The Fair Value of Data Under Heterogeneous Privacy Constraints in Federated Learning

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Abstract

Modern data aggregation often involves a platform collecting data from a network of users. Users are now requesting that the data they provide is protected with a guarantee of privacy. With privacy options for users, platforms must solve the problem of how to allocate incentives to users to convince them to share their data. The main goal of this paper is to characterize a *fair* amount to compensate users for their data at a given privacy level. We propose an axiomatic definition of fairness, along the lines of the celebrated Shapley value. To the best of our knowledge, these are the first fairness concepts for data that explicitly consider privacy constraints. We also formulate a heterogeneous federated learning problem for the platform with privacy level options for users. By studying this problem, we investigate the amount of compensation users receive under fair allocations with different privacy levels, amounts of data, and degrees of heterogeneity. Under certain conditions, we characterize the optimal behavior of the platform when incentives are constrained to be fair, revealing that the optimal behavior of the platform can be separated into three regimes, depending on the privacy sensitivity of the users. When privacy sensitivity is low, the platform will set incentives to ensure that it collects all the data with the lowest privacy options. When the privacy sensitivity is above a given threshold, the platform will provide no incentives to users. Between these two extremes, the platform will set the incentives, so some fraction of the users chooses the higher privacy option and the other chooses the lower privacy option.

1 Introduction

From media to healthcare to transportation, the vast amount of data generated by people living their everyday lives has been used to great effect to solve difficult problems across many domains. For example, nearly all machine learning algorithms, including those based on deep learning rely heavily on data. Many of the largest companies to ever exist center their business around the precious resource of data. This includes directly selling access to data to others for profit, selling targeted advertisements based on data, or by exploiting data through data-driven engineering, to better develop and market products. Simultaneously, as users become more privacy conscious, online platforms are increasingly providing *privacy level* options for users. Platforms may provide incentives to users to influence their privacy level decisions. This manuscript investigates how platforms can fairly compensate users for their data contribution at a given privacy level.

Consider a platform offering geo-location services with three user privacy level options:

i) Users send no data to the platform — all data processing is local and private.

ii) An intermediate option with federated learning (FL) for privacy. Data remains with the users, but the platform can ask for gradients with respect to a particular loss function, or data statistics.

iii) A non-private option, where the platform can collect any relevant data from a user device.

If users choose option (i), the platform does not stand to gain from using that data in other tasks. If the user chooses (ii), the platform is better off, but still has limited access to the data via FL and may not be able to fully leverage its potential. Therefore, the platform wants to incentivize users to choose option (iii).

This may be done by providing services, discounts or money to users that choose this option. Effectively, by choosing an option, users are informally selling (or not selling) their data to platforms.

Due to the lack of a formal exchange, it can be difficult to understand if this sale of user data is *fair*. Are platforms making the cost of choosing private options like (i) or (ii) too high? Is the value of data much higher than the platform is paying?

A major shortcoming of the current understanding of data value is that in many cases, it fails to explicitly consider a critical factor in an individual's decision to share data—privacy. This work puts forth two rigorous notions of the fair value of data in Section 3 that explicitly include privacy and make use of the axiomatic framework of the *Shapley value* from game theory (Shapley, 1952).

Compelled by the importance of data in our modern economy and a growing social concern about privacy, this paper presents frameworks for quantifying the fair value of private data. Specifically, we consider a setting where users are willing to provide their data to a platform in exchange for some sort of payment and under some privacy guarantees depending on their level of privacy requirements. The platform is responsible for running the private learning algorithm on the gathered data and making the fair payments with the objective of maximizing its utility including statistical accuracy and total amount of payments. Our goal is to understand fair mechanisms for this procedure as depicted in Fig. 1.

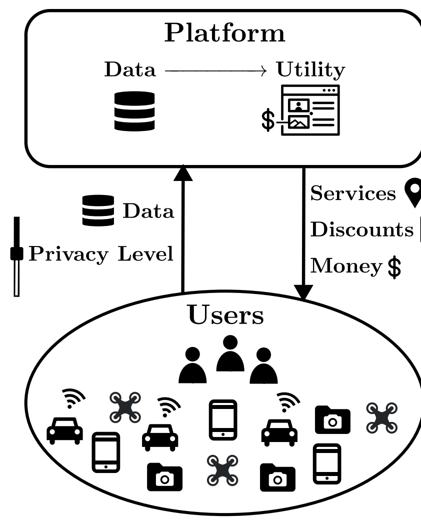


Figure 1: Depiction of interactions between platform and users. Users generate data with phones, cameras, vehicles, and drones. This data goes to the platform but requires some level of privacy. The platform uses this data to generate utility, often by using the data for learning tasks. In return, the platform may provide the users with payments in the form of access to services, discounts on products, or monetary compensation.

1.1 Related Work

Economics With widespread use of the internet, interactions involving those that have data and those that want it have become an important area of study (Balazinska et al., 2011), and a practical necessity (Spiekermann et al., 2015b). Among these interactions, the economics of data from privacy conscious users has received significant attention in Acquisti et al. (2016) and Wieringa et al. (2021). The economic and social implications of privacy and data markets are considered in Spiekermann et al. (2015a). In Acemoglu et al. (2019) the impact of data externalities is investigated. The leakage of data leading to the suppression of its market value is considered.

Privacy Currently, popular forms of privacy include federated learning (Kairouz et al., 2021) and differential privacy (DP) (Dwork, 2008; Bun & Steinke, 2016) either independently or in conjunction with one another. Our work uses a flexible framework that allows for a rage of different privacy models to be considered.

Optimal Data Acquisition One line of literature studies *data acquisition*, where platforms attempt to collect data from privacy conscious users. Ghosh & Ligett (2013) consider the case of uniform privacy guarantees (homogeneous DP), where users have unique minimum privacy constraints, focusing on characterizing equilibria. Ghosh & Roth (2011) allows for heterogeneous DP guarantees with the goal to design a dominant strategy truthful mechanism to acquire data and estimate the sum of users' binary data. In Fallah et al. (2022) the authors consider an optimal data acquisition problem in the context of private mean estimation in two different local and central heterogeneous DP settings. It is assumed that players care about both the estimation error of the common estimator generated by the platform and any payments made to them by the platform in their decision making. By assuming linear privacy sensitivity represented by scalars and drawn from a distribution, they devise a mechanism for computing the near-Bayes optimal privacy levels to provide to the players. Cummings et al. (2023) focuses on the central setting, under both the linear privacy sensitivity and the privacy constraints model, offering insights into the optimal solution. Hu & Gong (2020) goes beyond linear estimation to consider FL, where each user has a unique privacy sensitivity function parameterized by a scalar variable. Users choose their privacy level, and the platform pays them via a proportional scheme. For linear privacy sensitivity functions, an efficient way to compute the Nash equilibrium is derived. Roth & Schoenebeck (2012); Chen et al. (2018); Chen & Zheng (2019) also follow Ghosh & Roth (2011) and design randomized mechanisms that use user data with a probability that depends on their reported privacy sensitivity value.

Fairness In Jia et al. (2019), Ghorbani & Zou (2019) and Ghorbani et al. (2020) a framework for determining the fair value of data is proposed. These works extend the foundational principles of the Shapley value (Shapley, 1952), which was originally proposed as a concept for utility division in coalitional games to the setting of data. Our work takes this idea further and explicitly includes privacy in the definition of the fair value of data, ultimately allowing us to consider private data acquisition in the context of fairness constraints. Finally, we note that we consider the concept of fairness in data valuation, not algorithmic fairness, which relates to the systematic failure of machine learning systems to account for data imbalances.

1.2 Main Contributions

The main contribution of this work is the development of a rigorous notion of fairness in the context of user data acquisition with privacy. While the existing literature has investigated how a platform should design incentives for users to optimize its utility, the definitions of fairness that we propose in this work can offer another way to evaluate these mechanisms. We summarize the main contributions as follows:

- We present an axiomatic notion of fairness that is inclusive of the platforms and the users in Theorem 1. The utility to be awarded to each user and the platform is uniquely determined, providing a useful benchmark for comparison.
- In the realistic scenario that fairness is considered between users, Theorem 2 defines a notion of fairness based on axioms, but only places restriction on relative amounts distributed to the players. This creates an opportunity for the platform to optimize utility under fairness constraints.
- Section 4 contains an example inspired by online platform advertisement to heterogeneous users. We use our framework to fairly allocate payments, noticing how those payments differ among different types of users, and how payments change as the degree of heterogeneity increases or decreases.
- Finally, Section 5 explores the platform mechanism design problem. In Theorem 3 we establish that there are three distinct regimes in which the platform's optimal behavior differs depending on the common privacy sensitivity of the users. When privacy sensitivity is low, the platform will set incentives to ensure that it collects all the data with the lowest privacy options. When the privacy sensitivity is above a given threshold, the platform will provide no incentives to users. Between these two extremes, the platform will set the incentives so some fraction of the users choose the higher privacy option and some choose the lower privacy option.

Notation Throughout this manuscript lowercase boldface $\mathbf{x}$ and uppercase boldface $\mathbf{X}$ symbols denote vectors and matrices respectively. $\mathbf{X} \odot \mathbf{Y}$ represents the element-wise product of $\mathbf{X}$ and $\mathbf{Y}$. We use $\mathbb{R}_{\geq 0}$ for non-negative reals. Finally, $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i \; \forall i$.

2 PROBLEM SETTING

2.1 Privacy Levels and Utility Functions

Definition 1. A *heterogeneous privacy framework* on the space of random function $A : \mathcal{X}^N \to \mathcal{Y}$ is:

1. A set of *privacy levels* $\mathcal{E} \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}$, representing the amount of privacy of each user.
2. A constraint set $\mathcal{A}(\boldsymbol{\rho}) \subseteq \{A : \mathcal{X}^N \to \mathcal{Y}\}$, representing the set of random functions that respect the privacy levels $\rho_i \in \mathcal{E}$ for all $i \in [N]$. If a function $A \in \mathcal{A}(\boldsymbol{\rho})$ then we call it a $\boldsymbol{\rho}$-private algorithm.

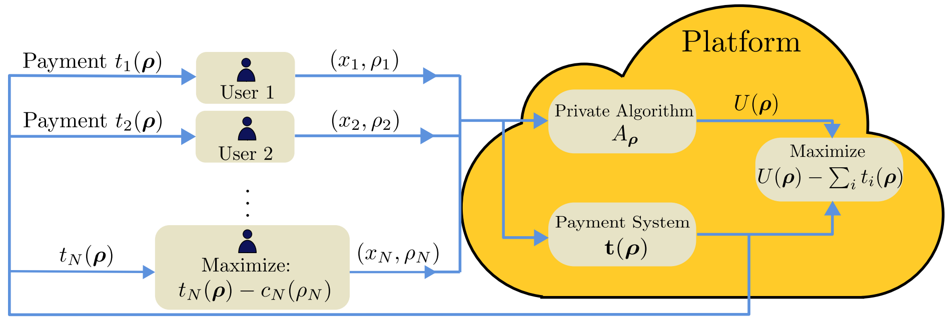


Figure 3: Users send their data x_i and a privacy level ϵ_i to the central platform in exchange for payments $t_i(\rho_i; \boldsymbol{\rho}_{-i})$. The central platform extracts utility from the data at a given privacy level and optimizes incentives to maximize the difference between the utility and the sum of payments $U(\boldsymbol{\rho}) - \sum_i t_i(\rho)$.

We maintain this general notion of privacy framework because different notions of privacy can be useful in different situations. For example, the lack of rigour associated with notions such as FL, may make it unsuitable for high security applications, but it may be very useful in protecting users against data breaches on servers, by keeping their data local. One popular choice with rigorous guarantees is DP:

Definition 2. Pure heterogeneous $\boldsymbol{\epsilon}$-DP, is a heterogeneous privacy framework with $\mathcal{E} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ and the constraint set $\mathcal{A}(\boldsymbol{\epsilon}) = \{A : \Pr(A(\mathbf{x}) \in S) \leq e^{\epsilon_i} \Pr(A(\mathbf{x}') \in S)\}$ for all measurable sets S.

Henceforth we will use the symbol $\boldsymbol{\epsilon}$ to represent privacy level when we are specifically referring to DP as our privacy framework, but if we are referring to a general privacy level, we will use $\boldsymbol{\rho}$. Fig. 2, depicts another heterogeneous privacy framework. $\rho_i = 0$ means the user will keep their data fully private, $\rho_i = 1$ is an intermediate privacy option where user data is obfuscated and only transmitted in part (perhaps via FL) and finally if $\rho_i = 2$, the users send a sufficient statistic for their data to the platform. The platform applies an $\boldsymbol{\rho}$-private algorithm $A_{\boldsymbol{\rho}} : \mathcal{X}^N \mapsto \mathcal{Y}$ to process the data, providing privacy level ρ_i to data x_i. The output of the algorithm $y = A_{\boldsymbol{\rho}}(\mathbf{x})$ is used by the platform to derive utility U, which depends on the privacy level $\boldsymbol{\rho}$. For example, if the platform is estimating the mean of a population, the utility could depend on the mean square error of the private estimator.

Differences from prior work This formulation differs from the literature of optimal data acquisition (i.e., Fallah et al. (2022)), where *privacy sensitivity* is reported by users, and the platform chooses privacy levels ρ_i based on this sensitivity. Their formulation allows for a relatively straightforward application of notions like incentive compatibility and individual rationality from mechanism design theory. In this work, however, we wish to emphasize that in reality, users choose a privacy level, rather than report the somewhat nebulously defined privacy sensitivity. Despite this difference, the notions of fairness described in the following section can be applied more broadly.

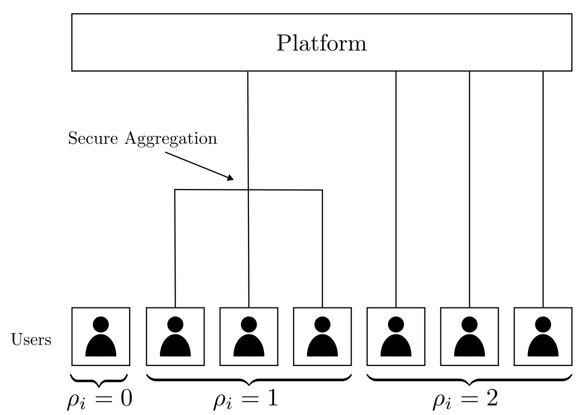


Figure 2: Users choose between three levels of privacy. If $\rho_i = 0$, users send no data to the platform. If $\rho_i = 1$, a user's model is securely combined with other users who also choose $\rho_i = 1$, and the platform receives only the combined model. If $\rho_i = 2$, users send their relevant information directly to the platform.

2.2 The Data Acquisition Problem

The platform generates a transferable and divisible utility $U(\boldsymbol{\rho})$ from the user data. In exchange, the platform distributes a portion of the utility $t_i(\rho_i; \boldsymbol{\rho}_{-i})$ to user i, where $\boldsymbol{\rho}_{-i}$ denotes the vector of privacy levels $\boldsymbol{\rho}$ with the ith coordinate deleted. These incentives motivates users to lower their privacy level, but each user will also have some sensitivity to their data being shared, modelled by a *sensitivity function* $c_i : \mathcal{E} \to [0, \infty)$, $c_i(0) = 0$. The behavior of users can be modelled with the help of a utility function:

$$u_i(\boldsymbol{\rho}) = t_i(\rho_i, \boldsymbol{\rho}_{-i}) - c_i(\rho_i). \tag{1}$$

The payment to user i will tend to increase with a lower privacy level, as the platform can better exploit the data, but their sensitivity c_i will increase with ρ_i, creating a trade-off. By specifying $t_i(\rho_i; \boldsymbol{\rho}_{-i})$, the platform effectively creates a game among the users. This situation is depicted in Fig. 3. Each user's action is the level of privacy that they request for the data they share. Users (players) select their privacy level ρ_i by considering their utility function u_i and the potential actions of the other players. From the perspective of the platform, the goal is to design the payments $t_i(\rho_i; \boldsymbol{\rho}_{-i})$ such that it maximizes the difference between the utility it receives and the payments made to the players i.e., $U(\boldsymbol{\rho}) - \mathbb{1}^T \mathbf{t}(\boldsymbol{\rho})$. One way to formulate this problem is to consider maximizing this difference at equilibrium points:

$$\begin{aligned} \underset{\mathbf{t}(\cdot), \mathcal{P}}{\text{maximize}} \quad & U(\mathcal{P}) - \mathbb{1}^T \mathbf{t}(\mathcal{P}) \\ \text{subject to} \quad & \mathcal{P} \in \text{NE}(\mathbf{t}). \end{aligned} \tag{2}$$

where we have used the shorthand $f(\mathcal{P}) = \mathbb{E}_{\boldsymbol{\rho} \sim \mathcal{P}}\left[f(\boldsymbol{\rho})\right]$. NE($\mathbf{t}$) denotes the set of Nash Equilibrium strategies induced by the payment function $\mathbf{t}$, which is the vector with payment function t_i at index i. Recall that the Nash Equilibrium is a stable state of a system such that no user can gain by a unilateral change of strategy if the strategies of the other users remain unchanged. We allow these equilibrium points to be mixed strategies over the privacy space, such that $\mathcal{P}$ represents a distribution over the privacy space $\mathcal{E}$.

Restrictions must be placed on $\mathbf{t}$, otherwise it can be made arbitrarily negative. *Individual rationality* is a common condition in mechanism design that says that a user can be made no worse off by participation. In Section 5, we consider a fairness constraint.

2.3 Model Limitations

Known sensitivity functions To solve equation 2, the platform requires the privacy sensitivity c_i of each user, and our solution in Section 5 depends on this information. This can be justified when platforms interact with businesses. For example an AI heath platform may interact with insurance companies and hospitals and can invest significant resources into studying each of its partners. Another example is advertisement platforms and sellers. Another justification is that the privacy sensitivity c_i is learned by the platforms over time, and we are operating in a regime where the estimates of c_i have converged. An interesting future direction could be investigating this learning problem.

Data-correlated sensitivity In Section 5 we treat the sensitivity function c_i as fixed and known, but a practical concern is that c_i may depends on the data x_i. Say x_i is biological data pertaining to a disease. Those users with the diseases may have higher c_i. Without taking this into account, the collected data will be biased. If our utility function is greatly increased by those users who do have the disease though, they may receive far more payment, compensating for this correlation. We leave a investigation of data-correlated sensitivity and fairness to future work.

Known transferable and divisible utility Solving equation 2 also requires knowledge of the utility function. In some cases, the platform may dictate the utility entirely on its own, perhaps to value a diverse set of users. In other cases, like in the estimation setting of Example 1, it may represent a more concrete metric, like a risk function that is easily computed. In some cases, however, the utility function may not be easily computed. For example it may depend on the revenue of a company's product, or the downstream performance of a deep network. We also note that $t_i(\rho_i; \boldsymbol{\rho}_{-i})$ may not represent a monetary transfer. Individuals are often compensated for data via discounts or access to services. A shortcoming of our model is that we assume a divisible and transferable utility, which may fail to capture these nuances of compensation.

Informed and Strategic Users We also assume that users can compute and play their equilibrium strategy, which is a standard assumption in game theory. Practically this also means that the platform must be transparent about the incentives, fully publishing this information to the users.

3 Axiomatic Fairness with Privacy

A natural question to ask is: What is a fair way to distribute utility back to the users as incentives? In this work, we view the users and platforms as a coalition, pooling their resources to generate utility. This coalitional perspective is not a complete characterization of the complex dynamics between users and platforms, but we argue that it is still a useful one. In our definition of fairness, we are interested in the intrinsic value of the data. That is, not the *market value* that users are willing to sell for (potentially depressed), but rather, how much of the utility generated comes from the data. This information is particularly useful to economists, regulators, and investors, who are interested in characterizing the value of data as capital for the purposes of analysis, taxation and investment respectively. The answer to this fairness question turns out to be connected to the celebrated Shapley value (Shapley, 1952). Following an axiomatic approach to fairness, the Shapley value describes how to fairly divide utility among a coalition. In this section we develop an axiomatic Shapley value-based approach to fairness for users providing private data to platforms.

3.1 Platform as a Coalition Member

We define a coalition of users and a platform as a collection of s users, with $0 \leq s \leq N$ and up to 1 platform. Let $a \in \{0, 1\}$ represent the action of the platform. Let $a = 1$ when the platform chooses to join the coalition, and $a = 0$ otherwise. Let $U(\boldsymbol{\rho})$ be as defined in Section 2. We augment the utility to take into account that the utility is zero if the platform does not participate, and define $\boldsymbol{\rho}_S$ as follows:

$$U(a, \boldsymbol{\rho}) := \begin{cases} U(\boldsymbol{\rho}) & a = 1 \\ 0 & a = 0 \end{cases}, \quad [\boldsymbol{\rho}_S]_i := \begin{cases} \rho_i & i \in S \\ 0 & \text{else} \end{cases}. \tag{3}$$

Let $\phi_p(a, \boldsymbol{\rho})$ and $\phi_i(a, \boldsymbol{\rho})$, $i \in [N]$ represent the "fair" amount of utility awarded to the platform and each user i respectively, given a and $\boldsymbol{\rho}$, otherwise described as the "value" of a user. Note that these values depend implicitly on both the private algorithm $A_{\boldsymbol{\rho}}$ and the utility function U, but for brevity, we avoid writing this dependence explicitly. The result of Hart & Mas-Colell (1989) show that these values are unique and well defined if they satisfy the following three axioms:

A.i) **(Fairness)** For any $i, j \in [N] : U(a, \boldsymbol{\rho}_{S\cup\{i\}}) = U(a, \boldsymbol{\rho}_{S\cup\{j\}}) \; \forall S \subset [N]\backslash\{i, j\} \implies \phi_i(a, \boldsymbol{\rho}) = \phi_j(a, \boldsymbol{\rho})$.

In addition, for any user $i \in [N]$, $U(1, \boldsymbol{\rho}_{S\cup\{i\}}) - U(1, \boldsymbol{\rho}_S) = 0 \;\; \forall S \subset [N]\backslash\{i\} \implies \phi_i(a, \boldsymbol{\rho}) = 0$.

A.ii) **(Efficiency)** The sum of values is the total utility $U(a, \boldsymbol{\rho}) = \phi_p(a, \boldsymbol{\rho}) + \sum_i \phi_i(a, \boldsymbol{\rho})$.

A.iii) **(Additivity)** Let $\phi_p(a, \boldsymbol{\rho})$ and $\phi_i(a, \boldsymbol{\rho})$ be the value of the platform and users respectively for the utility function U, under the $\boldsymbol{\rho}$-private $A_{\boldsymbol{\rho}}$. Let V be a separate utility function, also based on the output of $A_{\boldsymbol{\rho}}$, and let $\phi'_p(a, \boldsymbol{\rho})$ and $\phi'_i(a, \boldsymbol{\rho})$ be the utility of the platform and individuals with respect to V. Then under the utility $U + V$, the value of user i is $\phi_i(a, \boldsymbol{\rho}) + \phi'_i(a, \boldsymbol{\rho})$ and the value of the platform is $\phi_p(a, \boldsymbol{\rho}) + \phi'_p(a, \boldsymbol{\rho})$.

Theorem 1. *Let $\phi_p(a, \boldsymbol{\epsilon})$ and $\phi_i(a, \boldsymbol{\epsilon})$ satisfying axioms* (A.i-iii) *represent the portion of total utility awarded to the platform and each user i from utility $U(a, \boldsymbol{\epsilon})$. Then they are unique and take the form:*

$$\phi_p(a, \boldsymbol{\rho}) = \frac{1}{N+1} \sum_{S \subseteq [N]} \frac{1}{\binom{N}{|S|}} U(a, \boldsymbol{\rho}_S), \tag{4}$$

$$\phi_i(a, \boldsymbol{\rho}) = \frac{1}{N+1} \sum_{S \subseteq [N]\backslash\{i\}} \frac{1}{\binom{N}{|S|+1}} \left(U(a, \boldsymbol{\rho}_{S\cup\{i\}}) - U(a, \boldsymbol{\rho}_S) \right). \tag{5}$$

Theorem 1 is proved in Appendix A.2. We now consider a simple setting where we can apply this result.

Example 1. Let X_i represent independent and identically distributed data of user i respectively, with $\Pr(X_i = 1/2) = p$ and $\Pr(X_i = -1/2) = 1 - p$, with $p \sim \text{Unif}(0, 1)$. The platform's goal is to construct an $\boldsymbol{\epsilon}$-DP estimator for $\mu := \mathbb{E}[X_i] = p - 1/2$ that minimizes Bayes risk. There is no general procedure for finding the Bayes optimal $\boldsymbol{\epsilon}$-DP estimator, so restrict our attention to $\boldsymbol{\epsilon}$-DP linear-Laplace estimators of the form:

$$A(\mathbf{X}) = \mathbf{w}(\boldsymbol{\epsilon})^T \mathbf{X} + Z, \tag{6}$$

where $Z \sim \text{Laplace}(1/\eta(\boldsymbol{\epsilon}))$. In Fallah et al. (2022) the authors argue that unbiased linear estimators are nearly optimal in a minimax sense for bounded random variables. We assume a squared error loss $L(a, \mu) = (a - \mu)^2$ and let $\mathcal{A}_{\text{lin}}(\boldsymbol{\epsilon})$ be the set of $\boldsymbol{\epsilon}$-DP estimators satisfying equation 6. Then, we define:

$$A_{\boldsymbol{\epsilon}} = \underset{A \in \mathcal{A}_{\text{lin}}(\boldsymbol{\epsilon})}{\arg\min}\ \mathbb{E}[L(A(\mathbf{X}), \mu)] \qquad r(\boldsymbol{\epsilon}) = \mathbb{E}[L(A_{\boldsymbol{\epsilon}}(\mathbf{X}), \mu)]. \tag{7}$$

In words, $A_{\boldsymbol{\epsilon}}$ is an $\boldsymbol{\epsilon}$-DP estimator of the form equation 6, where $\mathbf{w}(\boldsymbol{\epsilon})$ and $\eta(\boldsymbol{\epsilon})$ are chosen to minimize the Bayes risk of the estimator, and $r(\boldsymbol{\epsilon})$ is the risk achieved by $A_{\boldsymbol{\epsilon}}$. Since the platform's goal is to accurately estimate the mean of the data, it is natural for the utility $U(\boldsymbol{\epsilon})$ to depend on $\boldsymbol{\epsilon}$ through the risk function $r(\boldsymbol{\epsilon})$. Note that if U is monotone decreasing in $r(\boldsymbol{\epsilon})$, then U is monotone increasing in $\boldsymbol{\epsilon}$. Let us now consider the case of $N = 2$ users, choosing from an action space of $\mathcal{E} = \{0, \epsilon'\}$, for some $\epsilon' > 0$. Furthermore, take U to be an affine function of $r(\boldsymbol{\epsilon})$: $U(\boldsymbol{\epsilon}) = c_1 r(\boldsymbol{\epsilon}) + c_2$. For concreteness, take $U(\mathbf{0}) = 0$ and $\sup_{\boldsymbol{\epsilon} \in \mathbb{R}} U(\boldsymbol{\epsilon}) = 1$. Note that this ensures that U is monotone increasing in $\boldsymbol{\epsilon}$, and is uniquely defined (exact calculations are available in Appendix A.1). Consider the example of a binary privacy space $\mathcal{E} = \{0, \infty\}$. By equation 33, the utility can be written in matrix form as:

$$\mathbf{U} = \begin{bmatrix} 0 & 2/3 \\ 2/3 & 1 \end{bmatrix}. \tag{8}$$

Note from equation 4 and equation 5, it is clear that $\phi_p(0, \boldsymbol{\epsilon}) = \phi_i(0, \boldsymbol{\epsilon}) = 0$. Let $\boldsymbol{\Phi}_p$ and $\boldsymbol{\Phi}_i^{(1)}$ represent the functions $\phi_p(1, \boldsymbol{\epsilon})$ and $\phi_i(1, \boldsymbol{\epsilon})$ in matrix form akin to $\mathbf{U}$. Then using equation 4 and equation 5, we find that the fair allocations of the utility are given by:

$$\boldsymbol{\Phi}_p = \begin{bmatrix} 0 & 1/3 \\ 1/3 & 5/9 \end{bmatrix}, \ \boldsymbol{\Phi}_1^{(1)} = \begin{bmatrix} 0 & 1/3 \\ 0 & 2/9 \end{bmatrix}, \ \boldsymbol{\Phi}_2^{(1)} = \begin{bmatrix} 0 & 0 \\ 1/3 & 2/9 \end{bmatrix}. \tag{9}$$

3.2 Fairness Among Users

Though we can view the interactions between the platform and the users as a coalition, due to the asymmetry that exists between the platform and the users, it also makes sense to discuss fairness among the users alone. In this case, we can consider an analogous set of axioms that involve only the users.

- B.i) **(Fairness)** For any $i, j \in [N] : U(\boldsymbol{\rho}_{S \cup \{i\}}) = U(\boldsymbol{\rho}_{S \cup \{j\}})\ \ \forall S \subset [N] \backslash \{i, j\} \implies \phi_i(\boldsymbol{\rho}) = \phi_j(\boldsymbol{\rho})$. In addition, for any user $i \in [N]$, $U(\boldsymbol{\rho}_{S \cup \{i\}}) - U(\boldsymbol{\rho}_S) = 0\ \ \forall S \subset [N] \backslash \{i\} \implies \phi_i(\boldsymbol{\rho}) = 0$.
- B.ii) **(Pseudo-Efficiency)** The sum of values is the total utility $\alpha(\boldsymbol{\rho}) U(\boldsymbol{\rho}) = \sum_i \phi_i(\boldsymbol{\rho})$. Where if $U(\boldsymbol{\rho}) = U(\tilde{\boldsymbol{\rho}})$ then $\alpha(\boldsymbol{\rho}) = \alpha(\tilde{\boldsymbol{\rho}})$ and $0 \le \alpha(\boldsymbol{\rho}) \le 1$.
- B.iii) **(Additivity)** Let $\phi_i(\boldsymbol{\rho})$ be the value of users for the utility function U, under the $\boldsymbol{\epsilon}$-private algorithm $A_{\boldsymbol{\rho}}$. Let V be a separate utility function, also based on the output of the algorithm $A_{\boldsymbol{\epsilon}}$, and let $\phi_i'(\boldsymbol{\rho})$ be the utility of the users with respect to V. Then under the utility $U + V$, the value of user i is $\phi_i(\boldsymbol{\rho}) + \phi_i'(\boldsymbol{\rho})$.

A notable difference between these axioms and (A.i-iii) is that the efficiency condition is replaced with pseudo-efficiency. Under this condition, the platform may determine the sum of payments awarded to the players, but this sum should in general depend only on the utility itself, and not on how that utility is achieved.

Theorem 2. *Let $\phi_i(\boldsymbol{\rho})$ satisfying axioms* (B.i-iii) *represent the portion of total utility awarded to each user i from utility $U(\boldsymbol{\rho})$. Then for $\alpha(\boldsymbol{\rho})$ that satisfies axiom* (B.ii) *ϕ_i takes the form:*

$$\phi_i(\boldsymbol{\rho}) = \frac{\alpha(\boldsymbol{\rho})}{N} \sum_{S \subseteq [N] \backslash \{i\}} \frac{1}{\binom{N-1}{|S|}} \left(U(\boldsymbol{\rho}_{S \cup \{i\}}) - U(\boldsymbol{\rho}_S) \right). \tag{10}$$

The proof of Theorem 2 can be found in Appendix A.2.

Example 2. Consider the utility function defined in equation 8, for the $N = 2$ user mean estimation problem with $\mathcal{E} = \{0, \infty\}$. By Theorem 2 the fair allocation satisfying (B.i-iii) must be of the form:

$$\boldsymbol{\Phi}_1^{(2)} = \mathbf{A} \odot \begin{bmatrix} 0 & 2/3 \\ 0 & 1/2 \end{bmatrix}, \quad \boldsymbol{\Phi}_2^{(2)} = \mathbf{A} \odot \begin{bmatrix} 0 & 0 \\ 2/3 & 1/2 \end{bmatrix}, \ \mathbf{A} = \mathbf{A}^T, \ \ 0 \le [\mathbf{A}]_{ij} \le 1. \tag{11}$$

Computational Complexity While at first glance it may seem that both notions of privacy have an exponential computational complexity of $N\,|\mathcal{E}|^N$, this is really only true for a worst-case exact computation of these quantities. In practice, U typically has some kind of structure that makes the problem much more tractable. In Ghorbani & Zou (2019), Jia et al. (2019), Wang & Jia (2023) and Lundberg & Lee (2017) special structures are used to compute these types of sums with significantly lower complexities, particularly in cases where the U is related to the accuracy of a deep network.

4 Fair Incentives In Federated Learning

FL is a distributed learning process used when data is either too large or too sensitive to be directly transferred in full to the platform. Instead of combining all the data together and learning at the platform, each user performs some part of the learning locally and the results are aggregated at the platform, providing some level of privacy. Recently, Donahue & Kleinberg (2021) consider a setting where heterogeneous users voluntarily opt-in to federation. A natural question to ask is: how much less valuable to the platform is a user that chooses to federate with others as compared to one that provides full access to their data? This section provides some interesting insights towards answering this question.

Let each user $i \in [N]$ have a unique mean and variance $(\theta_i, \sigma_i^2) \sim \Theta$, where Θ is some global joint distribution. To motivate this example, let θ_i represent some information about the user critical for advertising. We wish to learn θ_i as accurately as possible to maximize our profits, by serving the best advertisements possible to each user. User i draws n_i samples i.i.d. from its local distribution $\mathcal{D}_i(\theta_i, \sigma_i^2)$, that is, some distribution with mean θ_i and variance σ_i^2. Let $s^2 = \mathrm{Var}(\theta_i)$ represent the variance between users and $r^2 = \mathbb{E}[\sigma_i^2]$ represent the variance within a user's data. When $s^2 \gg \frac{r^2}{n_i}$ the data is very heterogeneous, and it is generally not helpful to include much information from the other users when estimating θ_i, however, if $s^2 \ll \frac{r^2}{n_i}$, the situation is reversed, and information from the other users will be very useful.

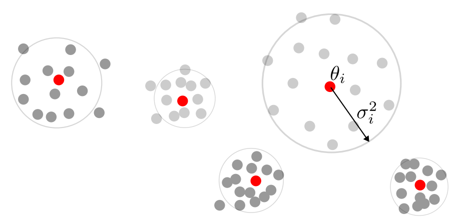


Figure 4: Each user $i \in [N]$ has mean and variance $(\theta_i, \sigma_i^2) \sim \Theta$, where Θ is a global joint distribution. Let $t^2 = \mathrm{Var}(\theta_i)$ and $s^2 = \mathbb{E}[\sigma_i^2]$ for all i. In this case s^2 is large relative to r^2, and the data is very heterogeneous.

The goal of the platform is to construct estimators $\hat{\theta}_i^p$ that minimize the expected mean squared-error of each estimate, while respecting the privacy level vector $\boldsymbol{\rho}$:

$$\mathrm{EMSE}_i(\boldsymbol{\rho}) := \mathbb{E}\left[\left(\hat{\theta}_i^p(\boldsymbol{\rho}) - \theta_i\right)^2\right]. \tag{12}$$

Fig. 2 summarizes our FL formulation. Users can choose from a 3-level privacy space $\mathcal{E} = \{0, 1, 2\}$. In this case the privacy space is not related to DP, but instead encodes how users choose to share their data with the platform. Let N_j be the number of users that choose privacy level j. When $\rho_i = 2$, user i provides its local estimator $\hat{\theta}_i$ directly to the platform. When $\rho_i = 1$, user i's local estimator is securely aggregated with all other users that choose this same privacy:

$$\hat{\theta}^f = \frac{1}{N_1} \sum_{i:\epsilon_i = 1} \hat{\theta}_i, \tag{13}$$

and the platform receives access to $\hat{\theta}^f$, rather than the local estimators. As before, $\rho_i = 0$ means user i chooses not to provide any information to the platform. Note that the error in estimating θ_i depends not just on the privacy level of the ith user ρ_i, but on the entire privacy vector. Let the users be ordered such that ρ_i is a non-increasing sequence. Then for each i the platform constructs estimators of the form:

$$\hat{\theta}_i^p = w_{i0}\hat{\theta}^f + \sum_{j=1}^{N_2} w_{ij}\hat{\theta}_j, \tag{14}$$

where, $\sum_j w_{ij} = 1$ for all i. In Proposition 5, found in Appendix A.3, we calculate the optimal choice of w_{ij} which depends on $\boldsymbol{\rho}$. From these estimators, the platform generates utility $U(\boldsymbol{\rho})$. The optimal w_{i0} and w_{ij} in equation 14 are well defined in a Bayesian sense if $\rho_i > 0$ for some i, but this does not make sense when $\boldsymbol{\rho} = \mathbf{0}$. We can get around this by defining $\text{EMSE}_i(\mathbf{0}) := r^2 + 2s^2$. For the purposes of our discussion, we assume the following logarithmic utility function:

$$U(\boldsymbol{\rho}) := \sum_{i=1}^{n} a_i \log\left(\frac{(r^2 + 2s^2)}{\text{EMSE}_i(\boldsymbol{\rho})}\right). \tag{15}$$

a_i represents the relative importance of each user. Since some users may be willing to spend more than others, the platform may care more about computing their θ_i more accurately, adding another layer of heterogeneity.

4.1 Fair Payments Under Optional Federation: Numerical Study

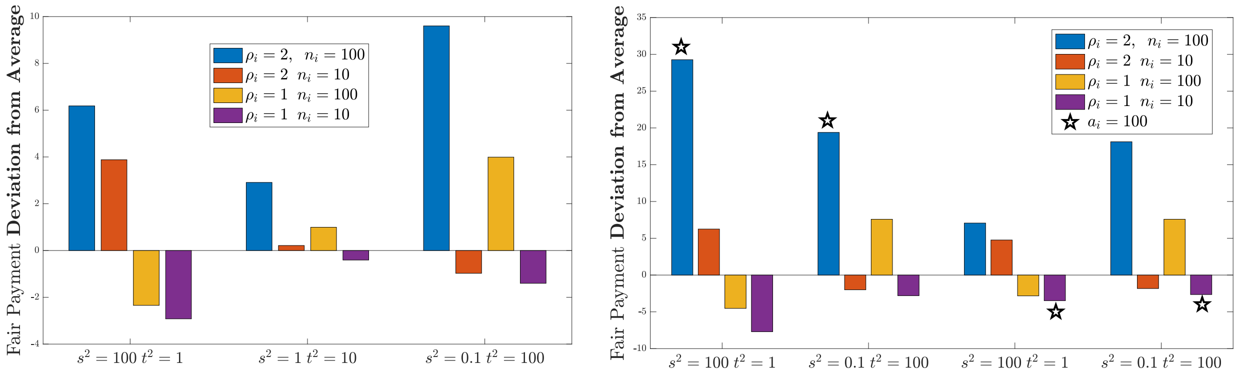
In this section, we focus on our definition of fairness in Theorem 2. Let there be $N = 10$ users. $N_1 = 5$ of these users opt for federating ($\rho_i = 1$), $N_2 = 4$ directly provide their data to the platform ($\rho_i = 2$), and finally, $N_0 = 1$ users chooses to not participate ($\rho_i = 0$). Without loss of generality, we assume $\alpha(\boldsymbol{\rho}) = 1$, and the results of this section can be scaled accordingly.

Different Amounts of Data Fig 5a plots the difference from an equal distribution of utility, i.e., how much each user's utility differs from $U(\boldsymbol{\rho})/N$. We assume $a_i = 1$ for all users. In the bars furthest to the left, where $s^2 = 100$ and $r^2 = 1$, we are in a very heterogeneous environment. Intuitively, this means that a user j will have data that may not be helpful for estimating θ_i for $j \neq i$, thus those users that choose $\rho_i = 2$ are paid the most, since at the very least, the information they provide can be used to target their own θ_i. Likewise, users that federate obfuscate where their data is coming from, making their data less valuable (since their own θ_i cannot be targeted), so users with $\rho_i = 1$ are paid less than an even allocation. On the right side, we have a regime where $s^2 = 0.1$ and $r^2 = 100$, meaning users are similar and user data more exchangeable. Now users with larger n_i are paid above the average utility per user, while those with lower n_i are paid less. Users with $\rho_i = 2$ still receive more than those with $\rho_i = 1$ when n_i is fixed, and this difference is significant when $n_i = 100$. In the center we have an intermediate regime of heterogeneity, where $s^2 = 1$ and $r^2 = 10$. Differences in payments appear less pronounced, interpolating between the two extremes.

More Valuable Users Fig 5b is like Fig 5a, except now in each set of graphs, exactly one user has $a_i = 100$, meaning that estimating θ_i for user i is 100 times more important than the others. Looking at the two leftmost sets of bars in Fig 5b we see that when user i with $\rho_i = 2$ and $n_i = 100$ is the most important one, when s^2 is large compared to r^2, it is user i who receives most of the benefit in terms of its payment but when s^2 is smaller, other users also benefit. This can be intuitively explained as follows: if users are very heterogeneous, other users $j \neq i$ do not have data that is helpful for determining θ_i, thus they do not benefit when user i has a larger a_i. Likewise, when s^2 is small compared to r^2 not just user i benefits, but also all those users that contribute more data, as those users with $\rho_i = 1$ and $n_i = 100$ are also paid over the average utility per user. Another key point is the similarity between the second and fourth set of graphs. This tells an interesting story: when users are not very heterogeneous, regardless of which user is has $a_i = 100$, it is those users with large n_i that will benefit.

5 Fairness Constraints: Data Acquisition

We now use our concrete definition of fairness to constrain the platform to a class of fair payments based on Theorem 2. The platform can choose the fraction of utility that it keeps α, but the incentives it provides to users must be distributed in a fair way. This type of constraint can be viewed as a form of regulation on a



(a) (b)

Figure 5: (a) Plot of *difference from the average* utility per user $U(\boldsymbol{\rho})/N$ for each of the four different types of users, for three different regimes of s^2 and r^2, with heterogeneity decreasing from left to right. In left (most heterogeneous) plot users who choose $\rho_i = 2$ are more valuable compared to those that choose $\rho_1 = 1$. In the center there is an intermediate regime, where all users are paid closer to the average, with users with more data being favored slightly. In the rightmost graph, with little heterogeneity users with more data are paid more, and privacy level has a lesser impact on the payments.
(b) In each case there is one user i with $a_i = 100$ (indicated with a star), while all other users $j \neq i$ have $a_j = 1$ (a_i represents the relative importance of the user in the utility function). In the two leftmost set of bars, we see that the user with $\rho_i = 2$ and $n_i = 100$ receives by far the most payment, when heterogeneity is high, but this becomes less dramatic as heterogeneity decreases. This shows that when users are very heterogeneous, if a_i is large for only user i, most of the benefit in terms of additional payments should go to user i. Likewise, comparing the second from the left and the rightmost plots we see little difference, showing that the opposite is true in the homogeneous case: any user can benefit from any other user having a large a_i.

platform. A natural question to ask is: What will the platform do when subject to this fairness constraint? This section addresses this problem by investigating the incentives of a platform designing a mechanism under the constraint of fairness.

Consider $N \geq 2$ users each with identical statistical marginal contribution, i.e., for any i, j we have $S \subseteq [N]\backslash\{i,j\}$, $U(\boldsymbol{\rho}_{S\cup\{i\}}) = U(\boldsymbol{\rho}_{S\cup\{j\}})$. The platform is restricted to making fair payments satisfying axioms (B.i-iii) with the additional constraint that $\alpha(\boldsymbol{\rho}) = \alpha \in [0,1]$. Users choose one of two available privacy levels $\rho_i \in \mathcal{E}^N$, with $\mathcal{E} = \{\rho_1', \rho_2'\}$ and $\rho_2' > \rho_1'$. We can write the utility of the user i as

$$u(\rho_i, \boldsymbol{\rho}_{-i}) = \alpha\phi(\rho_i; \boldsymbol{\rho}_{-i}) - c\mathbb{1}\{\rho_i = \rho_2'\}\,. \tag{16}$$

Users gain utility from incentives provided by the platform but incur a cost of c if they choose the less private option. For now, we assume c is the same for all users; later we discuss the case where c is different. Note that we can drop the index of ϕ_i due to the assumption of equal marginal contribution. To enrich the problem, we allow users to employ a mixed strategy denoted by $\mathbf{p} = [p, (1-p)]^T$, where users choose the ρ_1' with probability p and ρ_2' with probability $1-p$. This is justified because we expect users to repeatedly interact with platforms and sample from their mixed strategy and ultimately converge to their expected utility.

The platform is also trying to maximize the fraction of the total expected utility $U(\mathbf{p}) := \mathbb{E}_{\boldsymbol{\rho}\sim\mathbf{p}}\left[U(\boldsymbol{\rho})\right]$ that it keeps as in equation 2. The platform's goal is to choose a payment value α such that it optimizes:

$$\begin{aligned} \underset{\alpha}{\text{maximize}} \quad & (1-\alpha)U(\mathbf{p}^*(\alpha)) \\ \text{subject to} \quad & \mathbf{p}^*(\alpha) \in \mathrm{NE}(\alpha). \end{aligned} \tag{17}$$

The objective is simplified compared to equation 2 by exploiting the pseudo-efficiency axiom, which says that the sum of payments is α times the total utility. The constraint in equation 17 implicitly encodes the user behavior governed by equation 16, and will change with the privacy sensitivity c. Theorem 3 characterizes the solution of equation 17 for different values of c. To make equation 17 amenable to insightful analysis, we make some mild assumptions.

Assumption 1. *The utility U is monotone:* $\boldsymbol{\rho}_S^{(2)} \geq \boldsymbol{\rho}_S^{(1)} \implies U(\boldsymbol{\rho}_S^{(2)}) > U(\boldsymbol{\rho}_S^{(1)}) \ \forall S \subseteq [N]$.

Assumption 2. *The utility U has diminishing returns. Let $n_{private}(\boldsymbol{\rho}_S)$ represent the number of elements of $i \in S$ such that $\rho_i = \rho_1'$, i.e., the number of users choosing the higher privacy option. Furthermore, define $\Delta_i U(\boldsymbol{\rho}_S) := U(\boldsymbol{\rho}_S^{(i+)}) - U(\boldsymbol{\rho}_S)$, where $\boldsymbol{\rho}_S^{(i+)}$ is equal to $\boldsymbol{\rho}_S$ except $\rho_i^{(i+)} = \rho_2'$. In other words, $\Delta_i U(\boldsymbol{\rho}_S)$ is the marginal increase in utility when the ith user switches to the lower privacy option. Then U satisfies:*

$$n_{private}(\boldsymbol{\rho}_S^{(1)}) \geq n_{private}(\boldsymbol{\rho}_S^{(2)}) \implies \Delta_i U(\boldsymbol{\rho}^{(1)}) > \Delta_i U(\boldsymbol{\rho}^{(2)}). \tag{18}$$

It is helpful to define the *expected relative payoff*, where the expectation is taken with respect to the actions of the other players. When all other users choose a mixed strategy $\mathbf{p}$, the expected relative payoff is defined as:

$$\gamma(p) := \phi(\rho_2'; \mathbf{p}) - \phi(\rho_1'; \mathbf{p}) = \mathbb{E}_{\substack{\rho_j \sim \mathbf{p} \\ j \neq i}} \left[\phi(\rho_2'; \boldsymbol{\rho}_{-i}) - \phi(\rho_1'; \boldsymbol{\rho}_{-i})\right]. \tag{19}$$

For convenience, we have defined γ in terms of the scalar p, rather than the vector $\mathbf{p} = [p, \ (1-p)]^T$. This quantity represents the expected gain in incentive (normalized to make it invariant to α) if a user switches to a less private level from the more private level given everyone else plays the mixed strategy $\mathbf{p}$.

Theorem 3. *Consider a binary privacy level game with N users and a platform. If U satisfies Assumptions 1 and 2, and the platform payments are fair as defined in Theorem 2 with constant α then the optimal α^* can be divided into three regimes depending on c. The boundaries of these regions are $\gamma_{max} := \max_p \gamma(p)$ and some $c_{th} < \gamma_{max}$ such that:*

1. *When $c > \gamma_{max}$, $\alpha^* = 0$ is the maximizer of 17.*
2. *When $c_{th} < c < \gamma_{max}$ then α^* is the smallest $\alpha \in [0,1]$ such that $p^*(\alpha) \in \gamma^{-1}(c/\alpha)$.*
3. *When $c < c_{th}$: α^* is the smallest $\alpha \in [0,1]$ such that $p(\alpha) = 0$, where*

$$c_{th} = \max\left\{c \left| \frac{1 - c/\gamma_{min}}{1-\alpha} - \frac{U(p^*(\alpha))}{U(0)} \geq 0 \ \ \forall \alpha < c/\gamma_{min} \right.\right\}. \tag{20}$$

Theorem 3 can be interpreted as follows. If privacy sensitivity is above γ_{max} for the given task, it is not worth the effort of the platform to participate. On the other hand, if privacy sensitivity is less than c_{th}, the platform should set α to be as small as possible, while still ensuring that all users choose the low privacy setting. Finally, if privacy sensitivities lie somewhere in between, α^* should be chosen based on the γ function, and generally will lead to a mixed strategy with some proportion of users choosing each of the two options.

Comparison to other works Two key novelties of our work is that we (1) consider a constraint of fairness and (2) have users choose a privacy level, rather than report their privacy sensitivity. This is different from Fallah et al. (2022), and Cummings et al. (2023), which rely on incentive compatibility, and have users report their privacy parameters. In Fallah et al. (2022), a computationally efficient algorithm is proposed for computing user payments and privacy levels to assign users. Both of these works consider a mean estimation problem, where users have i.i.d. samples, and so also have the "equal marginal contibution" assumption that we have. Distinct from our model, users have an additional term in their utility where they benefit from reduced error in the estimation problem. These works focus on maximizing the platform utility, and it is very clear that the payments deviate significantly from the fair ones that satisfy the fairness axioms. Hu & Gong (2020) is perhaps the work most relevant to ours. They consider an incentive design problem where the platform fixes the total sum of payments R and the amount each user receives is proportional to their privacy level ρ_i, which the users choose. This proportional scheme, while potentially viewed as a type of fairness, does not satisfy our axioms. For a particular utility function, they develop a computationally efficient algorithm to compute the equilibrium privacy levels ρ_i based on the privacy sensitivities of the users and the total sum of payments R. In all of these works, users have a linear privacy sensitivity function with rate c_i. Though this seems different from our binary privacy problem, there is a direct correspondence here since we allow mixed strategies, so in expectation, our sensitivity is also reduced to a linear function of the mixed strategy: i.e., $\mathbb{E}\left[c_i \mathbb{1}\left\{\rho_i = \rho_2'\right\}\right] = c_i \Pr(\rho_i = \rho_2')$.

5.1 Mechanism Design: Mean Estimation Example

Let's look at the problem we discussed in Example 1 and 2 with the fair payments that we calculated in Section 2, and examine how it behaves under mechanism design. Figure 6 depicts the solution to equation 17. As predicted by Theorem 3, we find that the solution is clearly divided into three regions. Equation 20 tells us that $c_{th} = \frac{1}{3}$ and $\gamma_{max} = \frac{2}{3}$, matching our observations in Fig. 6. In the first region when $c \leq \frac{1}{3}$ the platform is able to capture most of the utility for itself, paying less of it out to the users. We also see that throughout this regime, the total utility is maximized, as predicted by the theory. For $c \in [\frac{1}{3}, \frac{2}{3}]$, the total utility begins to decrease, as users no longer have enough incentive to always choose the less private option. Finally, for $c \geq \frac{2}{3}$, the platform no longer attempts to incentivize the users, and the total utility falls to zero.

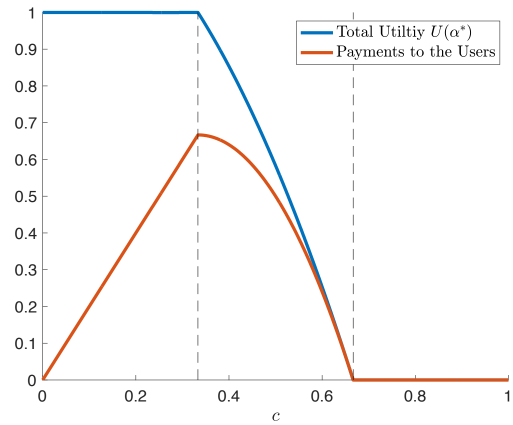


Figure 6: Utility of users and platform when platform solves equation 17. The solution has three separate regions as predicted by Theorem 3.

5.2 Considering Different Privacy Sensitivities

The computational burden in solving equation 17 is in characterizing the constraint, since the objective reduces to a one-dimensional optimization over $\alpha \in [0, 1]$. In the previous section, with the knowledge that the game is symmetric, we are able to easily characterize the equilibria as a function of α. If the c_i's are all different, for arbitrary utility functions, the problem essentially reduces to finding the equilibria in a general game. To make this tractable, we will need some assumptions. In Hu & Gong (2020), the specific choice of utility function and payments makes computation of the equilibrium tractable. If we have only two groups of users with different c_i that act together, and a finite privacy space, we can appeal to tools for enumerating equlibira in matrix games Avis et al. (2010). In this case if the privacy space is also binary, then the equilibria have an analytical solution, which we provide in Appendix C. Similar to the symmetric case, there are 3 cases for each of the two users as well as corresponding thresholds that depend on c_1 and c_2 respectively, resulting in 9 total cases. For example, in the case where payment is below the threshold of both users, neither participate at the low-privacy level, when the payment is high enough both participate at the low privacy level, and for the remaining intermediate cases, either only one user chooses the low privacy option, or there is some asymmetric mixed strategy. Below, we numerically investigate this case:

This problem differs from equation 17 because the equilibrium is governed by asymmetric users. For example, if user 1 and user 2 have privacy sensitivity c_1 and c_2 respectively, we have

$$u_1(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1^T \mathbf{\Phi}_1^{(2)} \mathbf{p}_2 - [0\ c_1]^T \mathbf{p}_1, \quad u_2(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1^T \mathbf{\Phi}_2^{(2)} \mathbf{p}_2 - [0\ c_2]^T \mathbf{p}_2. \tag{21}$$

Consider a setting where there are only two users (these can be thought of as representing two *groups* of users) with utility function u_1 and u_2 listed above. Thus, when the platform is trying to optimize it's own utility, it must take into consideration that these two groups will play different strategies.

$$\begin{aligned} \underset{\alpha}{\text{maximize}} \quad & \mathbf{p}_1^T \mathbf{U} \mathbf{p}_2 - (1-\alpha) \mathbf{p}_1^T \mathbf{U} \mathbf{p}_2 \\ \text{subject to} \quad & (\mathbf{p}_1, \mathbf{p}_2) \in \text{NE}(\alpha). \end{aligned} \tag{22}$$

Fig. 7 plots the results of simulating the solution of 22. It shows that there is one region when c_1 and c_2 are both small and close together ($< 1/3$), the platform chooses α to collect data from both users. If the difference is large, even in this region, the users may be asymmetrically engaged. When $c_1 > c_2 > 1/3$, the platform chooses α such that only user 2 chooses to participate, even if the difference is very small, and vice versa if $c_2 > c_1 > 1/3$, as before, when $c_1, c_2 > 2/3$ the sensitivity to too high and the platform can no longer offer enough payment to the users.

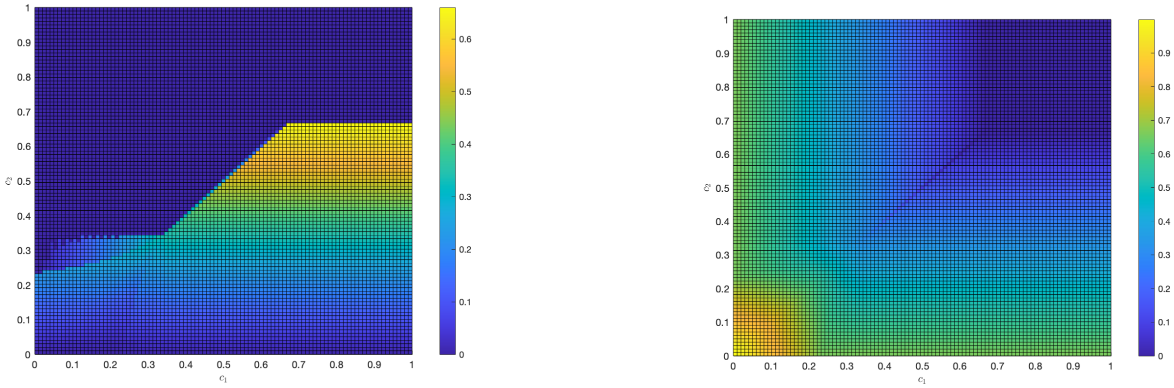


Figure 7: (Left) The payments to user 2 from the platform for a range of c_1, c_2. (Right) The platform's share of utility for the optimal α^* payments for a range of values c_1, c_2.

Broader Impact Statement One of the unique defining characteristics of data is that its generation process is inherently distributed, so no single entity exists to advocate for data sellers. In the past, platforms have been able to extract data from users, often with little to no compensation in return. As public consciousness around privacy changes, a nuanced relationship around privacy between platforms and users must develop. Transparency and understanding the value of user data is an important step in empowering regulators, consumers, and platforms.

- Users making strategic decisions about when they share their data stand to gain from incentives.
- For regulators, understanding the amount of value that flows through the interactions between platforms can enable better policies around data. Frameworks similar to those discussed in Theorem 1 and 2 can be a starting point in understanding exactly how much this value is.
- For platforms, understanding which data tasks are economically viable, and how they allocate incentive is important. Our discussion in Section 5, and our three regimes help shed light on this.

6 Conclusion

This paper introduces two formal definitions of fair payments in the context of acquisition of private data. The first treats the users and the platform together and uses axioms like those of the Shapley value to determine a unique fair distribution of utility. In the second, we define a notion of fairness between the users only, leading to a definition of fairness that admits a range of values, of which the platform is free to choose the most favorable. By formulating a federated mean estimation problem, we show that heterogeneous users can have significantly different contributions to the overall utility, and that a fair incentive, according to our second notion, must take into account the amount of data, privacy level as well as the degree of heterogeneity.

While previous literature has investigated how platforms should design incentives for users in order to optimize its utility, the definitions of fairness we propose offers another important way to evaluate the fairness of these mechanisms. This is a critical step towards future research in ensuring that data acquisition mechanisms are *both* fair for users and efficient for platforms.

Though we provide a characterization of optimal fair mechanisms when privacy sensitivity is the same across users, designing mechanisms that consider fairness with heterogeneous privacy sensitivities, with an arbitrary number of users N is an important question that remains, since in practice the platform interacts with large and diverse groups of users. Furthermore, there is subjectivity in the choice of axioms, and other choices may lead to meaningful notions of fairness worthy of study. We have also assumed a non-divisible and transferable utility, but in many cases, users are paid for their data in the form of access to services. Investigating the impact of this will also be important for the practical application of a comprehensive theory for fairness.

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