{1}{9H^{1/2}L(2r/\lambda_2+8r^2/\lambda_2^2)^{1/4}}$ and $\eta \leq \frac{n^{1/3}}{18LH(2r/\lambda_2+4r^2/\lambda_2^2)^{1/3}}$ we can eliminate terms with the multiplicative factor $\sum_{q=1}^{H}\mathbb{E}\|\sum_{i=1}^{n}h_i^q(X_t^i)/n\|^2$ in the inequality (31):

$$\begin{aligned}
\sum_{t=0}^{T-1}\mathbb{E}[f(\mu_{t+1})] \leq \sum_{t=0}^{T-1}\mathbb{E}[f(\mu_t)] - \sum_{i=0}^{T-1}\frac{\eta H}{n}\mathbb{E}\|\nabla f(\mu_t)\|^2 &+ \frac{4L^2H\eta^3\sigma^2T}{n} + \frac{4LH^2\eta^2\sigma^2T}{n^2} + \frac{16\eta^3L^2H^3T\sigma^2}{n}(2r/\lambda_2+8r^2/\lambda_2^2) \\
&+ \frac{384L^4H^3\eta^5T\sigma^2}{n}(2r/\lambda_2+8r^2/\lambda_2^2) \\
&+ \frac{384L^3H^4\eta^4\sigma^2T}{n^2}(2r/\lambda_2+8r^2/\lambda_2^2).
\end{aligned}$$

After rearranging terms and dividing by $\frac{\eta TH}{n}$ we get that

$$\begin{aligned}
\frac{\sum_{i=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2}{T} \leq \frac{n}{\eta TH}\mathbb{E}[f(\mu_0)-f(\mu_t)] &+ 4L^2\eta^2\sigma^2 + \frac{4LH\eta\sigma^2}{n} \\
&+ 16\eta^2L^2H^2\sigma^2(2r/\lambda_2+8r^2/\lambda_2^2) \\
&+ 384L^4H^2\eta^4\sigma^2(2r/\lambda_2+8r^2/\lambda_2^2) \\
&+ \frac{384L^3H^3\eta^3\sigma^2}{n}(2r/\lambda_2+8r^2/\lambda_2^2).
\end{aligned}$$

Next we use $\eta \leq 1/n$ and $\eta \leq \frac{1}{6HL}$:

$$\begin{aligned}\frac{\sum_{i=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2}{T} &\leq \frac{n}{\eta T H}\mathbb{E}[f(\mu_0)-f(\mu_t)] + \frac{4L^2\eta\sigma^2}{n} + \frac{4LH\eta\sigma^2}{n} \\ &\quad + \frac{16\eta L^2H^2\sigma^2}{n}(2r/\lambda_2 + 4r^2/\lambda_2^2) \\ &\quad + \frac{384L^4H^2\eta^3\sigma^2}{n}(2r/\lambda_2 + 8r^2/\lambda_2^2) \\ &\quad + \frac{384L^3H^3\eta^3\sigma^2}{n}(2r/\lambda_2 + 8r^2/\lambda_2^2) \\ &\leq \frac{n}{\eta T H}\mathbb{E}[f(\mu_0)-f(\mu_t)] + \frac{4L^2\eta\sigma^2}{n} + \frac{4LH\eta\sigma^2}{n} \\ &\quad + \frac{16\eta L^2H^2\sigma^2}{n}(2r/\lambda_2 + 8r^2/\lambda_2^2) \\ &\quad + \frac{11L^2\eta\sigma^2}{n}(2r/\lambda_2 + 8r^2/\lambda_2^2) \\ &\quad + \frac{11LH\eta\sigma^2}{n}(2r/\lambda_2 + 8r^2/\lambda_2^2)\end{aligned}$$

.

Recall that $\eta = \frac{n}{\sqrt{T}}$ to get:

$$\begin{aligned}\frac{\sum_{i=0}^{T-1}\mathbb{E}\|\nabla f(\mu_t)\|^2}{T} &\leq \frac{1}{\sqrt{T}H}\mathbb{E}[f(\mu_0)-f(\mu_t)] + \frac{4L^2\sigma^2}{\sqrt{T}} + \frac{4LH\sigma^2}{\sqrt{T}} \\ &\quad + \frac{16L^2H^2\sigma^2}{\sqrt{T}}(2r/\lambda_2 + 8r^2/\lambda_2^2) \\ &\quad + \frac{11L^2\sigma^2}{\sqrt{T}}(2r/\lambda_2 + 8r^2/\lambda_2^2) + \frac{11LH\sigma^2}{\sqrt{T}}(2r/\lambda_2 + 8r^2/\lambda_2^2) \\ &\leq \frac{1}{\sqrt{T}H}\mathbb{E}[f(\mu_0)-f(\mu_t)] + \frac{388H^2\max(1,L^2)\sigma^2}{\sqrt{T}}(\frac{r^2}{\lambda_2^2}+1).\end{aligned}$$

Notice that all assumptions and upper bounds on $\eta$ are satisfied if

$$\eta \leq \frac{1}{180nH\max(1,L)(\frac{r^2}{\lambda_2^2}+1)}, \tag{32}$$

which is true

$$T \geq 32400n^4H^2\max(1,L^2)(\frac{r^2}{\lambda_2^2}+1)^2. \tag{33}$$

□

# I ANALYSIS WHEN THE DATA IS PARTITIONED AMONG THE AGENTS

Our analysis can be extended to the case where agents do not necessarily have access to the entire data. More formally, let $f(x) = \sum_{i=1}^n f_i(x)/n$, where $f_i(x)$ is a loss function computed over the data available to the agent $i$. We will require the following assumptions(Which match the assumptions in Lian et al. (2017)):

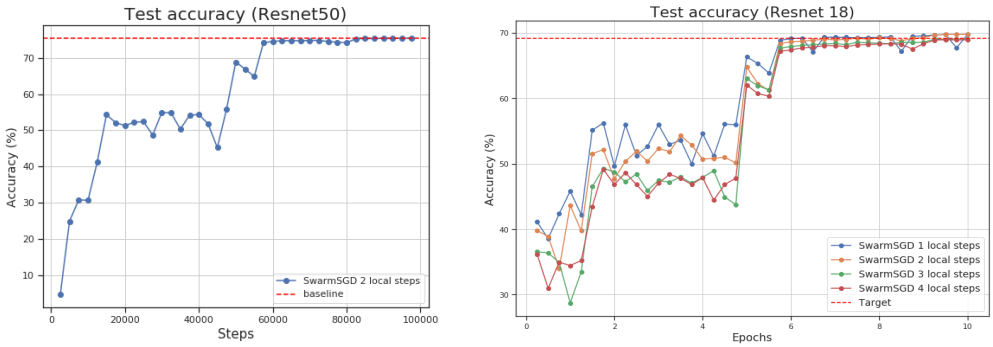
1. For each agent $i$, the gradient $\nabla f_i(x)$ is $L$-Lipschitz continuous for some $L > 0$, i.e. for all $x, y \in \mathbb{R}^d$:
$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x-y\|. \tag{34}$$
2. For each agent $i$ and $x \in \mathbb{R}^d$:
$$\mathbb{E}[\widetilde{g}_i(x)] = \nabla f_i(x). \tag{35}$$
3. For each agent $i$ and $x \in \mathbb{R}^d$ there exist $\varsigma^2$ such that:
$$\mathbb{E}\|\widetilde{g}_i(x) - \nabla f_i(x)\|^2 \leq \varsigma^2. \tag{36}$$

4. For each $x \in \mathbb{R}^d$ there exist $\rho^2$ such that:

$$\sum_{i=1}^{n} \|\nabla f_i(x) - \nabla f(x)\|^2/n \leq \rho^2. \tag{37}$$

Under assumptions 34 and 35 we are able to show that Theorem 4.1 still holds. In the case of Theorem 4.2 , if all the assumptions above hold(we do not need assumption 5 in this case), we are able to derive similar result by replacing $\sigma^2$ with $O(\varsigma^2 + \rho^2)$ in the convergence bound.

## J Additional Experimental Results




(a) Convergence of ResNet50/ImageNet versus number of gradient steps. SwarmSGD is able to recover the baseline top accuracy.

(b) Convergence versus number of local steps for ResNet18 on ImageNet. All variants recover the target accuracy, but we note the lower convergence of variants with more local steps.

Figure 2: Additional convergence results for ImageNet dataset.

We validated our analysis, by applying the algorithm to training deep neural networks for image classification and machine translation.

**Target System and Implementation.** We run SwarmSGD on the CSCS Piz Daint supercomputer, which is composed of Cray XC50 nodes, each with a Xeon E5-2690v3 CPU and an NVIDIA Tesla P100 GPU, using a state-of-the-art Aries interconnect. Please see (Piz) for hardware details.

We implemented SwarmSGD in Pytorch and TensorFlow using NCCL/MPI respectively. Basically, each node implements a computation thread, and a communication thread, each of which stores a copy of the model. The "live" copy, which is being updated with gradients, is stored by the computation thread. Periodically, the threads synchronize their two models. When interacting, the two nodes exchange model information via their communication threads. Our implementation closely follows the non-blocking Swarm algorithm description.

We used SwarmSGD to train ResNets on the classic CIFAR-10/ImageNet datasets, and a Transformer Vaswani et al. (2017) on the WMT17 dataset (English-Germa). The code will be made available upon publication.

**Hyperparameters.** The only additional hyperparameter is the total number of epochs we execute for. Once we have fixed the number of epochs, *we do not alter the other training hyperparameters*: in particular, the learning rate schedule, momentum and weight decay terms are identical to sequential SGD, for each individual model. Practically, if sequential SGD trains ResNet18 in 90 epochs, decreasing the learning rate at 30 and 60 epochs, then SwarmSGD with 32 nodes and multiplier 2 would $90 * 2/32 \simeq 5.6$ epochs per node, decreasing the learning rate at 2 and 4 epochs.

Specifically, for the ImageNet experiments, we used the following hyper-parameters. For ResNet18 and ResNet50, we ran for 240 total parallel epochs using 32 parallel nodes. The first communicated every 3 local steps, whereas the second communicated every 2 local steps. We used the same hyper-parameters (initial learning rate 0.1, annealed at 1/3 and 2/3 through training, and standard weight-decay and momentum parameters).

For the WMT17 experiments, we ran a standard Transformer-large model, and executed for 10 *global* epochs at 16, 32, and 64 nodes. We ran a version with multiplier 1 (i.e. $10/\text{NUM_NODES}$

epochs per model) and one with multiplier 1.5 (i.e. $15/\text{NUM_NODES}$ epochs per model) and registered the BLEU score for each.

**Baselines.** We consider the following baselines:

- **Data-parallel SGD:** Here, we consider both the small-batch (strong scaling) version, which executes a global batch size of 256 on ImageNet/CIFAR experiments, and the large-batch (weak-scaling) baseline, which maximizes the batch per GPU. For the latter version, the learning rate is tuned following Goyal et al. (2017).
- **Local SGD:** Stich (2018); Lin et al. (2018) We follow the implementation of Lin et al. (2018), communicating globally every 5 SGD steps (which was the highest setting which provided good accuracy on the WMT task).
- **Previous decentralized proposals:** We experimented also with D-PSGD Lian et al. (2017), AD-PSGD Lian et al. (2018), and SGP Assran et al. (2018). Due to computational constraints, we did not always measure their end-to-end accuracy. Our method matches the sequential / large-batch accuracy for the models we consider within $1\%$. We note that the best performing alternative (AD-PSGD) is known to drop accuracy relative to the baselines, e.g. (Assran et al., 2018).

**Results.** The accuracy results for ImageNet experiments are given in Table 1 and Figures 2(a) and 2(b). As is standard, we follow Top-1 validation accuracy versus number of steps.

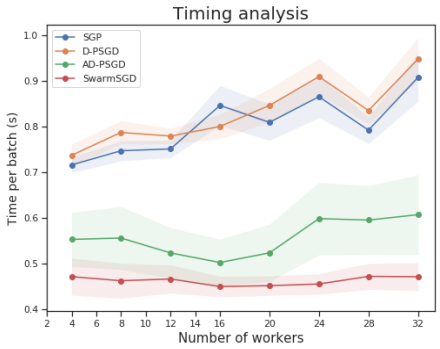
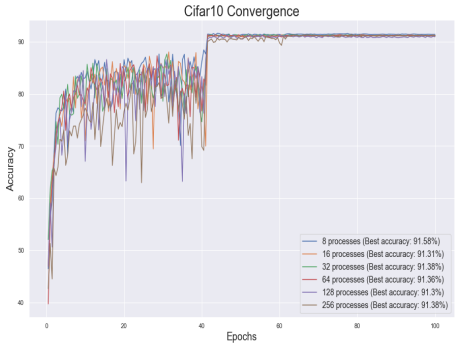



Figure 3: Average time per batch for previous methods, compared to SwarmSGD, on ResNet18/ImageNet, across 1000 repetitions with warm-up. Notice that 1) the time per batch of SwarmSGD stays *constant* relative to the number of nodes; 2) it is lower than any other method. This is due to the *reduced communication frequency*. Importantly, the base value on the y axis of this graph (0.4) is the average computation time per batch. Thus, everything above 0.4 represents the average communication time for this model.

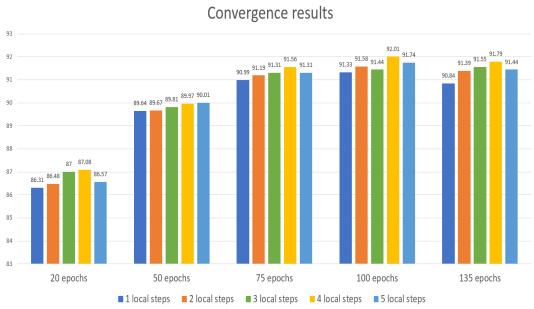
**Communication cost.** We now look deeper into SwarmSGD's performance. For this, we examine in Figure 3 the average time per batch of different methods when executed on our testbed. The base value on the y axis (0.4s) is exactly the *average time per batch*, which is the same across all methods. Thus, the extra values on the y axis equate roughly to the communication cost of each algorithm. The results suggest that the communication cost can be up to twice the batch cost (SGP and D-PSGD). Moreover, this cost is *increasing* when considered relative to the number of workers (X axis), for all methods except SwarmSGD.

This reduced cost is justified simply because our method *reduces communication frequency:* it communicates less often, and therefore the average cost of communication at a step is lower. We can therefore conclude that our method is *scalable*, in the sense that its communication cost remains constant relative to the total size of the system. Figure 2(b) shows the convergence versus time for ResNet18 on the ImageNet dataset, at 32 nodes, with 3 local steps per node, and $\sim$ 7 epochs per model.

**Convergence versus Steps and Epochs.** Figure 5 shows and discusses the results of additional ablation studies with respect to the number of nodes/processes and number of local steps / total epochs on the CIFAR-10 dataset / ResNet20 model. In brief, the results show that the method still preserves convergence even at very high node counts (256), and suggest a strong correlation between accuracy and the number of epochs executed per model. The number of local steps executed also impacts accuracy, but to a much lesser degree.



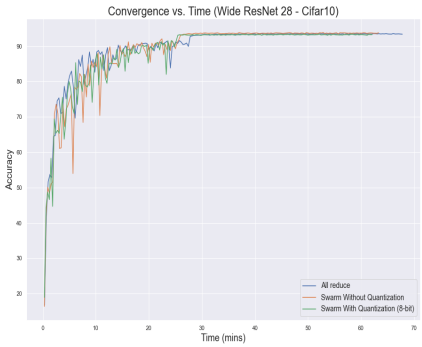

(a) Convergence versus number epochs (per model) for CIFAR-10/ResNet20, at node counts between 8 and 256. We note that the algorithm converges and recovers SGD accuracy (91.35% Top-1) for all node counts, although there are oscillations at high node counts.




(b) Accuracy versus local epochs and local steps for CIFAR-10/ResNet20. The original schedule for this model has 300 epochs, and this experiment is executed on 8 nodes. If the convergence scaling were perfect, $300/8 = 37.5$ epochs would have been sufficient to converge. However, in this case we need an epoch multiplier of 2, leading to 75 epochs to recover full accuracy (which in this case is $91.35\%$).

Figure 4: Additional convergence results for CIFAR-10 dataset, versus number of nodes (left), and local steps (right).

(a) Convergence versus number of steps for the quantized variant.




(b) Convergence versus time .

Figure 5: Convergence results for quantized 2xResNet28 trained on the CIFAR-10 dataset, versus iterations (left), and time (right).

**Quantization.** Finally, we show convergence and speedup for a WideResNet-28 model with width factor 2, trained on the CIFAR-10 dataset. We note that the epoch multiplier factor in this setup is 1, i.e. Swarm (and its quantized variant) execute exactly the same number of epochs as the baseline. Notice that the quantized variant provides approximately $10\%$ speedup in this case, for a $< 0.3\%$ drop in Top-1 accuracy.