Dynamic Structure Estimation from Bandit Feedback using Nonvanishing Exponential Sums

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Abstract

This work tackles the dynamic structure estimation problems for periodically behaved discrete dynamical system in the Euclidean space. We assume the observations become sequentially available in a form of bandit feedback contaminated by a sub-Gaussian noise. Under such fairly general assumptions on the noise distribution, we carefully identify a set of recoverable information of periodic structures. Our main results are the (computation and sample) efficient algorithms that exploit asymptotic behaviors of exponential sums to effectively average out the noise effect while preventing the information to be estimated from vanishing. In particular, the novel use of the Weyl sum, a variant of exponential sums, allows us to extract spectrum information for linear systems. We provide sample complexity bounds for our algorithms, and we experimentally validate our theoretical claims on simulations of toy examples, including Cellular Automata.

1 Introduction

System identification has been of great interest in controls, economics, and statistical machine learning (cf. Tsiamis & Pappas (2019); Tsiamis et al. (2020); Lale et al. (2020); Lee (2022); Kakade et al. (2020); Ohnishi et al. (2021); Mania et al. (2020); Simchowitz & Foster (2020); Curi et al. (2020); Hazan et al. (2018); Simchowitz et al. (2019); Lee & Zhang (2020)). In particular, estimations of periodic information, including eigenstructures for linear systems, under noisy and partially observable environments, are essential to a variety of applications such as biological data analysis (e.g., Hughes et al. (2017); Sokolove & Bushell (1978); Zielinski et al. (2014); also see Furusawa & Kaneko (2012) for how gene oscillation affects differentiation of cells), earthquake analysis (e.g., Rathje et al. (1998); Sabetta & Pugliese (1996); Wolfe (2006); see Allen & Kanamori (2003) for the connections of the frequencies and magnitude of earthquakes), chemical/asteroseismic analysis (e.g., Aerts et al. (2018)), and communication and information systems (e.g., Couillet & Debbah (2011); Derevyanko et al. (2016)), just to name a few.

Specifically, providing statistical guarantees for extractions of periodic information under perturbations following a general distribution is quite fundamental as it is connected to the information theory of communication capacities. Indeed, there has been an interest of studying novel paradigms for coding, transmitting, and processing information sent through optical communication systems Turitsyn et al. (2017). When signals are coded digitally, erroneous signal transmission is no longer modeled with a simple Gaussian distribution.

On the other hand, those fundamental estimation problems with statistical sample complexity guarantees are often treated within the literature of learning under partial observability (cf. Menda et al. (2020); Tsiamis & Pappas (2019); Tsiamis et al. (2020); Lale et al. (2020); Lee (2022); Adams et al. (2021); Bhouri & Perdikaris (2021); Ouala et al. (2020); Uy & Peherstorfer (2021); Subramanian et al. (2022); Bennett & Kallus (2021); Lee et al. (2020)). Inheriting accumulated results of the controls literature, however, most of the works on provably correct methods for partially observable dynamical systems consider additive Gaussian noise and controllable/observable linear system; some works further restrict the spectral radius of the linear system to be strictly less than one. In fact, if the observation is contaminated by a noise following a more general distribution, a concentration of measures should be adopted (cf. Shalev-Shwartz & Ben-David (2014)); and this approach suffers from a risk of making structural information of the underlying dynamics vanish as well.

In this paper, we tackle this periodic structure estimation problem for *nearly* periodically behaved discrete dynamical systems (cf. Arnold (1998); we allow systems that are not *exactly* periodic) with sequentially available bandit feedback. Due to the presence of noise and partial observability, our problem setups do not permit the recovery of the full set of period/eigenvalues information in general; as such we ask the following question: *what subset of information on dynamic structures can be statistically efficiently estimated?* This work successfully answers this question by identifying and mathematically defining recoverable information, and proposes algorithms for efficiently extracting such information.

The technical novelty of our approach is highlighted by the careful adoption of the asymptotic bounds on the exponential sums that effectively cancel out noise effects while preserving the information to be estimated. When the dynamics is driven by a linear system, the use of the Weyl sum Weyl (1916), a variant of exponential sums, enables us to extract more detailed information. To our knowledge, this is the first attempt of employing asymptotic results of the Weyl sum for statistical estimation problems, and further studies on the relations between statistical estimation theory and exponential sums (or even other number theoretical results) are of independent interests.

Finally, although beyond the scope of this work, it is worth mentioning that our algorithms for periodic structure estimation can be seamlessly applied to periodic/dynamic bandit problems (see Lattimore & Szepesvári (2020) for an overview of learning from bandit feedback); by using our algorithms for *explore* phase to determine the periodic structure behind the dynamically changing parameter, one can then *commit* to arms in a certain way to maximize rewards (see Appendix E). Before concluding the introduction, we nevertheless strongly emphasize that our focus of this work is on the periodic structure estimation from bandit feedback rather than the general parameter estimation problems of linear dynamical systems or regret minimization problems.

With these motivations and summary of our approach in place, we present our dynamical system model below.

Dynamic structure in bandit feedback. We define $D \subset \mathbb{R}^d$ as a (finite or infinite) collection of arms to be pulled. Let $(\eta_t)_{t=1}^{\infty}$ be a noise sequence. Let $\Theta \subset \mathbb{R}^d$ be a set of latent parameters. We assume that there exists a *dynamical system* f on Θ, equivalently, a map $f : \Theta \to \Theta$. At each time step $t \in \{1, 2, \dots\}$, a learner pulls an arm $x_t \in D$ and observes a reward

$$r_t(x_t) := f^t(\theta)^\top x_t + \eta_t,$$

for some $\theta \in \Theta$. In other words, the hidden parameters for the rewards may vary over time but follow only a rule f with initial value θ. The function r_t could be viewed as the specific instance of partial observation (cf. Ljung (2010)).

Our contributions. The contributions of this work are three folds: First, we mathematically identify and define a recoverable set of periodic/eigenvalues information when the observations are available in a form of bandit feedback. The feedback is contaminated by a sub-Gaussian noise, which is more general than those usually considered in system identification work. Second, we present provably correct algorithms for efficiently estimating such information; this constitutes the first attempt of adopting asymptotic results on the Weyl sum. Lastly, we implemented our algorithms for toy examples to experimentally validate our claims.

Notations. Throughout this paper, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}$, $\mathbb{Z}_{>0}$, $\mathbb{Q}$, $\mathbb{Q}_{>0}$, and $\mathbb{C}$ denote the set of the real numbers, the nonnegative real numbers, the natural numbers ($\{0, 1, 2, \dots\}$), the positive integers, the rational numbers, the positive rational numbers, and the complex numbers, respectively. Also, $[T] := \{1, 2, \dots T\}$ for $T \in \mathbb{Z}_{>0}$. The Euclidean norm is given by $\|x\|_{\mathbb{R}^d} = \sqrt{\langle x, x \rangle_{\mathbb{R}^d}} = \sqrt{x^\top x}$ for $x \in \mathbb{R}^d$, where $(\cdot)^\top$ stands for transposition. $\|M\|$ and $\|M\|_F$ are the spectral norm and Frobenius norm of a matrix M respectively, and $\mathscr{I}(M)$ and $\mathscr{N}(M)$ are the image space and the null space of M, respectively. If a is a divisor of b, it is denoted by $a|b$. The floor and the ceiling of a real number a is denoted by $\lfloor a \rfloor$ and $\lceil a \rceil$, respectively. Finally, the least common multiple and the greatest common divisor of a set $\mathcal{L}$ of positive integers are denoted by $\mathrm{lcm}(\mathcal{L})$ and $\gcd(\mathcal{L})$, respectively.

2 Related work

Our work may be viewed as a special instance of system identifications for partially observed dynamical systems. Existing works for sample complexity analysis of partially observed linear systems include Tsiamis & Pappas (2019); Tsiamis et al. (2020); Lale et al. (2020); Lee (2022); Hazan et al. (2018); Simchowitz et al. (2019); Lee & Zhang (2020), under either controlled or autonomous settings. Most of those works consider additive Gaussian noise and make controllability and/or observability assumptions (for autonomous case, transition with Gaussian noise with positive definite covariance is required). While Mhammedi et al. (2020) considers nonlinear observation, it still assumes Gaussian noise and controllability. The work Hazan et al. (2018) considers adversarial noise but with limited budget; we mention its *wave-filtering* approach is interesting and our use of exponential sums could also be viewed as *filtering*. Also, the work Simchowitz et al. (2019) considers control inputs and bounded semi-adversarial noise, which is another set of strong assumptions; however, it is interesting to ask if we can do better when control inputs are allowed in a future work. We also mention that most of the existing literature for linear systems assume the spectral radius is smaller than one. Instead, we consider systems with a sub-Gaussian observation noise while allowing the observation to be made as bandit feedback. Our work includes an algorithm for the case where the nearly periodically behaved system of interest is linear, for which we successfully employ the Weyl sum. The application of our mathematical approach to wider scenarios such as the case with unknown but fixed observation matrix under mild conditions is beyond the scope of this work but is an important future direction of research.

Secondly, our model of bandit feedback is commonly studied within stochastic linear bandit literature (cf. Abe & Long (1999); Auer (2003); Dani et al. (2008); Abbasi-Yadkori et al. (2011)). Also, as we consider the dynamically changing system states (or *reward vectors*), it is closely related to adversarial bandit problems (e.g., Bubeck & Cesa-Bianchi (2012); Hazan (2016)). Recently, some studies on non-stationary rewards have been made (cf. Auer et al. (2019); Besbes et al. (2014); Chen et al. (2019); Cheung et al. (2022); Luo et al. (2018); Russac et al. (2019); Trovo et al. (2020); Wu et al. (2018)) although they do not deal with periodically behaved dynamical system properly (see discussions in Cai et al. (2021) as well). For discrete action settings, Oh et al. (2019) proposed the periodic bandit, which aims at minimizing the total regret. Also, if the period is known, Gaussian process bandit for periodic reward functions was proposed Cai et al. (2021) under Gaussian noise assumption. While our results could be extended to the regret minimization problems by employing our algorithms for estimating the periodic information before committing to arms in a certain way, we emphasize that our primary goal is to estimate such periodic information in provably efficient ways. We thus mention that our work is orthogonal to the recent studies on regret minimization problems for non-stationary environments (or in particular, periodic/seasonal environments). Refer to Lattimore & Szepesvári (2020) for bandit algorithms that are not covered here.

Lastly, we mention that there exist many period estimation methods (e.g., Moore et al. (2014); Tenneti & Vaidyanathan (2015)), and in the case of zero noise, this becomes a trivial problem. Furthermore, one of the most famous work for dynamic structure estimation of general nonlinear systems is that of Takens' Takens (1981); however, it cannot be straightforwardly extended to the case of perturbed observations.

3 Problem definition

In this section, we describe our problem setting. In particular, we introduce some definitions on the properties of dynamical system.

3.1 Nearly periodic sequence

First, we define a general notion of nearly periodic sequence:

Definition 3.1 (Nearly periodic sequence)**.** Let $(\mathcal{X}, d)$ be a metric space. Let $\mu \geq 0$ and let $L \in \mathbb{Z}_{>0}$. We say a sequence $(y_t)_{t=1}^{\infty} \subset \mathcal{X}$ is *μ-nearly periodic* of length L if $d(y_{s+Lt}, y_s) < \mu$ for any $s, t \in \mathbb{Z}_{>0}$. We also call L the length of the μ-nearly period.

Intuitively, there exist L balls of diameter μ in $\mathcal{X}$ and the sequence $y_1, y_2, \ldots$ moves in the balls in order if $(y_t)_{t=1}^{\infty}$ is μ-nearly periodic of length L. Obviously, nearly periodic sequence of length L is also nearly periodic sequence of length mL for any $m \in \mathbb{Z}_{>0}$. We say a sequence is periodic if it is 0-nearly periodic. We introduce a notion of aliquot nearly period to treat estimation problems of period:

Definition 3.2 (Aliquot nearly period)**.** Let $(\mathcal{X}, d)$ be a metric space. Let $\rho > 0$ and $\lambda \geq 1$. Assume a sequence $\{y_t\}_{t=1}^{\infty} \subset \mathcal{X}$ is *μ-nearly periodic* of length L for some $\mu \geq 0$ and $L \in \mathbb{Z}_{>0}$. A positive integer ℓ is a (ρ, λ)*-aliquot nearly period* ((ρ, λ)*-anp*) of $(y_t)_{t=1}^{\infty}$ if $\ell | L$ and the sequence $(y_t)_{t=1}^{\infty}$ is $(\rho + 2\lambda\mu)$-nearly periodic.

We may identify the (ρ, λ)-anp with a $2\lambda\mu$-nearly period under an error margin ρ. When we estimate the length L of the nearly period of unknown sequence $(y_t)_t$, we sometimes cannot determine the L itself, but an aliquot nearly period.

Example 3.1. A trajectory of finite dynamical system is always periodic and it is the most simple but important example of (nearly) periodic sequence. We also emphasize that if we know the upper bound of the number of underlying space, the period is bounded above by the upper bound as well. These facts are summarized in Proposition 3.1. The cellular automata on finite cells is a specific example of finite dynamical systems. We will treat LifeGame Conway et al. (1970), a special cellular automata, in our simulation experiment (see Section 5).

Proposition 3.1. *Let $f : \Theta \to \Theta$ be a map on a set Θ. If $|\Theta| < \infty$, then for any $t \geq |\Theta|$ and $\theta \in \Theta$, $f^{t+L}(\theta) = f^t(\theta)$ for some $1 \leq L \leq |\Theta|$.*

Proof. Since $|\{\theta, f(\theta), \ldots, f^{|\Theta|}(\theta)\}| > |\Theta|$, there exist $0 \leq i < j \leq |\Theta|$ such that $f^i(\theta) = f^j(\theta)$ by the pigeon hole principal. Thus, $f^t(\theta) = f^{j-i+t}(\theta)$ for all $t \geq |\Theta|$. □

If a linear dynamical system generates a nearly periodic sequence, we can show the linear system has a specific structure as follows:

Proposition 3.2. *Let $M : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map. Let $\mathbb{C}^d = \oplus_\alpha V_\alpha$ be the decomposition via generalized eigenspaces of M, where α runs over the eigenvalues of M and $V_\alpha := \mathscr{N}((\alpha I - M)^d)$. Assume that there exists $\mu \geq 0$, for any $\theta \in \mathbb{R}^d$, $(M^t\theta)_{t=t_0}^{\infty}$ is μ-nearly periodic for some $t_0 \in \mathbb{N}$. Let $\theta = \sum_\alpha \theta_\alpha \in \oplus_\alpha V_\alpha$. Then, each eigenvalue α such that $\theta_\alpha \neq 0$ satisfies $|\alpha| \leq 1$, in addition, if $|\alpha| = 1$ and $\theta_\alpha \neq 0$, $M\theta_\alpha = \alpha\theta_\alpha$.*

Proof. We note that $\{M^t v\}_{t \geq 0}$ is bounded for any $v \in \mathbb{R}^d$ by the assumption on M. Thus, M cannot have an eigenvalue of magnitude greater than 1. We show the α is in the form of $\alpha = e^{i2\pi q}$ for some $q \in \mathbb{Q}$ if $|\alpha| = 1$. Suppose that $\alpha = e^{i2\pi\gamma}$ for an irratiolnal number $\gamma \in \mathbb{R}$. Then an eigenvector w for θ_α with $\|w\| > \mu$, $\{M^t w\}_{t > t_0}$ cannot become a μ-periodic sequence. Thus, we conclude $\alpha = e^{i2\pi q}$ for some $q \in \mathbb{Q}$. Next, we show $M\theta_\alpha = \alpha\theta_\alpha$ if $|\alpha| = 1$ and $\theta_\alpha \neq 0$. Suppose $(M - \alpha I)\theta_\alpha \neq 0$. Since $(M - \alpha I)^d \theta_\alpha = 0$, there exists $1 \leq d' < d$ such that $(M - \alpha I)^{d'+1}\theta_\alpha = 0$ but $(M - \alpha I)^{d'}\theta_\alpha \neq 0$. Let $w' := (M - \alpha I)^{d'-1}\theta_\alpha$. Then, we see that $(M - \alpha I)^2 w' = 0$ but $(M - \alpha I)w' \neq 0$. By direct computation, we see that

$$\|M^t w'\| = \|(M - \alpha I + \alpha I)^t w'\| \geq t\|(M - \alpha I)w'\| - \|w'\|.$$

Thus, we have $\|M^t w'\| \to \infty$ as $t \to \infty$, which contradicts the fact that $\{M^t w'\}_{t \geq 0}$ is a bounded sequence. The last statement is obvious. □

Let W be a linear subpace of $\mathbb{C}^d$ generated by the trajectory $\{M\theta, M^2\theta, \ldots\}$ and denote $\dim(W)$ by d_0. Note that restriction of M to W induces a linear map from W to W. We denote by M_θ the induced linear map from W to W. Let $W = \oplus_{\alpha \in \Lambda} W_\alpha$ be the decomposition via the generalized eigenspaces of M_θ, where Λ is the set of eigenvalues of M_θ and $W_\alpha := \mathscr{N}((\alpha I - M_\theta)^{d_0})$. We define

$$W_{=1} := \oplus_{|\alpha|=1} W_\alpha,$$
$$W_{<1} := \oplus_{|\alpha|<1} W_\alpha.$$

Then, we have the following statement as a corollary of Proposition 3.2:

Corollary 3.3. *There exist linear maps $M_1, M_{<1} : W \to W$ such that*

1. *$M_\theta = M_1 + M_{<1}$,*

2. *$M_1 M_{<1} = M_{<1} M_1 = O$,*

3. *M_1 is diagonalizable and any eigenvalue of M_1 is of magnitude 1, and*

4. *any eigenvalue of $M_{<1}$ is of magnitude smaller than 1.*

Proof. Let $p : W \to W_{=1}$ be the projection and let $i : W_{=1} \to W$ be the inclusion map. We define $M_1 := iM_\theta p$. We can construct $M_{<1}$ in the similar manner and these matrices are desired ones. $\square$

Example 3.2. Let G be a finite group and let ρ be a finite dimensional representation of G, namely, a group homomorphism $\rho : G \to \mathrm{GL}_m(\mathbb{C})$, where $\mathrm{GL}_m(\mathbb{C})$ is the set of complex regular matrices of size m. Fix $g \in G$. Let $B \in \mathbb{C}^{n\times n}$ be a matrix whose eigenvalues have magnitude smaller than 1. We define matrix of size $m + n$ by

$$M := \begin{pmatrix} \rho(g) & 0 \\ 0 & B \end{pmatrix}.$$

Then, $(M^t x)_{t=t_0}^{\infty}$ is a μ-nearly periodic sequence for any $\mu > 0$, $x \in \mathbb{C}^m$, and sufficiently large t_0. Moreover, we know that the length of the nearly period is $|G|$. We treat the permutation of variables in $\mathbb{R}^d$ in the simulation experiment (see Section 5), namely the case where G is the symmetric group $\mathfrak{S}_d$ and ρ is a homomorphism from G to $\mathrm{GL}_d(\mathbb{C})$ defined by $\rho(g)((x_j)_{j=1}^d) := (x_{g(j)})_{j=1}^d$, which is the permutation of variable via g.

3.2 Problem setting

Here, we state our problem settings. We use the notation introduced in the previous sections. First, we summarize our technical assumptions as follows:

Assumption 1 (Conditions on arms)**.** *The set of arms D contains the unit hypersphere.*

Assumption 2 (Assumptions on noise)**.** *The noise sequence $\{\eta_t\}_{t=1}^{\infty}$ is conditionally R-sub-Gaussian ($R \in \mathbb{R}_{\geq 0}$), i.e., given t,*

$$\forall \lambda \in \mathbb{R},\ \mathbb{E}\left[e^{\lambda \eta_t} | \mathcal{F}_{t-1}\right] \leq e^{\frac{\lambda^2 R^2}{2}},$$

and $\mathbb{E}[\eta_t|\mathcal{F}_{t-1}] = 0$, $\mathrm{Var}[\eta_t|\mathcal{F}_{t-1}] \leq R^2$, where $\{\mathcal{F}_\tau\}_{\tau\in\mathbb{N}}$ is a set of ascending family and we assume that $x_1, \ldots, x_{\tau+1}, \eta_1, \ldots, \eta_\tau$ are measurable with respect to $\mathcal{F}_\tau$.

Assumption 3 (Assumptions on dynamical systems)**.** *There exists $\mu > 0$ such that for any $\theta \in \Theta$, the sequence $(f^t(\theta))_{t=t_0}^{\infty}$ is μ-nearly periodic of length L for some $t_0 \in \mathbb{N}$. We denote by B_θ the radius of the smallest ball containing $\{f^t(\theta)\}_{t=0}^{\infty}$.*

Remark 3.4. Assumption 1 excludes the lower bound arguments of the minimally required samples for our work since taking sufficiently large vector (arm) makes the noise effect negligible. Considering more restrictive conditions for discussing lower bounds is out of scope of this work.

Then, our questions are described as follows:

- Can we estimate the length L from the collection of rewards $(r_t(x_t))_{t=1}^{T}$ efficiently ?
- If we assume the dynamical system is linear, can we further obtain the eigenvalues of f from a collection of rewards ?
- How many samples do we need to provably estimate the length L or eigenvalues of f ?

We will answer these questions in the following sections and via simulation experiments.

Algorithm 1 Period estimation (DFT)

Input: Current time $t_0 \in \mathbb{Z}_{>0}$; T_p; $\epsilon > 0$; $L_{\max} > 1$; orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ of $\mathbb{R}^d$
Output: Estimated length $\hat{L}$

$\ell \leftarrow 1$; $\beta \leftarrow 1$
for $m = 1, \ldots, d$ **do**
  for $t = t_0, t_0 + 1, \ldots, t_0 + T_p - 1$ **do**
    Sample arm $\mathbf{u}_m$ and observe $r_t(\mathbf{u}_m)$
  end for
  while $\ell \cdot \beta \leq L_{\max}$ **do**
    $\ell \leftarrow \ell + 1$
    for $(s, b) = (0,1), (0,2), \ldots, (0, \ell - 1), (1,1), (1,2), \ldots, (\beta - 1, \ell - 1)$ **do**
      if $\left|\mathcal{R}\left((r_{t_0+s+\beta t}(\mathbf{u}_m))_{t=1}^{\lfloor T_p/\beta \rfloor}; b/\ell\right)\}\right|^2 > \epsilon$ **then**
        $\beta \leftarrow \beta\ell$
        $\ell \leftarrow 1$
        Break
      end if
    end for
  end while
  $t_0 \leftarrow t_0 + T_p$
end for
$\hat{L} \leftarrow \beta$

4 Algorithms and theory

With the above settings in place, we present a (computationally efficient) algorithm for each presented problem, and show its sample complexity for estimating certain information.

4.1 Period estimation

Here, we describe an algorithm for period estimation followed by its theoretical analysis. The overall procedure is summarized in Algorithm 1. To analyze the sample complexity of this estimation algorithm, we first introduce an exponential sum that plays a key role:

Definition 4.1. For a positive rational number $q \in \mathbb{Q}_{>0}$ and T complex numbers $a_1, \ldots, a_T \in \mathbb{C}$, we define

$$\mathcal{R}\big((a_t)_{t=1}^T; q\big) := \frac{1}{T} \sum_{j=1}^{T} a_j e^{i2\pi q j}.$$

For a μ-nearly periodic sequence $\mathbf{a} := (a_t)_{t=1}^\infty$ of length L, we define the supremum of the standard deviations of the L sequential data of $\mathbf{a}$:

$$\sigma_L(\mathbf{a}) := \sup_{t_0 \geq 1} \sqrt{\frac{1}{L} \sum_{t=t_0}^{t_0+L-1} \left| a_t - \frac{1}{L} \sum_{j=t_0}^{t_0+L-1} a_j \right|^2}.$$

The exponential sum $\mathcal{R}(\cdot;\cdot)$ can extract a divisor of the nearly period of a μ-nearly periodic sequence if μ is "sufficiently smaller" than the variance of the sequence even when the sequence is contaminated by noise; more precisely, we have the following lemma:

Lemma 4.1. *Let $\mathbf{a} := (a_j)_{j=1}^\infty$ be a μ-nearly periodic sequence of length L. Then, we have the following statements:*

1. *if $L > 1$, then there exists $s \in \mathbb{Z}_{>0}$ with $s < L$ such that*

$$\left|\mathcal{R}\left((a_j)_{j=1}^T; s/L\right)\right| > \sqrt{\frac{\sigma^2 - 2\mu\sigma}{L}} - \mu - \frac{L \sup_{t\geq 1}|a_t|}{T}, \tag{4.1}$$

2. *if β is not a divisor of L, then for any $\alpha \in \mathbb{Z}_{>0}$,*

$$\left|\mathcal{R}\left((a_j)_{j=1}^T; \alpha/\beta\right)\right| < \mu + \frac{L^2 \mathcal{C}_0(\mu + \sup_{t\geq 1}|a_t|)}{T}, \tag{4.2}$$

where $\mathcal{C}_0 := 1 + 2/\sqrt{2}\pi(3/4)^{\pi^2/6} = 1.72257196806914....$

Proof. As $(a_t)_{t=1}^\infty$ is μ-almost periodic, there exist $(b_t)_{t=1}^\infty$ of period L and $(c_t)_{t=1}^\infty$ with $\sup_{t\geq 1}|c_t| < \mu$ such that $a_t = b_t + c_t$.

First, we prove (4.1). Let

$$\tilde{b} := \frac{1}{L}\sum_{t=1}^{L} b_t, \quad \hat{b}_q := \frac{1}{L}\sum_{t=1}^{L} b_t e^{i2\pi tq}, \quad (q \in \mathbb{Q}).$$

We claim that

$$L \cdot \sup\{|\hat{b}_{s/L}|^2\}_{s=1}^{L-1} > \sum_{s=1}^{L-1} |\hat{b}_{s/L}|^2 = \frac{1}{L}\sum_{t=1}^{L}|b_t - \tilde{b}|^2 \geq \sigma^2 - 2\mu\sigma. \tag{4.3}$$

In fact, the first inequality is obvious. The equality follows from the Plancherel formula for a finite abelian group (see, for example, (Serre, 1977, Excercise 6.2)). As for the last inequality, take arbitrary $t_0 \in \mathbb{Z}_{>0}$ and define $\tilde{a} = L^{-1}\sum_{t=t_0}^{t_0+L-1} a_t$ and $\tilde{c} = L^{-1}\sum_{t=t_0}^{t_0+L-1} c_t$. Then, we have

$$\begin{aligned}
\frac{1}{L}\sum_{t=1}^{L}|b_t - \tilde{b}|^2 &\geq \frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|a_t - \tilde{a}|^2 + \frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|c_t - \tilde{c}|^2 - \frac{2}{L}\sum_{t=t_0}^{t_0+L-1}|a_t - \tilde{a}| \cdot |c_t - \tilde{c}| \\
&\geq \frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|a_t - \tilde{a}|^2 + \frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|c_t - \tilde{c}|^2 \\
&\quad - 2\sqrt{\frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|a_t - \tilde{a}|^2} \cdot \sqrt{\frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|c_t - \tilde{c}|^2} \\
&\geq \frac{1}{L}\sum_{t=t_0}^{t_0+L-1}|a_t - \tilde{a}|^2 - 2\mu\sigma.
\end{aligned}$$

Here, we used the Cauchy-Schwartz inequality in the second inequality. Since t_0 is arbitrary, we have (4.3). Let $s \in \operatorname{argmax}_{s=1,\ldots,L-1}|\hat{b}_{s/L}|^2$ and let $q := s/L$. Let $T = LT' + \gamma$ for some $T', \gamma \in \mathbb{N}$ with $0 \leq \gamma < L$. Then, we have

$$\begin{aligned}
\left|\mathcal{R}\left((a_j)_{j=1}^T; q\right)\right| &\geq \left|\frac{1}{T'}\sum_{t=0}^{T'-1}\left[\frac{1}{L}\sum_{s=1}^{L} a_{Lt+s} e^{i2\pi qs}\right]\right| - \frac{L \sup_{t\geq 1}|a_t|}{T} \\
&> |\hat{b}_q| - \mu - \frac{L \sup_{t\geq 1}|a_t|}{T} \\
&\geq \sqrt{\frac{\sigma^2 - 2\mu\sigma}{L}} - \mu - \frac{L \sup_{t\geq 1}|a_t|}{T}.
\end{aligned}$$

Next, we prove (4.2). As in the same computation as above, we have

$$\begin{aligned}
\left|\mathcal{R}\left((a_j)_{j=1}^{T};\alpha/\beta\right)\right| &\le \frac{LT'}{T}\left|\frac{1}{T'}\sum_{t=0}^{T'-1} e^{i2\pi\alpha Lt/\beta}\left[\frac{1}{L}\sum_{s=1}^{L} a_{Lt+s}e^{i2\pi\alpha s/\beta}\right]\right| + \frac{L\sup_{t>0}|a_t|}{T} \\
&< \frac{LT'}{T}\left|\frac{1}{T'}\sum_{t=0}^{T'-1} e^{i2\pi\alpha Lt/\beta}\right|\cdot\left|\frac{1}{L}\sum_{s=1}^{L} b_s e^{i2\pi\alpha s/\beta}\right| + \mu + \frac{L\sup_{t>0}|a_t|}{T} \\
&< \frac{2}{\left|1-e^{i2\pi\alpha L/\beta}\right|}\cdot\left(\mu+\sup_{t\ge 1}|a_t|\right)\cdot\frac{L}{T} + \mu + \frac{L\sup_{t>0}|a_t|}{T}.
\end{aligned}$$

Since $|1-e^{i2\pi a}| \ge \sqrt{2}(1-a^2)^{\pi^2/6}a$ for $a\in(0,1)$ by (Chesneau & Bagul, 2020, Proposition 3.2), we have

$$\begin{aligned}
\left|\mathcal{R}\left((a_j)_{j=1}^{T};\alpha/\beta\right)\right| &< \left(\mu+\sup_{t\ge 1}|a_t|\right)\frac{2\beta L}{\sqrt{2}(3/4)^{\pi^2/6}T} + \mu + \frac{L\sup_{t\ge 1}|a_t|}{T} \\
&< \mu + \frac{L\beta\mathcal{C}_0(\mu+\sup_{t\ge1}|a_t|)}{T}.
\end{aligned}$$

□

Then, we obtain the explicit lower bound of the samples for period estimation:

Proposition 4.2. *Let* $\mathbf{a} := (a_t)_{t=1}^{\infty}$ *be a* μ*-nearly periodic sequence of length* L*. Fix a positive integer* $L_{\max} > 1$ *with* $L \le L_{\max}$*,* $\delta\in(0,1)$*,* $\xi\in(0,1)$*, and* $\sigma_0 > 0$*. Let* $(\eta_t)_{t=1}^{\infty}$ *be a noise sequence satisfying Assumption 2. Put* $\gamma := 1/(1+\sqrt{4L_{\max}+1})$ *and* $\lambda := \mu/(\sigma_0\gamma)$*. We define*

$$\varepsilon := \sigma_0\gamma\xi.$$

If $\mu/(\gamma\xi) < \sigma_0 \le \sigma_L(\mathbf{a})$*, then, for any*

$$T \ge \frac{8L_{\max}R^2\log(4/\delta)}{\sigma_0^2(\xi-\lambda)^2} + \frac{36L_{\max}^{5/2}\sup_{t\ge1}|a_t|}{\sigma_0(\xi-\lambda)},$$

the set of rational numbers

$$S_{T,\varepsilon} := \left\{q\in\mathbb{Q}\cap(0,1) : qL\in\mathbb{Z}_{>0} \text{ and } \left|\mathcal{R}\left((a_t+\eta_t)_{t=1}^{T};q\right)\right| > \varepsilon\right\}$$

is non-empty with probability at least $1-\delta$*.*

If we apply several collections of rewards $(r_t(x_t))_{t=1}^{T}$ for sufficiently large T indicated in Proposition 4.2, we obtain various divisors of L. Finally, we provide the precise inputs and output of Algorithm 1 in the following Theorem:

Theorem 4.3. *Suppose Assumptions 1, 2 and 3 hold. Let* $r\in[0,1)$ *be a non-negative real number, and suppose* $\rho > 0$ *and* $\delta\in(0,1)$ *are given. Fix a positive integer* $L_{\max} > 1$ *with* $L\le L_{\max}$*. We define*

$$\varepsilon := \frac{\rho}{6\sqrt{d}L_{\max}}. \tag{4.4}$$

Assume that $r\varepsilon \ge \mu$*. Let* T_p *be an integer satisfying*

$$T_p \ge \frac{72dAL_{\max}^2}{\rho^2(1-r)^2} + \frac{108B_\theta\sqrt{d}L_{\max}^3}{\rho(1-r)}, \tag{4.5}$$

where $A := R^2\log(4dL_{\max}^2\log L_{\max}/\delta)$*. Then, the output* $\hat{L}$ *of Algorithm 1 is a* $(\rho,\sqrt{d})$*-anp of* $(f^t(\theta))_{t=t_0}^{\infty}$ *with probability at least* $1-\delta$*.*

If μ is sufficiently small, we may set r as a small positive number, in particular $r=0$ if the system is periodic.

Remark 4.4. If random arm selection is adopted rather than the orthogonal basis, it may underestimate an error margin on some dimensions, which could lead to the nearly period with much larger error margin than expected; considering failure probability of such a case may potentially produce a variant of our algorithm.

Algorithm 2 Eigenvalue estimation

Input: Effective sample size $N \in \mathbb{Z}_{>0}$; threshold $\gamma(N)$
Output: Matrix $A_1(N)\hat{A}_0(N)^\dagger$

Independently draw random unit vectors $\tilde{x}_m, \;\; m \in [d]$, from uniform distribution over the unit sphere in $\mathbb{R}^d$.
Wait N time steps.
for $t = N+1, \cdots, N+2Nd^2$ **do**
    $m_0 \leftarrow \{(t-N-1) \mod 2d^2\} + 1$
    $m \leftarrow \lceil m_0/2d \rceil$
    Sample arm $x_t = \tilde{x}_m$ and observe $\tilde{r}_t := r_t(\tilde{x}_m)$.
end for
Construct matrix $A_0(N)$ and $A_1(N)$ as in (4.6), respectively.
Obtain the low rank approximation $\hat{A}_0(N)$ of $A_0(N)$ via SVD with the threshold $\gamma(N)$.
Output $A_1(N)\hat{A}_0(N)^\dagger$.

4.2 Eigenvalue estimation

If the underlying system has certain structures, more detailed information about the system is expected to be obtained. In this section, we assume the following condition, linearity of the underlying dynamical system f on Θ, in addition to Assumption 1, 2, and 3:

Assumption 4 (Linear dynamical systems)**.** *The dynamical system $f : \Theta \to \Theta$ is linear and is represented by a matrix $M \in \mathbb{R}^{d\times d}$.*

Let $\mathbb{C}^d = \oplus_\alpha V_\alpha$ be the decomposition via generalized eigenspaces of M, where α runs over the eigenvalues of M and $V_\alpha := \mathscr{N}((\alpha I - M)^d)$. We describe $\theta = \sum_\alpha \theta_\alpha$ with $\theta_\alpha \in V_\alpha$. We remark that an eigenvalue α of M such that $\theta_\alpha \neq 0$ is in the form of $e^{2\pi i \ell/L}$ unless $|\alpha| < 1$ by Proposition 3.2.

Our objective is to estimate some of, if not all of, the eigenvalues of M with high probability within some error that decreases by the sample size. To this end, we define the meaningful subset of eigenvalues of M.

Definition 4.2 ((θ, k)-distinct eigenvalues)**.** For a vector $\theta \in \mathbb{C}^d$ and $k \in \mathbb{Z}_{>0}$, we define a (θ, k)-distinct eigenvalue by an eigenvalue β of M^k such that $|\beta| = 1$ and $\theta_\beta \neq 0$.

In our case, starting from a vector θ, the effect of the eigenvalues that are not of (θ, d)-distinct eigenvalues of M may not be observable. Basically, once being able to ignore the effects of eigenvalues of magnitudes less than 1, the system becomes nearly periodic and we aim at estimating (θ, d)-distinct eigenvalues as we obtain more samples.

Our eigenvalue estimation algorithm is summarized in Algorithm 2; it maintains the following matrices. For $N \in \mathbb{Z}_{>0}$, d random unit vectors $\tilde{x}_1, \ldots, \tilde{x}_d$, and $s = 0, 1$, we define the matrix $A_s(N) \in \mathbb{C}^{d\times d}$ so that its (k, ℓ) element is given by

$$\sum_{j=0}^{N-1} r_{2(k-1)d+sd+2d^2j+N+\ell}(\tilde{x}_k) e^{\frac{i2\pi j^2}{4L}}. \tag{4.6}$$

That is, after N steps, the reward multiplied by $\exp(i2\pi j^2/4L)$ is placed from the top row of A_0 and then the top row of A_1, followed by the second rows of them, and so on. Then, those values are summed up for every $2d^2$ steps or every jth cycle. Here, after throwing away N samples, the effects of eigenvalues of magnitude less than 1 become negligible, and the trajectory becomes nearly periodically behaved under Assumption 4. The rest of the samples is used to average out the observation noise while maintaining some meaningful information about M.

The aforementioned exponential sum can be characterized by the Weyl-type sum of matrices, a key machinery for our algorithm, which we define below:

Definition 4.3. Let W be a linear space and let $M_1, \ldots, M_N$ for $N \in \mathbb{Z}_{>0}$ be linear maps on W. For $L \in \mathbb{Z}_{>0}$, we define

$$\mathscr{W}((M_1, \ldots, M_N)) := \frac{1}{N} \sum_{j=0}^{N-1} M_{j+1} e^{\frac{i2\pi j^2}{4L}}.$$

Remark 4.5. Let $E_{s,n} := (\eta_{2di+N+sd+j+1+2d^2n})_{i,j=0,\ldots,d-1}$ be a noise matrix for $s = 0, 1$. Let

$$X := \begin{pmatrix} \tilde{x}_1^\top M^1 \\ x_2^\top M^{2d+1} \\ \vdots \\ \tilde{x}_d^\top M^{2(d-1)d+1} \end{pmatrix} \text{ and } K := (M\theta, \ldots, M^d\theta).$$

Then, $A_s(N)$ has an alternative description as follows:

$$A_s(N) = X\mathscr{W}\left((M^{2d^2j})_{j=0}^{N-1}\right) M^{sd+N-1}K + \mathscr{W}\left((E_{s,j})_{j=0}^{N-1}\right). \tag{4.7}$$

As in Proposition 4.7 below, the Weyl-type sum has a crucial property. Define $\kappa \geq 1$ by

$$\kappa := \inf_P \left\{\|P\|\|P^{-1}\| : P^{-1}MP = J_M\right\},$$

where J_M is a Jordan normal form of M. Also, we define $\Delta \in (0, 1]$ to be a value such that, for any eigenvalue α of M satisfying $|\alpha| < 1$, $|\alpha| \leq 1 - \Delta$ (define $\Delta = 1$ if no such eigenvalue exists). Note by Proposition 3.2, the existence of such spectral gap is guaranteed without any further assumptions.

Roughly speaking, Algorithm 2 estimates "$A_1(N)A_0(N)^{-1} = XM^dX^{-1}$". Of course, the formula in "..." is not valid as X, K, and the Weyl-type sum are not necessarily invertible and we cannot recover full information of M^d in general. However, we can still reconstruct information of M^d restricted on the eigenspaces for (θ, d)-distinct eigenvalues.

To see this, we introduce the Weyl sum and its lower bound:

Lemma 4.6 (Lower bound on the Weyl sum Bourgain (1993); Oh)**.** *Define the Weyl sum by*

$$\mathscr{W}(N, b, q) := \sum_{j=0}^{N} e^{i2\pi\left(j\frac{2b}{4q} + j^2\frac{1}{4q}\right)},$$

for some $b \in \mathbb{N}$, $q \in \mathbb{Z}_{>0}$, $b < q$ and $N \in \mathbb{Z}_{>0}$. Then, for $N \geq 16q^2$, it holds that

$$|\mathscr{W}(N, b, q)| \in \Omega\left(\frac{N}{\sqrt{q}}\right).$$

Proof. It is immediate from Proposition 3.1 in Oh because $\gcd(1, 4q) = 1$, $4q \equiv 0 \mod 4$, and $2b$ is even. ☐

We define $C_{\mathrm{ws}}(L) > 0$ by

$$C_{\mathrm{ws}}(L) := \inf\left\{C > 0 : C^{-1} < \left|\frac{1}{N}\mathscr{W}(N, b, L)\right| < C \text{ for any } N \geq 16L^2, 0 \leq b < L\right\}.$$

Then, we have the following proposition:

Proposition 4.7. *Let M_1 and $M_{<1}$ be matrices as in Corollary 3.3. We define linear maps on W by*

$$Q_N(M_1) := \mathscr{W}\left((M_1^{2d^2j})_{j=0}^{N-1}\right)$$
$$Q_N(M_{<1}) := \mathscr{W}\left((M_{<1}^{2d^2j})_{j=0}^{N-1}\right).$$

Then, we have the following statement:

1. $\mathscr{W}\left((M_\theta^{2d^2j})_{j=0}^{N-1}\right) = Q_N(M_1) + Q_N(M_{<1})$,

2. *for any* $N \geq 16L^2$, $Q_N(M_1)$ *is invertible on* $W_{=1}$,

3. *for any* $r \geq 0$ *and for any* $N \geq 16L^2$, $\|M_1^r Q_N(M_1)|_{W_{=1}}\|, \|(M_1^r Q_N(M_1))|_{W_{=1}}^{-1}\| \leq \kappa C_{\mathrm{ws}}(L)$,

4. *for any* $r \geq 2(d-1)$, *we have*

$$\|M_{<1}^r Q_N(M_{<1})\| \leq \frac{d^2\kappa e^{-\Delta(r-d+1)}}{N\Delta^{d-1}}.$$

Proof. We prove 1. By the properties 1 and 2 in Corollary 3.3, we have $\mathscr{W}((M^{2d^2j})_{j=0}^{N-1}) = Q_N(M_1) + Q_N(M_{<1})$. Next, we prove 2. When we regard $Q_N(M_1)|_{W_{=1}}$ as a linear map on $W_{=1}$, it is represented as a diagonal matrix $\mathrm{diag}(\mathscr{W}(N,b_1,L),\ldots,\mathscr{W}(N,b_m,L))$, where $m = \dim W_{=1}$. Therefore, by Lemma 4.6, $Q_N(M_1)$ is a bijective linear map on $W_{=1}$. Next, we prove 3. We estimate $\|M_1^r Q_N(M_1)\|$. Let $p : \mathbb{C}^d \to W$ be the orthogonal projection and let $i : W \to \mathbb{C}^d$ be the inclusion map. Let $\tilde{M}_1 := iM_1p \in \mathbb{C}^{d\times d}$. Then, we have

$$\|Q_N(M_1)\| = \|Q_N(\tilde{M}_1)\| \leq \kappa C_{\mathrm{ws}}(L).$$

Next, we estimate $\|M_{<1}^r Q_N(M_{<1})\|$. Let $\tilde{M}_{<1} := iM_{<1}p$. Then, $iM_{<1}^r Q_N(M_{<1})p = \mathscr{W}((\tilde{M}_{<1}^{2d^2j+r})_{j=0}^{N-1})$ and $\|iM_{<1}^r Q_N(M_{<1})p\| = \|M_{<1}^r Q_N(M_{<1})\|$. Let P be a regular matrix such that

$$\tilde{M}_{<1} = PJP^{-1},$$

where J is the Jordan normal form. Then, for $r \geq 2(d-1)$, we see that

$$\begin{aligned}
\|M_{<1}^r Q_N(M_{<1})\| &= \|iM_{<1}^r Q_N(M_{<1})p\| \\
&\leq \frac{\kappa}{N}\sum_{j=0}^{N-1}\|J^{2d^2j+r}\| < \frac{d^2\kappa}{N}\sum_{j=0}^{\infty}\binom{2d^2j+r}{d-1}(1-\Delta)^{2d^2j+r-d+1} \\
&\leq \frac{d^2\kappa(1-\Delta)^{r-d+1}}{N\Delta^{d-1}} \leq \frac{d^2\kappa e^{-\Delta(r-d+1)}}{N\Delta^{d-1}}.
\end{aligned}$$

The second last inequality is proved as follows: for $|a| < 1$, $n \geq 1$ and $r \geq m \geq 0$,

$$\begin{aligned}
\left|\sum_{j=0}^{\infty}\binom{nj+r}{m}a^{nj-m+r}\right| &= \left|\frac{1}{m!}\frac{d^m}{dx^m}\sum_{j=0}^{\infty}x^{nj+r}\bigg|_{x=a}\right| \\
&\leq \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}\frac{r!}{(r-m+j)!}|a|^{r-m+j}\left|\frac{d^j}{dx^j}\frac{1}{1-x^n}\bigg|_{x=a}\right| \\
&\leq \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}\frac{r!}{(r-m+j)!}\frac{j!|a|^{r-m+j}}{(1-|a|)^j} = \sum_{j=0}^{m}\binom{r}{j}\frac{|a|^{r-m+j}}{(1-|a|)^j},
\end{aligned}$$

where, the last inequality follows from

$$\left|\frac{d^m}{dx^m}\frac{1}{1-x^n}\right| = \left|\sum_{\zeta^n=1}\frac{m!\zeta^{1-n}}{n(\zeta-x)^m}\right| \leq \frac{m!}{(1-|x|)^m}.$$

Thus, we have

$$\sum_{j=0}^{m}\binom{r}{j}\frac{|a|^{r-m+j}}{(1-|a|)^j} \leq |a|^{r-m}\sum_{j=0}^{r}\binom{r}{j}\left(\frac{|a|}{1-|a|}\right)^j \leq \frac{|a|^{r-m}}{(1-|a|)^m}.$$

□

Proposition 4.7 plays an essential role in our analysis and guarantees that the information in $A_s(N)$ about (θ, d)-distinct eigenvalues does not vanish while noise effects are canceled out. Now, we state our main theoretical result for the eigenvalue estimation algorithm. Before stating the theorem, we introduce lower rank approximation via the singular value threshold.

Definition 4.4. Let $A : \mathbb{R}^{m\times n}$ be a matrix. Let $A = U\,(D\ O)\,V^*$ be a singular valued decomposition of A where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are orthogonal matrices and $D := \mathrm{diag}(\sigma_1, \ldots, \sigma_r, \mathbf{o}^\top)$ is a diagonal matrix with nonnegative components. Let $\gamma > 0$. We define a low rank approximation A_γ of A via the singular value threshold γ by the matrix A_γ defined by $A_\gamma = U\,(D_\gamma\ O)\,V^*$, where, $D_\gamma := \mathrm{diag}(\mathbf{1}_{[\gamma,\infty)}(\sigma_1)\sigma_1, \ldots, \mathbf{1}_{[\gamma,\infty)}(\sigma_r)\sigma_r, \mathbf{o}^\top)$ and $\mathbf{1}_I$ is defined to be the characteristic function supported on $I \subset \mathbb{R}$.

Given this definition, we are ready to present the following main result:

Theorem 4.8. *Suppose Assumptions 1, 2, 3, and 4 hold. Given $\delta \in (0, 1]$, let the effective sample size*

$$N \geq \max\left\{16L^2, \frac{-(d-1)\log\Delta}{\Delta} + \frac{\log(B_\theta\kappa^2) + d + 6}{\Delta} + d\right\}, \tag{4.8}$$

and $\gamma(N) = (\sqrt{4d^2R^2\log(4d^2/\delta)} + 1)/\sqrt{N}$. Then, there exists a matrix A whose eigenvalues are zeros except for (θ, d)-distinct eigenvalues of M, such that the output of Algorithm 2, i.e. $A_1(N)\hat{A}_0(N)^\dagger$, satisfies, with probability at least $1-\delta$, that

$$\left\|A - A_1(N)\hat{A}_0(N)^\dagger\right\| \leq C\left(\frac{R^2(\log(1/\delta) + 1) + 1}{\sqrt{N}}\right). \tag{4.9}$$

Here, $\hat{A}_0(N)^\dagger$ is the Moore-Penrose pseudo inverse of a lower rank approximation of $A_0(N)$ via the singular value threshold $\gamma(N)$. The constant $C > 0$ depends on θ, M, d, $(\tilde{x}_m)_{m=1}^d$, and $C_{\mathrm{ws}}(L)$.

We mention that by using the results shown in Song (2002), the bound on spectral norm (4.9) can be translated to the bounds on eigenvalues, where the constant depends on the form of A. As described in Theorem 4.8, the constant is not the absolute constant for any problem instance but depends on several factors; however, for the same execution, this rate is useful to judge how many samples one collects to estimate eigenvalues.

Remark 4.9. We note that we can reconstruct the $(\theta, 1)$-distinct eigenvalues of M via Algorithm 2 using the following trick: Fix non-negative integer $r \geq 0$. Take $d + r$ random unit vectors $\tilde{x}_1, \ldots, \tilde{x}_{d+r}$. Then, for $s = 0, 1$, we may define a matrix $A_s(N; r) \in \mathbb{C}^{(d+r)\times(d+r)}$ so that its (k, ℓ) element is given by

$$\sum_{j=0}^{N-1} r_{2(k-1)(d+r)+s(d+r)+2(d+r)^2 j+N+\ell}(\tilde{x}_k)e^{\frac{i2\pi j^2}{4L}}.$$

Then, we see that Algorithm 2 outputs a matrix $\tilde{A}(r)$ that well approximates the $(\theta, d{+}r)$-distinct eigenvalues of M since the matrix $A_s(N; r)$ coincides with $A_s(N)$ in the case when we replace M and θ with $M(r)$ and $\theta(r)$ defined by

$$M(r) := \begin{pmatrix} M & O \\ O & O \end{pmatrix} \in \mathbb{C}^{(d+r)\times(d+r)}, \theta(r) := \begin{pmatrix} \theta \\ \mathbf{o} \end{pmatrix} \in \mathbb{C}^{d+r}.$$

Let r be an integer such that $r + d$ is prime to L and fix a positive integer m such that $m(r + d) \equiv 1 \mod L$. Then, the eigenvalues of $\tilde{A}(r)^m$ is close to those of $(\theta, 1)$-distinct eigenvalues if we take sufficiently large N.

5 Simulated experiments

In this section, we present simulated experiments that complement the theoretical claims. In particular, we conducted period estimations for an instance of LifeGame Conway et al. (1970), which is a special case

Figure 1: Left: Illustration of a period eight instance of LifeGame; (top) original transitions. (down) an instance of noisy observation. Right: μ-nearly periodic dynamics (5.1).

of cellular automata Von Neumann et al. (1966), and for a nearly periodic toy system, and an eigenvalue estimation for a linear system, where some dimensions are for permutations and the rest is for shrinking.

Period estimation: LifeGame. We use a specific instance of LifeGame which is illustrated in Figure 1. As shown on the top eight pictures, starting from certain configuration of cells, it shows transitions of period eight. The sample size is computed as the smallest integer satisfying (4.5), and the threshold ε is given by (4.4). To prevent the dimension from becoming too large, we used five cells that correctly display period eight; that is $d = 5$. Noise η_t is given by i.i.d. Gaussian with proxy $R = 0.3$, and the down eight pictures of Figure 1 are some instances of noisy observations. We tested 50 different random seeds (i.e., 1, 51, 101, 151, ... , 2451), and computed the error rate (the number of runs producing a wrong estimate othar than the fundamental period eight, which is divided by 50); and it was zero.

Period estimation: Simple μ-nearly periodic system. We consider the following μ-nearly periodic system that circulates over a circle with small variations:

$$r_{t+1} = \mu\left(\alpha\frac{r_t - 1}{\mu} - \lceil\alpha\frac{r_t - 1}{\mu}\rceil\right) + 1, \qquad \theta_{t+1} = \theta_t + \frac{2\pi}{L}, \tag{5.1}$$

where r and θ are the radius and angle, and $\alpha \notin \mathbb{Q}$. We use $\mu = 0.001$, $L = 5$, and $\alpha = \pi$. Noise η_t is drawn i.i.d. from the uniform distribution within $[-R, R]$ for $R = 0.3$. We tested 50 different random seeds (i.e., 1, 51, 101, 151, ... , 2451), and computed the error rate; and it was zero.

Eigenvalue estimation: Permutation and shrink. We use $d = 5$, and $M \in \mathbb{R}^{5\times5}$ is made such that 1) the first four dimensions are for permutation (i.e., each of row and column of 4×4 sub-matrix has only one nonzero element that is one.), and 2) the last dimension is simply shrinking; we gave 0.7 for $(5, 5)$-element of M. Initial vector θ_0 and each arm $\tilde{x}_m$, $m \in [d]$, are uniformly sampled from the unit sphere in $\mathbb{R}^5$. The value L is computed by $4! = 24$. We used the smallest integer N that satisfies (4.8), multiplied by $C_{\mathrm{sim}} > 0$. The results are shown in Table 1; it is observed that the more samples we use the more accurate the estimates become to $(\theta_0, 5)$-distinct eigenvalues of M. Noise η_t is drawn i.i.d. from the uniform distribution within $[-R, R]$ for $R = 0.3$, and the Table 1 is of the random seed 1234. We also tested 50 different random seeds (i.e., 1, 51, 101, 151, ... , 2451) for $C_{\mathrm{sim}} = 30$, and computed the mean absolute error between the true $(\theta_0, 5)$-distinct eigenvalues and their nearest estimated values; it was 0.0019, which is sufficiently small.

6 Discussion and conclusion

This work proposed novel algorithms for estimating periodic information about the dynamical systems from bandit feedback contaminated by a sub-Gaussian noise. We present a potentially important list of future works below.

Choice of hyperparameters such as $L_{\max}, N$: If those hyperparameters are correctly chosen for nearly periodic systems, the outputs of our algorithms are consistent and reasonable across runs, following the theoretical insights; therefore, practically, one can check if the hyperparameters are properly chosen by

Table 1: Results for the eigenvalue estimation. The top-most row shows the true eigenvalues of M^5, and the second row shows its $(\theta_0, 5)$-distinct eigenvalues. From the third to sixth row, it shows the estimated eigenvalues for different values of C_{sim}.

eigenvalues of M^5	1.000	1.000	$-0.500 - 0.866i$	$-0.500 + 0.866i$	0.168
$(\theta_0, 5)$	1.000	0	$-0.500 - 0.866i$	$-0.500 + 0.866i$	0
when $C_{\mathrm{sim}} = 1$	$1.000 + 0.008i$	0	$-0.489 - 0.860i$	$-0.510 + 0.869i$	0
when $C_{\mathrm{sim}} = 5$	$0.999 - 0.000i$	0	$-0.495 - 0.867i$	$-0.501 + 0.862i$	0
when $C_{\mathrm{sim}} = 10$	$1.000 + 0.003i$	0	$-0.497 - 0.867i$	$-0.501 + 0.864i$	0
when $C_{\mathrm{sim}} = 30$	$1.000 + 0.001i$	0	$-0.500 - 0.866i$	$-0.500 + 0.864i$	0

running the algorithms. On the other hand, studying if it is possible to theoretically ensure correctness of hyperparameters or to identify non-periodic systems is an important future work.

Extension to general spectrum information estimation problems: We only impose nearly periodicity on the dynamical systems on top of the linearity for the eigenvalue estimation problem. Nevertheless, additional studies are needed to allow other forms of observations (not limited to bandit feedback) and non-periodic systems to consider eigenvalues with arguments of 2π times irrational numbers. Actually, the asymptotic bounds of the Weyl sum themselves are still valid when it is sufficiently well approximated by a rational number Bourgain (1993); Oh. Moreover, allowing control inputs would make eigenvalues of magnitudes smaller than one efficiently recoverable.

Extensions to random dynamical systems: This work studied deterministic dynamical systems; however, we conjecture that, under the condition that the variance of trajectories generated by a random dynamical system (RDS) (cf. Arnold (1998)) is sufficiently small, the similar estimation procedure is adopted for such RDSs. It is important to study if the estimation problem becomes easier when the system is driven by a particular noise as it could be treated as a random control input. Also, studying other dynamic structures such as the Lyapunov exponents for nonlinear systems is an interesting future work.

Study of statistical estimation leveraged by other number theoretical results: The proper use of exponential sums enables us to average out the noise while preserving particular information. Studying when this separation is feasible for different sets of information, noise, and class of problems should be important.

Optimality of the results: Although we gave a sufficient number of samples for provably guaranteeing (approximate) correctness of the estimates, it is unclear if our sample complexity is what one can best achieve under our problem settings. Studying if the sample complexity bound is tight should be further investigated.

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A Proof of Proposition 4.2

Proof.

$$
\begin{aligned}
B &:= L_{\max} \sup_{t>0} |a_t|, \\
C &:= \sqrt{4R^2 \log(4/\delta)}, \\
D &:= L_{\max}^2 \mathcal{C}_0 \left(\mu + \sup_{t\geq 1} |a_t| \right).
\end{aligned}
$$

Then, $\sqrt{T} > 0$ satisfies the following inequality

$$
\sqrt{T} \geq \max \left\{ \frac{C + \sqrt{C^2 + 4(\varepsilon - \mu)B}}{2(\varepsilon - \mu)}, \frac{C + \sqrt{C^2 + 4(\varepsilon - \mu)D}}{2(\varepsilon - \mu)} \right\}
$$

if and only if

$$
\begin{aligned}
\varepsilon &\leq 2\varepsilon - \mu - \frac{L_{\max} \sup_{t>0} |a_t|}{T} - \sqrt{\frac{4R^2 \log(4/\delta)}{T}}, \text{ and} \\
\varepsilon &\geq \mu + \frac{L_{\max}^2 \mathcal{C}_0 (\mu + \sup_{t\geq 1} |a_t|)}{T} + \sqrt{\frac{4R^2 \log(4/\delta)}{T}}.
\end{aligned}
$$

Since $L_{\max} \geq 2$, $D \geq B$, $\mu < \gamma \sup_{t\geq 1} |a_t|$, $\mathcal{C}_0(1+\gamma) < 3$, and

$$
\begin{aligned}
\varepsilon - \mu &= \frac{\sigma_0(1-\lambda)}{\sqrt{L_{\max}}} \cdot \frac{\sqrt{L_{\max}}}{1 + \sqrt{4L_{\max} + 1}} \\
&\geq \frac{\sigma_0(1-\lambda)}{2\sqrt{2}\sqrt{L_{\max}}},
\end{aligned}
$$

$$
\begin{aligned}
&\max \left\{ \frac{C + \sqrt{C^2 + 4(\varepsilon - \mu)B}}{2(\varepsilon - \mu)}, \frac{C + \sqrt{C^2 + 4(\varepsilon - \mu)D}}{2(\varepsilon - \mu)} \right\} \\
&= \frac{C + \sqrt{C^2 + 4(\varepsilon - \mu)D}}{2(\varepsilon - \mu)} \\
&\leq 2\sqrt{\frac{L_{\max} C^2}{4(\varepsilon - \mu)^2} + \frac{D}{(\varepsilon - \mu)}} \\
&\leq 2\sqrt{\frac{2L_{\max}^2 C^2}{\sigma_0^2 (\xi - \lambda)^2} + \frac{6\sqrt{2} L_{\max}^{5/2} \sup_{t\geq 1} |a_t|}{\sigma_0(\xi - \lambda)}} \\
&\leq 2\sqrt{\frac{2L_{\max}^2 C^2}{\sigma_0^2 (\xi - \lambda)^2} + \frac{9 L_{\max}^{5/2} \sup_{t\geq 1} |a_t|}{\sigma_0(\xi - \lambda)}},
\end{aligned}
$$

where we used the formula $\sqrt{1 - 2\gamma} = 2\gamma\sqrt{L_{\max}}$. Since $2\varepsilon \leq \sqrt{(\sigma_0^2 - 2\mu\sigma_0)/L_{\max}}$, the statement follows from Lemma 4.1. □

B Proof of Theorem 4.3

Proof. Let $\mathscr{K}$ be the set of all combinations (m, β, s, q) such that $m \in [d]$, $\beta \in [L_{\max}]$, $s \in \{0, 1, \ldots, \beta - 1\}$ and $\{q = \alpha/\ell : \alpha \in \{1, \ldots, \ell - 1\}\}$. We remark that

$$\begin{aligned}
|\mathscr{K}| &\leq \sum_{m=1}^{d} \sum_{\beta=1}^{L_{\max}} \sum_{\alpha \leq L_{\max}/\beta} \sum_{\alpha=1}^{\ell-1} \sum_{s=0}^{\beta-1} 1 = d \sum_{\beta=1}^{L_{\max}} \sum_{\ell \leq L_{\max}/\beta} \beta(\ell - 1) \\
&\leq d \sum_{\beta=1}^{L_{\max}} \frac{L_{\max}}{2} \left(\frac{L_{\max}}{\beta} - 1 \right) \\
&\leq \frac{dL_{\max}^2 \log L_{\max}}{2}.
\end{aligned}$$

Also, let $\mathcal{E}_\kappa^{T_p}$ be the event such that, for the combination $\kappa \in \mathscr{K}$,

$$\left| \frac{1}{\lfloor T_p/\beta \rfloor} \sum_{j=0}^{\lfloor T_p/\beta \rfloor - 1} \left\{ \eta_{t_0+\beta j+s} e^{\frac{i2\pi\alpha j}{\ell}} \right\} \right| \leq \sqrt{\frac{4R^2 \log\left(4dL_{\max}^2 \log L_{\max}/\delta\right)}{\lfloor T_p/\beta \rfloor}}.$$

Because the error sequence satisfies conditionally R-sub-Gaussian, using Lemma D.1 and from the fact that any subsequence of the filtration $\{\mathcal{F}_\tau\}$ is again a filtration, we obtain, for each $\kappa \in \mathscr{K}$,

$$\Pr\left[\mathcal{E}_\kappa^{T_p}\right] \geq 1 - \frac{\delta}{dL_{\max}^2 \log L_{\max}}.$$

Define $\mathcal{E}^{T_p} := \bigcap_{\kappa \in \mathscr{K}} \mathcal{E}_\kappa^{T_p}$. Then, it follows from the Fréchet inequality that

$$\begin{aligned}
\Pr\left[\mathcal{E}^{T_p}\right] &= \Pr\left[\bigcap_{\kappa \in \mathscr{K}} \mathcal{E}_\kappa^{T_p}\right] \geq |\mathscr{K}| \left(1 - \frac{\delta}{dL_{\max}^2 \log L_{\max}}\right) - (|\mathscr{K}| - 1) \\
&= 1 - \frac{\delta |\mathscr{K}|}{dL_{\max}^2 \log L_{\max}} \geq 1 - \delta.
\end{aligned}$$

Let $\widehat{L}$ be the output of Algorithm 1. We show that, with probability $1 - \delta$, $\widehat{L}$ is $(\rho, \sqrt{d})$-anp of L. In fact, suppose $\widehat{L}$ is not $(\rho, \sqrt{d})$-anp. We note that $\widehat{L} < L$. There exists $s \in \{0, \ldots, \widehat{L} - 1\}$ and $t_1, t_2 \in \mathbb{Z}_{>0}$,

$$\|f^{s+\widehat{L}t_1}(\theta) - f^{s+\widehat{L}t_2}(\theta)\| > \rho + 2\sqrt{d}\mu.$$

Let $m' \in \operatorname{argmax}_{m=1,\ldots,d} \left| f^{s+\widehat{L}t_1}(\theta)^\top \mathbf{u}_m - f^{s+\widehat{L}t_2}(\theta)^\top \mathbf{u}_m \right|$. Put $a_t := f^{s+\widehat{L}t}(\theta)^\top \mathbf{u}_{m'}$. Then, for any $t, t' \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
|a_t - a_{t'}| &\geq |a_{t_1} - a_{t_2}| - 2\mu \\
&\geq \frac{\|f^{s+\widehat{L}t_1}(\theta) - f^{s+\widehat{L}t_2}(\theta)\|}{\sqrt{d}} - 2\mu \\
&> \frac{\rho}{\sqrt{d}}.
\end{aligned}$$

Thus, by definition, we have

$$\sigma_{L/\widehat{L}}((a_t)_t) \geq \frac{\rho}{2\sqrt{dL/\widehat{L}}} \geq \frac{\rho}{2\sqrt{dL_{\max}}}.$$

Let $\sigma_0 = \rho/2\sqrt{dL_{\max}}$ and $\xi := \gamma^{-1}/3\sqrt{L_{\max}}$. Then, $\mu/(\gamma\xi) < \sigma_0 \leq \sigma_{L/\widehat{L}}((a_t)_t)$, and

$$\begin{aligned}
&\frac{8L_{\max}R^2\log(4/\delta)}{\sigma_0^2(\xi-\lambda)^2} + \frac{36L_{\max}^{5/2}\sup_{t\geq 1}|a_t|}{\sigma_0(\xi-\lambda)} \\
&\leq \frac{32\xi^{-2}dL_{\max}^2R^2\log(4/\delta)}{\rho^2(1-r)^2} + \frac{72\xi^{-1}\sqrt{d}L_{\max}^3\sup_{t\geq 1}|a_t|}{\rho(1-r)} \\
&\leq \frac{72dL_{\max}^2R^2\log(4/\delta)}{\rho^2(1-r)^2} + \frac{108\sqrt{d}L_{\max}^3\sup_{t\geq 1}|a_t|}{\rho(1-r)}.
\end{aligned}$$

The last inequality follows from $\xi \geq 2/3$. Thus, by Proposition 4.2, the algorithm finds $\beta > \widehat{L}$ in m'-th loop, and the output becomes an integer larger than $\widehat{L}$, which is contradiction. $\square$

C Proof of Theorem 4.8

Here, we provide the proof of Theorem 4.8.

C.1 Proof

Let

$$\begin{aligned}
K &:= (M\theta, \ldots, M^d\theta) : \mathbb{C}^d \to W, \\
E_s(N) &:= \mathscr{W}\left((E_{s,j})_{j=0}^{N-1}\right) : \mathbb{C}^d \to \mathbb{C}^d.
\end{aligned}$$

For a linear map $\mathcal{M} : W \to W$, we define linear maps:

$$\begin{aligned}
X(\mathcal{M}) &:= \begin{pmatrix} \tilde{x}_1^\top \mathcal{M}^1 \\ \tilde{x}_1^\top \mathcal{M}^{2d+1} \\ \vdots \\ \tilde{x}_d^\top \mathcal{M}^{2(d-1)d+1} \end{pmatrix} : W \to \mathbb{C}^d, \\
Q_N(\mathcal{M}) &:= \mathscr{W}\left((\mathcal{M}^{2d^2 j})_{j=0}^{N-1}\right) : W \to W.
\end{aligned}$$

For $s = 0, 1$, we define linear maps on $\mathbb{C}^d$ by

$$\begin{aligned}
A_s(N;\mathcal{M}) &:= X(\mathcal{M})Q_N(\mathcal{M})\mathcal{M}^{sd+N-1}K + E_s(N), \\
B_s(N;\mathcal{M}) &:= X(\mathcal{M})Q_N(\mathcal{M})\mathcal{M}^{sd+N-1}K.
\end{aligned}$$

We note that $A_s(N; M_\theta)$ is identical to $A_s(N)$ defined in (4.7). We impose the following assumption on $X(M_\theta)$:

Assumption 5. *The kernel of the linear map $X(M_1)$ is the same as $\mathcal{N}(M_1)$.*

Note that this assumption holds with probability 1 if we randomly choose $\tilde{x}_1, \ldots, \tilde{x}_d$ (see Lemma C.7).

The following lemma provides an explicit description of $B_0(N; M_1)^\dagger$.

Lemma C.1. *Suppose Assumption 5 holds. Assume $N \geq 16L^2$. Let*

$$\begin{aligned}
U_1 &:= \mathscr{I}(B_0(N;M_1)) \subset \mathbb{C}^d, \\
U_2 &:= \mathscr{N}(B_0(N;M_1)) \subset \mathbb{C}^d, \\
U_3 &:= \mathscr{I}(M_1) = W_{=1} \subset W.
\end{aligned}$$

Let $i : U_1 \to \mathbb{C}^d$ be the inclusion map and $p : \mathbb{C}^d \to U_2^\perp$ the orthogonal projection. Then, restriction of $X(M_1)$ (resp. M_1K) to U_3 (resp. $U_2^\perp$) induces an isomorphism onto U_1 (resp. U_3). If we denote the isomorphism by $\tilde{X}$ (resp, $\tilde{K}$). Then, $B_0(N; M_1)^\dagger$ is given by

$$B_0(N;M_1)^\dagger = p^*\tilde{K}^{-1}M_1|_{U_3}^{2-N}Q_N(M_1)|_{U_3}^{-1}\tilde{X}^{-1}i^*.$$

Proof. Since we have $\mathscr{N}(M_1) = \mathscr{N}(M_1^{r+1})$ for all $r \geq 0$ and Assumption 5, surjectivity of K by Proposition C.5, and bijectivity of $Q_N(M_1)$ on U_3 by Proposition 4.7, we have

$$\begin{aligned} U_1 &= \mathscr{I}(X(M_1)|_{U_3}), \\ U_2 &= \mathscr{N}(M_1 K). \end{aligned}$$

Thus, the restriction of $X(M_1)$ (resp. K) to U_3 (resp. $U_2^{\perp}$) induces an isomorphism onto U_1 (resp. U_3). Let us denote the isomorphism by $\tilde{X}$ (resp. $\tilde{K}$). Then, the last statement follows from Proposition C.4. □

Lemma C.2. *Assume $N \geq 16L^2$. Let*

$$A := B_1(N; M_1) B_0(N; M_1)^{\dagger}.$$

Then, A is independent of N and its eigenvalues are zeros except for (θ, d)-distinct eigenvalues of M.

Proof. We use the notation as in Lemma C.1. By Lemma C.1, we obtain

$$B_1(N; M_1) B_0(N; M_1)^{\dagger} = i \tilde{X} M_1^d \tilde{X}^{-1} i^*,$$

which is independent of N and its eigenvalues are zeros except for (θ, d)-distinct eigenvalues of M. □

The following result will be used in the proof of Theorem 4.8 (but not essential).

Lemma C.3. *We have*

$$\begin{aligned} \|X(M_1)\| &\leq \kappa \sqrt{d}, \\ \|X(M_{<1})\| &\leq \kappa (d+1) 2^d. \end{aligned}$$

Proof. Considering the Jordan normal form, the first inequality is obvious. As for the second inequality, we define $J \in \mathbb{R}^{d \times d}$ by the nilpotent matrix

$$J := \begin{pmatrix} 0 & 1 & & O \\ & \ddots & \ddots & \\ & & & 1 \\ O & & & 0 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \kappa^{-1} \|X(M_1)\| &\leq \sum_{r=1}^{d} \|(I + J)\|_1 \\ &= \sum_{r=1}^{d} \sum_{j=0}^{d} \binom{r}{j} (d - j) \\ &= \sum_{j=0}^{d} (d - j) \sum_{r=1}^{d} \binom{r}{j} \\ &= d + \sum_{j=1}^{d} (d - j) \sum_{r=1}^{d} \binom{r}{j} \\ &= d + \sum_{j=1}^{d} (d - j) \binom{r+1}{j+1} \\ &= d + (d+1) 2^d - (d+1) d - (d+1) \\ &< (d+1) 2^d. \end{aligned}$$

□

Proof of Theorem 4.8. Let A be the matrix introduced in Lemma C.2. Let

$$\gamma(N) := \frac{\sqrt{4d^2R^2\log(4d^2/\delta)}+1}{\sqrt{N}}.$$

Let M_1 and $M_{<1}$ be matrices as in Corollary 3.3. Then, by Proposition 3.2, we see that

$$\begin{aligned}A_s(N,M) &= A_s(N,M_1) + A_s(N,M_{<1}),\\ B_s(N,M) &= B_s(N,M_1) + B_s(N,M_{<1}).\end{aligned}$$

We denote by $\hat{A}_s(N;M)$ (resp, $\hat{B}_s(N;M)$) the low rank approximation via the singular value threshold $\gamma(N)$ (see Definition 4.4). By direct computations, we have

$$\begin{aligned}&\|A - A_1(N;M)\hat{A}_0(N;M)^\dagger\|\\ &\le \|A_1(N;M)\|\cdot\|\hat{A}_0(N;M)^\dagger - B_0(N;M_1)^\dagger\| + \|A_1(N;M) - B_1(N;M_1)\|\cdot\|B_0(N;M_1)^\dagger\|\\ &\le (\|B_1(N;M_1)\| + \|B_1(N;M_{<1})\| + \|E_1(N)\|)\cdot\|\hat{A}_0(N;M)^\dagger - B_0(N;M_1)^\dagger\|\\ &\qquad + \|B_1(N;M_{<1}) + E_1(N)\|\cdot\|B_0(N;M_1)^\dagger\|.\end{aligned}$$

By Proposition 4.7, for $s=0,1$ and for $N \ge \max\{2d, 16L^2\}$, we have

$$\begin{aligned}\|B_s(N;M_1)\| &\le \|X(M_1)\|\cdot\|M_1^{N+sd-1}Q_N(M_1)\|\cdot\|K\|\\ &\le \kappa C_{\mathrm{ws}}(L)\|X(M_1)\|\|K\|\\ &\le Bd\kappa^2 C_{\mathrm{ws}}(L)\\ \|B_s(N;M_{<1})\| &\le \|X(M_{<1})\|\cdot\|M_{<1}^{N+sd-1}Q_N(M_{<1})\|\cdot\|K\|\\ &\le \|X(M_{<1})\|\cdot\|K\|\cdot\frac{d^2\kappa e^{-\Delta(N+sd-d)}}{N\Delta^{d-1}}\\ &\le B\kappa\sqrt{d}(d+1)2^d\cdot\frac{d^2\kappa e^{-\Delta(N+sd-d)}}{N\Delta^{d-1}}\\ &\le \frac{B\kappa^2 e^{d+6-\Delta(N+sd-d)}}{N\Delta^{d-1}},\end{aligned}$$

where we used $\|K\| \le \sqrt{d}B$ (Assumption 3), and $\|X(M_{<1})\| \le \kappa(d+1)2^d$ (Lemma C.3), and $\sqrt{d}(d+1)d^2 2^d < e^{d+6}$. By Lemma C.1 with Proposition 4.7, we see that

$$\|B_0(N;M_1)^\dagger\| \le \kappa C_{\mathrm{ws}}(L)\|\tilde{X}^{-1}\|\cdot\|\tilde{K}^{-1}\|.$$

By using Lemma D.1 and union bounds, we obtain

$$\max(\|E_1(N)\|_F, \|E_2(N)\|_F) \le \gamma(N) - \frac{1}{\sqrt{N}},$$

with probability at least $1-\delta$. Assume that

$$N \ge \frac{-(d-1)\log\Delta}{\Delta} + \frac{\log(B\kappa^2)+d+6}{\Delta} + d.$$

Then, we see that $\|B_s(N;M_{<1})\| \le 1/N$. Thus, by Lemma C.6, with probability at least $1-\delta$, we have

$$\begin{aligned}\|\hat{A}_0(N;M)^\dagger - B_0(N;M_1)^\dagger\| &\le 8(\|E_0(N)\| + \|B_0(N;M_{<1})\|)\cdot\|B_0(N,M_1)^\dagger\|^2(\sqrt{d}+1)\\ &\le 8\gamma(N)\cdot\|B_0(N,M_1)^\dagger\|^2(\sqrt{d}+1)\\ &\le 8(1+\sqrt{d})\kappa^2 C_{\mathrm{ws}}(L)^2\|\tilde{X}^{-1}\|^2\cdot\|\tilde{K}^{-1}\|^2\gamma(N).\end{aligned}$$

Therefore, there exists $C>0$ depending on $\|X(M)\|$, $\|K\|$, κ, $C_{\mathrm{ws}}(L)$, $\|\tilde{X}^{-1}\|$, $\|\tilde{K}^{-1}\|$, and d such that

$$\|A - A_1(N;M)\hat{A}_0(N;M)^\dagger\| < C\left(\frac{R^2(\log(1/\delta)+1)+1}{\sqrt{N}}\right),$$

with probability at least $1-\delta$. $\square$

C.2 Miscellaneous

We provide an expression of Moore-Penrose pseudo inverse:

Proposition C.4. *Let $A : \mathbb{C}^m \to \mathbb{C}^n$ be a linear map. Let $i : \mathscr{I}(A) \to \mathbb{C}^n$ be the inclusion map and let $p : \mathbb{C}^m \to \mathscr{N}(A)^\perp$ be the orthogonal projection. Let $\tilde{A} := A|_{\mathscr{N}(A)} : \mathscr{N}(A)^\perp \to \mathscr{I}(A)$ be an isomorphism. Then, the Moore-Penrose pseudo inverse $A^\dagger$ coincides with $i^*\tilde{A}^{-1}p^*$.*

Proof. Let $B := p^*\tilde{A}^{-1}i^*$. We remark that $A = i\tilde{A}p$, $i^*i = \mathrm{id}, pp^* = \mathrm{id}$, we see that $ABA = A$, $BAB = B$, $(AB)^* = AB$, and $BA = (BA)^*$. By the uniqueness of Moore-Penrose pseudo inverse, $B = A^\dagger$. $\square$

Proposition C.5. *Let $A \in \mathbb{C}^{d\times d}$ be a matrix and let $v \in \mathbb{C}^d$ be a vector. Let $V \subset \mathbb{C}^d$ be a linear subspace generated by $\{A^j v\}_{j=1}^\infty$. Then, $V = \mathscr{I}\left(Av, A^2v, \ldots, A^dv\right)$.*

Proof. Put $W = \mathscr{I}\left((Av, A^2v, \ldots, A^dv)\right)$. The inclusion $W \subset V$ is obvious, we prove the opposite inclusion. It suffices to show that $A^jv \in W$ for any positive integer $j > d$. By the Cayley-Hamilton theorem, $A^d = \sum_{j=1}^d c_jA^{d-j}$ for some $c_j \in \mathbb{C}$. Thus, by induction A^j is a linear combination of $A, A^2, \ldots, A^d$, namely, $A^jv \in W$. $\square$

We give several lemmas here.

Lemma C.6 (Perturbation bounds of the Moore-Penrose inverse)**.** *Suppose $A \in \mathbb{C}^{d\times d}$ is a matrix. Let $E \in \mathbb{C}^{d\times d}$ be a matrix satisfying*

$$\exists C > 0, \quad \forall N \in \mathbb{Z}_{>0}, \quad \|E\| \leq C\frac{1}{\sqrt{N}},$$

and let $\hat{A}_E \in \mathbb{C}^{d\times d}$ be the low rank approximation of $A+E$ via SVD with the singular value threshold $C/\sqrt{N}$. Then, we obtain

$$\left\|A^\dagger - \hat{A}_E^\dagger\right\| \leq \frac{8C\|A^\dagger\|^2\left(\sqrt{d}+1\right)}{\sqrt{N}}.$$

Proof. Let $\sigma_{\min} = \|A^\dagger\|^{-1}$ be the minimal singular value of A. From (Meng & Zheng, 2010, Theorem 1.1) (or originally Wedin (1973)) and from the fact

$$\left\|\hat{A}_E - A\right\| \leq \|E\| + \frac{\sqrt{d}C}{\sqrt{N}},$$

we obtain

$$\left\|A^\dagger - \hat{A}_E^\dagger\right\| \leq \frac{1+\sqrt{5}}{2}\max\left\{\left\|A^\dagger\right\|^2, \left\|\hat{A}_E^\dagger\right\|^2\right\}\left(\sqrt{d}+1\right)\frac{C}{\sqrt{N}}.$$

For N such that $1 \leq N \leq 4C^2/\sigma_{\min}^2$, we have $\left\|\hat{A}_E^\dagger\right\| \leq \sqrt{N}/C \leq 2/\sigma_{\min}$. Suppose singular values $\sigma_k,\ k \in [d]$, of A, and $\hat{\sigma}_k,\ k \in [d]$, of $A+E$ are sorted in descending order. Then, it holds that

$$|\sigma_k - \hat{\sigma}_k| \leq \|E\|.$$

Therefore, for $N > 4C^2/\sigma_{\min}^2$, the minimum singular value $\hat{\sigma}_d$ is greater than $\sigma_{\min}/2$. In this case, $\hat{A}_E = A + E$, and it follows that $\left\|\hat{A}_E^\dagger\right\| \leq 2/\sigma_{\min}$. Hence, for all $N \geq 1$, we obtain

$$\left\|\hat{A}_E^\dagger\right\| \leq \frac{2}{\sigma_{\min}},$$

from which, it follows that

$$\left\|A^\dagger - \hat{A}_E^\dagger\right\| \leq \frac{2C\left(1+\sqrt{5}\right)\left(\sqrt{d}+1\right)}{\sigma_{\min}^2\sqrt{N}} \leq \frac{8C\|A^\dagger\|^2\left(\sqrt{d}+1\right)}{\sqrt{N}}.$$

□

Lemma C.7 (Null space of random matrix)**.** *Suppose $M_k \in \mathbb{R}^{d\times d},\ k\in[d]$, have the same null space. Then, the null space of*

$$X := \begin{pmatrix} x_1^\top M_1 \\ x_2^\top M_2 \\ \vdots \\ x_d^\top M_d \end{pmatrix},$$

where $x_k,\ k\in[d]$, are independently drawn unit vector from the uniform distribution over the unit hypersphere, is the same as those of M_k with probability one.

Proof. Given any $k-1$ dimensional linear subspace in $\mathcal{N}(M_k)^\perp$ for any $k\in[d]$, it holds that the probability that $x_k^\top M_k$ lies on that space is zero. Therefore, by union bound, and by the fact that the row space is the orthogonal complement of the null space, we obtain the result. □

D Azuma-Hoeffding inequality for exponential sum

Lemma D.1. *Let $\{X_j\}_{j=1}^n$ for $n\in\mathbb{Z}_{>0}$ be sub-Gaussian martingale difference with variance proxy R^2 and a filtration $\{\mathcal{F}_j\}$. Also, let $\{a_j\}\subset\mathbb{C}$ be a sequence of complex numbers satisfying $|a_j|\leq 1$ for all $j\in[n]$. Then, the followings hold, where $*[\cdot]$ stands for $\mathfrak{Re}[\cdot]$ or $\mathfrak{Im}[\cdot]$.*

$$\Pr\left[\frac{1}{n}\left|*\left[\sum_{j=1}^n a_jX_j\right]\right| \leq \sqrt{\frac{2R^2\log\left(2/\delta\right)}{n}}\right] \geq 1-\delta,$$
$$\Pr\left[\frac{1}{n}\left|\sum_{j=1}^n a_jX_j\right| \leq \sqrt{\frac{4R^2\log\left(4/\delta\right)}{n}}\right] \geq 1-\delta.$$

Proof. For a filtration $\{\mathcal{F}_i\}_{i\leq n}$, we have

$$\mathbb{E}\left[e^{\lambda\mathfrak{Re}\left[\sum_{j=1}^n a_jX_j\right]}\right] \leq \mathbb{E}\left[e^{\lambda\mathfrak{Re}\left[\sum_{j=1}^{n-1} a_jX_j\right]}\mathbb{E}\left[e^{\lambda\mathfrak{Re}[a_nX_n]}\middle|\mathcal{F}_{n-1}\right]\right] \leq e^{\frac{\lambda^2R^2}{2}}\mathbb{E}\left[e^{\lambda\mathfrak{Re}\left[\sum_{j=1}^{n-1} a_jX_j\right]}\right],$$

where the first inequality follows from the assumption of filtration, and the second inequality follows from

$$\mathbb{E}\left[e^{\lambda\mathfrak{Re}[a_nX_n]}\middle|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[e^{\lambda X_n\mathfrak{Re}[a_n]}\middle|\mathcal{F}_{n-1}\right] \leq e^{\frac{\lambda^2R^2}{2}}.$$

By induction, we obtain

$$\mathbb{E}\left[e^{\lambda\mathfrak{Re}\left[\sum_{j=1}^n a_jX_j\right]}\right] \leq e^{\frac{n\lambda^2R^2}{2}}.$$

By using Markov inequality and the union bound, it follows that

$$\Pr\left[\frac{1}{n}\left|\mathfrak{Re}\left[\sum_{j=1}^n a_jX_j\right]\right| > \epsilon\right] \leq 2e^{-\frac{n\epsilon^2}{2R^2}}.$$

Similarly, we have

$$\Pr\left[\frac{1}{n}\left|\mathfrak{Im}\left[\sum_{j=1}^{n} a_j X_j\right]\right| > \epsilon\right] \le 2e^{-\frac{n\epsilon^2}{2R^2}}.$$

Therefore, we obtain

$$\begin{aligned}
&\Pr\left[\frac{1}{n}\left|\sum_{j=1}^{n} a_j X_j\right| \le \sqrt{\frac{4R^2\log(4/\delta)}{n}}\right] \\
&\ge \Pr\left[\frac{1}{n}\left|\mathfrak{Re}\left[\sum_{j=1}^{n} a_j X_j\right]\right| \le \sqrt{\frac{2R^2\log(4/\delta)}{n}}\right] \\
&\quad + \Pr\left[\frac{1}{n}\left|\mathfrak{Im}\left[\sum_{j=1}^{n} a_j X_j\right]\right| \le \sqrt{\frac{2R^2\log(4/\delta)}{n}}\right] - 1 \\
&\ge \left(1 - \frac{\delta}{2}\right) + \left(1 - \frac{\delta}{2}\right) - 1 \ge 1 - \delta,
\end{aligned}$$

where the first inequality follows from the Fréchet inequality. □

E Applications to bandit problems

We briefly cover the applicability of our proposed algorithms to bandit problems (e.g., regret minimization) which is mentioned in the introduction.

A naive approach is an explore-then-commit type algorithm (cf. Robbins (1952); Anscombe (1963)). One employs our algorithm to estimate a nearly period, followed by a certain periodic bandit algorithm such as the work in Cai et al. (2021) to obtain an asymptotic order of regret. Caveat here is, because our estimate is only an aliquot nearly period, one may need to take into account the regret caused by this misspecification when running bandit algorithms (e.g., ρ and μ may lead to (small) linear regret). Avoiding this small linear regret would require the system to be 0-nearly periodic and that there exists a sufficiently large *gap* ensuring μ-nearly period with sufficiently small μ implies 0-nearly period.

If one aims at designing an anytime algorithm, the straightforward application of our algorithms may not give near optimal asymptotic rate of *expected* regret because the failure probability of periodic structure estimations cannot be adjusted later. To remedy this, one can employ our algorithm repetitively, and gradually increase the span of such procedure. Importantly, samples from separated spans can contribute to the estimate together when the *surplus* beyond a multiple of period is properly dealt with. Since failure probability decreases exponentially with respect to sample size, we conjecture that increasing the span for bandit algorithm by a certain order will lead to the same rate (up to logarithm) of *expected* regret of the adopted bandit algorithm.

F Simulation setups and results

Throughout, we used the following version of Julia Bezanson et al. (2017); for each experiment, the running time was less than a few minutes.

```
Julia Version 1.6.3
Platform Info:
OS: Linux (x86_64-pc-linux-gnu)
CPU: Intel(R) Core(TM) i7-6850K CPU @ 3.60GHz
WORD_SIZE: 64
LIBM: libopenlibm
```

Table 2: Hyperparameters used for period estimation of LifeGame.

LifeGame hyperparameter	Value	Algorithm hyperparameter	Value
height	12	accuracy for estimation ρ	0.98
width	12	failure probability bound δ	0.2
observed dimension	5	maximum possible period $L_{\max}$	10
observation noise proxy R	0.3		
ball radius B	$\sqrt{5}$		

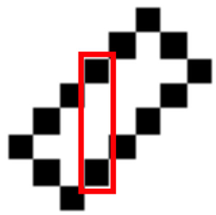

Figure 2: The area we focus on for the cellular automata experiment.

```
LLVM: libLLVM-11.0.1 (ORCJIT, broadwell)
Environment:
JULIA_NUM_THREADS = 12
```

We also used some tools and functionalities of Lyceum Summers et al. (2020). The licenses of Julia and Lyceum are [The MIT License; Copyright (c) 2009-2021: Jeff Bezanson, Stefan Karpinski, Viral B. Shah, and other contributors: https://github.com/JuliaLang/julia/contributors], and [The MIT License; Copyright (c) 2019 Colin Summers, The Contributors of Lyceum], respectively.

In this section, we provide simulation setups, including the details of parameter settings.

F.1 Period estimation: LifeGame

The hyperparameters of LifeGame environment and the algorithm are summarized in Table 2. Note $\mu = 0$ because it is a periodic transition. Here, we used 12×12 blocks of cells and we focused on the five blocks surrounded by the red rectangle in Figure 2. The transition rule is given by

1. If the cell is alive and two or three of its surrounding eight cells are alive, then the cell remains alive.
2. If the cell is alive and more than three or less than two of its surrounding eight cells are alive, then the cell dies.
3. If the cell is dead and exactly three of its surrounding eight cells are alive, then the cell is revived.

F.2 Period estimation: Simple μ-nearly periodic system

The dynamical system

$$r_{t+1} = \mu\left(\alpha\frac{r_t - 1}{\mu} - \lceil\alpha\frac{r_t - 1}{\mu}\rceil\right) + 1, \qquad \theta_{t+1} = \theta_t + \frac{2\pi}{L},$$

is μ-nearly periodic. See Figure 3 for the illustrations when $\mu = 0.2,\ L = 5,\ \alpha = \pi$. It is observed that there are five *clusters*. We mention that this system is not exactly periodic. The hyperparameters of this system and the algorithm are summarized in Table 3.

Table 3: Hyperparameters used for μ-nearly periodic system.

System hyperparameter	Value	Algorithm hyperparameter	Value
dimension	2	accuracy for estimation ρ	0.3
μ	0.001	failure probability bound δ	0.2
α	π	maximum possible nearly period $L_{\max}$	8
true length L	5		
observation noise proxy R	0.3		
ball radius B	2		

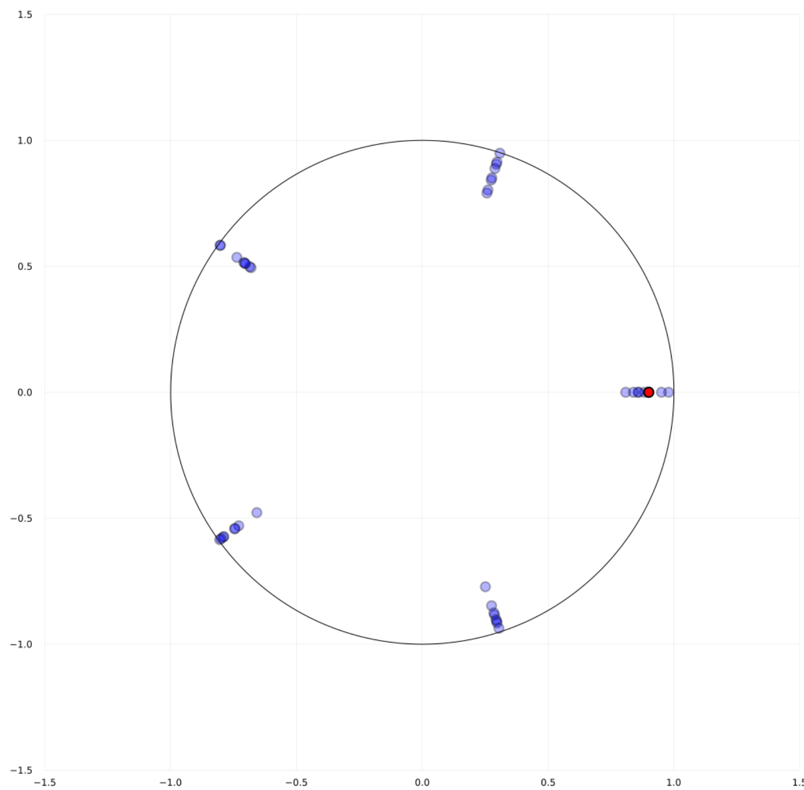

Figure 3: An example of μ-nearly periodic system.

F.3 Eigenvalue estimation

We used the matrix M given by

$$M := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 \end{pmatrix}.$$

The first 4×4 block matrix is for permutation. After N steps, it is expected that the last dimension shrinks so that the system becomes nearly periodic. It follows that $4! = 24$ is a multiple of the length L. Eigenvalues of M^5 are given by $1.000,\ 1.000,\ -0.500-0.866i,\ -0.500+0.866i,\ 0.168$, and the $(\theta_0, 5)$-distinct eigenvalues are $1.000,\ \ -0.500-0.866i,\ \ -0.500+0.866i$.

Table 4: Hyperparameters used for eigenvalue estimation.

Hyperparameter	Value	Hyperparameter	Value
κ	6	a nearly period L	24
Δ	0.1	failure probability bound δ	0.2
dimension	5	observation noise proxy R	0.3
ball radius B	1		

The hyperparameters of the environment and the algorithm are summarized in Table 4. Note we don't necessarily need κ, B, and Δ to run the algorithm as long as the effective sample size is sufficiently large; we used the values (satisfying the conditions) in Table 4 for simplicity.