Global Convergence to Local Minmax Equilibrium in Classes of Nonconvex Zero-Sum Games

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Abstract

We study gradient descent-ascent learning dynamics with timescale separation in unconstrained continuous action zero-sum games where the minimizing player faces a nonconvex optimization problem and the maximizing player optimizes a Polyak-Łojasiewicz (PŁ) or strongly-concave (SC) objective. In contrast to past work, we assess convergence in relation to game-theoretic equilibria instead of only notions of stationarity. In pursuit of this goal, we prove that the only locally stable points of the continuous-time limiting system correspond to strict local minmax equilibria in each class of games. For the class of nonconvex-PŁ zero-sum games, we exploit timescale separation to construct a potential function that when combined with the stability characterization and an asymptotic saddle avoidance result gives a global asymptotic almost-sure convergence guarantee to a set of the strict local minmax equilibrium. For the class of nonconvex-SC zero-sum games, we show the surprising property that the function of the game can be made a potential with timescale separation. Combining this insight with the stability characterization allows us to generalize methods for efficiently escaping saddle points from nonconvex optimization to this class of zero-sum games and obtain a global finite-time convergence guarantee for gradient descent-ascent with timescale separation to approximate local minmax equilibria.

1 Introduction

We study continuous action zero-sum games of the form

$$\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} f(x,y)$$

where $f \in C^2(\mathcal{X}\times\mathcal{Y},\mathbb{R})$ and $\mathcal{X}=\mathbb{R}^{d_1}$ and $\mathcal{Y}=\mathbb{R}^{d_2}$ denote the individual action spaces and $d=d_1+d_2$. In particular, we focus on unconstrained, continuous strategy space zero-sum games in which $f(\cdot,y)$ is potentially nonconvex in $x\in\mathcal{X}$ and $f(x,\cdot)$ satisfies the Polyak-Łojasiewicz (PŁ) condition [47] or is strongly-concave (SC) in $y\in\mathcal{Y}$. We refer to these classes of games as nonconvex-PŁ and nonconvex-SC zero-sum games, respectively.

This general formulation has a broad spectrum of applications such as fair classification [45], distributionally robust optimization [44, 48, 53], and adversarial training [38]. Consequently, there has been a surge of interest in recent years toward developing methods for solving these problems efficiently. So far, existing work on gradient-based learning in nonconvex-PŁ/SC zero-sum games has exclusively

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focused on providing global convergence rates to approximate stationary points with no attention given to the characterization in terms of game-theoretic equilibrium concepts [33, 34, 36, 45, 48, 62].

In contrast, a common theme in the study of general nonconvex-nonconcave zero-sum games is to assess the *types* of stationary points an algorithm locally converges toward in terms of their higher order structure [10, 15, 16, 23, 40, 41, 58, 64]. The purpose of this analysis is generally to determine whether commonly deployed algorithms can guarantee local convergence to *only* game-theoretic equilibria or to design algorithms that achieve this objective.

The goal of this paper is to close the gap between the two problem classes and determine whether gradient-based learning algorithms in nonconvex-PŁ/SC zero-sum games can be shown to globally converge to *only* game-theoretically meaningful equilibria (local minmax/Stackelberg or Nash equilibria). We focus our attention on the canonical gradient descent-ascent learning dynamics with timescale separation between players and a stochastically perturbed variant. In these algorithms, timescale separation is manifested in the different (yet constant) learning rates of the maximizing and minimizing players. The descriptions of these systems, which we refer to as τ-GDA and τ-PGDA where the ratio of learning rates denoted by $\tau > 0$ parameterizes the timescale separation between players, are provided in Algorithm 1 and Algorithm 2, respectively. Simply put, τ-GDA (Algorithm 1) corresponds to each player following their individual gradient in a noiseless setting, while τ-PGDA (Algorithm 2) describes each player following their individual (potentially stochastic) gradient with noise injections. Notably, both τ-GDA and τ-PGDA only require first-order gradients and as a result are computationally efficient methods for machine learning problems formulated as zero-sum games.

1.1 Contributions

We show that τ-GDA and τ-PGDA have *global* convergence guarantees in nonconvex-PŁ/SC zero-sum games to the natural game-theoretic solution concept for this problem class of strict local minmax equilibria.[1] The contributions of this work are summarized as follows.

1) In Theorem 1, we prove the only critical points that are locally stable with respect to the τ-GDA continuous-time limiting system are strict local minmax in nonconvex-PŁ/SC zero-sum games.

2) In Theorem 2, we combine Theorem 1 with a potential function construction and an asymptotic saddle avoidance result to prove that τ-GDA (Algorithm 1) globally asymptotically converges to strict local minmax equilibria almost surely in nonconvex-PŁ/SC zero-sum games.

3) In Theorem 3, for the class of nonconvex-SC zero-sum games, we provide high-probability finite-time rates showing that τ-PGDA (Algorithm 2) globally converges to ε–local minmax equilibria with a complexities of $\widetilde{\mathcal{O}}(\varepsilon^{-4})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ in stochastic and deterministic problems, respectively.

To our knowledge, our results provide the broadest existing global convergence guarantee for gradient-based algorithms to *game-theoretically meaningful* equilibria in zero-sum continuous games.

1.2 Practical Motivation

The study of nonconvex-PŁ/SC zero-sum games has often been motivated by machine learning problems formulated as games. We remark that given the problem formulations, it is natural from both game-theoretic and machine learning perspectives to seek notions of minmax equilibria. Indeed, notions of stationarity are not guaranteed to reflect a meaningful solution to the underlying problem. Consequently, it is critical to give convergence guarantees to minmax equilibrium as we pursue in this work. We now provide examples from the literature of machine learning applications that are relevant to the class of games we consider. Note that the following application problems are illustrative in nature and do not immediately fall into the classes of games we study. However, as we elaborate on shortly, each of the problems can be transformed into unconstrained nonconvex-PŁ/SC zero-sum games while retaining the optimization objectives after simple, standard modifications.

Example 1. *In fair classification [45] and learning from multiple distributions, the objective is to minimize the maximum loss over multiple categories. An example formulation is the problem*

$$\min_{w\in\mathbb{R}^d} \max_{i\in\{1,\dots,K\}} \ell_i(w)$$

[1] Strict local minmax equilibria characterized by gradient-based sufficient conditions are also known as differential Stackelberg equilibria in the literature [15, 16] and we use the terms interchangeably in this work.

where $\ell_i(w)$ represents the loss on category i with w denoting neural network parameters. A reformulation of this problem [45] with $\mathcal{T}$ is the simplex in $\mathbb{R}^K$ is given by the zero-sum game

$$\min_{w\in\mathbb{R}^d} \max_{t\in\mathcal{T}} \sum_{i=1}^N t_i \ell_i(w). \tag{1}$$

Example 2. *To train a neural network classifier robust against adversarial attacks, a common approach is to formulate training as a robust minmax optimization problem of the form*

$$\min_{w\in\mathbb{R}^d} \sum_{i=1}^N \max_{\delta_i:|\delta_i|_\infty\leq\epsilon} \ell(x_i+\delta_i, y_i, w),$$

where $\ell(x_i+\delta_i, y_i, w)$ represents the loss on sample x_i perturbed by δ_i with budget ϵ and label y_i as a function of the parameter weights w [38]. A reformulation of this problem [45] is given by

$$\min_{w\in\mathbb{R}^d} \sum_{i=1}^N \max_{t\in\mathcal{T}} \sum_{j=1}^K t_j \ell(\widehat{x}_{ij}, y_i, w) \tag{2}$$

where $\widehat{x}_{ij}$ is the result of a targeted attack on the sample x_i seeking to change the output of the network to label j and $\mathcal{T}$ is the simplex in $\mathbb{R}^K$.

Example 3. *Distributionally robust optimization often results in a zero-sum game of the form*

$$\min_{w\in\mathbb{R}^d} \max_{t\in\mathcal{T}} \sum_{i=1}^N t_i \ell_i(w) - r(t) \tag{3}$$

where $\ell_i(x)$ is the loss of a model w on the i–th data point, $\mathcal{T}$ is the simplex in $\mathbb{R}^N$, and $r(t)$ is carefully selected convex regularizer [44, 48, 53].

The machine learning problem formulations in (1)–(3) from Examples 1–3 represent nonconvex-concave zero-sum games in which the strategy space of the minimizing player is unconstrained and the strategy space of the maximizing player is constrained to the simplex. These problems are naturally adapted to the unconstrained nonconvex-PŁ/SC zero-sum game setting considered in this paper by removing the constraint on the strategy space of the maximizing player and including a suitable PŁ/SC regularization penalty on the choice variable of the maximizing player.

2 Related Work

We now cover the most relevant related work with further discussion provided in Appendix A.

Nonconvex-Nonconcave Zero-Sum Games. A common theme in analyzing gradient descent-ascent with or without timescale separation in nonconvex-nonconcave zero-sum games has been to assess the local stability around critical points of the continuous-time limiting system and draw connections to the differential Nash and Stackelberg equilibrium notions (see Definition 3) [10, 15, 16, 23, 40, 42, 43, 63]. Importantly, unless the timescale separation is chosen very carefully, the stable critical points of gradient descent-ascent may not be game-theoretically meaningful [15, 23]. We obtain stronger stability characterizations in our analysis of the continuous-time system using the structure of nonconvex-PŁ/SC zero-sum games. Further discussion of this topic is given in Section 4.

Nonconvex-PŁ and Nonconvex-SC Zero-Sum Games. Table 1 provides a comprehensive comparison between our results and existing results for gradient descent variants in nonconvex optimization and nonconvex-PŁ/SC zero-sum games. We leave discussion of research on these classes of games with other algorithmic methods to Appendix A. The key distinction between this paper and past work is that instead of assessing convergence in terms of only reaching an approximate stationary point of the dynamics or a surrogate function, we obtain convergence guarantees in regards to differential Stackelberg equilibria (equivalently strict local minmax equilibria). Despite this being a much stricter and meaningful notion of solving the problem, we show that global asymptotic convergence guarantees remain obtainable in nonconvex-PŁ zero-sum games (Theorem 2). Moreover, for the subclass of nonconvex-SC zero-sum games, we provide novel global finite-time convergence guarantees to differential Stackelberg equilibria with rates comparable to existing results for finding a local minimum in nonconvex optimization or any stationary point in this class of games (Theorem 3).

Escaping Saddle Points in Nonconvex Optimization. A key contribution of this work is in regards to escaping saddles of the dynamics in classes of nonconvex zero-sum games. We build on methods from nonconvex optimization to obtain analogous results. In general, saddle avoidance results for variants of gradient descent in nonconvex optimization are asymptotic or finite-time. The former states that almost surely the algorithm does not converge to saddles points [28, 29], while the latter

Table 1: The gradient complexity of gradient descent (GD), gradient descent-ascent with timescale separation (τ-GDA), and alternating (including multi-step) gradient descent-ascent (AGDA) or perturbed variants (PGD, τ-PGDA) in deterministic and stochastic nonconvex optimization, nonconvex-PŁ zero-sum games, and nonconvex-SC zero-sum games. We state the complexity in terms of the ε tolerance of the guarantee and the dimension d with the notation $\widetilde{\mathcal{O}}(\cdot)$ hiding logarithmic factors in ε and d.

Problem	Algorithm & Reference	Complexity		Guarantee
		Deterministic	Stochastic	
Nonconvex Optimization	GD [24]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-4})$	ε–Stationarity
	PGDA [24]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-4})$	ε–Local Min
Nonconvex SC Zero-Sum	τ-GDA, τ-AGDA [33]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-4})$	ε–Stationarity
	Theorem 3 (τ-PGDA)	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-4})$	ε–**DSE**
Nonconvex PŁ Zero-Sum	AGDA [45],[62, Appendix D]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	–	ε–Stationarity
	Theorem 2 (τ-GDA)	Asymptotic	–	**DSE**

gives rates of escape to conclude convergence to approximate local minimum [17, 20, 24]. A key assumption in the aforementioned works is what is known as the strict saddle property, which ensures directions of escape exists from a saddle point. We make an analogous assumption refined for games following past works [40]. Further discussion of how we extend the methods from nonconvex optimization to zero-sum games along with the technical challenges is included in Section 6.

3 Preliminaries

In the zero-sum games we study, we refer to the minimizing player controlling x as player 1 and the maximizing player controlling y as player 2. The set of players is denoted by $\mathcal{I} = \{1, 2\}$. We consider objective functions $f \in C^2(\mathcal{Z}, \mathbb{R})$ where the joint strategy space is denoted by $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ where $\mathcal{X} = \mathbb{R}^{d_1}$ and $\mathcal{Y} = \mathbb{R}^{d_2}$ denote the individual action spaces and $d = d_1 + d_2$. We often denote a joint strategy using the shorthand notation $z = (x, y) \in \mathcal{Z}$.

Notation. We denote ∇f as the total derivative of f, $\nabla_i f$ as the derivative of f with respect to the choice variable of player i, $\nabla_{ij} f$ as the partial derivative of $\nabla_i f$ with respect to the choice variable of player j, and $\nabla_i^2 f$ as the partial derivative of $\nabla_i f$ with respect to the choice variable of player i. We let $\|\cdot\|$ denote the 2-norm of vectors unless otherwise specified, $\mathrm{spec}(\cdot)$ denote the set of eigenvalues of a matrix, $\mathrm{Re}(\cdot)$ denote the real part of a complex number, and $\mathbb{C}_-^\circ$ and $\mathbb{C}_+^\circ$ denote the open left-half and right-half complex plane, respectively. Let $\lambda_{\min}(A)$ denote the eigenvalue of A with the minimum real part, and $\lambda_{\max}(A)$ the eigenvalue of A with the maximum real part. Finally, we indicate A is positive and negative definite using the notation $A \succ 0$ and $A \prec 0$, respectively.

Classes of Games. We study and analyze both nonconvex-PŁ and nonconvex-SC zero-sum games. To begin, we state a standard smoothness assumption that is needed throughout.

Assumption 1. *Given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, $\nabla_1 f(x, y)$ and $\nabla_2 f(x, y)$ are L_1 and L_2 Lipschitz, respectively. That is,*

$$\|\nabla_1 f(x, y) - \nabla_1 f(x', y')\| \leq L_1(\|x - x'\| + \|y - y'\|),$$
$$\|\nabla_2 f(x, y) - \nabla_2 f(x', y')\| \leq L_2(\|x - x'\| + \|y - y'\|).$$

This assumption immediately implies that the vector of individual gradients denoted by

$$g(x, y) := (\nabla_1 f(x, y), -\nabla_2 f(x, y))$$

is also Lipschitz with parameter $L = L_1 + L_2$.

We now define a nonconvex-PŁ zero-sum game [45]. This class of games allows for the objective to be nonconvex in $x \in \mathcal{X}$, but it needs to satisfy the Polyak-Łojasiewicz (PŁ) condition [47] in $y \in \mathcal{Y}$.

Definition 1 (Nonconvex-PŁ Game)**.** *Consider a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$. The game is called nonconvex-PŁ if $f(x, \cdot)$ is μ-PŁ with respect to the argument $y \in \mathcal{Y}$. That is, for $\mu > 0$ and for all $z = (x, y) \in \mathcal{Z}$,*

$$\|\nabla_2 f(x, y)\|^2 \geq 2\mu(\max_{y' \in \mathcal{Y}} f(x, y') - f(x, y)),$$

Algorithm 1 τ-GDA

Input: $x_0 \in \mathbb{R}^{d_1}$, $y_0 \in \mathbb{R}^{d_2}$
for $k = 0, 1, \ldots$ **do**
  $x_{k+1} \leftarrow x_k - \gamma \nabla_1 f(x_k, y_k)$
  $y_{k+1} \leftarrow y_k + \gamma\tau \nabla_2 f(x_k, y_k)$
end for

Algorithm 2 τ-PGDA

Input: $x_0 \in \mathbb{R}^{d_1}$, $y_0 \in \mathbb{R}^{d_2}$
for $k = 0, 1, \ldots$ **do**
  Sample $\theta_{i,k} \sim \mathcal{D}_i, \forall\, i \in \mathcal{I}$; $(\xi_{1,k}, \xi_{2,k}) \sim \mathcal{N}(0, (r^2/d)I)$
  $x_{k+1} \leftarrow x_k - \gamma(g_1(x_k, y_k; \theta_{1,k}) + \xi_{1,k})$,
  $y_{k+1} \leftarrow y_k + \gamma\tau(g_2(x_k, y_k; \theta_{2,k}) + \tau^{-1}\xi_{2,k})$
end for

where for any fixed $x \in \mathcal{X}$, $\max_{y' \in \mathcal{Y}} f(x, y')$ has a nonempty solution set and a finite optimal value.

In nonconvex-PŁ zero-sum games, the solution set of $\operatorname{argmax}_{y' \in \mathcal{Y}} f(x^*, y')$ for any fixed $x^* \in \mathcal{X}$ may not be a singleton and the maximizing player's objective may be nonconcave. However, by Definition 1 it follows immediately that any (x^*, y^*) such that $\|\nabla_2 f(x^*, y^*)\| = 0$ satisfies $f(x^*, y^*) = \max_{y' \in \mathcal{Y}} f(x^*, y')$ so any stationary point with respect to y is a global maximum.

The following provides a definition for what we call a nonconvex-SC zero-sum game. In this class of games, the function that defines the game may be nonconvex in $x \in \mathcal{X}$, but it must be SC in $y \in \mathcal{Y}$.

Definition 2 (Nonconvex-SC Game)**.** *Consider a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$. The game is nonconvex-SC if $f(x, \cdot)$ is μ-strongly-concave with respect to the argument $y \in \mathcal{Y}$. That is, given any $x \in \mathcal{X}$ and for all $y, y' \in \mathcal{Y}$,*

$$f(x, y') \leq f(x, y) + \langle \nabla_2 f(x, y), y' - y \rangle - \frac{\mu}{2}\|y' - y\|_2^2.$$

It is worth noting that nonconvex-SC zero-sum games are nonconvex-PŁ zero-sum games, but the converse does not necessarily hold as a result of the relationship between PŁ and SC functions [25].

Learning Dynamics. We study gradient descent-ascent with timescale separation and a perturbed variant. Let γ be the learning rate of player one and $\tau > 0$ be the timescale parameterization. The deterministic τ-GDA dynamics we study are presented in Algorithm 1 and the potentially stochastic variant with injected noise perturbations called τ-PGDA is given in Algorithm 2 (see Section 6 for the relevant notation). The key distinction between this study of gradient descent-ascent with timescale separation and past work is how we assess convergence as we now begin to formalize.

Stationarity Notions. We call *critical points* strategy profiles at which the individual gradient of each player is equal to zero; that is $(x, y) \in \mathcal{Z}$ such that $\nabla_1 f(x, y) = 0$ and $\nabla_2 f(x, y) = 0$. Critical points correspond to stationary points of the τ-GDA dynamics. The nonconvex-PŁ/SC zero-sum game literature has generally assessed convergence in terms of the complexity of finding an *ε-critical point* (see, e.g., [45, 62]). That is, joint strategies $(x, y) \in \mathcal{Z}$ such that $\|\nabla_1 f(x, y)\| \leq \varepsilon$ and $\|\nabla_2 f(x, y)\| \leq \varepsilon$. A related and common notion of convergence in this body of work (see e.g., [33]) is that of finding an ϵ critical-point of the function $\max_{y \in \mathcal{Y}} f(\cdot, y)$. This criterion amounts to seeking to achieve the condition $\|\nabla_1 \max_{y \in \mathcal{Y}} f(x, y)\| \leq \varepsilon$.

In contrast, we assess convergence with connections to the equilibrium notions that are commonly studied in the nonconvex-nonconcave zero-sum game literature. Since either stationarity notion may lack any game-theoretic meaning, we consider a strictly harder notion of solving a game.

Equilibrium Notions. The typical solution concept in game theory when an implicit or explicit order of play is present in the structure of the game is the (local) Stackelberg (equivalently minmax in zero-sum games) equilibrium concept [6].[2] Informally, in nonconvex-PŁ/SC zero-sum games, a local minmax equilibrium corresponds to a strategy pair $(x^*, y^*) \in \mathcal{Z}$ such that x^* is a local minimum of the function $f(x, y_*(x))$ where $y_*(x) \in \operatorname{argmax}_{y \in \mathcal{Y}} f(x, y)$ is a local maximum of the function $f(x^*, y)$. When the function f is bounded or when $f(\cdot, y)$ is bounded and $f(x, \cdot)$ is strongly concave, a minmax equilibrium is guaranteed to exist.

We characterize the local minmax (Stackelberg) equilibrium notion in terms of sufficient conditions on player costs as is typical in learning in games (see, e.g., [15, 16, 23, 40, 58, 64]). Toward presenting

[2] By implicit order of play, we mean the min and max order are not interchangeable.

this definition, we denote by $J(x, y)$ the Jacobian of the individual gradient vector $g(x, y)$ given by

$$J(x, y) = \begin{bmatrix} \nabla_1^2 f(x, y) & \nabla_{12} f(x, y) \\ -\nabla_{12}^\top f(x, y) & -\nabla_2^2 f(x, y) \end{bmatrix}. \tag{4}$$

Let $\mathtt{S}_1(\cdot)$ denote the Schur complement of $(\cdot)$ with respect to the $d_2 \times d_2$ block in $(\cdot)$. The following definition is characterized by sufficient conditions for a local minmax equilibrium in zero-sum games.

Definition 3 (Differential Stackelberg/Strict Local Minmax Equilibrium [16, 23])**.** *The joint strategy $(x^*, y^*) \in \mathcal{Z}$ is a differential Stackelberg equilibrium when the conditions $\nabla_1 f(x^*, y^*) = 0$, $\nabla_2 f(x^*, y^*) = 0$ and $\mathtt{S}_1(J(x^*, y^*)) \succ 0$, $\nabla_2^2 f(x^*, y^*) \prec 0$ hold.*

We remark that in nonconvex-PŁ/SC zero-sum games, $\mathtt{S}_1(J(x^*, y^*))$ as in Definition 3 is well-defined at any critical point (x^*, y^*) since $\det(\nabla_2^2 f(x^*, y^*)) \neq 0$ (see Lemma 8 in Appendix B.3).

4 Local Stability Analysis

To characterize the convergence of τ-GDA, we begin by studying its continuous-time limiting system

$$\dot{z} = -\Lambda_\tau g(z) \tag{5}$$

where $\dot{z} = (\dot{x}, \dot{y})$, $\Lambda_\tau = \mathrm{blockdiag}(I_{d_1}, \tau I_{d_2})$, and $g(z)$ is the vector of individual gradients. The Jacobian of this system is given by

$$J_\tau(z) = \Lambda_\tau J(z). \tag{6}$$

We analyze the stability of the continuous-time system around critical points $z^* = (x^*, y^*)$ as a function of the timescale separation τ using the Jacobian $J_\tau(z^*)$ in this section toward drawing conclusions about the stability and convergence of the discrete time system τ-GDA. A critical point is said to be locally (exponentially) stable when the spectrum of $-J_\tau(z^*)$ is in the open left-half complex plane $\mathbb{C}_-^\circ$ (cf. Theorem 4, Appendix B.2). Simply put, a critical point z^* is locally exponentially stable if and only if the real parts of the eigenvalues of $-J_\tau(z^*)$ are strictly negative. Throughout, we use the broader term "stable" to mean the following.

Definition 4 (Stability)**.** *A critical point $z^* = (x^*, y^*) \in \mathcal{Z}$ is locally exponentially stable for $\dot{z} = -\Lambda_\tau g(z)$ if and only if $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ ($\equiv \mathrm{spec}(J_\tau(z^*)) \subset \mathbb{C}_+^\circ$).*

Stability with respect to the continuous-time τ-GDA dynamics guarantees that the system asymptotically converges at an exponential rate to the critical point in a local neighborhood. Moreover, given a suitable choice of learning rates, equivalent insights hold for the discrete-time dynamics [8].

Stability in Nonconvex-Nonconcave Zero-Sum Games. Given the implications regarding convergence, a number of papers in the past several years study the stability of τ-GDA around critical points and the connections to differential Nash and Stackelberg equilibrium in zero-sum games [10, 15, 16, 23, 40]. However, this body of research focuses on general nonconvex-nonconcave zero-sum games. In general across the spectrum of nonconvex-nonconcave zero-sum games, the stable critical points of τ-GDA coincide with the set of differential Stackelberg equilibria only when the timescale separation $\tau \to \infty$ [23]. Given that such a choice of timescale separation requires the learning rate $\gamma \to 0$ in order to retain stability of the discrete-time system, it is not clear how to derive a practical algorithm from this insight.

Toward remedying this problem, Fiez and Ratliff [15] provide stability results in terms of the timescale separation concerning a given critical point, rather than across the space of nonconvex-nonconcave zero-sum games. In particular, they prove a stability and instability result as a function of the timescale separation in the τ-GDA dynamics. The stability results say that given a differential Stackelberg equilibrium z^*, there exists a finite $\tau^* \in (0, \infty)$ that can be constructed such that z^* is stable for all $\tau \in (\tau^*, \infty)$. On the other hand, the instability results says that given a critical point which is not a differential Stackelberg equilibrium, there exists a finite $\tau_0 \in (0, \infty)$ that can be constructed such that z^* is not stable for all $\tau \in (\tau_0, \infty)$.

Stability in Nonconvex-PŁ/SC Zero-Sum Games. To our knowledge, the connection between the stability (and instability) of critical points with respect to τ-GDA dynamics and game-theoretic equilibrium notions in the semi-structured problems of nonconvex-PŁ and nonconvex-SC zero-sum

games has not been fully characterized. We show in the following result that when a nonconvex-nonconcave game is specialized to a nonconvex-PŁ zero-sum game as from Definition 1, significantly more general stability characterizations can be obtained by exploiting the fact that $\nabla_2^2 f(x^*, y^*) \prec 0$ at any critical point $z^* = (x^*, y^*)$ (see Lemma 8 in Appendix B.3). Notably, any critical point $z^* = (x^*, y^*)$ that is not a differential Stackelberg equilibrium is not stable ("unstable") for all $\tau > 0$. In other words, τ-GDA does not admit spurious stable points in nonconvex-PŁ zero-sum games. This is in stark contrast to the known fact that 1-GDA admits spurious stable points in the more general class of nonconvex-nonconcave games as has been shown in previous literature [10, 23, 40]. Moreover, if z^* is a differential Stackelberg equilibrium, then z^* is stable for all τ larger than the minimum τ_* for which z^* is stable and such a finite τ_* is guaranteed to exist. This result implies that in practice, one can select a finite value of τ to run τ-GDA with, and all stable critical points (if they exist) will be differential Stackelberg equilibria, and if τ is scaled up and the set of stable points grows, then only differential Stackelberg equilibria can be introduced to the set.

Theorem 1. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ where $f \in C^2(\mathcal{Z}, \mathbb{R})$. Then, the following hold: 1) Any critical point $z^* \in \mathcal{Z}$ that is not a differential Stackelberg equilibrium is unstable with respect to τ-GDA for all $\tau \in (0, \infty)$; 2) If z^* is a differential Stackelberg equilibrium, then $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ for all $\tau \in [\tau_*, \infty)$ where τ_* is the minimum $\tau \in (0, \infty)$ such that $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ and a finite τ_* is guaranteed to exist.*

In comparison to the stability results for nonconvex-nonconcave zero-sum games of Fiez and Ratliff [15], we obtain the stronger results in nonconvex-PŁ zero-sum games that (i) $\tau_0 = 0$ for any critical point that is not a differential Stackelberg equilibrium and (ii) a differential Stackelberg equilibrium is never unstable after it becomes stable as a function of the timescale separation τ.

Remark 1. *The proof of Theorem 1 also reveals novel insights into the local stability of τ-GDA around critical points in nonconvex-nonconcave zero-sum games. In particular, given any critical point $z^* = (x^*, y^*)$ in a nonconvex-nonconcave zero-sum game $(f, -f)$ where $f \in C^2(\mathcal{Z}, \mathbb{R})$ such that $\nabla_2^2 f(x^*, y^*) \prec 0$, if z^* is not a differential Stackelberg equilibrium, then it is unstable with respect to τ-GDA for all $\tau \in (0, \infty)$. This observation follows from the proof of Theorem 1 relying on the fact that $\nabla_2^2 f(x^*, y^*) \prec 0$ holds at any critical point in nonconvex-PŁ/SC zero-sum games.*

5 Global Asymptotic Convergence Analysis

Theorem 1 in the previous section completely characterizes the local behavior of the continuous-time τ-GDA dynamics around critical points. However, ultimately we care about the convergence properties of the discrete-time τ-GDA dynamics. We begin our study of this system by providing a global asymptotic analysis in this section of the deterministic τ-GDA dynamics presented in Algorithm 1.

We prove in Theorem 2 of this section that the deterministic τ-GDA (Algorithm 1) almost surely converges to a differential Stackelberg equilibrium in nonconvex-PŁ zero-sum games, a class of games that subsumes nonconvex-SC zero-sum games. Despite this result being asymptotic in nature, to our knowledge it is the most general class of zero-sum games in which a global convergence guarantee to established game-theoretic equilibria has been given.

To begin, observe that the continuous-time τ-GDA dynamics do not necessarily correspond to a gradient flow of a function, which can be observed from the fact that the Jacobian J_τ is not guaranteed to be symmetric. As a consequence, there may exist complex limiting behavior beyond convergence to a critical point such as non-trivial limit cycles or periodic orbits. To rule out this phenomenon, we prove that there exists a bounded potential function that decreases along the iterates of τ-GDA and only stops decreasing at critical points of τ-GDA. In what follows, $y_*(x)$ denotes an element of $\mathrm{argmax}_{y \in \mathcal{Y}} f(x, y)$ and we define $\kappa := L_1/\mu$. Moreover, as an implication of Assumption 1 and Definition 1, $\nabla \max_{y \in \mathcal{Y}} f(\cdot, y)$ is $L_4 = (1 + \kappa L_2)$ Lipschitz (see Lemma 7 in Appendix B.3).

Lemma 1. *Consider a nonconvex-PŁ zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. Suppose that $\tau > 7\kappa^2$, and $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ then, for any $\Gamma \in (0, 1/7]$, $\Phi(x, y) = f(x, y_*(x)) - \Gamma f(x, y)$ is a potential function for τ-GDA.*

The function $f(x, y_*(x))$ can be seen as the function the x player would minimize if the y player was playing a best-response $y_*(x)$. The potential function $\Phi(x, y)$ essentially captures that along trajectories of τ-GDA, either the value of $f(x, y_*(x))$ should decrease, or the value of $f(x, y_*(x)) - f(x, y)$ should decrease since the y-player converges at a fast rate to $y_*(x)$ given the time-scale

separation. Indeed, this potential function implicitly guarantees that the maximizing player *tracks* the best response set, and that the minimizing player essentially ends up minimizing $f(x, y_*(x))$ as desired. The choice of τ and the learning rate γ allow us to guarantee that this occurs.

Given the potential function in Lemma 1, we can conclude that Algorithm 1 converges to critical points. The critical points may correspond to stable points of the continuous-time τ-GDA dynamics (Definition 4) or saddle points of the continuous-time τ-GDA dynamics, which are defined as follows.

Definition 5 (Saddle Point)**.** *The critical point $z^* = (x^*, y^*) \in \mathcal{Z}$ is a saddle point of the τ-GDA continuous-time dynamics $\dot{z} = -\Lambda_\tau g(z)$ if $\mathrm{Re}(\lambda_{\max}(-J_\tau(z^*))) \geq 0$ and a strict saddle if $\mathrm{Re}(\lambda_{\max}(-J_\tau(z^*))) > 0$ and $\mathrm{Re}(\lambda) \neq 0 \; \forall \; \lambda \in \mathrm{spec}(-J_\tau(z^*))$.*

Recall that Theorem 1 indicates the only stable points of the continuous-time τ-GDA dynamics are differential Stackelberg equilibria. Thus, if we can show that the discrete-time τ-GDA dynamics avoid saddle points of the continuous-time τ-GDA dynamics, then we can conclude that Algorithm 1 converges to only differential Stackelberg equilibria. The following result of Mazumdar et al. [40] states that the discrete-time τ-GDA dynamics avoid strict saddle points of the continuous-time system.[3]

Lemma 2 (Theorem 4.1 [40])**.** *Consider a zero-sum game defined by the nonconvex-nonconcave function $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. The set of initial conditions $z \in \mathcal{Z}$ from which τ-GDA with $\gamma < \min\{\frac{1}{L}, \frac{1}{\tau L}\}$ converges to strict saddle points of $\dot{z} = -\Lambda_\tau g(z)$ is of measure-zero.*

Lemma 2 can be combined with the existence of a potential function (Lemma 1) and the guarantee all stable points of the τ-GDA continuous-time system are strict local minmax (Theorem 1) to arrive at a desirable global convergence guarantee for τ-GDA when given the following assumption.

Assumption 2 (Strict Saddle Property)**.** *Given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, we assume every saddle point of the continuous-time τ-GDA dynamics given by the system $\dot{z} = -\Lambda_\tau g(z)$ from* (5) *is a strict saddle point for the choice of $\tau \in (0, \infty)$.*

Assumption 2 can be seen as an analogue for gradient dynamics in nonconvex zero-sum games to the commonly assumed strict saddle property in nonconvex optimization [17, 20, 24]. A key distinction is that in nonconvex optimization, strict saddles of the function and strict saddles of the gradient dynamics are equivalent, whereas this property does not carry over to the zero-sum game setting and instead the assumption needs to be explicitly made on saddle points of the gradient dynamics. Given this assumption, we are now ready to the state main result of this section.

Theorem 2. *Consider a nonconvex-PŁ zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ that satisfies Assumptions 1–2. Then, τ-GDA with $\tau > 7\kappa^2$, and $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ asymptotically converges to the set of strict local minmax that are stable for $\dot{z} = -\Lambda_\tau g(z)$ almost surely. That is, for almost all initial conditions, τ-GDA will converge to a strict local minmax point.*

If we fix a τ satisfying the assumptions of the above theorem, then if we consider the set of stable critical points for $\dot{z} = -\Lambda_\tau g(z)$, we know that by Theorem 1 this set only contains differential Stackelberg (and hence, strict local minmax) points. It is precisely this set of strict local minmax to which τ-GDA converges almost surely. Note that a stable differential Stackelberg equilibrium is ensured to exist for the choice of τ and γ. Moreover, by Theorem 1, as we increase τ, no new spurious non-minmax points are introduced to the set of stable critical points; only additional strict local minmax points can be added to this set.

To conclude this section, we state a global complexity bound for finding an ε-critical point. We remark that while Theorem 2 implies a global convergence guarantee to a strict local minmax equilibrium, a global convergence rate to the same set does not easily follow. This is due to the fact that obtaining such a result requires the dynamics to escape saddle points efficiently, a topic dealt with in the section that follows. On the other hand, a convergence rate as we now state to an approximate critical point follows rather directly from the potential function given in Lemma 1.

[3] The strict saddle avoidance result of Mazumdar et al. [40, Theorem 4.1] holds more generally for simultaneous gradient descent with heterogeneous learning rates in n-player general-sum nonconvex games. For simplicity, we only state the result in the context of this work. We also remark that Daskalakis and Panageas [10, Theorem 2.2] show an analogous strict saddle avoidance result for τ-GDA in nonconvex-nonconcave zero-sum games, but restricted to $\tau = 1$, hence the reason it is not referenced in Lemma 2.

Corollary 1. *Consider a nonconvex-PŁ zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ that satisfies Assumptions 1–2. Then, τ-GDA from any initialization with $\tau > 7\kappa^2$, $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$, has at least one iterate that is an ε-critical point after $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ iterations.*

Before moving on, we note that τ-GDA locally converges exponentially fast around stable strict local minmax equilibrium in nonconvex-nonconcave zero-sum games [15] and thus this in class of games.

6 Global Finite-Time Convergence Analysis

In this section, we show that in nonconvex-SC zero-sum games, not only does a τ-GDA variant provably converge in an almost sure asymptotic sense to strict local minmax, but there is also finite-time convergence guarantee. The difficulty in showing this stems from possibility that the game dynamics can get stuck in a region around a non-game theoretically meaningful strict saddle. This problem has been observed in the nonconvex single player setting [12] with gradient descent and trivially extends to τ-GDA in the zero-sum game setting. A body of work in nonconvex optimization has developed algorithmic methods that address this issue [9, 14, 17, 20, 24] including by injecting artificial noise to perturb the updates away from strict saddle points. Yet, to our knowledge, such methods have not yet been extended to the zero-sum game setting. Notably, escaping strict saddles is more of nuanced issue in this setting since strict saddles of the dynamics are not always strict saddles of the function defining the game and vice versa. This being said, we now study τ-PGDA (Algorithm 2), which is τ-GDA with noise injected into the updates, and prove in Theorem 3 this update escapes strict saddles efficiently and consequently globally converges to approximate strict local minmax equilibrium efficiently. We show the result for the stochastic case in which players have unbiased stochastic gradients. The deterministic case is immediate by taking the stochasticity to zero.

We now consider the setting that players do not have access to their exact gradients $\nabla_1 f(x, y)$ and $\nabla_2 f(x, y)$. Instead, for any (x, y) a gradient query to an oracle returns $g_1(x, y; \theta_1)$ and $g_2(x, y; \theta_2)$ to the two players respectively, where each θ_i is a random variable drawn from distribution $\mathcal{D}_i$ with $i \in \mathcal{I}$ and the stochastic gradients satisfy the following assumptions.[4]

Assumption 3. *For any $(x, y) \in \mathcal{Z}$ and each $i \in \mathcal{I}$, $\mathbb{E}_{\theta_i \sim \mathcal{D}_i}[g_i(x, y; \theta_i)] = \nabla_i f(x, y)$ and $\forall\, t \in \mathbb{R}$:*

$$\mathbb{P}(\|g_i(x, y; \theta) - \nabla_i f(x, y)\| \geq t) \leq 2\exp(-\tfrac{t^2}{2\sigma_i^2}).$$

Assumption 4. *For each $i \in \mathcal{I}$ and any $\theta_i \in \operatorname{supp}(\mathcal{D}_i)$, $g_i(\cdot, \cdot; \theta_i)$ is $\widetilde{\ell}_i$-Lipschitz: that is, for all $(x_1, y_1), (x_2, y_2) \in \mathcal{Z}$, $\|g_i(x_1, y_1; \theta_i) - g_i(x_2, y_2; \theta_i)\| \leq \widetilde{\ell}_i \|(x_1, y_1) - (x_2, y_2)\|$.*

We also need the following higher-order smoothness assumption in this section.

Assumption 5. *Given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, $\nabla^2 f(\cdot, \cdot)$ is β-Lipschitz with respect to the induced 2-norm. That is, for all $z, z' \in \mathcal{Z}$, $\|\nabla^2 f(z) - \nabla^2 f(z')\| \leq \beta \|z - z'\|$.*

The following assumption is needed to show there is a direction of escape from a strict saddle point.

Assumption 6. *Given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, at any strict saddle $z^* \in \mathcal{Z}$ of the continuous-time τ-GDA dynamics given by the system $\dot{z} = -\Lambda_\tau g(z)$ from (5), the eigenvalue $\lambda_{\min}(J_\tau(z))$ is simple, that is the algebraic and geometric multiplicity are equal to one.*

The stability characterization allows us to define a natural approximate differential Stackelberg equilibrium notion. In particular, the following definition is based on an observation which follows from Theorem 1 that if a critical point is stable with respect to τ-GDA for the choice of $\tau \in (0, \infty)$, then it must be a differential Stackelberg equilibrium.

Definition 6. *A point $z^* \in \mathcal{Z}$ of a nonconvex–SC zero-sum game $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumptions 1–2 and 5 is an ε-differential Stackelberg equilibrium (ε-local minmax) if $\|\nabla_1 f(z^*)\| \leq \varepsilon$ and $\|\nabla_2 f(z^*)\| \leq \varepsilon$. and $\operatorname{Re}(\lambda_{\min}(J_\tau(z^*))) \geq -\sqrt{\varepsilon\beta}$.*

Let $\ell := L_1 + 3\kappa' L_2$ where $\kappa' := \frac{L_2}{\mu}$ and observe that $\lambda_{\max}(\nabla^2 f(x, y)) \leq \ell$. In the remainder, we work in the case for which $\ell \geq \sqrt{\beta\varepsilon}$; otherwise the problem of finding ε-local minmax is straightforward since all ε-stationary points will be ε-local minmax points.

[4] Note that we are overloading notation by using $g_1(x, y; \theta_1)$ and $g_2(x, y; \theta_2)$ to denote stochastic individual gradients for each player, whereas previously $g(x, y)$ has denoted the vector of each players gradients.

A key step toward developing the main result of this section is constructing a potential function. In the existing literature on convergence in the nonconvex-SC literature [35, 59, 62], the proposed potential function has a structure closely related to the potential function in Lemma 1; in particular, it takes the form $f(x, y) + r(x, y)$ where $r(x, y)$ is a tracking term of the form $\|y - y_*(x)\|^2$ or $f(x, y_*(x))$. Yet, perhaps surprisingly, despite the fact that f depends on the sequences generated by two separate gradient updates—one from minimizing f and one from maximizing it—the descent lemma below (Lemma 3) shows that the function f can be decomposed into a component that is decreasing along trajectories of τ-PGDA (Algorithm 2) and a component that exhibits a possible increase due to randomness in the stochastic gradients and injected noise. The primary reason for the decrease is large gradients in addition to the structure of the maximizing player's problem and the timescale separation which we exploit in the proof to ensure sufficient decrease of f along trajectories.

Lemma 3 (Descent Lemma)**.** *Consider a nonconvex-SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumptions 1–5. If $\frac{4}{\mu\tau} \leq \gamma \leq 1/\ell$, then with probability at least $1 - \delta$ for any $\delta \in (0, 1)$, $f(x_k, y_k)$ is a potential function for τ-PGDA in the stochastic and deterministic settings.*

The descent lemma enables us to argue that with high probability, either the function value decreases a sufficient amount or the iterates remain in a small region. Using this property of τ-PGDA together with the descent lemma, we then show that if two sequences that are *coupled* (i.e., two sequences starting from the same initial condition and having equivalent noise except along the direction of escape) and are in the region around a saddle point, then with high probability, at least one of them escapes. The steps in showing that τ-PGDA (Algorithm 2) escapes saddles with high probability largely follow those in Jin et al. [24], with several modifications due to the fact that we consider a zero-sum game as opposed to a single player optimization problem. The challenge in the zero-sum game setting as compared to the single player optimization setting (beyond analyzing two sequences of iterates) is the eigenvalues of the local linearization from (6) may be complex-valued. In the key step of many of the proofs on saddle avoidance, escaping from the *stuck regions* around a saddle (with high probability) is achieved by analyzing the local linearization around the saddle point, which crucially, is symmetric. Hence, the problem reduces to analyzing the escape time along the eigendirection of the minimum eigenvalue of the Hessian which relies heavily on the orthogonality of eigenvectors of symmetric matrices. In zero-sum games, the linearization is no longer symmetric and these key results have to modified. We highlight these distinctions more clearly in Appendix E.

Putting these results together, we have the following finite time guarantee on convergence to ε-differential Stackelberg equilibria (ε–local minmax points). Let $f^* = \min_{x,y} f(x, y)$ be the minimum value of the function f, which we assume is finite. These results provide novel convergence guarantees to not just ε-stationary points but those that are game theoretically meaningful.

Theorem 3. *Consider a nonconvex-SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumptions 1–6. For any $\varepsilon > 0$ and $\delta \in (0, 1)$, there exists γ and τ such that with probability $1 - \delta$, starting from any $z \in \mathcal{Z}$, at least half the τ-PGDA iterates will be ε-differential Stackelberg equilibria after $\widetilde{\mathcal{O}}(\varepsilon^{-4})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ iterations in the stochastic and deterministic settings, respectively.*

7 Discussion

To the best of our knowledge, our results are the first to guarantee global convergence of gradient-based algorithms to game theoretically meaningful equilibria in such a general class of games. We believe that a more detailed analysis could make use of the potential function defined for nonconvex-PŁ in Section 5 to prove similar finite-time results for τ-PGDA, though the proof appears to be tedious and hence, we leave this for future work. We also believe future work can be done to obtain global guarantees in this class of games to game-theoretically meaningful equilibria without gradient information and also in the setting with constraints extending recent saddle avoidance results for the single player nonsmooth, constrained setting.

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   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
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   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See the stated assumptions in the paper and the appendix.
   (b) Did you include complete proofs of all theoretical results? [Yes] See the appendix.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]