GRAPH JOINT ATTENTION NETWORKS

Anonymous authors
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ABSTRACT

Graph attention networks (GATs) have been recognized as powerful tools for learning in graph structured data. However, how to enable the attention mechanisms in GATs to smoothly consider both structural and feature information is still very challenging. In this paper, we propose Graph Joint Attention Networks (JATs) to address the aforementioned challenge. Different from previous attention-based graph neural networks (GNNs), JATs adopt novel joint attention mechanisms which can automatically determine the relative significance between node features and structural coefficients learned from graph subspace, when computing the attention scores. Therefore, representations concerning more structural properties can be inferred by JATs. Besides, we theoretically analyze the expressive power of JATs and further propose an improved strategy for the joint attention mechanisms that enables JATs to reach the upper bound of expressive power which every message-passing GNN can ultimately achieve, i.e., 1-WL test. JATs can thereby be seen as most powerful message-passing GNNs. The proposed neural architecture has been extensively tested on widely used benchmarking datasets, including Cora, Cite, and Pubmed and has been compared with state-of-the-art GNNs for node classification tasks. Experimental results show that JATs achieve state-of-the-art performance on all the testing datasets.

1 INTRODUCTION

Many real-world data can be modeled as a graph, where a set of nodes (vertices), edges, and bag-of-words features respectively represent data instances, instance-instance interrelationships, and contents characterizing the nodes. For example, scientific articles in a research domain can be modeled as a graph, where nodes, edges, and node features respectively represent published articles, citations, and index information of the articles. Besides, social network users and interacted biological units can also be similarly represented as graphs possessing different structural and descriptive information. As graph data are widely available and they are related to various analytical tasks, learning in graphs has been a hot-spot in machine learning community.

There have been a number of approaches proposed to effectively learn in graph structured data. Amongst them, graph convolutional networks (GCNs) have shown to be powerful in learning low-dimensional representations for various subsequent analytical tasks. Different from those empirical convolutional neural networks (CNNs) which have achieved a great success in learning in image, vision, and natural language data (Krizhevsky et al., 2012; Xu et al., 2014), and whose convolution operators are always defined to process a grid-like data structure, GCNs attempt to formulate convolution operators aggregating the node features according to the observed graph structure, and learn the information propagation through different neural architectures. Meaningful representations which capture discriminative node features as well as intricate graph structure can thereby be learned by GCNs. There have been several sophisticated GCNs proposed in the recent. According to the ways through which GCNs make use of graph topology to define convolution operators for feature aggregation, GCNs can generally be categorized as spectral, and spatial ones (Wu et al., 2020).

Spectral GCNs define the convolutional layer for aggregating neighbor features based on the spectral representation of the graph. For example, Spectral CNN (Bruna et al., 2013) constructs the convolution layer based on the eigen-decomposition of graph Laplacian in the Fourier domain. However, such layer is computationally demanding. Aiming to reduce such computational burden, several approaches adopting the convolution operators which are based on simplified/approximate spectral graph theory are proposed. First, parameterized filters with smooth coefficients are introduced for

Spectral CNN to allow it to consider spatially localized nodes in the graph (Henaff et al., 2015). Chebyshev expansion is then introduced in (Defferrard et al., 2016) to approximate graph Laplacian rather than directly perform eigen-decomposition of it. Finally, the graph convolution filter is further simplified by only considering connected neighbors of each node (Kipf & Welling, 2017) so as to further make spectral GCNs computationally efficient.

In contrast, spatial GCNs define the convolution operators for feature aggregation directly making use of local structural properties of the central node. The key of spatial GCNs is consequently how to design an appropriate function for aggregating the effect brought by the features of candidate neighbors selected according to a proper sampling strategy. To achieve this, it sometimes requires to learn a weight matrix according to node degree (Duvenaud et al., 2015), utilize the power of transition matrix to preserve the neighbor importance (Atwood & Towsley, 2016), extract the normalized neighbors (Niepert et al., 2016), or sample a fixed number of neighbors (Hamilton et al., 2017; Zhang et al., 2020).

As a representative spatial GCN, Graph attention network (GAT) (Veličković et al., 2018; Zhang et al., 2020) has shown a promising performance in various graph learning tasks. What makes GATs effective in learning graph representations is they adopt the attention mechanism, which has been successfully used in machine reading and translation (Cheng et al., 2016; Luong et al., 2015), and video processing (Xu et al., 2015), to compute the node-feature-based attention weights (attention scores) between a central node and its one-hop neighbors (including the central node itself). Then, GATs utilize the attention scores to obtain a weighted aggregation of node features which are propagated to the next layer. As a result, those neighbors possessing similar features may impact more on the center node, and meaningful representations can be inferred by GATs.

Although GATs have been experimentally verified as powerful tools for various learning tasks in graph data, they still confront several challenges. First, there are no appropriate attention mechanisms which can automatically identify the relative significance between the latent structure and node features while computing the attention scores. As a result, current attention mechanisms for GATs cannot effectively capture the joint effect brought by the underlying graph structure and node features for seamlessly impacting the message-passing in the neural architecture. Second, whether the expressive power of GNNs adopting the attention mechanisms which can effectively acquire the aforementioned joint effect may reach the upper bound of message-passing GNNs has not been theoretically investigated. To address the mentioned challenges, in this paper, we propose novel attention-based GNNs, dubbed Graph Joint Attention Networks (JATs). Different from previous works, the attention mechanisms adopted by JATs are able to automatically capture the relative significance between structural coefficient learned from graph subspace, and node features, so that higher attention scores may be learned by those neighbors which are topologically and contextually correlated. JATs are consequently able to smoothly adjust attention scores according to the contemporary structure and node features, and truly capture the joint attention on structural and contextual information propagated in the neural network. Besides, we theoretically analyze the expressive power of JATs and further propose an improved strategy which enables JATs to distinguish all distinct graph structures as 1-dimensional Weisfeiler-Lehman test (1-WL test) does. This means JATs can reach the upper bound w.r.t. expressive power which all message-passing GNNs can ultimately achieve. JATs have been extensively tested on three widely used datasets, i.e., Cora, Citeseer, and Pubmed, and have been compared with a number of strong baselines. The experimental results show that JATs achieve the state-of-the-art performance.

The rest of the paper is organized as follows. In Section 2, we elaborate the proposed JATs, and compare JATs with other GNNs. In Section 3, we prove the limitation w.r.t. expressive power of the joint attention mechanisms presented in Section 2. A strategy is then proposed to improve JATs to reach the upper bound of expressive power which all message-passing GNNs can at most achieve. The comprehensive experiments which are used to validate the effectiveness of JATs are presented in Section 4. Finally, we summarize the contributions of the paper and propose future works potentially improving JATs.

2 JOINT ATTENTION-BASED GRAPH NEURAL NETWORKS

In this section, we elaborate the proposed JATs. Mathematical preliminaries and notations used in the paper are firstly illustrated. How JATs learns the structural coefficients which are used in the

joint attention mechanisms is then introduced. Following that, the joint attention layer, which is the cornerstone of JATs is elaborated. At last, we compare the proposed JATs with their counterparts.

2.1 NOTATIONS AND PRELIMINARIES

Throughout this paper, we assume a graph $G = \{V, E\}$ containing N nodes, $|E|$ edges, where V and E respectively represent the node and edge set. We use $\mathbf{A} \in \{0,1\}^{N\times N}$ and $\mathbf{X} \in \{0,1\}^{N\times D}$ to represent graph adjacency matrix and node feature matrix, respectively. $\mathcal{N}_i$ denotes the union of node i and its one-hop neighbors. $\mathbf{W}^l$ and $\{\mathbf{h}_i^l\}_{i=1,...N}$ denote the weight matrix and features of node i at lth layer of JATs, respectively, and $\mathbf{h}^0$ is set to be the input feature, i.e., $\mathbf{X}$. For the nodes in $\mathcal{N}_i$, their feature vectors form a multiset $M_i = (S_i, \mu_i)$, where $S_i = \{s_1, ...s_n\}$ is the ground set of M_i and $\mu_i : S_i \to \mathbb{N}^\star$ is the multiplicity function giving the multiplicity of each s in S_i.

2.2 LEARNING STRUCTURAL COEFFICIENTS FROM GRAPH SUBSPACE

It is well known that topology is the corner stone of the graph. How to utilize such structural information to compute the attention scores naturally has a profound impact on the performance of attention-based GNNs. Empirical attention based GNNs, e.g., GAT, compute the attention scores between connected neighbors only using node features, but overlook the structural correlation between pairwise nodes. To allow attention-based GNNs to capture the higher-order structures in the graph, we propose JATs to learn the topological coefficients from the graph subspace. Inspired by subspace clustering (Elhamifar & Vidal, 2013), we may formulate the learning of structural coefficients as follows. Given N nodes in the graph drawn from multiple linear subspaces $\{\mathcal{S}_i\}_{i=1,...K}$, one can represent a node in a subpace as a linear combination of other nodes. If each row in $\mathbf{A}$ is treated as the structural information of each node, one can simply represent it using other nodes (other rows in $\mathbf{A}$) as one equation, i.e., $\mathbf{A}_{i,:} = \mathbf{C}_{i,:}\mathbf{A}$, where $\mathbf{C}$ denotes the structural coefficient matrix as $\mathbf{A}$ is used. It has been shown in previous works (Ji et al., 2014) that under the assumption the subsapces are independent, by minimizing certain norm of $\mathbf{C}$, $\mathbf{C}$ may have a block diagonal structure (after finite permutations). In other words, $\mathbf{C}_{ij} \neq 0$ if and only if two nodes, v_i and v_j are in the same subspace. So, we can utilize $\mathbf{C}$ to learn the structural correlations between neighbors in the graph. And the above learning task can be formulated as the following optimization problem:

$$\begin{aligned} &\text{minimize } \|\mathbf{C}\|_p \\ &\text{subject to } \mathbf{A} = \mathbf{C}\mathbf{A}, \mathbf{C}_{ii} = 0, \end{aligned} \tag{1}$$

where $\|\cdot\|_p$ stands for a certain matrix norm, and the zero constraint on the diagonal of $\mathbf{C}$ may prevent trivial solutions when $\|\cdot\|_p$ is the norm considering sparsity. To make the data corruption explainable, the equality constraint in Eq. (1) is often relaxed as a regularization term and the learning of structural coefficients can be reformulated as follows:

$$\begin{aligned} &\text{minimize } \|\mathbf{C}\|_p + \beta\|\mathbf{A} - \mathbf{C}\mathbf{A}\|_F^2 \\ &\text{subject to } \mathbf{C}_{ii} = 0. \end{aligned} \tag{2}$$

By minimizing Eq. (2), one may identify those nodes which are in the same graph subspace, and the structural correlations between neighboring nodes can therefore be inferred. As the above learning problem can be effectively solved via gradient descent, JATs can optimize Eq. (2) together with the training of the neural architecture. Also, we use l_1 norm for $\mathbf{C}$ to force JATs to learn sparse structural coefficients.

2.3 JOINT ATTENTION LAYER

Having made the structural coefficients available, we are now presenting joint attention layer, which is the core module for building JATs and will be used in our experiments. Different from the attention layers utilized by other GNNs, the joint attention layer in the proposed framework adopts novel attention mechanisms, which may automatically identify the relative significance between input features and structural information. Appropriate attention scores between central node and its neighbors can be obtained and meaningful representations can be inferred by JATs.

Given a set of node features $\{\mathbf{h}_i^l\}_{i=1,...N}$, each $\mathbf{h}_i^l \in R^{F^l}$, the joint attention layer of JATs is to map them into F^{l+1} dimensional space $\{\mathbf{h}_i^{l+1}\}_{i=1,...N}$, according to the correlations of input features

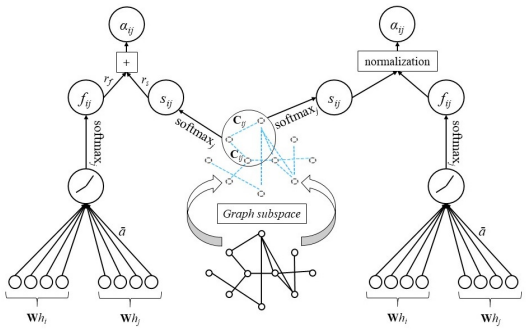


Figure 1: Graphical illustration of the joint attention mechanisms used in each layer of JATs. Left: Joint attention mechanism using *Implicit direction* strategy. Right: Joint attention mechanism using *Explicit direction* strategy. Both two mechanisms consider structural coefficients learned from graph adjacency.

and graph topology. JAT first attempts to compute the correlation between two connected nodes, say v_i and v_j. To do so, we directly adopt the feature-based attention mechanism utilized by previous GATs (Veličković et al., 2018):

$$f_{ij} = \frac{\exp(\text{LeakyReLU}(\tilde{\mathbf{a}}^T(\mathbf{W}^l\mathbf{h}_i^l \parallel \mathbf{W}^l\mathbf{h}_j^l)))}{\sum_{k\in\mathcal{N}_i}\exp(\text{LeakyReLU}(\tilde{\mathbf{a}}^T(\mathbf{W}^l\mathbf{h}_i^l \parallel \mathbf{W}^l\mathbf{h}_k^l)))}, \tag{3}$$

where $\tilde{\mathbf{a}} \in \mathbb{R}^{2F^{l+1}}$ is a vector of parameters of the feedforward layer, $\parallel$ stands for the concatenation function, and $\mathbf{W}^l$ is $F^{l+1} \times F^l$ variable matrix for feature mapping. Based on Eq. (3), JAT may capture the feature correlations between connected nodes (first-order neighbors) by computing the similarities w.r.t. features mapped to next layer.

However, determining the attention scores solely based on node features leads a GNN to overlook the structural information hidden in the graph. To overcome this issue, we propose JAT to learn structural coefficients as mentioned in Section 2.2 and utilize the novel joint attention mechanisms to appropriately compute the attention scores. Given learnable structural coefficient, $\mathbf{C}_{ij}$ between two nodes, JAT obtains the structural correlation as follows:

$$s_{ij} = \frac{\exp(\mathbf{C}_{ij})}{\sum_{k\in\mathcal{N}_i}\exp(\mathbf{C}_{ik})}. \tag{4}$$

Given f_{ij} and s_{ij}, JAT has two strategies (attention mechanisms) to compute the final attention scores. The first mechanism is dubbed *Implicit direction*. It aims at computing the attention scores whose relative significance between structural and feature correlations can be automatically acquired. To do so, JAT introduces two learnable parameters (g_f and g_s) used to determine the relative significance and such significance can be obtained as follows:

$$r_i = \frac{\exp(g_i)}{\sum_{k=s,f}\exp(g_k)}, i = s, f. \tag{5}$$

Given r_i, JAT is able to compute the attention score based on *Implicit direction* strategy:

$$\alpha_{ij} = \frac{r_f \cdot f_{ij} + r_s \cdot s_{ij}}{\sum_{k\in\mathcal{N}_i}[r_f \cdot f_{ik} + r_s \cdot s_{ik}]} = r_f \cdot f_{ij} + r_s \cdot s_{ij}. \tag{6}$$

Given the attention mechanism shown in Eq. (6), α_{ij} attempts to capture the weighted mean attention in terms of structural and feature correlations between neighbors. Compared with the attention mechanism solely based on features of one-hop neighbors, α_{ij} computed by Eq. (6) may be softly adjusted according to the implicit impact brought by the structural coefficients. Moreover, the relative significance r_i can also be automatically inferred by JAT as it is involved into the back propagation process. More smooth and appropriate attention scores can thereby be computed by JAT for learning meaningful representations.

To enhance the structural impact, JAT has another strategy, named as *Explicit direction*, to compute attention scores between neighbors. Given f_{ij} and s_{ij}, the attention scores obtained via *Explicit direction* strategy is defined as follows:

$$\alpha_{ij} = \frac{f_{ij} \cdot s_{ij}}{\sum_{k \in \mathcal{N}_i} f_{ik} \cdot s_{ik}} \tag{7}$$

Compared with Eq. (6), the structural coefficient explicitly influences the magnitude of f_{ij}, so that those node pairs which are structurally irrelevant are impossible to assign with high attention weights. Utilizing *Explicit direction* strategy, JAT becomes more structure dependent when performing message passing in its neural architecture.

Having obtained the attention scores, JAT is now able to compute a linear combination of features corresponding to each node and its neighbors as the layer-wise output, which will be either propagated to the higher layer, or be used as the final representations for subsequent learning tasks. The mentioned output features can be computed as follows:

$$\mathbf{h}_i^{l+1} = \sum_{j \in \mathcal{N}_i} \alpha_{ij} \mathbf{W} \mathbf{h}_j^l. \tag{8}$$

In Fig. 1, the joint attention mechanisms proposed in this paper are graphically illustrated. And we may use a particular number of joint attention layers using the proposed attention mechanisms to construct JATs. In practice, we also adopt the multi-head attention strategy (Vaswani et al., 2017) to stabilize the learning process. JATs either concatenate the node features from multiple hidden layers as the input for next layer, or compute the average of node features obtained from multiple units of output layers as the final node representations. The details on how to implement multi-head attention in graph neural networks can be checked in (Veličković et al., 2018).

2.4 Comparison to related graph neural networks

Based on the structure of the introduced joint attention layer, it is found that Graph Joint Attention Networks are quite different from previous related neural architectures and have several advantages.

When compared with spectral based GNNs, the proposed JATs provide a dynamic way to update layer-wise representations. In each layer, the updating function for node representation in prevalent spectral GNNs, e.g., (Kipf & Welling, 2017) can generally be formulated as $\mathbf{h}_i' = \sigma(\sum_{j \in \mathcal{N}_i} [\mathbf{D}^{-\frac{1}{2}} \mathbf{A} \mathbf{D}^{-\frac{1}{2}}]_{ij} \mathbf{h}_j \mathbf{W})$, where $\mathbf{A}$ and $\mathbf{D}$ are adjacency matrix and degree matrix, respectively, $\mathbf{W}$ is the learnable weight matrix for mapping the node features to the next layer, and $\mathbf{h}_j$ represents the feature vector of jth node. As it shows, the message passing in spectral GNNs is defined according to the normalized adjacency matrix and it is static when spectral GNNs performs back propagation. In contrast, JATs compute the attention score (α_{ij}) for each pair of neighbors based on structural coefficient and node features, which naturally leads each layer in the neural network to pay more attention to those node pairs sharing similar local structures and features. Higher-order representations concerning such analogous neighbors in the graph can therefore be learned by JATs.

When compared with spatial GNNs, especially those attention based ones (Zhang et al., 2020; Veličković et al., 2018), JATs adopt adaptive attention mechanisms which smoothly determine the relative significance between graph structure and node features, to compute more appropriate attention scores for representation learning. Compared with GATs (Veličković et al., 2018), JATs are able to compute attention scores for first order neighbors according to both structural coefficients and node features, so that the attention-based message passing considers more structural information. JATs are also different from those spatial GNNs which update node representations via sampling a fixed size of neighbors. For example, to reduce the computational demand, GraphSAGE (Hamilton et al., 2017) performs the task of graph representation learning via max pooling. Another attention based framework, ADSF (Zhang et al., 2020) also takes into consideration structural information when computing attention scores. However, ADSF cannot infer the relative significance between utilized topological information and node features, and ADSF considers the best k neighbors for computing attention scores. Unlike JATs, such spatial GNNs sampling neighbors for representation updating cannot fully utilize the structural information provided by proximal nodes in the graph. Due to the flexibility of the proposed attention mechanisms, JATs can be readily combined with any paradigm which can learn the topological correlations, just like the structural coefficients used in this paper. This property enhances the applicability of the proposed JATs.

3 MORE EXPRESSIVELY POWERFUL JATS

Study on the expressive power of various GNNs has drawn much attention in the recent. It concerns whether a given GNN can distinguish different structures where vertices possessing various vectorized features. It has been found that what the neighborhood aggregation functions, or readout operators of all message-passing GNNs, including GCNs, GATs, and other related aim at are analogous to what 1-dimensional Weisfeiler-Lehman test (1-WL test), which is injective and iteratively operated in Weisfeiler-Lehman algorithm (Weisfeiler & Leman, 1968; Xu et al., 2018; Zhang & Xie, 2020), does. In other words, both aggregation/readout functions in message-passing GNNs and 1-WL test attempt to distinguish structures which are different in some ways. As a result, all message-passing GNNs are as most powerful as 1-WL test (Xu et al., 2018). The theoretical validation of the expressive power of a given GNN thereby lies in whether those adopted aggreation/readout functions are homogeneous to 1-WL test.

Having the effective framework evaluating the expressive power of a given message-passing GNN, one may naturally be interested in whether the proposed JATs can distinguish all different graph structures as 1-WL test does. As we mainly consider node classification tasks, in this section, we investigate the expressive power of the neighborhood aggregation function concerning the joint attention mechanisms in Eqs. (6) and (7). We firstly show that the neighborhood aggregation function utilized by JATs still fails to discriminate some graph structures possessing certain topological properties. Then, we propose a simple but effective strategy for the joint attention mechanisms to enable JATs to successfully distinguish all those graph structures that previously cannot be discriminated.

For the neighborhood aggregation utilizing the attention mechanism shown in Eq. (6), we have the following theorem pointing out the conditions under which the aggregation function fails to distinguish different structures.

Theorem 1 *Assume the feature space $\mathcal{X}$ is countable and the aggregation function concerns the attention mechanism in Eq. (6) is represented as $h(c, X) = \sum_{x \in X} \alpha_{cx} g(x)$, where c is the feature of center node, $X \in \mathcal{X}$ is a multiset containing the feature vectors from nodes in $\mathcal{N}_i$, $g(\cdot)$ is a function for mapping input feature X, and α_{cx} is the attention score between $f(c)$ and $f(x)$. For all g and the joint attention mechanism in Eq. (6), $h(c_1, X_1) = h(c_2, X_2)$ if and only if $c_1 = c_2$, $X_1 = \{S, \mu_1\}$, $X_2 = \{S, \mu_2\}$, and $\sum_{y=x, y \in X_1} f_{c_1 y} - \sum_{y=x, y \in X_2} f_{c_2 y} = q[\sum_{y=x, y \in X_2} s_{c_2 y} - \sum_{y=x, y \in X_1} s_{c_1 y}]$, for $q = \frac{r_s}{r_f}$ and $x \in S$. In other words, h will map different multiset into the same embedding iff the multisets have same central node feature, same underlying set, and the difference in feature-based attention scores is proportional ($\frac{r_s}{r_f}$) to the opposite of that in attention weights corresponding to structural coefficients.*

We leave the proof of all the theorems and corollaries in the appendix. For the aggregation function utilizing the attention mechanism shown in Eq. (7), we have the following theorem indicating the structures which cannot be correctly distinguished.

Theorem 2 *Under the same assumptions shown in Theorem 1, for all g and the joint attention mechanism in Eq. (7), $h(c_1, X_1) = h(c_2, X_2)$ if and only if $c_1 = c_2$, $X_1 = \{S, \mu_1\}$, $X_2 = \{S, \mu_2\}$, and $q \cdot \sum_{y=x, y \in X_1} \phi(\mathbf{C}_{c_1 x}) = \sum_{y=x, y \in X_2} \phi(\mathbf{C}_{c_2 y})$, for $q > 0$ and $x \in S$, where $\phi(\cdot)$ is an function for mapping values to $\mathbb{R}^+$. In other words, h will map different multiset into the same embedding iff the multisets have same central node feature, same node features whose corresponding mapped structural coefficients are proportional.*

Theorems 1 and 2 indicate that the joint attention mechanisms may still fail to distinguish some graph structures, although the aggregation functions adopting the joint attention mechanisms may be more expressively powerful than empirical GATs. As node features and graph structure are heterogeneous, intuitively, sub-structures satisfying the mentioned conditions should be infrequent. This may well explain why those attention-based GNNs concerning including structural factors into the computation of attention scores may experimentally perform better than GATs. However, when distinct multisets with corresponding structural properties meet the conditions mentioned in Theorems 1 and 2, the joint attention mechanisms in Eqs. (6) and (7) cannot correctly distinguish such multisets. Thus, JATs fail to reach the upper bound of expressive power of all message-passing GNNs, i.e., 1-WL test.

However, we are able to readily improve the expressive power of JATs to meet the condition of 1-WL test by slightly modifying the joint attention mechanisms. The modified attention mechanisms are defined as follows:

$$\alpha_{ij} = \begin{cases} \alpha_{ij} & j \in \mathcal{N}_i, j \neq i, \\ \alpha_{ij} + \epsilon \cdot \frac{1}{|\mathcal{N}_i|} & j = i, \epsilon > 0, \end{cases} \tag{9}$$

where α_{ij} is the attention weight obtained by either Eq. (6) or (7). Then, the newly obtained attention scores can be used to aggregate the node features passed to the higher layers. Next, we prove that such improved attention mechanisms reach the upper bound of message-passing GNNs via showing they can distinguish those structures possessing the properties mentioned in Theorems 1 and 2.

Corollary 1 *Let $\mathcal{T}$ be the attention-based aggregator shown in Eq. (8) that considers the attention mechanism in Eq. (6) or (7) and operates on a multiset $H \in \mathcal{H}$, where $\mathcal{H}$ is a node feature space mapped from the countable input feature space $\mathcal{X}$. A $\mathcal{H}$ exists so that utilizing α_{ij} in Eq. (9), $\mathcal{T}$ can distinguish all different multisets in aggregation that it previously cannot discriminate.*

Based on Eq. (9), one may use original JATs to perform different classification tasks in graph data by setting $\epsilon = 0$. The expressive power of JATs can immediately reach the upper bound of message-passing GNNs when ϵ is set as a positive value. Theoretically, the expressive power of JATs is stronger than state-of-the-art attention-based GNNs, e.g., GATs (Veličković et al., 2018), and ADSF (Zhang et al., 2020).

4 Experiments and analysis

In this section, we evaluate the proposed Graph Joint Attention Networks (JATs) against a variety of prevalent baselines, on three widely used network datasets.

4.1 Experimental set-up

To validate the effectiveness of JATs, we compare them with a number of state-of-the-art baselines, including Gaussian fields and harmonic functions (GF) (Zhu et al., 2003), Manifold regularization (Mani-reg) (Belkin et al., 2006), Deepwalk (Perozzi et al., 2014), Semi-supervised graph embedding (Planetoid) (Yang et al., 2016), Graph convolutional networks (GCN) (Kipf & Welling, 2017), GCN with Chebyshev filters (Chebyshev) (Defferrard et al., 2016), mixture model CNN (MoNet) (Monti et al., 2017), Graph attention networks (GAT) (Veličković et al., 2018), Bayesian GCN (BGCN) (Hasanzadeh et al., 2020), and Adaptive structural fingerprints (ADSF) (Zhang et al., 2020). Based on the experimental results reported in the previous works, these baselines may represent the most advanced techniques for learning in graph structured data.

Three widely-used document networks, which are Cora, Citeseer, and Pubmed (Sen et al., 2008; Lu & Getoor, 2003), are used in our experiments to validate the performance of different approaches. The effectiveness of all methods is validate via allowing them to perform semi-supervised node classification (transductive learning) in all the benchmarking sets and the classified nodes are evaluated using $Accuracy$ ($Micro-F_1$). To compare JATs impartially with other baselines, we closely follow the experimental paradigms used in previous works (Veličković et al., 2018; Kipf & Welling, 2017; Yang et al., 2016). We leave the details of testing datasets and experimental scenarios in appendix due to space limitation.

4.2 Results on node classification

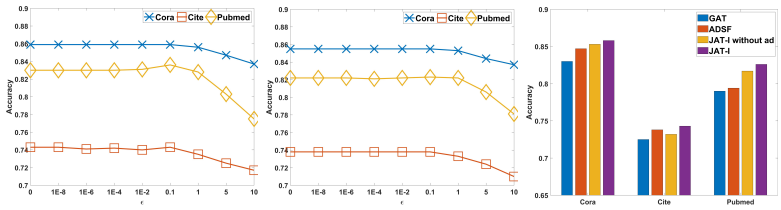
We obtain the average classification accuracy over 10 runs of JATs and compare it with other state-of-the-art approaches. The corresponding results are summarized in Table 1. As the table shows, JATs utilizing two different joint attention mechanisms outperform all the baselines on all the three testing datasets. Specifically, the average $Accuracy$ on node classification obtained by JAT-I (JAT using *Implicit direction* strategy) achieves 85.8%, 74.3%, and 82.8% on Cora, Citeseer, and Pubmed, respectively. The presented experimental results demonstrate that the proposed method is one of the most effective GNNs for learning in graph structured data.

Table 1: Average $Accuracy$ on different testing datasets

Approaches	**Cora**	**Citeseer**	**Pubmed**
GF (Zhu et al., 2003)	68.0%	45.3%	63.0%
Mani-reg (Belkin et al., 2006)	59.5%	60.1%	70.7%
Deepwalk (Perozzi et al., 2014)	67.2%	43.2%	65.3%
Planetoid (Yang et al., 2016)	75.7%	64.7%	77.2%
Chebyshev (Defferrard et al., 2016)	81.2%	69.8%	74.4%
GCN (Kipf & Welling, 2017)	81.5%	70.3%	79.0%
MoNet (Monti et al., 2017)	81.7%	–	78.8%
GAT (Veličković et al., 2018)	83.0%	72.5%	79.0%
BGCN Hasanzadeh et al. (2020)	82.2%	70.0%	–
ADSF (Zhang et al., 2020)	84.7%	73.8%	79.4%
JAT-E	**85.5±0.4%**	**73.8±0.4%**	**82.0±0.3%**
JAT-I	**85.8±0.5%**	**74.3±0.4%**	**82.8±0.4%**

4.3 Ablation Study

Besides testing the performance of transductive learning in network data, we further investigate how different settings of JATs may impact their performance. Specifically, we firstly investigate how different settings of ϵ may impact the performance of JATs. The sensitivity test on $\epsilon = [10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}, 10^{-1}, 1, 5, 10]$ is plotted in Fig. 2 (a) and (b). As the figures depict, both two versions of JATs perform robustly when $\epsilon \leq 1$ and the improved strategy shown in Eq. (9) may boost the performance of JATs in some datasets, e.g., Pubmed. Thus, we recommend to set $\epsilon \leq 1$ for both two versions of JATs. Such setting may ensure JATs leverage the joint attention mechanisms as well as preserve the expressive power, when classifying nodes in graph structured data. Then, to further investigate whether the strategy of automatically determining the relative significance between feature and structural attention (Eqs. (5) and (6)) may truly improve the performance on transductive learning, we compare JAT-I with GAT, ADSF, and JAT utilizing the attention strategy shown in Eq. (6) but setting $r_f = r_s = 0.5$ (JAT-I w/o ad). As Fig. 2 (c) shows, JAT-I can outperform all the baselines on all the testing datasets as it is able to automatically determine which attention, i.e., structure or node features is more important when computing the final attention scores for aggregating features for the central node. The presented results indicate the effectiveness of the strategy of automatic determination adopted by JATs.



(a) JAT-I (b) JAT-E (c) JAT-I with/without ad

Figure 2: Sensitivity test on ϵ and Automatic determination of S/F significance

5 Conclusion

In this paper, we propose novel attention-based GNNs, dubbed Graph Joint Attention Networks (JATs). Different from previous related approaches, JATs adopt novel joint attention mechanisms that can smoothly infer the relative significance between graph structure and node features, so that structural properties and node features are appropriately preserved in node representations. Besides, the expressive power of JATs is theoretically analyzed and the improved strategy to ensure JATs to be most powerful message-passing GNNs is also proposed. In future, we will further improve the effectiveness of JATs by considering different structural properties hidden in the network data and explore JATs' applicability by using them in multi-view and heterogeneous network data.

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PROOF OF THEOREM 1

To prove Theorem 1, we need to consider the two directions of the iff conditions (Zhang & Xie, 2020). If given $c_1 = c_2$, $S_1 = S_2$, and $\sum_{y=x,y\in X_1} f_{c_1y} - \sum_{y=x,y\in X_2} f_{c_2y} = q[\sum_{y=x,y\in X_2} s_{c_2y} - \sum_{y=x,y\in X_1} s_{c_1y}]$, for $q = \frac{r_s}{r_f}$, for the aggregation function utilizing the joint attention mechanism shown in Eq. (6), we have:

$$\begin{aligned} &h(c_i, X_i) = \sum_{x\in X_i} \alpha_{c_ix} g(x), \\ &\alpha_{c_ix} = r_f \cdot f_{c_ix} + r_s \cdot s_{c_ix}, \\ &f_{c_ix} = \frac{\exp(m_{c_ix})}{\sum_{x\in X_i} \exp(m_{c_ix})}, s_{c_ix} = \frac{\exp(\mathbf{C}_{c_ix})}{\sum_{x\in X_i} \exp(\mathbf{C}_{c_ix})}, \end{aligned} \tag{10}$$

where m_{c_ix} represents the feature similarity between c_i and x. Given Eq. (10), we may directly derive $h(c_1, X_1)$ and $h(c_2, X_2)$:

$$\begin{aligned} h(c_1, X_1) &= \sum_{x\in X_1} \alpha_{c_1x} g(x) = \sum_{x\in X_1} [r_f \cdot f_{c_1x} + r_s \cdot s_{c_1x}] \cdot g(x) \\ h(c_2, X_2) &= \sum_{x\in X_2} \alpha_{c_2x} g(x) = \sum_{x\in X_2} [r_f \cdot f_{c_2x} + r_s \cdot s_{c_2x}] \cdot g(x) \end{aligned} \tag{11}$$

Considering $c_1 = c_2$, $S_1 = S_2$, and $\sum_{y=x,y\in X_1} f_{c_1y} - \sum_{y=x,y\in X_2} f_{c_2y} = q[\sum_{y=x,y\in X_2} s_{c_2y} - \sum_{y=x,y\in X_1} s_{c_1y}]$, for $q = \frac{r_s}{r_f}$, we directly derive $h(c_1, X_1) = h(c_2, X_2)$.

If we are given $h(c_1, X_1) = h(c_2, X_2)$, we are able to prove that the conditions mentioned in the theorem are necessary by showing contradictions occur when they are not satisfied. If $h(c_1, X_1) = h(c_2, X_2)$, we have:

$$\begin{aligned} &h(c_1, X_1) - h(c_2, X_2) = \\ &\sum_{x\in X_1} [r_f \cdot f_{c_1x} + r_s \cdot s_{c_1x}] \cdot g(x) - \sum_{x\in X_2} [r_f \cdot f_{c_2x} + r_s \cdot s_{c_2x}] \cdot g(x) = 0 \end{aligned} \tag{12}$$

Firstly, assuming $S_1 \neq S_2$, for any $g(\cdot)$, we thereby have:

$$\begin{aligned} &h(c_1, X_1) - h(c_2, X_2) = \sum_{x\in S_1\cap S_2} [\sum_{y=x,y\in X_1} [r_f \cdot f_{c_1y} + r_s \cdot s_{c_1y}] \\ &- \sum_{y=x,y\in X_2} [r_f \cdot f_{c_2y} + r_s \cdot s_{c_2y}]] \cdot g(x) \\ &+ \sum_{x\in S_1\setminus S_2} \sum_{y=x,y\in X_1} [r_f \cdot f_{c_1y} + r_s \cdot s_{c_1y}] \cdot g(x) \\ &- \sum_{x\in S_2\setminus S_1} \sum_{y=x,y\in X_2} [r_f \cdot f_{c_2y} + r_s \cdot s_{c_2y}] \cdot g(x) = 0 \end{aligned} \tag{13}$$

As Eq. (13) holds for any $g(\cdot)$, we may define another function $g'(\cdot)$ as follows:

$$\begin{aligned} &g(x) = g'(x), \text{for } x \in S_1 \cap S_2 \\ &g(x) = g'(x) - 1, \text{for } x \in S_1 \setminus S_2 \\ &g(x) = g'(x) + 1, \text{for } x \in S_2 \setminus S_1 \end{aligned} \tag{14}$$

If Eq. (13) holds, we also have:

$$
\begin{aligned}
h(c_1, X_1) - h(c_2, X_2) = & \sum_{x \in S_1 \cap S_2} [\sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] \\
& - \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}]] \cdot g'(x) \\
& + \sum_{x \in S_1 \setminus S_2} \sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] \cdot g'(x) \\
& - \sum_{x \in S_2 \setminus S_1} \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}] \cdot g'(x) \\
= & \sum_{x \in S_1 \cap S_2} [\sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] \\
& - \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}]] \cdot g(x) \\
& + \sum_{x \in S_1 \setminus S_2} \sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] \cdot [g(x) - 1] \\
& - \sum_{x \in S_2 \setminus S_1} \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}] \cdot [g(x) + 1] = 0
\end{aligned} \tag{15}
$$

As Eq. (13) equals Eq. (15), we have:

$$
\sum_{x \in S_1 \setminus S_2} \sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] + \sum_{x \in S_2 \setminus S_1} \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}] = 0 \tag{16}
$$

Obviously, the above equation does not hold as the terms in the summation operator are positive. Thus, $S_1 \neq S_2$ is not true. We may now assume $S_1 = S_2 = S$. Eliminating the irrational terms in Eq. (13), we have:

$$
\sum_{x \in S_1 \cap S_2} [\sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] - \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}]] \cdot g(x) = 0 \tag{17}
$$

Thus we know each term in the summation equals zero:

$$
\sum_{y=x, y \in X_1} [r_f \cdot f_{c_1 y} + r_s \cdot s_{c_1 y}] - \sum_{y=x, y \in X_2} [r_f \cdot f_{c_2 y} + r_s \cdot s_{c_2 y}] = 0 \tag{18}
$$

As structural coefficients and node feature similarity are heterogeneous, we may assume $\sum_{y=x, y \in X_1} r_s \cdot s_{c_1 y} = \sum_{y=x, y \in X_2} r_s \cdot s_{c_2 y}$. Eq. (18) can be simplified and rewritten as:

$$
\frac{\mu_1(x)}{\mu_2(x)} = \frac{\exp(m_{c_2 x}) \sum_{x \in S_1} \sum_{y=x, y \in X_1} \exp(m_{c_1 x})}{\exp(m_{c_1 x}) \sum_{x \in S_2} \sum_{y=x, y \in X_2} \exp(m_{c_2 x})} \tag{19}
$$

It is obvious that LHS of Eq. (19) is a rational number. However, the RHS of Eq. (19) can be an irrational number. We may consider $S = \{s, s_0\}$ and assume $c_1 = s_0$, $c_2 = s$. We may also assume the feature similarity between central node and others as follows:

$$
\begin{aligned}
& m_{c_1 x} = 1, \text{for } x \in S \\
& m_{c_2 s} = 1, m_{c_2 s_0} = 2
\end{aligned} \tag{20}
$$

Consider $x = s$, we have:

$$
\frac{\mu_1(s)}{\mu_2(s)} = \frac{|X_1|}{|X_2| - n + ne}, \tag{21}
$$

where n stands for the number of s_0 in X_2. It is obvious that the above equality does not hold as the RHS is an irrational number, while LHS is a rational number. Thus $c_1 \neq c_2$ is false. Given $c_1 = c_2$, Eq. (18) can be rewritten as:

$$
\sum_{y=x, y \in X_1} r_f \cdot f_{c_1 y} - \sum_{y=x, y \in X_2} r_f \cdot f_{c_2 y} + \sum_{y=x, y \in X_1} r_s \cdot s_{c_1 y} - \sum_{y=x, y \in X_2} r_s \cdot s_{c_2 y} = 0 \tag{22}
$$

To ensure above equation holds, we have:

$$\sum_{y=x,y\in X_1} f_{c_1y} - \sum_{y=x,y\in X_2} f_{c_2y} = \frac{r_s}{r_f}[\sum_{y=x,y\in X_2} s_{c_2y} - \sum_{y=x,y\in X_1} s_{c_1y}] \tag{23}$$

Further denoting $\frac{r_s}{r_f} = q$, we have $\sum_{y=x,y\in X_1} f_{c_1y} - \sum_{y=x,y\in X_2} f_{c_2y} = q[\sum_{y=x,y\in X_2} s_{c_2y} - \sum_{y=x,y\in X_1} s_{c_1y}]$. □

PROOF OF THEOREM 2

To prove Theorem 2, we can follow the procedure which is used to prove Theorem 1. If given $c_1 = c_2$, $S_1 = S_2$, and $q \cdot \sum_{y=x,y\in X_1} \phi(\mathbf{C}_{c_1y}) = \sum_{y=x,y\in X_1} \phi(\mathbf{C}_{c_2y})$, for $q > 0$, we may directly replace $\phi(\cdot)$ using $\exp\{\cdot\}$. For the aggregation function using the joint attention mechanism in Eq. (7) we have:

$$\begin{aligned} h(c_i, X_i) &= \sum_{x\in X_i} \alpha_{c_ix} g(x), \\ \alpha_{c_ix} &= \frac{f_{c_ix} \cdot s_{c_ix}}{\sum_{x\in X_i} f_{c_ix} \cdot s_{c_ix}}, \\ f_{c_ix} &= \frac{\exp(m_{c_ix})}{\sum_{x\in X_i} \exp(m_{c_ix})}, s_{c_ix} = \frac{\exp(\mathbf{C}_{c_ix})}{\sum_{x\in X_i} \exp(\mathbf{C}_{c_ix})}, \end{aligned} \tag{24}$$

where m_{c_ix} represents the feature similarity between c_i and x. Given Eq. (24), we may directly derive $h(c_1, X_1)$ and $h(c_2, X_2)$:

$$\begin{aligned} h(c_1, X_1) &= \sum_{x\in X_1} \alpha_{c_1x} g(x) = \sum_{x\in X_1} [\frac{f_{c_1x} \cdot s_{c_1x}}{\sum_{x\in X_1} f_{c_1x} \cdot s_{c_1x}}] \cdot g(x) \\ h(c_2, X_2) &= \sum_{x\in X_2} \alpha_{c_2x} g(x) = \sum_{x\in X_2} [\frac{f_{c_2x} \cdot s_{c_2x}}{\sum_{x\in X_2} f_{c_2x} \cdot s_{c_2x}}] \cdot g(x) \end{aligned} \tag{25}$$

Given $S_1 = S_2$ and $c_1 = c_2$, $h(c_2, X_2)$ can be rewritten as:

$$\begin{aligned} h(c_2, X_2) &= \sum_{x\in S_2} [\frac{f_{c_2x} \cdot \sum_{y=x,y\in X_2} s_{c_2y}}{\sum_{x\in S_2} f_{c_2x} \cdot \sum_{y=x,y\in X_2} s_{c_2y}}] \cdot g(x) \\ &= \sum_{x\in S_2} [\frac{\frac{\exp(m_{c_2x})}{\sum_{y\in X_2} \exp(m_{c_2y})} \cdot \sum_{y=x,y\in X_2} \frac{\exp(\mathbf{C}_{c_2y})}{\sum_{y\in X_2} \exp(\mathbf{C}_{c_2y})}}{\sum_{x\in S_2} \frac{\exp(m_{c_2x})}{\sum_{y\in X_2} \exp(m_{c_2y})} \sum_{y=x,y\in X_2} \frac{\exp(\mathbf{C}_{c_2y})}{\sum_{y\in X_2} \exp(\mathbf{C}_{c_2y})}}] \cdot g(x) \\ &= \sum_{x\in S_2} \frac{\exp(m_{c_2x}) \sum_{y=x,y\in X_2} \exp(\mathbf{C}_{c_2y})}{\sum_{x\in S_2} \exp(m_{c_2x}) \sum_{y=x,y\in X_2} \exp(\mathbf{C}_{c_2y})} \cdot g(x) \end{aligned} \tag{26}$$

Given $q \cdot \sum_{x\in X_1} \exp\{\mathbf{C}_{c_1x}\} = \sum_{y=x} \exp\{\mathbf{C}_{c_2y}\}$, Eq. (26) is equivalent to:

$$h(c_2, X_2) = \sum_{x\in S_1} \frac{q \cdot \exp(m_{c_2x}) \sum_{y=x,y\in X_1} \exp(\mathbf{C}_{c_1y})}{\sum_{x\in S_1} q \cdot \exp(m_{c_2x}) \sum_{y=x,y\in X_1} \exp(\mathbf{C}_{c_1y})} \cdot g(x) \tag{27}$$

Considering $c_1 = c_2$, we have $h(c_1, X_1) = h(c_2, X_2)$.

If we are given $h(c_1, X_1) = h(c_2, X_2)$, we are able to prove the conditions mentioned in the theorem are necessary by showing contradictions occur when they are not satisfied. If $h(c_1, X_1) = h(c_2, X_2)$, we have:

$$\begin{aligned} &h(c_1, X_1) - h(c_2, X_2) = \\ &\sum_{x\in X_1} [\frac{f_{c_1x} \cdot s_{c_1x}}{\sum_{x\in X_1} f_{c_1x} \cdot s_{c_1x}}] \cdot g(x) - \sum_{x\in X_2} [\frac{f_{c_2x} \cdot s_{c_2x}}{\sum_{x\in X_2} f_{c_2x} \cdot s_{c_2x}}] \cdot g(x) = 0 \end{aligned} \tag{28}$$

Firstly, assuming $S_1 \neq S_2$, for any $g(\cdot)$, we thereby have:

$$\begin{aligned}
h(c_1, X_1) - h(c_2, X_2) = & \sum_{x \in S_1 \cap S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})} \\
& - \frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot g(x) \\
& + \sum_{x \in S_1 \setminus S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}] \cdot g(x) \\
& - \sum_{x \in S_2 \setminus S_1} [\frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot g(x) = 0
\end{aligned} \tag{29}$$

As Eq. (29) holds for any $g(\cdot)$, we may define another function $g'(\cdot)$ as shown in Eq. (14). If Eq. (29) holds, we also have:

$$\begin{aligned}
h(c_1, X_1) - h(c_2, X_2) = & \sum_{x \in S_1 \cap S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})} \\
& - \frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot g'(x) \\
& + \sum_{x \in S_1 \setminus S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}] \cdot g'(x) \\
& - \sum_{x \in S_2 \setminus S_1} [\frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot g'(x) \\
= & \sum_{x \in S_1 \cap S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})} \\
& - \frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot g(x) \\
& + \sum_{x \in S_1 \setminus S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}] \cdot [g(x) - 1] \\
& - \sum_{x \in S_2 \setminus S_1} [\frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot [g(x) + 1] = 0
\end{aligned} \tag{30}$$

As Eq. (29) equals Eq. (30), we have:

$$\begin{aligned}
& \sum_{x \in S_1 \setminus S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}] \\
& + \sum_{x \in S_2 \setminus S_1} [\frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] = 0
\end{aligned} \tag{31}$$

Obviously, the above equation does not hold as softmax function is positive. Thus, $S_1 \neq S_2$ is not true. We may now assume $S_1 = S_2 = S$. Eliminating the irrational terms in Eq. (29), we have:

$$\begin{aligned}
& \sum_{x \in S_1 \cap S_2} [\frac{\exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})}{\sum_{x \in S_1} \exp(m_{c_1 x}) \sum_{y=x, y \in X_1} \exp(\mathbf{C}_{c_1 y})} \\
& - \frac{\exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}{\sum_{x \in S_2} \exp(m_{c_2 x}) \sum_{y=x, y \in X_2} \exp(\mathbf{C}_{c_2 y})}] \cdot g(x) = 0
\end{aligned} \tag{32}$$

Thus we know each term in the summation equals zero:

$$\frac{\exp\left(m_{c_1x}\right)\sum_{y=x,y\in X_1}\exp\left(\mathbf{C}_{c_1y}\right)}{\sum_{x\in S_1}\exp\left(m_{c_1x}\right)\sum_{y=x,y\in X_1}\exp\left(\mathbf{C}_{c_1y}\right)}-\frac{\exp\left(m_{c_2x}\right)\sum_{y=x,y\in X_2}\exp\left(\mathbf{C}_{c_2y}\right)}{\sum_{x\in S_2}\exp\left(m_{c_2x}\right)\sum_{y=x,y\in X_2}\exp\left(\mathbf{C}_{c_2y}\right)}=0 \tag{33}$$

Eq. (33) is equivalent to:

$$\frac{\sum_{y=x,y\in X_1}\exp\left(\mathbf{C}_{c_1y}\right)}{\sum_{y=x,y\in X_2}\exp\left(\mathbf{C}_{c_2y}\right)}=\frac{\exp\left(m_{c_2x}\right)\sum_{x\in S_1}\exp\left(m_{c_1x}\right)\sum_{y=x,y\in X_1}\exp\left(\mathbf{C}_{c_1y}\right)}{\exp\left(m_{c_1x}\right)\sum_{x\in S_2}\exp\left(m_{c_2x}\right)\sum_{y=x,y\in X_2}\exp\left(\mathbf{C}_{c_2y}\right)} \tag{34}$$

We may consider $S=\{s,s_0\}$ and assume $c_1=s_0$, $c_2=s$. We may also assume the feature similarity between central node and others as follows:

$$\begin{aligned} m_{c_1x}&=1, \text{for } x\in S\\ m_{c_2s}&=1, m_{c_2s_0}=2 \end{aligned} \tag{35}$$

Consider $x=s$, we have:

$$\frac{\sum_{s\in X_1}\exp\left(\mathbf{C}_{c_1s}\right)}{\sum_{s\in X_2}\exp\left(\mathbf{C}_{c_2s}\right)}=\frac{e[e\sum_{s\in X_1}\exp\left(\mathbf{C}_{c_1s}\right)+e\sum_{s_0\in X_1}\exp\left(\mathbf{C}_{c_1s_0}\right)]}{e[e\sum_{s\in X_2}\exp\left(\mathbf{C}_{c_2s}\right)+e^2\sum_{s_0\in X_2}\exp\left(\mathbf{C}_{c_2s_0}\right)]} \tag{36}$$

As the learning of $\mathbf{C}$ is independent of feature mapping, and the computation of attention coefficients, $\exp\left(\mathbf{C}_{cx}\right)$ can be any positive value. By setting the exponential values in the above equation as a, which is a positive value. We have $\frac{\mu_1(s)}{\mu_2(s)}=\frac{|X_1|}{|X_2|-n+ne}$. Similar with Eq. (21), $c_1\neq c_2$ is not true. Since $c_1=c_2=c$, Eq. (34) can be rewritten as:

$$\frac{\sum_{y=x,y\in X_1}\exp\left(\mathbf{C}_{cy}\right)}{\sum_{y=x,y\in X_2}\exp\left(\mathbf{C}_{cy}\right)}=\frac{\sum_{x\in S_1}\exp\left(m_{cx}\right)\sum_{y=x,y\in X_1}\exp\left(\mathbf{C}_{cy}\right)}{\sum_{x\in S_2}\exp\left(m_{cx}\right)\sum_{y=x,y\in X_2}\exp\left(\mathbf{C}_{cy}\right)}=const>0. \tag{37}$$

By setting $const$ as $\frac{1}{q}$ and $\exp\left(\mathbf{C}_{cy}\right)=\phi(\mathbf{C}_{cy})$, we have $q\sum_{y=x,y\in X_1}\phi(\mathbf{C}_{cy})=\sum_{y=x,y\in X_2}\phi(\mathbf{C}_{cy})$. □

PROOF OF COROLLARY 1

For any two distinct multisets X_1 and X_2 that cannot be distinguished by $\mathcal{T}$, according to **Theorem 1**, we denote $X_1=(S,\mu_1)$, $X_2=(S,\mu_2)$, $c\in S$. We also assume $\sum_{y=x,y\in X_1}f_{c_1y}-\sum_{y=x,y\in X_2}f_{c_2y}=q[\sum_{y=x,y\in X_2}s_{c_2y}-\sum_{y=x,y\in X_1}s_{c_1y}]$, for $q=\frac{r_s}{r_f}$. When using the attention mechanism in Eq. (6) for the aggregation function, it is easy to verify $\sum_{x\in X_1}\alpha_{cx}f(x)=\sum_{x\in X_2}\alpha_{cx}f(x)$. When using the improved mechanism shown in Eq. (9), we have $\sum_{x\in X_1}\alpha_{cx}f(x)-\sum_{x\in X_2}\alpha_{cx}f(x)=\epsilon(\frac{1}{|X_1|}-\frac{1}{|X_2|})$, where $|X_1|=|\mathcal{N}_1|$, and $|X_2|=|\mathcal{N}_2|$. Since $|X_1|\neq|X_2|$, $\sum_{x\in X_1}\alpha_{cx}f(x)-\sum_{x\in X_2}\alpha_{cx}f(x)\neq 0$, meaning the improved joint attention mechanism can distinguish those distinct multiset which $\mathcal{T}$ fails to distinguish. Similarly, we can prove the aggregation function using the improved attention mechanism concerning Eq. (7) also can distinguish those distinct multisets which $\mathcal{T}$ fails to distinguish. □

DETAILED EXPERIMENTAL SET-UP AND ADDITIONAL TESTS

In this section, how the experiments used to validate the effectiveness of the proposed JATs are set up is introduced with more details.

DATASET DESCRIPTION

Three widely used document networks, which are Cora, Citeseer, and Pubmed (Sen et al., 2008; Lu & Getoor, 2003), are used in our experiments to validate the effectiveness of different GNNs. In these three networks, vertices, edges, and vertex features represent the documents, citations between pairwise documents, and the bag-of-words representations of the documents, respectively. Besides, each node in these three document networks has a class label. Following the transductive setting in (Yang et al., 2016; Veličković et al., 2018), we use only 20 vertices per class for training, but all

Table 2: Characteristics of the testing datasets used in our experiments

	N	$\|E\|$	D	C	**Training Nodes**	**Validation Nodes**	**Testing Nodes**
Cora	2708	5429	1433	7	140	500	1000
Citeseer	3327	4732	3703	6	120	500	1000
Pubmed	19717	44338	500	3	60	500	1000

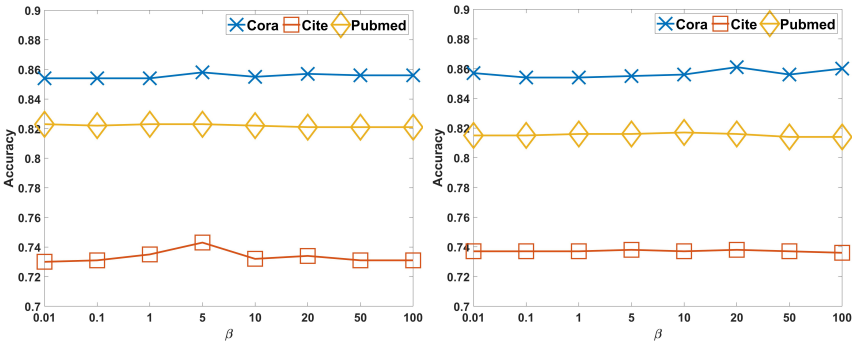
feature vectors. The datasets are split into three parts: training, validation, and testing. Algorithm performance will be evaluated on the test split. The statistics of these benchmarking datasets are summarized in Table 2, where N, $|E|$, D, and C denote the number of vertices, edges, vertex features, and number of classes in each dataset, respectively.

EXPERIMENTAL SCENARIOS

The effectiveness of different methods is validated via allowing them to perform semi-supervised learning (transductive learning) in the aforementioned graph datasets. To compare the proposed JATs impartially with other baselines, we closely follow the experimental paradigms used in previous works (Veličković et al., 2018; Kipf & Welling, 2017; Yang et al., 2016). Specifically, only 20 vertices in each class are labeled in the training stage, and all the graph data, including vertex features and edges, can be accessible for the training of neural networks. The predictive ability of the approaches is validated using 1000 vertices in the test split. Algorithm performance is evaluated using $Accuracy$ ($Micro - F_1$).

For network structure of JAT, following (Veličković et al., 2018), we use a similar two-layer deep neural network. The first layer consists of eight attention heads each computing eight features, followed by an exponential linear unit nonlinearity. The second layer is used for classification composed of attention heads and a softmax activation. For Cora and Citeseer, a single attention head is used, while for Pubmed, eight output attention heads are used, to compute C features. Regularization is applied within the model to cope with the small training set sizes. During training, L_2 regularization with $\lambda = 0.0005$ is used for all the testing datasets. And dropout with $p = 0.6$ is applied for both layers' inputs.

All the networks are initialized using Glorot initialization and trained to minimize cross-entropy on the training nodes plus the structural coefficients (Eq. (2)) using the Adam SGD optimizer (Kingma & Ba, 2014). A learning rate of 0.01 is used for all the datasets and the maximum number of epochs is set to 10000. More details on the experimental scenarios can be checked in (Veličković et al., 2018).



(a) JAT-I (b) JAT-E

Figure 3: Sensitivity test on β

SENSITIVITY TEST ON β

For sensitivity test on β, which determines the relative significance of graph self-expression (Eq. (2)), we set $\beta = [0.01, 0.1, 1, 5, 10, 20, 50, 100]$ and let two versions of JAT to classify nodes on all

the three datasets. The corresponding results have been plotted in Fig. 3 (a) and (b). As the figures depict, both two versions of JAT may obtain a robust performance when different settings of β are used. Based on the the obtained results, we set $\beta = 5$ for JATs when they perform the tasks of semi-supervised node classification on various datasets.