GENERALIZATION BOUNDS FOR FEDERATED LEARNING: FAST RATES, UNPARTICIPATING CLIENTS AND UNBOUNDED LOSSES

Anonymous authors
Paper under double-blind review

ABSTRACT

In federated learning, the underlying data distributions may be different across clients. This paper provides a theoretical analysis of generalization error of federated learning, which captures both heterogeneity and relatedness of the distributions. In particular, we assume that the heterogeneous distributions are sampled from a meta-distribution. In this two-level distribution framework, we characterize the generalization error not only for clients participating in the training but also for unparticipating clients. We first show that the generalization error for unparticipating clients can be bounded by participating generalization error and participating gap caused by clients' sampling. We further establish fast learning bounds of order $\mathcal{O}(\frac{1}{mn} + \frac{1}{m})$ for unparticipating clients, where m is the number of clients and n is the sample size at each client. To our knowledge, the obtained fast bounds are state-of-the-art in the two-level distribution framework. Moreover, previous theoretical results mostly require the loss function to be bounded. We derive convergence bounds of order $\mathcal{O}(\frac{1}{\sqrt{mn}} + \frac{1}{\sqrt{m}})$ under unbounded assumptions, including sub-exponential and sub-Weibull losses.

1 INTRODUCTION

In federated learning, a common model is trained based on the collaboration of the participating clients holding local data samples (McMahan et al., 2017). Typically, the underlying distributions vary across clients since the data-generating processes are affected by the local environment. Federated learning is heterogeneous in the scenario where local distributions are different (Wang et al., 2021). Most existing experimental and theoretical results focus on the convergence of optimization on training datasets (Li et al., 2020b; Karimireddy et al., 2020; Mitra et al., 2021; Mishchenko et al., 2022; Yun et al., 2022). The generalization error, which is more natural and important in machine leanring, seems not to have been carefully examined in heterogeneous federated learning.

As a key performance indicator of the machine learning model, generalization error measures the performance of a trained model by its population risk with the corresponding distribution. However, existing generalization results are generally derived for clients participating in the training, which only captures the performance of the learned model on seen distributions during training (Mohri et al., 2019; Chen et al., 2021; Masiha et al., 2021).

In practice, the probability that a client participates in the federated training is affected by many factors such as the reliability of network connections or the availability of the client. The realistic participation ratio may be slow and a variety of clients never have a chance to participate during the training process (Kairouz et al., 2021; Li et al., 2020a; Yuan et al., 2021). Though the training process is operated only on participating clients, the trained model will be used by both unparticipating and participating clients. Since the data distributions of unparticipating clients are different from that of participating clients, it is natural and emergent to ask the following question:

Would the unparticipating clients benefit from the model trained by participating clients?

To answer this question theoretically, we take the participation gap into account in the analysis of generalization error, which is generally ignored by existing works.

In addition to the ignored participating gap, existing theoretical results on the generalization error of heterogeneous federated learning have two more limitations to our knowledge. First, all previous learning rates in probability form are of the order $\mathcal{O}(\frac{1}{\sqrt{mn}})$, where m is the number of clients and n is the sample size at each client (Mohri et al., 2019). We note that faster rates of order $\mathcal{O}(\frac{1}{mn})$ are derived in (Chen et al., 2021). However, their learning rates are in expectation form. Faster learning rates in probability form haven't been derived even only for participating clients. The guarantees in-expectation form reflect the average performance of the model trained based on the randomly sampled datasets. The theoretical bounds in probability form, which we focus on in this paper, reflect the performance of a single sampling on datasets (Klochkov & Zhivotovskiy, 2021; Kanade et al., 2022; Sefidgaran et al., 2022a). Second, most previous generalization bounds are derived by assuming that the loss function is bounded. However, there are a variety of learning problems that do not satisfy this assumption. This includes regression problems where unbounded noise is added to labels (Kuchibhotla & Patra, 2022; Kuchibhotla & Chakrabortty, 2018; Zhang & Zhou, 2018), clustering tasks with heavy-tailed distribution (Paul et al., 2021; Vellal et al., 2022), domain adaptation, and so on. Notable exception works in this direction include (Barnes et al., 2022) and (Sefidgaran et al., 2022b). However, their results are established under the assumption that local clients are homogeneous, which is highly restrictive in the general federated scenario.

In this paper, we assume that data distributions of participating and unparticipating clients are drawn from a meta-distribution P. We argue that this assumption is reasonable in practice. For instance, in cross-device federated learning, the number of total clients is generally large and it is natural to assume that there exists a meta-distribution (Reisizadeh et al., 2020; Wang et al., 2021). In this learning scenario, we assume that the total number of clients is M. Among all these M clients, only m clients have a chance to participate in the training phase, which means that the training process only involves the m distributions $\{D_i\}_{i=1}^m$. Note that the total number M and the number of unparticipating clients/distributions is generally larger than m (Hu et al., 2022; Xu & Wang, 2020; Yang et al., 2020). Practically, the model is trained based on datasets $\{S_i\}_{i=1}^m$, where S_i is the dataset located in client i and is sampled from D_i. This two-level framework not only captures the heterogeneity of clients' distributions but also reflects the relatedness of the distributions. Thanks to this framework, we are allowed to characterize the generalization performance of both participating distributions and unparticipating distributions. A similar framework has been used by several recent literature (Yuan et al., 2021; Reisizadeh et al., 2020; Wang et al., 2021). However, these works mainly focus on the optimization performance or only involve experimental results on the generalization. The objective of this work is to provide theoretical results on generalization error in this framework. Our contributions are summarized as follows.

- We provide a systematic analysis of the generalization error of federated learning in the two-level framework, which captures the missed participating gap in the existing works. This two-level framework captures both heterogeneity and relatedness of clients' distributions. Moreover, all learning bounds presented in this paper are in probability form instead of expectation form.

- We derive fast learning rates in the empirical risk minimization setting. The unparticipating error is bounded by two terms. One is participating error. The other is the participation gap results from missing clients in the training. Our participating bounds and unparticipating bounds are of order $\mathcal{O}(\frac{1}{mn})$ and $\mathcal{O}(\frac{1}{mn}+\frac{1}{m})$, respectively.

- We study the learning bounds for unbounded loss functions, including sub-gaussian, sub-exponential, and heavy-tailed losses. Small-ball methods and concentration inequalities for unbounded random variables are used in the unbounded setting. Our bounds are comparable with the existing results with bounded assumptions.

The rest of the paper is organized as follows. In Section 2, we describe the two-level distribution framework and provide basic theoretical results in this framework. In Section 3, we derive fast generalization bounds. In Section 4, we go beyond the bounded assumption and provide the generalization bounds for unbounded losses such as sub-exponential and sub-Weibull losses. In Section 5, we discuss related work in the directly of generalization analysis of heterogeneous federated learning. Finally, we conclude this paper in Section 6. All proofs are postponed to the appendix.

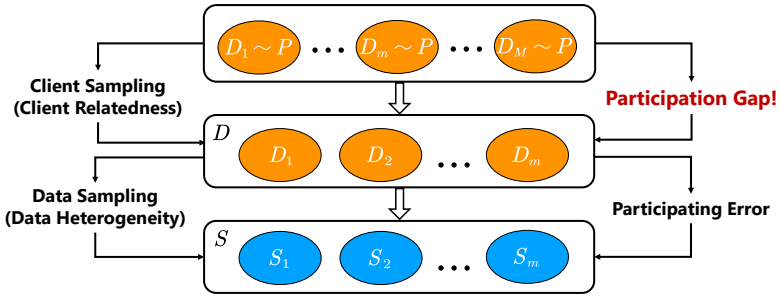

Figure 1: Illustration of the participation gap and participation error.

2 Two-Level Distribution Framework

Let $\mathcal{X}$ denote the input space and $\mathcal{Y}$ the output space. For simplicity, we denote $Z = (X, Y)$ the random variable with support $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$. Let $\mathcal{D}$ denote the set of all probability distributions on $\mathcal{Z}$ and P is a meta-distribution on $\mathcal{D}$. The assumption of meta-distribution is reasonable especially in cross-device federated learning scenario, where the local devices may be a large population of mobile phones. As shown in Figure 1, in this two-level distribution framework, we assume the total number of clients is M (may be infinity) and the number of clients participating in training is m. It is worth emphasizing that M is generally larger than m owing to unreliable network connections. We denote by D_i the distribution associated to client i and assume $\{D_1, \cdots, D_m\}$ are independently sampled from $\mathcal{D}$ according to P. Data sample $S_i = \{Z_i^j\}_{j=1}^n$ located on participating client i is made of n i.i.d realizations of Z following D_i. The global model is trained based on $\{S_i\}_{i=1}^m$ and will be used by all M clients. Two-level distribution framework allows us to measure the performance of the global model with respect to clients' distribution P, which quantifies both the participation gap (caused by client sampling) and participating error (caused by data sampling from participating distributions).

Let the hypothesis space $\mathcal{H}$ be a family of real-valued functions defined on $\mathcal{X}$. The loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ is a non-negative function. We denote the population risk $\mathcal{L}_P(h)$ by $\mathcal{L}_P(h) = \mathbb{E}_{D_i \sim P}\left[\mathbb{E}_{Z \sim D_i}\left[\ell(h(X), Y)\right]\right]$, where $h \in \mathcal{H}$ represents the global hypothesis shared by all local clients. The population risk minimizer h^* associated to population risk $\mathcal{L}_P(h)$ is define as $h^* = \arg\min_{h \in \mathcal{H}} \mathcal{L}_P(h)$. However, it is impossible to minimize $\mathcal{L}_P(h)$ directly because the exact meta distribution and client local distributions are unknown to us. We have access to only a finite number of clients and finite training data at each client. The global objective function defined as population risk is often optimized by the form of empirical risk minimization (ERM) objective function defined as: $\mathcal{L}_S(h) = \frac{1}{m}\sum_{i=1}^m \frac{1}{n}\sum_{j=1}^n \ell(h(X_i^j), Y_i^j)$, where (X_i^j, Y_i^j) represents the j-th training data point at i-th participating client. For simplicity, we denote $Z_i^j = (X_i^j, Y_i^j)$ the data point. Let $S_i = \{Z_i^j\}_{j=1}^n$ denotes the local training set at i-th participating client and $S = S_i \bigcup \cdots \bigcup S_m$ represent the global training set across all participating clients. The empirical risk minimizer $\widehat{h}$ condition on dataset S is define as $\widehat{h} = \arg\min_{h \in \mathcal{H}} \mathcal{L}_S(h)$. To analyze the generalization in our two-level framework, we further define semi-empirical distribution D and the corresponding semi-empirical risk $\mathcal{L}_D(h)$ by $D = \frac{1}{m}\sum_{i=1}^m D_i$ and $\mathcal{L}_D(h) = \frac{1}{m}\sum_{i=1}^m \mathbb{E}_{Z \sim D_i}\left[\ell(h, z)\right]$. We extend the previous definitions and denote by $\widehat{h}^*$ the semi-empirical risk minimizer $\widehat{h}^* = \arg\min_{h \in \mathcal{H}} \mathcal{L}_D(h)$.

The semi-excess risk for participating clients is defined as: $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)$. Semi-excess risk indicates the performance of the learned model $\widehat{h}$ on the unseen data associated with semi-empirical distribution D. The excess risk for unparticipating clients is defined as: $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$. Excess risk indicates the performance of the learned model $\widehat{h}$ on the unseen clients distributed according to P. Note that the the excess risk $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$ is defined across two-level distribution framework. It will be shown that, in our analysis, all upper bound of excess risk $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$ involves semi-excess risk $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)$ or its upper bound. To understand this framework better, we present our basic results of excess risk as follows:

Theorem 1 (Generalization error for unparticipating clients)**.** *For the* VC *major class*[1] $\mathcal{F}$ *with* VC *dimension* d*. If the loss function* ℓ *is bounded by* b*, it follows that with probability at least* $1-2\delta$*,*

$$\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*) \leq c_1 b\sqrt{\frac{d}{mn}} + b\sqrt{\frac{\ln(1/\delta)}{2mn}} + c_2 b\sqrt{\frac{d}{m}} + b\sqrt{\frac{\ln(1/\delta)}{2m}},$$

where c_1 *and* c_2 *are constants.*

Remark 1. *Assume the total number of clients is* M *and* P *is a concrete meta-distribution on* M *different clients' distributions. The global model* $\widehat{h}$ *is trained based on* m *participating clients. The excess risk measures the average performance of* $\widehat{h}$ *on total* M *clients, which include participating and unparticipating clients. Theorem 1 indicates that, increasing the number of participating clients* m *leads to the decrease of excess risk* $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$*. In cross-device federated learning, the number of participating clients* m *may be large enough such that the excess risk approaches zero. Based on these discussions, we can give a positive answer to the question asked in Introduction.* ***This is, from the perspective of average performance, unparticipating clients would benefit from the model trained by participating clients****.*

Remark 2. *Theorem 1 is established by combining the upper bounds of participating error and participation gap. In particular, the terms of order* $\mathcal{O}(\frac{1}{\sqrt{mn}})$ *correspond to participating generalization error. The remain terms of order* $\mathcal{O}(\frac{1}{\sqrt{m}})$ *correspond to* ***participation gap****. We point out that, in addition to* VC *major class (van der Vaart & Wellner, 1996), our proof framework can be applied to many interesting function classes with uniformly entropy number (Lei et al., 2016; Mendelson, 2003).*

3 Fast Learning Rates in Two-level Distribution Framework

In this section, we present fast learning rates in our two-level distribution framework. Recall that $\widehat{h}$ is the empirical risk minimizer and h^* is the population risk minimizer. Our goal is to bound the semi-excess risk for participating clients $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)$ and excess risk for unparticipating clients $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$. To get faster learning rates in our two-level distribution framework, we start by making some assumptions on loss function ℓ, hypothesis space $\mathcal{H}$, semi-empirical distribution D, and meta distribution P.

Assumption 1. *Loss function* ℓ *is L-Lipschitz in its first argument:* $|\ell(y_1, y) - \ell(y_2, y)| \leq L\,|y_1 - y_2|$.

Definition 1 (Bernstein condition)**.** *Let* μ *be a distribution supported on* $\mathcal{X} \times \mathcal{Y}$ *and let* ℓ *be a loss function with domain* $\mathcal{Y} \times \mathcal{Y}$*. The tuple* $(\mu, \ell, \mathcal{H}, h^*)$ *satisfies the* (β, B)*-Bernstein condition with parameter* $B > 0$ *if the following holds for any* $h \in \mathcal{H}$ *:*

$$\mathbb{E}\left(h(X) - h^*(X)\right)^2 \leq B\mathbb{E}\left[\ell(h(X), Y) - \ell(h^*(X), Y)\right]^\beta .$$

Assumption 2. *Theoretical analyses in our two-level distribution framework involve different types of Bernstein conditons:*

(a) The tuple $(D, \ell, \mathcal{H}, \widehat{h}^*)$ *satisfies the Bernstein condition with parameter* $B' \geq 1, 0 < \beta' \leq 1$*. That is, for any* $h \in \mathcal{H}$*,* $\frac{1}{m}\sum_{i=1}^m \mathbb{E}[h(X_i^1) - \widehat{h}^*(X_i^1)]^2 \leq B'(\mathcal{L}_D(h) - \mathcal{L}_D(\widehat{h}^*))^{\beta'}$*.*

(b) The tuple $(P, \ell, \mathcal{H}, h^*)$ *satisfies the Bernstein condition with parameter* $B'' \geq 1, 0 < \beta'' \leq 1$*. That is, for any* $h \in \mathcal{H}$*,* $\mathbb{E}_{D_i \sim P}[\mathbb{E}_{X \sim D_i}[h(X) - h^*(X)]^2] \leq B''(\mathcal{L}_P(h) - \mathcal{L}_P(h^*))^{\beta''}$*.*

It is well known that fast learning rates require extra assumptions. Bernstein condition is widely used to get fast learning rates in the learning theory community (Xu & Zeevi, 2020; van Erven et al., 2015; Wu et al., 2022). We emphasize that it is not too restrictive. For example, it is directly implied by the boundedness property of functions with any probability distribution (Bartlett et al., 2004). Moreover, regression problems with strictly convex loss function satisfy the Bernstein condition if the function class is convex (Lecué & Mendelson, 2013). Other examples include excess risk functions with minimizer of population risk when the loss function is strongly convex and Lipschitz (Klochkov &

[1]The definition of VC major class is provided in appendix A.

Zhivotovskiy, 2021). For our purposes, we need to check that both (a) and (b) in Assumption 2 hold. A typical example satisfying Assumption 2 is quadratic loss with convex function class $\mathcal{H}$. We provide some examples satisfying Assumption 2 in appendix. For more details, we refer to (Xu & Zeevi, 2020; Wu et al., 2022; van Erven et al., 2015).

Assumption 3 (Uniformly entropy number[2]). *Let $\mathcal{H}$ be a family of bounded functions with uniformly entropy number $\log \mathcal{N}(\epsilon, \mathcal{H}, \|\cdot\|_2)$. Assume that there exist positive numbers γ, d and p such that $\log \mathcal{N}(\epsilon, \mathcal{H}, \|\cdot\|_2) \leq d \log^p(\gamma/\epsilon)$ for any $0 < \epsilon \leq \gamma$.*

Assumption 3 is a mild assumption if the function classes are bounded. We list some popular function classes satisfying Assumption 3: (a) If the VC-dimension of $\mathcal{H}$ is finite, then $\mathcal{H}$ satisfies assumption 3. For instance, the function class of k-means methods has finite VC dimension. For more details, we refer the reader to (Devroye et al., 2013). (b) When we set $\epsilon \in (0, 1)$, then all the unit Euclidean ball $\mathcal{B} \subset \mathbb{R}^d$ satisfy assumption 3. (c) If $\mathcal{H}$ is a RKHS with kernel k and the rank of k is d, then $\mathcal{H}$ satisfies assumption 3.

3.1 Fast Learning rates for participating clients

In this subsection, we provide fast learning rates for participating clients in high probability. To obtain faster convergence rates, we focus on semi-excess risk. Let $\widehat{\mathcal{F}}^* := \{f : (X, Y) \mapsto \ell(h(X), Y) - \ell(\widehat{h}^*(X), Y), h \in \mathcal{H}\}$, where $\widehat{h}^*$ is the semi-empirical risk minimizer. We denote $\mathcal{R}_{mn}(\widehat{\mathcal{F}}^*, r)$ by the local empirical Rademacher complexity associated with semi-empirical distribution D: $\mathcal{R}_{mn}(\widehat{\mathcal{F}}^*, r) = \mathbb{E}_{S,\sigma}[\frac{1}{mn}\sup_{V(f)\leq r} \sum_{i=1}^m \sum_{j=1}^n \sigma_i^j f(Z_i^j)]$, where $V(f) = \frac{1}{m}\sum_{i=1}^m \mathbb{E}_{Z\sim D_i}\left[f(Z)\right]^2$.

Theorem 2 (Semi-excess risk for participating clients). *Let $\mathcal{F}$ be a family of functions bounded by b. Under assumptions 1, 3 and (a) of Assumption 2, when $mn \geq cd\log^p(mn)$, it follows that with probability at least $1 - \delta$,*

$$\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*) \leq c_1 \left(\frac{\log^p(mn)}{mn}\right)^{\frac{1}{2-\beta'}} + c_2 \left(\frac{\log(1/\delta)}{mn}\right)^{\frac{1}{2-\beta'}},$$

where c_1 and c_2 are constants depending on γ, p, L, β' and B_1, b, β' respectively.

Remark 3. *Theorem 2 shows that the convergnece rate of semi-empirical excess risk ranges from $\mathcal{O}(\frac{1}{\sqrt{mn}})$ to faster order $\mathcal{O}(\frac{1}{mn})$, which corresponds to $\beta' = 0$ and $\beta' = 1$, respectively. It indicates that, under Bernstein condition, the semi-empirical excess risk convergences faster when we increase number of clients m or the size n of local dataset. We emphasize that our bounds in Theorem 2 is in high probability form, which is more emergent and challenging, when compared to the previous results in expectation form (Chen et al., 2021; Fallah et al., 2021). The learning bounds in Theorem 2 are conducted for the empirical risk minimizer $\widehat{h}$. For the inexact minimizer of $\widehat{h}$, the proof technique and the final bounds only involve an extra optimization term. For more details about the approximation error, we refer to (Wang et al., 2021; Su et al., 2021; Khaled et al., 2019).*

3.2 Fast Learning rates for unparticipating clients

In this subsection we provide fast learning rates for **unparticipating clients** in high probability. To the best of our knowledge, this is the first result derived for unparticipating clients in heterogeneous federated learning. To develope theoretical guarantees in our two-level framework, we extend the previous local Rademacher complexity as follows. Let $\mathcal{F}^* := \{f : (X, Y) \mapsto \ell(h(X), Y) - \ell(h^*(X), Y), h \in \mathcal{H}\}$, where h^* is the population risk minimizer. We denote $\mathcal{R}_m(\mathcal{F}^*, r)$ by the local Rademacher complexity associated with meta-distribution P: $\mathcal{R}_m(\mathcal{F}^*, r) = \mathbb{E}\left[\mathbb{E}_{\boldsymbol{\sigma}}[\sup_{T(f)\leq r} \frac{1}{m}\sum_{i=1}^m \sigma_i f(X_i, Y_i)]\right]$, where $T(f) = \mathbb{E}_{D_i\sim P}\mathbb{E}_{Z_i\sim D_i}\left[f(X_i)\right]^2$.

Theorem 3 (Excess risk for unparticipating clients). *Let $\mathcal{F}$ be a family of functions taking values in $[0, b]$. Let $B_2 := \max(B''L^2, 1)$. Let $\varphi_m(r)$ be a sub-root function with fixed point r_m^* and assume $\varphi_m(r)$ satisfies, for any $r \geq r_m^*$, $\varphi_m(r) \geq B_2\mathcal{R}_m(\mathcal{F}^*, r)$. Then, under assumptions 1 and (b) of*

[2]The definition of uniformly entropy number is provided in appendix B.

Assumption 2, for any $\delta > 0, \alpha > 0$, with probability at least $1-\delta$, we have

$$\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*) \leq c_0(\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)) + c_1 \max((r_m^*)^{\frac{1}{2-\beta''}}, (r_m^*)^{\frac{1}{\beta''}}) + \left(\frac{c_2 \log(\frac{1}{\delta})}{m}\right)^{\frac{1}{2-\beta''}} + \frac{c_3 \log(\frac{1}{\delta})}{m},$$

where $c_0 = \frac{K}{K-\beta''}, c_1 = (2K)^{\frac{\beta''}{2-\beta''}} (10(1+\alpha))^{\frac{2}{2-\beta''}}, c_2 = 2^{\beta''+1} B_2^2 K^{\beta''}, c_3 = \frac{2b(1/3+1/\alpha)}{(2-\beta'')}$.

Remark 4. *Theorem 3 is developed across the two-level distribution framework, which brings extra challenges to the analysis. It is shown that the upper bound derived in 3 include semi-empirical excess risk term $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)$, which is an outcome of excess risk decomposition across two-level framework. The second term in the upper bound involves the fixed point r_m^* of sub-root function. As pointed by (Mendelson, 2002), it is non-trivial to construct a computational sub-root function. In the following, we tackle this challenge using the tool of uniformly entropy number.*

Theorem 4. *Let $\mathcal{F}$ be a family of functions bounded by b. Under assumptions 1, 3 and (b) of Assumption 2, when $m \geq cd\log^p(m)$, for any $\delta > 0$, it follows that with probability at least $1-\delta$,*

$$\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*) \leq c_0 \left(\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)\right) + c_1 \left(\frac{\log^p m}{m}\right)^{\frac{1}{2-\beta''}} + c_2 \left(\frac{\log(1/\delta)}{m}\right)^{\frac{1}{2-\beta''}},$$

where $c_0 = \frac{K}{K-\beta''}$, c_1 and c_2 are constants depending on γ, p, L, β'' and B_2, b, β'' respectively.

Remark 5. *The first term in Theorem 4 is semi-empirical excess risk term $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*)$, whose upper bound has been derived in Theorem 2. Recall that β' and β'' are constants defined in Assumption 2. In the cases where $\beta' = 1$ and $\beta'' = 1$, it can be shown that excess risk is of order $\mathcal{O}(\frac{1}{mn} + \frac{1}{m})$ with high probability. To present Theorem 4, we must construct a sub-root function that links the expected local Rademacher complexity associated with meta-distribution P and uniformly entropy number. However, the conventional Dudley's integral bounds are build under empirical constraints (Boucheron et al., 2013). We tackle this challenge by extending the techniques developed in (Lei et al., 2016) to our two-level distribution framework.*

4 Learning Rates for Sub-Weibull Losses

In this section, we provide generalization error bounds for unbounded losses in two-level distribution framework. In particular, we focus on loss functions satisfying sub-Weibull condition.

Definition 2 (Sub-Weibull random variables)**.** *A random variable X is said to be sub-Weibull if there is constant $\|X\|_{\psi_\alpha} < \infty$, such that $\mathbb{P}(|X| \geq t) \leq 2\exp(-t^\alpha/\|X\|_{\psi_\alpha}^\alpha)$, for all $t \geq 0$. Sub-Gaussian and sub-exponential random variables are two special cases of Sub-Weibull random variables, which corresponds to $\alpha = 2$ and $\alpha = 1$, respectively.*

The learning rates derived in two-level framework for sub-exponential losses are deferred to the appendices. In the following we use small-ball method to establish learning rates for more heavy-tailed losses, where two-side concentration inequalities may fail to hold.

This subsection aims at establishing generalization bounds for unbounded losses that have heavier tails than sub-exponential distribution. Since the two side inequalities for empirical process fail to hold when the losses are heavy-tailed, the analysis of heavy-tailed losses require new method to relate empirical risk and population risk. In this subsection, we establish generalization bounds for heterogeneous federated learning by extending the small-ball method from i.i.d setting to our two-level distribution framework. We consider the quadratic loss function in this section. The extension to general losses can be achived by using the techniques presented in (Mendelson, 2018).

In what follows, we denote by $\|h\|_{L_2(\mu)}$ for Banach spaces $L_2(\mathcal{X}, \mu)$. Recall that D is the semi-empirical distribution and P is meta-distribution. In particular, we have $\|h\|_{L_2(D)} = (\frac{1}{m}\sum_{i=1}^m \mathbb{E}_{X\sim D_i}[h(X)]^2)^{1/2}$ and $\|h\|_{L_2(P)} = (\mathbb{E}_{D_i\sim P}\mathbb{E}_{X\sim D_i}[h(X)]^2)^{1/2}$. For the sake of clear exposition, we first introduce the small-ball condition.

Assumption 4 (Small-ball condition)**.** *Let $\mathcal{H} \subset L_2(D)$ be a closed and convex class of functions and $\mathcal{H} - \mathcal{H} := \{h - h' : h, h' \in \mathcal{H}\}$.*

(a) There is a $\tau \geq 0$ for which $Q_{\mathcal{H}-\mathcal{H}}(\tau) > 0$, where $Q_{\mathcal{H}}(\tau) = \inf_{h\in\mathcal{H}} \mathbb{P}(|h| \geq \tau\|h\|_{L_2(D)})$.

(b) Let $Q_{\mathcal{H}}(\tau, P) = \inf_{h\in\mathcal{H}} \mathbb{P}\left(|\mathbb{E}_{X\sim D_i}[h(X)]| \geq \tau\|h\|_{L_2(P)}\right)$, where D_i represent local distribution at i-th participating client. There is a $\tau \geq 0$ for which $Q_{\mathcal{H}-\mathcal{H}}(\tau, P) > 0$.

Assumption 4, small-ball condition, has been assumed for i.i.d and dependent data-generating process. To obtain high-probability theoretical guarantees, concentration techniques are widely used in the analysis of generalization error (Boucheron et al., 2013). Intuitively, empirical risk will concentrate around population risk with high probability only when the loss function has well-behaved moments. However, this condition may fail to hold for heavy-tailed losses (Mendelson, 2015). Assumption 4 appears first in the work of (Mendelson, 2015). Losses with any sort of moment equivalence satisfy small-ball condition, which is weaker than concentration condition and can be used to model heavy-tailedness. For example, even weak condition $\|h\|_{L_2(P)} \leq c\|h\|_{L_1(P)}$ yields nontrivial small-ball estimate. Moreover, the equivalence between higher-order moments and second-order moment such as $\|h\|_{L_p(P)} \leq c\|h\|_{L_2(P)}$ also leads to small-ball condition (Lecué & Mendelson, 2016). Based on these observations, condition (b) of Assumption 4 is generally implied when we consider each local distribution D_i as a random variable according to client distribution P. Let us discuss condition (a) of Assumption 4 in more detail. Note that the establishment of this assumption 4 only requires $\mathcal{H} \subset L_2(D)$ with high probability, where D is the semi-empirical distribution. This requirement is not too restritive since the elements of D are i.i.d sampled from P. To our knowledge, this is the first time that small-ball condition is used under heterogeneous data generating assumption.

4.1 Learning rates for participating clients with small-ball condition

We first describe the basic idea of generalization analysis for participating clients. Recall that $\widehat{h}^*$ is the minimizer of semi-empirical risk $\mathcal{L}_D(h)$ in $\mathcal{H}$. In this subsection we focus on the measure $\|h - \widehat{h}^*\|^2_{L_2(D)}$, which represents the distance between h and $\widehat{h}^*$ with respect to semi-empirical distribution D. For quadratic loss and every $h \in \mathcal{H}$, we have

$$\mathcal{L}_S(h) - \mathcal{L}_S(\widehat{h}^*) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(h(X_i^j) - Y_i^j)^2 - (\widehat{h}^*(X_i^j) - Y_i^j)^2\right] \tag{1}$$

$$= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(h - \widehat{h}^*)^2(X_i^j) + \frac{2}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j), \tag{2}$$

where $\xi_i^j = \widehat{h}^*(X_i^j) - Y_i^j$. Since $\widehat{h}$ is the minimizer of empirical risk $\mathcal{L}_S(h)$, we have $\mathcal{L}_S(\widehat{h}) - \mathcal{L}_S(\widehat{h}^*) \leq 0$. If on an event $\|h - \widehat{h}^*\|_{L_2(D)}$ is large, then the summation of two terms in (2) is larger than 0 with high probability. It follows that with high probability $\|\widehat{h} - \widehat{h}^*\|_{L_2(D)}$ is small since $\mathcal{L}_S(\widehat{h}) - \mathcal{L}_S(\widehat{h}^*) \leq 0$.

Let $\{(X_i^j, Y_i^j)\}_{(i,j)=(1,1)}^{(m,n)}$ be global data samples whose elements $\{(X_i^j, Y_i^j)\}_{j=1}^n$ are i.i.d random pairs at i-th client. The analysis of the first term in (2) involves the following definition of Rademacher complexity.

Definition 3. *We define $\mathcal{H} - \widehat{h}^* = \{h - \widehat{h}^* : h \in \mathcal{H}\}$ and denote by B_2^m the $L_2(D)$ unit ball entered at $\widehat{h}^*$, that is $B_2^m = \{h \in \mathcal{H} : \|h - \widehat{h}^*\|_{L_2(D)} \leq 1\}$. For every $\eta > 0$, define*

$$\omega_{mn}(\mathcal{H} - \widehat{h}^*, \eta) := \inf\left\{s > 0 : \mathbb{E}\left[\sup_{h\in(\mathcal{H}-\widehat{h}^*)\cap sB_2^m}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j h(X_i^j)\right|\right] \leq \eta s,\right\}$$

where σ_i^j are Rademacher random variables.

The quantity $\omega_{mn}(\mathcal{H} - \widehat{h}^*, \eta)$ measures the Rademcher complexity of the localized function set $\{h - \widehat{h}^* : h \in \mathcal{H} \cap sB_2^m\}$. Note that $\omega_{mn}(\mathcal{H} - \widehat{h}^*, \eta)$ depends only on the hypothesis class $\mathcal{H}$ and global input samples drawn from semi-empirical distribution D.

Moreover, the analysis of the second term in (2) requires the following definition.

Definition 4. *For every $0 < \delta < 1$ and $\eta > 0$, define*

$$\kappa_{mn}(\eta, \delta) := \inf\left\{s > 0 : \mathbb{P}\left[\sup_{h\in(\mathcal{H}-\widehat{h}^*)\cap sB_2^m}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j\xi_i^j h(X_i^j)\right| \leq \eta s^2\right] \geq 1 - \delta,\right\}$$

where $\xi_i^j = h^*(X_i^j) - Y_i^j$ and σ_i^j are Rademacher random variables.

It is important to note that the quantity $\kappa_{mn}(\eta, \delta)$ measures the maximal correlation between localized function set $\{h(X_i^j)\}_{(i,j)=(1,1)}^{(m,n)}$ and noise vector $\{\sigma_i^j \xi_i^j\}_{(i,j)=(1,1)}^{(m,n)}$. Intuitively, $\kappa_{mn}(\eta, \delta)$ reflactes the noise level in learning problems.

Theorem 5. *Fix $\tau > 0$ for which $Q_{\mathcal{H}-\mathcal{H}}(2\tau) > 0$ and set $\eta < \tau^2 Q_{\mathcal{H}-\mathcal{H}}(2\tau)/32$. If every random variable $\left(\xi_i^j h(X_i^j) - \mathbb{E}[\xi_i^j h(X_i^j)]\right)$ for all $h \in \mathcal{H} - \widehat{h}^*$ is Sub-Weibull. For every $0 < \delta < 1$, $0 < \iota < \frac{1}{4}$ and sufficiently large mn, with probability at least $1 - \delta - \exp\left(-mnQ^2_{\mathcal{H}-\mathcal{H}}(2\tau)/2\right)$ one has*

$$\left\|\widehat{h} - \widehat{h}^*\right\|_{L_2(D)} \leq 2 \max\left\{\kappa_{mn}(\eta, \delta/4), \left(\frac{1}{mn}\right)^{-\frac{1}{4}+\iota}\right\}.$$

Remark 6. *To the best of our knowledge, Theorem 5 provide the first result on the generalization error of heterogeneous federated learning with heavy-tailed losses. It suggests that both hypothesis 'size' and noise level play important roles in the generalization error of heterogeneous learning problems.*

4.2 Learning rates for unparticipating clients with small-ball condition

In the analysis of generalization error for unparticipating clients, we focus on the measure $\|h - h^*\|^2_{L_2(P)}$, which represents the distance between h and h^* with respect to meta-distribution P. The analysis of generalization error for unparticipating clients follows a similar path to the previous analysis. Let $\{(X_i, Y_i)\}_{i=1}^m$ be dataset whose elements are sampled across the two-level framework, that is $\mathbb{E}[(X_i, Y_i)] = \mathbb{E}_{D_i \sim P}\mathbb{E}_{(X_i,Y_i)\sim D_i}[(X_i, Y_i)]$. We present the different definitions of Rademacher complexity terms in the following.

Definition 5. *We define $\mathcal{H} - h^* = \{h - h^* : h \in \mathcal{H}\}$ and denote by B_2 the $L_2(P)$ unit ball entered at h^*. For every $\eta > 0$, define*

$$\omega_m(\mathcal{H} - h^*, \eta) := \inf\left\{s > 0 : \mathbb{E}\left[\sup_{h \in (\mathcal{H}-h^*)\cap sB_2}\left|\frac{1}{m}\sum_{i=1}^m \sigma_i h(X_i)\right|\right] \leq \eta s,\right\}$$

where σ_i are Rademacher random variables.

The quantity $\omega_{mn}(\mathcal{H} - h^*, \eta)$ measures the localized complexity of $\{h - h^* : h \in \mathcal{H} \cap sB_2\}$.

Definition 6. *For every $0 < \delta < 1$ and $\eta > 0$, define*

$$\kappa_m(\eta, \delta) := \inf\left\{s > 0 : \mathbb{P}\left[\sup_{h \in (\mathcal{H}-h^*)\cap sB_2}\left|\frac{1}{m}\sum_{i=1}^m \sigma_i \xi_i \left[h(X_i)\right]\right| \leq \eta s^2\right] \geq 1 - \delta,\right\}$$

where $\xi_i = h^*(X_i) - Y_i$, σ_i are Rademacher random variables.

Theorem 6. *Fix $\tau > 0$ for which $Q_{\mathcal{H}-\mathcal{H}}(2\tau, P) > 0$ and set $\eta < \tau^2 Q_{\mathcal{H}-\mathcal{H}}(2\tau)/32$. If every random variable $(\xi_i h(X_i) - \mathbb{E}[\xi_i h(X_i)])$ for all $h \in \mathcal{H} - h^*$ is Sub-Weibull. For every $0 < \delta < 1$, $0 < \iota < \frac{1}{4}$ and sufficiently large m, with probability at least $1 - \delta - \exp\left(-mQ^2_{\mathcal{H}-\mathcal{H}}(2\tau, p)/2\right)$ one has*

$$\left\|\widehat{h}^* - h^*\right\|_{L_2(P)} \leq 2 \max\left\{\kappa_m(\eta, \delta/4), \left(\frac{1}{m}\right)^{-\frac{1}{4}+\iota}\right\}.$$

Remark 7. *Theorem 6 provide the first result on the generalization error of unparticipating clients in heterogeneous federated learning with heavy-tailed losses.*

5 Related Work

Generalization Error for Heterogeneous Federated Learning. Several attempts have been made in the analysis of generalization error for heterogeneous federated learning. We compare our results with most related works in Table 1. Complexity-based bounds for participating clients are derived in the work of (Mohri et al., 2019), who present high probability slow rates of order $\mathcal{O}(\frac{1}{\sqrt{mn}})$

Table 1: **Generalization Bounds for Heterogeneous Federated Learning**. SC, Pro, and Exp denote Strong convexity, In probability, and In expectation. Sub-expon denote sub-exponential losses.

Reference	Loss Function	Assumption	Part	Unpart	Type
Mohri et al. (2019)	Bounded	Bi-Classification	$\mathcal{O}(\frac{1}{\sqrt{mn}})$	/	Pro
Chen et al. (2021)	Bounded	Smoothness, SC	$\mathcal{O}(\frac{1}{mn})$	/	Exp
Fallah et al. (2021)	Bounded	Smoothness, SC	$\mathcal{O}(\frac{1}{mn})$	/	Exp
Our Results	Sub-expon	Lipschitz	$\mathcal{O}(\frac{1}{\sqrt{mn}})$	$\mathcal{O}(\frac{1}{\sqrt{mn}}+\frac{1}{\sqrt{m}})$	Pro
Our Results	Bounded	Bernstein Con	$\mathcal{O}(\frac{1}{mn})$	$\mathcal{O}(\frac{1}{mn}+\frac{1}{m})$	Pro
Our Results	Sub-Weibull	Small-ball	$\mathcal{O}\left(\frac{1}{(mn)^{\frac{1}{4}}}\right)$	$\mathcal{O}\left(\frac{1}{(mn)^{\frac{1}{4}}}+\frac{1}{m^{\frac{1}{4}}}\right)$	Pro

for bounded losses. Fast rates bounds are obtained based on stability tools in (Chen et al., 2021; Fallah et al., 2021). However, their results are in expectation form. Among the existing theoretical work, different measurements are used to model the heterogeneity of local distributions. These measurements include gradient dissimilarity and parameter dissimilarity of local optimal models. Here we argue that it is more natural to make an assumption from the perspective of the data-generating process. Therefore, in this paper, we assume that the local distributions are sampled from a higher meta-distribution.

A similar two-level distribution framework has been used in the analysis of meta-learning. However, the learning scenarios and objectives of federated learning are different from that of meta-learning. The goal of meta-learning is to choose an optimal hypothesis space $\mathcal{H}$ from the hypothesis space family $\mathbb{H}$. Ideally, the chosen hypothesis $\mathcal{H}$ should contain good hypothesis $h \in \mathcal{H}$ for each distribution D_i sampled from the meta distribution P. In this paper, we focus on the performance of common model $\widehat{h}$ trained by participating clients. The performance of the common model is measured by the population risk with respect to meta distribution P. Another line of research closely related to heterogeneous federated learning is domain adaptation/generalization. In this line, possibly the results in (Li et al., 2022) are most relevant to ours.

Generalization error for Unbounded losses. The unbounded assumption brings two major challenges to complexity-based generalization analysis. One is that the two-side concentration inequalities do not hold when the losses are heavy-tailed. The other is that the standard techniques used to upper bound the complexity of hypothesis space are developed for bounded losses. The straightforward way to avoid these two challenges is to assume there exists an envelope function with respect to the underlying distribution and hypothesis class (Adamczak, 2008; Lecué & Mendelson, 2012). Small-ball method is first proposed to replace the concentration tools for empirical process in (Mendelson, 2015) and further developed in (Mendelson, 2018). Inspired by the small-ball method, Offset Rademacher complexity-based method provides another replacement for two-side concentration inequality (Liang et al., 2015). However, most existing generalization bounds for unbounded losses are derived in the i.i.d setting. Roy et al. (2021) extend the small-ball method in the dependent data setting. In this paper, we focus on the heterogeneous federated learning scenario with unbounded losses, where the samples are independent but non-identically distributed.

6 Conclusion

We present a systematic generalization analysis of heterogeneous distributed learning. Our analysis captures the generalization performance of the learned model on both participating and unparticipating clients. To our knowledge, this is the first theoretical analysis under the assumption that the local distributions are sampled from a meta-distribution. We recover the current state of art guarantees without using bounded assumptions. Moreover, under the empirical risk minimization setting, we derive fast generalization rates in our two-level distribution setting.

REFERENCES

Radoslaw Adamczak. A tail inequality for suprema of unbounded empirical processes with applications to markov chains. *Electronic Journal of Probability*, 13:1000–1034, 2008.

L. P. Barnes, Alex Dytso, and H. Vincent Poor. Improved information theoretic generalization bounds for distributed and federated learning. In *IEEE International Symposium on Information Theory, ISIT 2022, Espoo, Finland, June 26 - July 1, 2022*, pp. 1465–1470. IEEE, 2022. doi: 10.1109/ISIT50566.2022.9834700. URL https://doi.org/10.1109/ISIT50566.2022.9834700.

Peter L Bartlett, Shahar Mendelson, and Petra Philips. Local complexities for empirical risk minimization. In *International Conference on Computational Learning Theory*, pp. 270–284. Springer, 2004.

Peter L Bartlett, Olivier Bousquet, and Shahar Mendelson. Local rademacher complexities. *The Annals of Statistics*, 33(4):1497–1537, 2005.

Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.

Shuxiao Chen, Qinqing Zheng, Qi Long, and Weijie J Su. A theorem of the alternative for personalized federated learning. *arXiv preprint arXiv:2103.01901*, 2021.

Luc Devroye, László Györfi, and Gábor Lugosi. *A probabilistic theory of pattern recognition*, volume 31. Springer Science & Business Media, 2013.

Richard M Dudley. Central limit theorems for empirical measures. *The Annals of Probability*, pp. 899–929, 1978.

Alireza Fallah, Aryan Mokhtari, and Asuman Ozdaglar. Generalization of model-agnostic meta-learning algorithms: Recurring and unseen tasks. *Advances in Neural Information Processing Systems*, 34:5469–5480, 2021.

David Haussler. Decision theoretic generalizations of the pac model for neural net and other learning applications. In *The Mathematics of Generalization*, pp. 37–116. CRC Press, 2018.

Miao Hu, Di Wu, Yipeng Zhou, Xu Chen, and Min Chen. Incentive-aware autonomous client participation in federated learning. *IEEE Transactions on Parallel and Distributed Systems*, 33 (10):2612–2627, 2022.

Peter Kairouz, H. Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista A. Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, Rafael G. L. D'Oliveira, Hubert Eichner, Salim El Rouayheb, David Evans, Josh Gardner, Zachary Garrett, Adrià Gascón, Badih Ghazi, Phillip B. Gibbons, Marco Gruteser, Zaïd Harchaoui, Chaoyang He, Lie He, Zhouyuan Huo, Ben Hutchinson, Justin Hsu, Martin Jaggi, Tara Javidi, Gauri Joshi, Mikhail Khodak, Jakub Konečný, Aleksandra Korolova, Farinaz Koushanfar, Sanmi Koyejo, Tancrède Lepoint, Yang Liu, Prateek Mittal, Mehryar Mohri, Richard Nock, Ayfer Özgür, Rasmus Pagh, Hang Qi, Daniel Ramage, Ramesh Raskar, Mariana Raykova, Dawn Song, Weikang Song, Sebastian U. Stich, Ziteng Sun, Ananda Theertha Suresh, Florian Tramèr, Praneeth Vepakomma, Jianyu Wang, Li Xiong, Zheng Xu, Qiang Yang, Felix X. Yu, Han Yu, and Sen Zhao. Advances and open problems in federated learning. *Found. Trends Mach. Learn.*, 14(1-2):1–210, 2021. doi: 10.1561/2200000083. URL https://doi.org/10.1561/2200000083.

Varun Kanade, Patrick Rebeschini, and Tomas Vaskevicius. Exponential tail local rademacher complexity risk bounds without the bernstein condition. *CoRR*, abs/2202.11461, 2022. URL https://arxiv.org/abs/2202.11461.

Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda Theertha Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *International Conference on Machine Learning*, pp. 5132–5143. PMLR, 2020.

Ahmed Khaled, Konstantin Mishchenko, and Peter Richtárik. First analysis of local gd on heterogeneous data. *arXiv preprint arXiv:1909.04715*, 2019.

Yegor Klochkov and Nikita Zhivotovskiy. Stability and deviation optimal risk bounds with convergence rate $o(1/n)$. In Marc'Aurelio Ranzato, Alina Beygelzimer, Yann N. Dauphin, Percy Liang, and Jennifer Wortman Vaughan (eds.), *Advances in Neural Information Processing Systems 34: Annual Conference on Neural Information Processing Systems 2021, NeurIPS 2021, December 6-14, 2021, virtual*, pp. 5065–5076, 2021. URL https://proceedings.neurips.cc/paper/2021/hash/286674e3082feb7e5afb92777e48821f-Abstract.html.

Arun K Kuchibhotla and Rohit K Patra. On least squares estimation under heteroscedastic and heavy-tailed errors. *The Annals of Statistics*, 50(1):277–302, 2022.

Arun Kumar Kuchibhotla and Abhishek Chakrabortty. Moving beyond sub-gaussianity in high-dimensional statistics: Applications in covariance estimation and linear regression. *arXiv preprint arXiv:1804.02605*, 2018.

Guillaume Lecué and Shahar Mendelson. General nonexact oracle inequalities for classes with a subexponential envelope. *The Annals of Statistics*, 40(2):832–860, 2012.

Guillaume Lecué and Shahar Mendelson. Learning subgaussian classes: upper and minimax bounds (2013). *Topics in Learning Theory-Societe Mathematique de France,(S. Boucheron and N. Vayatis Eds.)*, 2013.

Guillaume Lecué and Shahar Mendelson. Performance of empirical risk minimization in linear aggregation. *Bernoulli*, 22(3):1520–1534, 2016.

Yunwen Lei, Lixin Ding, and Yingzhou Bi. Local rademacher complexity bounds based on covering numbers. *Neurocomputing*, 218:320–330, 2016.

Da Li, Henry Gouk, and Timothy Hospedales. Finding lost dg: Explaining domain generalization via model complexity. *arXiv preprint arXiv:2202.00563*, 2022.

Tian Li, Anit Kumar Sahu, Ameet Talwalkar, and Virginia Smith. Federated learning: Challenges, methods, and future directions. *IEEE Signal Process. Mag.*, 37(3):50–60, 2020a. doi: 10.1109/MSP.2020.2975749. URL https://doi.org/10.1109/MSP.2020.2975749.

Tian Li, Anit Kumar Sahu, Manzil Zaheer, Maziar Sanjabi, Ameet Talwalkar, and Virginia Smith. Federated optimization in heterogeneous networks. *Proceedings of Machine Learning and Systems*, 2:429–450, 2020b.

Tengyuan Liang, Alexander Rakhlin, and Karthik Sridharan. Learning with square loss: Localization through offset rademacher complexity. In *Conference on Learning Theory*, pp. 1260–1285. PMLR, 2015.

Mohammad Saeed Masiha, Amin Gohari, Mohammad Hossein Yassaee, and Mohammad Reza Aref. Learning under distribution mismatch and model misspecification. In *IEEE International Symposium on Information Theory, ISIT 2021, Melbourne, Australia, July 12-20, 2021*, pp. 2912–2917. IEEE, 2021. doi: 10.1109/ISIT45174.2021.9517732. URL https://doi.org/10.1109/ISIT45174.2021.9517732.

Andreas Maurer and Massimiliano Pontil. Concentration inequalities under sub-gaussian and subexponential conditions. *Advances in Neural Information Processing Systems*, 34:7588–7597, 2021.

Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pp. 1273–1282. PMLR, 2017.

Shahar Mendelson. Improving the sample complexity using global data. *IEEE transactions on Information Theory*, 48(7):1977–1991, 2002.

Shahar Mendelson. A few notes on statistical learning theory. In *Advanced lectures on machine learning*, pp. 1–40. Springer, 2003.

Shahar Mendelson. Learning without concentration. *Journal of the ACM (JACM)*, 62(3):1–25, 2015.

Shahar Mendelson. Learning without concentration for general loss functions. *Probability Theory and Related Fields*, 171(1):459–502, 2018.

Konstantin Mishchenko, Grigory Malinovsky, Sebastian Stich, and Peter Richtárik. Proxskip: Yes! local gradient steps provably lead to communication acceleration! finally! *arXiv preprint arXiv:2202.09357*, 2022.

Aritra Mitra, Rayana Jaafar, George J Pappas, and Hamed Hassani. Linear convergence in federated learning: Tackling client heterogeneity and sparse gradients. *Advances in Neural Information Processing Systems*, 34:14606–14619, 2021.

Mehryar Mohri, Gary Sivek, and Ananda Theertha Suresh. Agnostic federated learning. In *International Conference on Machine Learning*, pp. 4615–4625. PMLR, 2019.

Debolina Paul, Saptarshi Chakraborty, Swagatam Das, and Jason Xu. Uniform concentration bounds toward a unified framework for robust clustering. *Advances in Neural Information Processing Systems*, 34:8307–8319, 2021.

David Pollard. *Convergence of stochastic processes*. Springer Science & Business Media, 2012.

Amirhossein Reisizadeh, Farzan Farnia, Ramtin Pedarsani, and Ali Jadbabaie. Robust federated learning: The case of affine distribution shifts. *Advances in Neural Information Processing Systems*, 33:21554–21565, 2020.

Abhishek Roy, Krishnakumar Balasubramanian, and Murat A Erdogdu. On empirical risk minimization with dependent and heavy-tailed data. *Advances in Neural Information Processing Systems*, 34:8913–8926, 2021.

Milad Sefidgaran, Romain Chor, and Abdellatif Zaidi. Rate-distortion theoretic bounds on generalization error for distributed learning. *CoRR*, abs/2206.02604, 2022a. doi: 10.48550/arXiv.2206.02604. URL https://doi.org/10.48550/arXiv.2206.02604.

Milad Sefidgaran, Romain Chor, and Abdellatif Zaidi. Rate-distortion theoretic bounds on generalization error for distributed learning. *arXiv preprint arXiv:2206.02604*, 2022b.

Lili Su, Jiaming Xu, and Pengkun Yang. Achieving statistical optimality of federated learning: Beyond stationary points. *arXiv preprint arXiv:2106.15216*, 2021.

Aad W. van der Vaart and Jon A. Wellner. *Weak Convergence*, pp. 16–28. Springer New York, New York, NY, 1996. ISBN 978-1-4757-2545-2. doi: 10.1007/978-1-4757-2545-2_3. URL https://doi.org/10.1007/978-1-4757-2545-2_3.

Tim van Erven, Peter D. Grünwald, Nishant A. Mehta, Mark D. Reid, and Robert C. Williamson. Fast rates in statistical and online learning. *CoRR*, abs/1507.02592, 2015. URL http://arxiv.org/abs/1507.02592.

Adithya Vellal, Saptarshi Chakraborty, and Jason Q Xu. Bregman power k-means for clustering exponential family data. In *International Conference on Machine Learning*, pp. 22103–22119. PMLR, 2022.

Jianyu Wang, Zachary Charles, Zheng Xu, Gauri Joshi, H Brendan McMahan, Maruan Al-Shedivat, Galen Andrew, Salman Avestimehr, Katharine Daly, Deepesh Data, et al. A field guide to federated optimization. *arXiv preprint arXiv:2107.06917*, 2021.

Xuetong Wu, Jonathan H Manton, Uwe Aickelin, and Jingge Zhu. Fast rate generalization error bounds: Variations on a theme. *arXiv preprint arXiv:2205.03131*, 2022.

Jie Xu and Heqiang Wang. Client selection and bandwidth allocation in wireless federated learning networks: A long-term perspective. *IEEE Transactions on Wireless Communications*, 20(2): 1188–1200, 2020.

Yunbei Xu and Assaf Zeevi. Towards problem-dependent optimal learning rates. *Advances in Neural Information Processing Systems*, 33:2196–2206, 2020.

Howard H Yang, Ahmed Arafa, Tony QS Quek, and H Vincent Poor. Age-based scheduling policy for federated learning in mobile edge networks. In *ICASSP 2020-2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 8743–8747. IEEE, 2020.

Niloofar Yousefi, Yunwen Lei, Marius Kloft, Mansooreh Mollaghasemi, and Georgios C Anagnostopoulos. Local rademacher complexity-based learning guarantees for multi-task learning. *The Journal of Machine Learning Research*, 19(1):1385–1431, 2018.

Honglin Yuan, Warren Morningstar, Lin Ning, and Karan Singhal. What do we mean by generalization in federated learning? *arXiv preprint arXiv:2110.14216*, 2021.

Chulhee Yun, Shashank Rajput, and Suvrit Sra. Minibatch vs local SGD with shuffling: Tight convergence bounds and beyond. In *The Tenth International Conference on Learning Representations, ICLR 2022, Virtual Event, April 25-29, 2022*. OpenReview.net, 2022. URL `https://openreview.net/forum?id=LdlwbBP2mlq`.

Lijun Zhang and Zhi-Hua Zhou. ℓ_1-regression with heavy-tailed distributions. *Advances in Neural Information Processing Systems*, 31, 2018.

A APPENDIX

Definition 7 (VC major class)**.** *For binary classification problem, uniformly entropy number can be bounded directly if the function class $\mathcal{H}$ has finite* VC *dimension (van der Vaart & Wellner, 1996). In regression problem, similar results can be derived if $\mathcal{H}$ has finite* VC *subgraph dimension (van der Vaart & Wellner, 1996), which is an extension concept of* VC *dimension.* VC *major class refer to the class either has finite* VC *dimension or* VC *subgraph dimension.*

A.1 PROOF OF THEOREM 1

We first show that semi-excess risk $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(h^*)$ can be upper bounded by supremum of the empirical process indexed by $\mathcal{H}$: $\mathcal{L}_D(\widehat{h}) - \mathcal{L}_D(\widehat{h}^*) \leq 2\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)|$. Recall that $\widehat{h}$ is the empirical minimizer and h^* is the population minimizer. We decompose excess risk $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$ in two-level framework: $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*) \leq 2\sup_{h\in\mathcal{H}} |\mathcal{L}_P(h) - \mathcal{L}_D(h)| + 2\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)|$, where the inequality uses $\mathcal{L}_S(\widehat{h}) - \mathcal{L}_S(h^*) \leq 0$ since $\widehat{h}$ is the minimizer of $\mathcal{L}_S(h)$. Note that the first term $\sup_{h\in\mathcal{H}} |\mathcal{L}_P(h) - \mathcal{L}_D(h)|$ in the upper bound of excess risk quantifies the participation gap. The second term $\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)|$ quantifies the generalization error for participating clients. In the following Theorem, we present basic upper bound of excess risk by taking VC major class as an example.

The goal of Theorem 1 is to present theoretical bounds of excess risk $\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*)$, we first show that excess risk can be bounded by participation gap $\sup_{h\in\mathcal{H}} \{|\mathcal{L}_P(h) - \mathcal{L}_D(h)|\}$ and participating error $\sup_{h\in\mathcal{H}} \{|\mathcal{L}_D(h) - \mathcal{L}_S(h)|\}$:

$$\begin{aligned}
\mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*) &\leq \mathcal{L}_P(\widehat{h}) - \mathcal{L}_P(h^*) - (\mathcal{L}_S(\widehat{h}) - \mathcal{L}_S(h^*)) \\
&= \mathcal{L}_P(\widehat{h}) - \mathcal{L}_D(\widehat{h}) + \mathcal{L}_D(h^*) - \mathcal{L}_P(h^*) + \mathcal{L}_D(\widehat{h}) - \mathcal{L}_S(\widehat{h}) + \mathcal{L}_S(h^*) - \mathcal{L}_D(h^*) \\
&\leq 2\sup_{h\in\mathcal{H}} \{|\mathcal{L}_P(h) - \mathcal{L}_D(h)|\} + 2\sup_{h\in\mathcal{H}} \{|\mathcal{L}_D(h) - \mathcal{L}_S(h)|\},
\end{aligned}$$

where the first inequality is due to $\mathcal{L}_S(\widehat{h}) - \mathcal{L}_S(h^*) \leq 0$. Based on these observations, we first provide theoretical bounds on the term $\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)|$. Then we move to the participating gap $\sup_{h\in\mathcal{H}} |\mathcal{L}_P(h) - \mathcal{L}_D(h)|$.

Let $\boldsymbol{\sigma} = \{\sigma_i^j\}_{i\in[m],j\in[n]}$ be a collection of independent Rademacher variables, this is random variables taking values uniformly in $\{+1, -1\}$. We define the generalized rademacher complexity $\mathcal{R}_{mn}(\mathcal{F})$ used in heterogeneous federated learning as is as follows:

$$\mathcal{R}_{mn}(\mathcal{F}) = \mathop{\mathbb{E}}_{S_1,\cdots,S_m,\boldsymbol{\sigma}} \left[\sup_{h\in\mathcal{H}} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_i^j \ell(h, Z_i^j) \right].$$

Lemma 1 (Generalization error for participating clinets)**.** *Let $\mathcal{F}$ be a family of functions related to hypothesis space $\mathcal{H}$: $\mathcal{F} = \{z \mapsto \ell(h, z) : h \in \mathcal{H}\}$. Assume that loss function ℓ is bounded by M. Then, for any $\delta \geq 0$, with probability at least $1 - \delta$, we have*

$$\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)| \leq 2\mathcal{R}_{mn}(\mathcal{F}) + M\sqrt{\frac{\ln(1/\delta)}{2mn}}.$$

Lemma 1 provide high probability theoretical guarantees for participating clients under bounded assumption. Compared to the classical iid setting in learning theory, the results in Lemma 1 in under independent but non-identically distributed setting.

Since each local distribution D_i is sampled independently from P, we regard local population risk $\{\mathbb{E}_{Z\sim D_i}[\ell(h, Z)]\}_{i\in[m]}$ as a collection of iid random variables. Let $\boldsymbol{\sigma} = \{\sigma_i\}_{i\in[m]}$ be a collection of independent Rademacher variables, the rademacher complexity used in the analysis of participating gap is defined as:

$$\mathcal{R}_m(\mathcal{F}) = \mathop{\mathbb{E}}_{D_1,\cdots,D_m,\boldsymbol{\sigma}} \left[\sup_{h\in\mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \mathbb{E}_{Z\sim D_i}[\ell(h, Z)] \right].$$

Lemma 2 (Participation gap)**.** *Let $\mathcal{F}$ be a family of functions related to hypothesis space $\mathcal{H}$: $\mathcal{F} = \{z \mapsto \ell(h, z) : h \in \mathcal{H}\}$. Assume that loss function ℓ is bounded by M. Then, for any $\delta \geq 0$, with probability at least $1 - \delta$, we have*

$$\sup_{h\in\mathcal{H}} |\mathcal{L}_P(h) - \mathcal{L}_D(h)| \leq 2\mathcal{R}_m(\mathcal{F}) + M\sqrt{\frac{\ln(1/\delta)}{2m}}.$$

The participation gap in Lemma 2 quantifies the error caused by missing clients during training. The Rademacher complexity term $\mathcal{R}_m(\mathcal{F})$ indicates the 'size' of hypothesis class, which can be future bounded by convering number or VC dimension of the choosen hypothesis. To show the explicit form of the rademacher complexity terms in Lemma 1 and Lemma 2, we take VC major class as an example.

PROOF OF LEMMA 1

Proof. We create sample set S' from S by changing one sample point, without loss generality, we assume the last data point Z_s^k in S is changed into $Z_s'^k$, then we define $\Phi(S)$ by $\Phi(S) = \sup_{h\in\mathcal{H}} \mathcal{L}_D(h) - \mathcal{L}_S(h)$ From the definition of S' and S, we have

$$\begin{aligned}
\Phi\left(S'\right) - \Phi(S) &= \sup_{h\in\mathcal{H}} \{\mathcal{L}_D(h) - \mathcal{L}_{S'}(h)\} - \sup_{h\in\mathcal{H}} \{\mathcal{L}_D(h) - \mathcal{L}_S(h)\} \\
&\leq \sup_{h\in\mathcal{H}} \{\{\mathcal{L}_D(h) - \mathcal{L}_{S'}(h)\} - \{\mathcal{L}_D(h) - \mathcal{L}_S(h)\}\} \\
&= \sup_{h\in\mathcal{H}} \{\mathcal{L}_S(h) - \mathcal{L}_{S'}(h)\} = \frac{1}{mn} \sup_{h\in\mathcal{H}} \ell\left(h, Z_s^k\right) - \ell\left(h, Z_s'^k\right)
\end{aligned}$$

By assuming that loss function f is bounded by M, then we have $|\Phi\left(S'\right) - \Phi(S)| \leq \frac{M}{mn}$. We apply McDiarmid's inequality and obtain: for any fixed $\delta \in (0, 1)$, it follows that with probability at least $1 - \delta$

$$\Phi(S) \leq \mathop{\mathbb{E}}_{S_1,\cdots,S_m} [\Phi(S)] + M\sqrt{\frac{\ln(2/\delta)}{2mn}}$$

Next we consider the expectation of $\Phi(S)$. By symmetrization, we have

$$\begin{aligned}
\mathop{\mathbb{E}}_{S_1,\cdots,S_m} [\Phi(S)] &= 2 \mathop{\mathbb{E}}_{S_1,\cdots,S_m,\sigma} \left[\sup_{h\in\mathcal{H}} \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij} \ell(h, Z_i^j)\right] \\
&= 2 \mathop{\mathbb{E}}_{S_1,\cdots,S_m} [\mathcal{R}_{mn}(\mathcal{F})]
\end{aligned}$$

where $\mathcal{R}_{mn}(\mathcal{F}) = \mathop{\mathbb{E}}_{\sigma}\left[\sup_{h\in\mathcal{H}} \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij} \ell(h, Z_i^j)\right]$. Replacing one data point of $S = S_1 \bigcup \cdots \bigcup S_m$ make $\mathcal{R}_{mn}(\mathcal{F})$ vary at most $\frac{M}{mn}$, then applying McDiarmid's inequality again, with probability at least $1 - \delta$ we have

$$\mathop{\mathbb{E}}_{S_1,\cdots,S_n} [\mathcal{R}_{mn}(\mathcal{F})] \leq \mathcal{R}_{mn}(\mathcal{F}) + M\sqrt{\frac{\ln(2/\delta)}{2mn}}$$

□

PROOF OF LEMMA 2

Proof of Lemma 2. We create distributions set D' from D by changing one client distribution. Without loss generality, we assume the k-th D_k in D is changed into D_k', then we define $\Phi(D)$ by

$$\Phi(D) = \sup_{h\in\mathcal{H}} \mathcal{L}_P(h) - \mathcal{L}_D(h).$$

From the definition of D' and D, we have

$$\begin{aligned}
&\Phi\left(D'\right)-\Phi(D) \\
&\quad=\sup_{h \in \mathcal{H}}\left\{\mathcal{L}_P(h)-\mathcal{L}_{D'}(h)\right\}-\sup_{h \in \mathcal{H}}\left\{\mathcal{L}_P(h)-\mathcal{L}_D(h)\right\} \\
&\quad\leq \sup_{h \in \mathcal{H}}\left\{\left\{\mathcal{L}_P(h)-\mathcal{L}_{D'}(h)\right\}-\left\{\mathcal{L}_P(h)-\mathcal{L}_D(h)\right\}\right\} \\
&\quad=\sup_{h \in \mathcal{H}}\left\{\mathcal{L}_D(h)-\mathcal{L}_{D'}(h)\right\}=\sup_{h \in \mathcal{H}} \frac{1}{m}\left(\mathbb{E}_{Z \sim D_k}[\ell(h, Z)]-\mathbb{E}_{Z \sim D_k'}[\ell(h, Z)]\right)
\end{aligned}$$

By assuming that loss function f is bounded by M, then we have $|\Phi\left(D'\right)-\Phi(D)| \leq \frac{M}{m}$. According to McDiarmid's inequality, for all $\delta \in (0,1)$, with probability at least $1-\delta$, we have

$$\Phi(D) \leq \mathop{\mathbb{E}}_{D_1, \cdots, D_m}[\Phi(D)]+M \sqrt{\frac{\ln (2 / \delta)}{2 m}}$$

Now we deal with the expectation of $\Phi(D)$. By symmetrization, we have

$$\mathop{\mathbb{E}}_{D_1, \cdots, D_m}[\Phi(D)]=\mathop{\mathbb{E}}_{D_1, \cdots, D_m}\left[\sup_{h \in \mathcal{H}} \mathcal{L}_P(h)-\mathcal{L}_D(h)\right] \quad \leq 2 \mathop{\mathbb{E}}_{D_1, \cdots, D_m, \sigma}\left[\sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \sigma_i \mathcal{L}_{D_i}(h)\right]$$

$$\begin{aligned}
\mathop{\mathbb{E}}_{D_1, \cdots, D_m}[\Phi(D)] &= 2 \mathop{\mathbb{E}}_{D_1, \cdots, D_m, \sigma}\left[\sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \sigma_i \mathcal{L}_{D_i}(h)\right] \\
&= 2 \mathop{\mathbb{E}}_{D_1, \cdots, D_m, \sigma}\left[\sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \sigma_i \mathbb{E}_{Z_i \sim D_i}[\ell(h, Z_i)]\right] \\
&\leq 2 \mathop{\mathbb{E}}_{Z_1, \cdots, Z_m, \sigma}\left[\sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \sigma_i[\ell(h, Z_i)]\right] \\
&= 2 \mathop{\mathbb{E}}_{Z_1, \cdots, Z_m} \mathcal{R}_m(\mathcal{L})
\end{aligned}$$

where $\mathcal{R}_m(\mathcal{L}) = \mathop{\mathbb{E}}_{\sigma}\left[\sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \sigma_i \ell(h, Z_i)\right]$. Replacing one point of $Z=\{Z_i\}_{i=1}^m$ make $\mathcal{R}_m(\mathcal{L})$ vary at most $\frac{M}{m}$, then applying McDiarmid's inequality again, with probability at least $1-\delta$ we have

$$\mathop{\mathbb{E}}_{Z_1, \cdots, Z_m}[\mathcal{R}_m(\mathcal{L})] \leq \mathcal{R}_m(\mathcal{L})+M \sqrt{\frac{\ln (2 / \delta)}{2 m}}$$

□

PROOF OF THEOREM 1

Proof. We use subgraph dimension(aka pseudo dimension) and Dudley's theorem to bound the empirical rademacher complexity $\mathcal{R}_{mn}(\mathcal{H})$. According to Dudley (1978); Pollard (2012); Haussler (2018), the uniform matric entropy of real valued function class $\mathcal{F}$ with subgraph dimension $\mathrm{VC}(\mathcal{F})=d$ can be bounded

$$\log \mathcal{N}(\epsilon, \mathcal{F},\|\cdot\|_2) \leq C d \log (1 / \epsilon)$$

where F is the envolop function of $\mathcal{F}$ Then by Dudley's theorem and van der Vaart & Wellner (1996), we have

$$\mathcal{R}_{mn}(\mathcal{F}) \leq C M \sqrt{\frac{d}{m n}}, \qquad \mathcal{R}_m(\mathcal{F}) \leq C M \sqrt{\frac{d}{m}}.$$

Combining the results of Lemma 1 and 2 leads to the final results. □

B PROOFS OF FAST RATES WITH BOUNDED LOSSES

Covering number can be used to give tighter estimate on the hypothesis size. Here we provide the definition of convering number and uniformly entropy number.

Definition 8 (Convering number)**.** *Let $(\mathcal{G}, \rho)$ be a metric space and $\mathcal{F} \subseteq \mathcal{G}$. For any $\epsilon \geq 0$, $\mathcal{F}_\epsilon$ is an ϵ-cover of $\mathcal{F}$ with respect of ρ if for all $f \in F$, we can find $f' \in \mathcal{F}_\epsilon$ such that $\rho(f, f') \leq \epsilon$. The covering number $\mathcal{N}(\epsilon, \mathcal{F}, \rho)$ is defined as the minimum size of an ϵ-cover:*

$$\mathcal{N}(\epsilon, \mathcal{F}, \rho) := \min\{|\mathcal{F}_\epsilon| : \mathcal{F}_\epsilon \text{ is an } \epsilon\text{-cover of } \mathcal{F} \text{ w.r.t } \rho\}.$$

Definition 9 (Uniformly entropy number)**.** *The entropy number is defined as the logarithm of the covering number. Let $(\mathcal{G}, \rho)$ be a normed space with $\rho(f, f') = \|f - f'\|$. Let F be an envelope function of $\mathcal{F}$ such that $|f(Z)| \leq F(Z)$, for all Z and f. We further define uniformly entropy number of $\mathcal{F}$ as: $\log \mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_2) = \sup_Q \log \mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_{L_2(Q)})$, where Q is taken over all probability measures with $0 \leq QF^2 \leq \infty$.*

B.1 EXAMPLES SATISFYING ASSUMPTION 2

B.1.1 EXAMPLE 1

In this subsection, we show that when the hypothesis class contains the $\widehat{h}^*$ and h^*, Assumption 2 holds. We denote by $\{\ell_h(X, Y) = \ell(h(X), Y) : h \in \mathcal{H}\}$.

$$\begin{aligned}
\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*})^2 &= \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}_{(X,Y)\sim D_i}\left[((h(X) - Y)^2 - (\widehat{h}^*(X) - Y)^2)^2\right] \\
&\leq \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}_{(X,Y)\sim D_i}\left[\left|(h(X) - Y)^2 - (\widehat{h}^*(X) - Y)^2\right|^2\right] \\
&= \frac{L^2}{m}\sum_{i=1}^{m} \mathbb{E}_{X\sim D_i}\left[\left(h(X) - \widehat{h}^*(X)\right)^2\right] = L^2\mathbb{E}_{X\sim D}(h - \widehat{h}^*)^2.
\end{aligned}$$

Next we show that

$$\begin{aligned}
\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*}) &= \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}_{(X,Y)\sim D_i}\left[(h(X) - Y)^2 - (\widehat{h}^*(X) - Y)^2\right] \\
&= \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}_{X\sim D_i}\left[(h(X) - \widehat{h}^*(X))^2\right] = \mathbb{E}_{X\sim D}(h - \widehat{h}^*)^2.
\end{aligned}$$

Thus, if $\widehat{h}^* \in \mathcal{H}$, we have

$$\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*})^2 \leq L^2\mathbb{E}_{X\sim D}(h - \widehat{h}^*)^2 = L^2\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*}).$$

Thus, the tuple $\left(D, \ell, \mathcal{H}, \widehat{h}^\star\right)$ satisfies the Bernstein condition with parameter $B_2 > 0$ such that $\mathbb{E}_{X\sim D}(h - \widehat{h}^*)^2 \leq B_2\mathbb{E}_{X\sim D}(\ell_h - \ell_{\widehat{h}^*})$. Similarly, we can prove that

$$\mathbb{E}_{(X,Y)\sim P}(\ell_h - \ell_{h^*})^2 \leq L^2\mathbb{E}_{X\sim P}(h - h^*)^2 = L^2\mathbb{E}_{(X,Y)\sim P}(\ell_h - \ell_{h^*}).$$

Thus, the tuple $(P, f, \mathcal{H}, h^\star)$ satisfies the Bernstein condition with parameter $B_1 > 0$ such that $\mathbb{E}_{X\sim P}(h(X) - h^*(X))^2 \leq B_1\mathbb{E}_{(X,Y)\sim P}(\ell_h - \ell_{h^*})$.

B.1.2 EXAMPLE 2

In this subsection, we show that even when the hypothesis class does not include $\widehat{h}^*$ and h^*, Assumption 2 still holds. If $\mathcal{F}$ is a nonempty, closed and convex subset of a Hilbert space with inner

produc $\langle f, g\rangle = \mathbb{E}_{(X,Y)\sim D}(f(X,Y)g(X,Y))$, $\widehat{h}^*$ is the projection of Y in the space $\mathcal{F}$. By the definition we have $\langle Y - \widehat{h}^*(X), h(X) - \widehat{h}^*(X)\rangle \leq 0$.

$$\begin{aligned}\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*}) &= \mathbb{E}_{(X,Y)\sim D}\left[(Y - h(X))^2 - (Y - \widehat{h}^*(X))^2\right] \\ &= \left\|(h(X) - \widehat{h}^*(X))^2\right\|_{L_2}^2 + \langle Y - \widehat{h}^*(X), \widehat{h}^*(X) - h(X)\rangle \\ &\geq \mathbb{E}_{X\sim D}(h - \widehat{h}^*)^2.\end{aligned}$$

Thus, for regression problems with square loss function, where $(\widehat{h}^* \notin \mathcal{H})$, we have

$$\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*})^2 \leq L^2\mathbb{E}_{X\sim D}(h - \widehat{h}^*)^2 = L^2\mathbb{E}_{(X,Y)\sim D}(\ell_h - \ell_{\widehat{h}^*}).$$

Similarly, we can prove that

$$\mathbb{E}_{(X,Y)\sim P}(\ell_h - \ell_{h^*})^2 \leq L^2\mathbb{E}_{X\sim P}(h - h^*)^2 = L^2\mathbb{E}_{(X,Y)\sim P}(\ell_h - \ell_{h^*}).$$

B.2 Proof of Theorem 2

Theorem 7 (Theorem 1 in (Yousefi et al., 2018)). *Let $\mathcal{F}$ be a family of functions satisfying $\forall f \in \mathcal{F}, \forall Z \in \mathcal{Z}, |f(Z)| \leq M$. If $\sup_{f\in\mathcal{F}} \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}[f(Z_i^j)]^2 \leq r$, then, for any $\delta \geq 0$ and any $\alpha \geq 0$, with probability at least $1-\delta$,*

$$\sup_{f\in\mathcal{F}}\left[\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[\mathbb{E}[f(Z_i^j)] - f(Z_i^j)\right]\right] \leq 2(1+\alpha)\mathbb{E}[\mathcal{R}_{mn}(\mathcal{F})] + \sqrt{\frac{8r\ln(1/\delta)}{mn}} + \left(1+\frac{2}{\alpha}\right)\frac{4M\ln(1/\delta)}{mn},$$

where $\mathcal{R}_{mn}(\mathcal{H}) = \mathop{\mathbb{E}}_{\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \sigma_i^j f(Z_i^j)\right]$.

Lemma 3 (Lemma B.1 in (Yousefi et al., 2018)). *Let $c_1, c_2 > 0$ and $s > q > 0$. Then the equation $x^s - c_1x^q - c_2 = 0$ has a unique positive solution x_0 satisfying*

$$x_0 \leq \left[c_1^{\frac{s}{s-1}} + \frac{sc_2}{s-q}\right]^{\frac{1}{s}}.$$

Moreover, for any $x \geq x_0$, we have $x^s \geq c_1x^q + c_2$.

Lemma 4 (Corollary 1 in Lei et al. (2016)). *Let $\mathcal{F}$ be a function class with $\sup_{f\in\mathcal{F}}\|f\|_\infty \leq b$. Assume that there exist three positive numbers γ, d, p such that $\log\mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_2) \leq d\log^p(\gamma/\epsilon)$ for any $0 < \epsilon \leq \gamma$, then for any $0 < r \leq \gamma^2$ and $n \geq \gamma^{-2}$ there holds that*

$$\mathbb{E}R_n\left\{f\in\mathcal{F} : Pf^2 \leq r\right\} \leq c(b,p,\gamma)\min\left[\left(\sqrt{\frac{dr\log^p(2\gamma r^{-1/2})}{n}} + \frac{d\log^p\left(2\gamma r^{-1/2}\right)}{n}\right)\right.$$
$$\left.\left(\frac{d\log p\left(2\gamma n^{1/2}\right)}{n} + \sqrt{\frac{rd\log^p(2\gamma n^{1/2})}{n}}\right)\right]$$

Proof of Theorem 2. First we define $\mathcal{F}^* := \{f : (X,Y) \mapsto \ell(h(X),Y) - \ell(\widehat{h}^*(X),Y), h \in \mathcal{H}\}$. Combining Assumption (a), for any $f \in \mathcal{F}^*$ we derive that

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m\mathbb{E}\left[f^2(X_i^1,Y_i^1)\right] &= \frac{1}{m}\sum_{i=1}^m\mathbb{E}\left[\ell(h(X_i^1),Y_i^1) - \ell(\widehat{h}^*(X_i^1),Y_i^1)\right]^2 \\ &\leq \frac{L^2}{m}\sum_{i=1}^m\mathbb{E}\left[h(X_i^1) - \widehat{h}^*(X_i^1)\right]^2 \\ &\leq B'L^2\left(\frac{1}{m}\sum_{i=1}^m\mathbb{E}\left[\ell(h(X_i^1),Y_i^1) - \ell(\widehat{h}^*(X_i^1),Y_i^1)\right]\right)^\beta.\end{aligned}$$

Let $V(f) := \frac{L^2}{m}\sum_{i=1}^m \mathbb{E}\left[h(X) - \widehat{h}^*(X)\right]^2$, $B = B'L^2$, then $\mathcal{F}^*$ is a β, B-Bernstein function class.

$$\begin{aligned}
\mathcal{R}(\mathcal{F}^*, r) &= \mathbb{E}_{X,\sigma}\left[\sup_{\substack{V(f_h)\le r\\ h\in\mathcal{H}}} \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \sigma_i^j f(X_i^j, Y_i^j)\right] \\
&= \mathbb{E}_{X,\sigma}\left[\sup_{\substack{V(f_h)\le r\\ h\in\mathcal{H}}} \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \sigma_i^j \ell(h(X_i^j), Y_i^j)\right] \\
&\le L\mathbb{E}_{X,\sigma}\left[\sup_{\substack{L^2\mathbb{E}_{X\sim D}(h-\widehat{h}^*)^2\le r\\ h\in\mathcal{H}}} \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \sigma_i^j h(X_i^j)\right] \\
&\le L\mathcal{R}(\mathcal{H}^*, r).
\end{aligned}$$

Condiser the function class $\mathcal{G}_r$ associated with $\mathcal{F}$:

$$\mathcal{G}_r := \left\{g := \frac{rf}{\max(r, V(f))}, f\in\mathcal{F}\right\}.$$

We denote V_r^+ by

$$V_r^+ = \sup_{g\in\mathcal{G}_r}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[\mathbb{E}[g(Z_i^j)] - g(Z_i^j)\right].$$

Let $K\ge 1, r\ge 0, 0<\beta\le 1$ and $B\ge 1$. We first prove that, if $V_r^+ \le \frac{r^{1/\beta}}{BK}$, then

$$\forall f\in\mathcal{F}\qquad \frac{1}{m}\sum_{i=1}^m\mathbb{E}[f(Z_i^1)] \le \frac{K}{(mn)(K-\beta)}\sum_{i=1}^m\sum_{j=1}^n\left[f(Z_i^j)\right] + \frac{r^{1/\beta}}{K}. \tag{3}$$

If $V(f)\le r$, then $g=f$. It follows that with the assumption $V_r^+\le\frac{r^{1/\beta}}{BK}$, we have

$$\frac{1}{m}\sum_{i=1}^m\mathbb{E}[f(Z_i^1)] \le \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[f(Z_i^j)\right] + \frac{r^{1/\beta}}{BK} \tag{4}$$

$$\le \frac{K}{(mn)(K-\beta)}\sum_{i=1}^m\sum_{j=1}^n\left[f(Z_i^j)\right] + \frac{r^{1/\beta}}{K}. \tag{5}$$

If $V(f)\ge r$, then $g = rf/V(f)$. It follows that with the assumption $V_r^+\le\frac{r^{1/\beta}}{BK}$, we have

$$\frac{1}{m}\sum_{i=1}^m\mathbb{E}[f(Z_i^1)] \le \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[f(Z_i^j)\right] + \frac{r^{\frac{1}{\beta}-1}V(f)}{BK} \tag{6}$$

$$\le \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[f(Z_i^j)\right] + \frac{r^{\frac{1}{\beta}-1}}{BK}\left(\frac{1}{m}\sum_{i=1}^m\mathbb{E}[f(Z_i^1)]\right)^\beta \tag{7}$$

$$\le \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[f(Z_i^j)\right] + \frac{r^{\frac{1}{\beta}-1}}{BK}\frac{1}{m}\sum_{i=1}^m\mathbb{E}[f(Z_i^1)] + \frac{1-\beta}{K}\frac{(r^{\frac{1}{\beta}-1})^{\frac{1}{1-\beta}}}{\frac{1}{1-\beta}}. \tag{8}$$

By the construction of $\mathcal{G}_r$, we can prove that $Pg^2\le r$. Combining the boundness of $\mathcal{G}_r$ and Theorem 7, with probability at least $1-\delta, \forall\delta\ge 0, \forall\alpha\ge 0$, we have

$$\sup_{g\in\mathcal{G}_r}\left[\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\left[\mathbb{E}[g(Z_i^j)] - g(Z_i^j)\right]\right] \le 2(1+\alpha)\mathcal{R}(\mathcal{G}_r) + \sqrt{\frac{8r\ln(1/\delta)}{mn}} + \left(1+\frac{2}{\alpha}\right)\frac{4M\ln(1/\delta)}{mn}$$

Next we apply "peeling" technique to bound $\mathcal{R}(\mathcal{G}_r)$. The following inequality can be obtained with probability at least $1-\exp^{-x}, \forall x \geq 0$,

$$\sup_{g\in\mathcal{G}_r}\left[\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\mathbb{E}[g(Z_i^j)]-g(Z_i^j)\right]\right] \leq \frac{10(1+\alpha)}{B}\sqrt{rr^*}+\sqrt{\frac{8rx}{mn}}+\left(1+\frac{2}{\alpha}\right)\frac{4Mx}{mn}$$

where r^* is the fixed point of sub-root function $\psi(r)$. Recall that the condition we get inequality (3) is $\sup_{g\in\mathcal{G}} Pg - P_m g \leq \frac{r^{1/\beta}}{BK}$

$$A=\frac{10(1+\alpha)}{B}\sqrt{r^*}+\sqrt{\frac{8x}{mn}}, C=\left(1+\frac{2}{\alpha}\right)\frac{4Mx}{mn}$$

Next, we need to solve

$$A\sqrt{r}+C \leq \frac{r^{1/\beta}}{BK}$$

Assume r_0 is the positive solution of $A\sqrt{r}+C=\frac{r^{1/\beta}}{BK}$. Then by Lemma 3, we have

$$\begin{aligned} r_0^{\frac{1}{\beta}} &\leq (ABK)^{\frac{2}{2-\beta}}+\frac{2BKC}{2-\beta} \\ &\leq (BK)^{\frac{2}{2-\beta}}\, 2^{\frac{\beta}{2-\beta}}\left[\left(\frac{10(1+\alpha)}{B}\right)^{\frac{2}{2-\beta}}(r^*)^{\frac{1}{2-\beta}}+\left(\frac{8x}{mn}\right)^{\frac{1}{2-\beta}}\right]+\left(1+\frac{2}{\alpha}\right)\frac{8BKMx}{(2-\beta)mn}. \end{aligned}$$

If $r^* \leq r_0$, we can take $r=r_0$. Combining inequality 3, we get

$$\begin{aligned} Pf &\leq \frac{K}{K-\beta}P_m f + r_0^{\frac{1}{\beta}}/(BK) \\ &\leq \frac{K}{K-\beta}P_m f + (BK)^{\frac{2}{2-\beta}}\, 2^{\frac{\beta}{2-\beta}}\left[\left(\frac{10(1+\alpha)}{B}\right)^{\frac{2}{2-\beta}}(r^*)^{\frac{1}{2-\beta}}+\left(\frac{8x}{mn}\right)^{\frac{1}{2-\beta}}\right] \\ &\quad + \left(1+\frac{2}{\alpha}\right)\frac{8Mx}{(2-\beta)mn}(2K)^{\frac{\beta}{2-\beta}}\,(10(1+\alpha))^{\frac{2}{2-\beta}}\max((r^*)^{\frac{1}{2-\beta}},(r^*)^{\frac{1}{\beta}}) \\ &\quad + \left(\frac{2^{\beta+3}B^2K^\beta x}{mn}\right)^{\frac{1}{2-\beta}}+\frac{(1+(2/\alpha))8Mx}{(2-\beta)mn} \end{aligned}$$

Moreover, $\psi(r)$ is set to satisfy the following conditon,

$$\psi(r) \geq BL\mathcal{R}(\mathcal{H}^*, r), B=B_2L^2,$$

$$\mathcal{R}(\mathcal{H}^*, r)=\mathbb{E}_{X,\sigma}\left[\sup_{\substack{L^2P_m(h-\widehat{h}^*)^2\leq r \\ h\in\mathcal{H}}}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j h(X_i^j)\right]$$

Combining 3 and solving quadratic equations, we get the following results

$$\begin{aligned} P_m(\ell_h-\ell_{\widehat{h}^*}) &\leq \frac{K}{K-\beta}P_m^n(\ell_h-\ell_{\widehat{h}^*})+(2K)^{\frac{\beta}{2-\beta}}20^{\frac{2}{2-\beta}}((r^*)^{\frac{1}{2-\beta}}\vee(r^*)^{\frac{1}{\beta}}) \\ &\quad + \left(\frac{2^{\beta+3}B^2K^\beta x}{mn}\right)^{\frac{1}{2-\beta}}+\frac{48LBbx}{(2-\beta)mn} \end{aligned}$$

Using the fact that $P_m^n(\ell_{\widehat{h}}-\ell_{\widehat{h}^*}) \leq 0$, we have

$$P_m(\ell_{\widehat{h}}-\ell_{\widehat{h}^*}) \leq (2K)^{\frac{\beta}{2-\beta}}20^{\frac{2}{2-\beta}}\max((r^*)^{\frac{1}{2-\beta}},(r^*)^{\frac{1}{\beta}})+\left(\frac{2^{\beta+3}B^2K^\beta x}{mn}\right)^{\frac{1}{2-\beta}}+\frac{48LBbx}{(2-\beta)mn}$$

By applying Lemma 4, in a similar way to Lei et al. (2016), we can get

$$\mathcal{L}_D(\widehat{h})-\mathcal{L}_D(\widehat{h}^*) \leq c_1\left(\frac{\log^p(mn)}{mn}\right)^{\frac{1}{2-\beta'}}+c_2\left(\frac{\log(1/\delta)}{mn}\right)^{\frac{1}{2-\beta'}},$$

□

B.3 PROOF OF THEOREM 3

Proof. We first apply Theorem 3.3 of Bartlett et al. (2005) to the class $\mathbb{E}_{z\sim D}\left[f_h(z) - f_{h^*}(z)\right]$.

$$\begin{aligned}
&\operatorname{Var}\left[\mathbb{E}_{z\sim D}\left[f_h(z) - f_{h^*}(z)\right]\right] \\
&\quad \leq \mathbb{E}_{D\sim P}\left[\mathbb{E}_{z\sim D}\left[f_h(z) - f_{h^*}(z)\right]\right]^2 \\
&\quad \leq \mathbb{E}_{D\sim P}\mathbb{E}_{z\sim D}\left[f_h(z) - f_{h^*}(z)\right]^2 \\
&\quad \leq L^2\mathbb{E}_{D\sim P}\mathbb{E}_{z\sim D}\left[h(x) - h^\star(x)\right]^2 \\
&\quad \leq BL^2\mathbb{E}_{D\sim P}\mathbb{E}_{z\sim D}\left[f_h(x,y) - f_{h^*}(x,y)\right]
\end{aligned}$$

Let $T(h) = L^2\mathbb{E}_{D\sim P}\mathbb{E}_{z\sim D}\left[h(x) - h^\star(x)\right]^2$, then we have $\forall h \in \mathcal{H}$

$$R_P\left[h - h^\star\right] \leq \frac{K}{K-1}R_D\left[h - h^\star\right] + \frac{c_1 K}{B}r^* + \frac{x\left(11LM + c_2 BK\right)}{m}$$

where $c_1 = 704, c_2 = 26, K > 1$.

$$\begin{aligned}
\mathop{\mathbb{E}}_{D,\sigma}\mathcal{R}_m(\mathcal{L}) &= \mathop{\mathbb{E}}_{D,\sigma}\left[\sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^m \sigma_i \mathbb{E}_{z\sim D_i}\left[f_h(z) - f_{h^*}(z)\right]; T(h) < r\right] \\
&\leq \mathop{\mathbb{E}}_{P^m,\sigma}\left[\sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^m \sigma_i\left[f_h(z_i) - f_{h^*}(z_i)\right]; T(h) < r\right] \\
&\leq \mathop{\mathbb{E}}_{P^m,\sigma}\left[\sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^m \sigma_i\left[f_h(z_i)\right]; T(h) < r\right] \\
&\leq L\mathop{\mathbb{E}}_{P_x^m,\sigma}\left[\sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^m \sigma_i\left[h(x_i)\right]; T(h) < r\right]
\end{aligned}$$

$$\psi(r) \geq BL\mathop{\mathbb{E}}_{P_x^m,\sigma}\left[\sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^m \sigma_i\left[h(x_i)\right]; T(h) < r\right]$$

when we set $h = \widehat{h}$

$$R_P\left[\widehat{h} - h^\star\right] \leq \frac{K}{K-1}R_D\left[\widehat{h} - h^\star\right] + \frac{c_1 K}{B}r^* + \frac{x\left(11LM + c_2 BK\right)}{m} \tag{9}$$

Next we will deal with $R_D\left[\widehat{h} - h^\star\right]$

$$R_D\left[\widehat{h} - h^\star\right] = R_D\left[\widehat{h}^* - h^\star + \widehat{h} - \widehat{h}^*\right] \leq R_D\left[\widehat{h} - \widehat{h}^*\right]. \tag{10}$$

This completes the proof.

□

B.4 PROOF OF THEOREM 4

Proof. Let $\mathcal{F}$ be the loss function class. We consider the functional $T(f) := Pf^2$ here. The structural result on covering numbers implies that

$$\log\mathcal{N}\left(\epsilon, \mathcal{F}, \|\cdot\|_2\right) \leq \log\mathcal{N}\left(\epsilon/L, \mathcal{H}, \|\cdot\|_2\right) \leq d\log^p(\gamma L/\epsilon).$$

Lemma 4 implies that the sub-root function can be chosen as

$$\psi(r) := c\left[\frac{d\log^p\left(2\gamma m^{1/2}\right)}{m} + \sqrt{\frac{rd\log^p\left(2\gamma m^{1/2}\right)}{m}}\right].$$

. In a similar way to Lei et al. (2016), let r_m^* be its fixed point then we know that

$$r_m^* = c\left[\frac{d\log^p\left(2\gamma m^{1/2}\right)}{m} + \sqrt{\frac{r^* d\log^p\left(2\gamma m^{1/2}\right)}{m}}\right].$$

Solving this equality gives $r_m^* \leq cdm^{-1}\log^p(m)$. It can be directly checked that any $f \in \mathcal{F}$ also satisfies $\|f\|_\infty \leq 4b^2$. □

C PROOFS OF THE RESULTS WITH SUB-EXPONENTIAL LOSSES

C.1 LEARNING RATES FOR SUB-EXPONENTIAL LOSSES

We state our results on the convergnece rate of generalization error for sub-exponential losses. First, we consider the participating clients. Let $S = \{Z_i^j\}_{(i,j=1,1)}^{(m,n)}$ be global data samples whose subsets $S_i = \{Z_i^j\}_{j=1}^n$ include i.i.d random variables at i-th client.

Theorem 8 (Participating error for sub-exponential losses)**.** *Suppose Z_i^j take valued in a Banach space $(\mathcal{Z}, \|\cdot\|)$ and each $\|Z_i^j\|$ is sub-exponential distributed. We denote by $\mathcal{F} = \{Z \mapsto \ell(h, Z) : h \in \mathcal{H}\}$ such that, $\forall f \in \mathcal{F}$ and $\forall z, z' \in \mathcal{Z}$, $|f(z) - f(z')| \leq L\|z - z'\|$. For any $\delta > 0$, if $mn \geq \ln(1/\delta) \geq \ln 2$, then with probability at least $1-\delta$, we have*

$$\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)| \leq 2\mathbb{E}[\mathcal{R}_{mn}(\mathcal{F})] + \max_{i\in[m]} 16eL\|\|Z_i^1\|\|_{\psi_1}\sqrt{\frac{2\ln(1/\delta)}{mn}},$$

where $\mathcal{R}_{mn}(\mathcal{F}) = \mathop{\mathbb{E}}_{\sigma}\left[\sup_{f\in\mathcal{F}} \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \sigma_i^j f(Z_i^j)\right]$.

Remark 8. *Theorem 8 can be used to bound semi-empirical excess by applying standard uniformly supremum of $h \in \mathcal{H}$. It is worth emphasizing that the bounds derived in Theorem 8 include Rademacher complexity term and $\|\|Z_i^1\|\|_{\psi_1}$ measuring the tails of input data samples $\{Z_i^j\}_{(i,j=1,1)}^{(m,n)}$. Intuitively, in regression problems, as the noise added to the lables increases, it is expected that participating error increase as well. This phenomenon is ignored under the previous bounded assumption on losses.*

Example 1 (Linear regression with unbounded loss)**.** *Let $\mathcal{Z} = (\mathcal{X}, \mathbb{R})$, where $\mathcal{X}$ is a Hilbert-space with norm $\|\cdot\|_H$. We denote by (X_i^j, Y_i^j) each sub-exponential random variables in H and $\mathbb{R}$ respectively. Let loss function ℓ be a 1-Lipschitz function (absolute function or Huber loss) and $\mathcal{F} = \{(x, y) \to f(x, y) = \ell(\langle w, x\rangle - y) : \|w\|_H \leq L\}$. If $mn \geq \ln(1/\delta) \geq \ln 2$, it follows with probability at least $1-\delta$*

$$\sup_{h\in\mathcal{H}} |\mathcal{L}_D(h) - \mathcal{L}_S(h)| \leq \frac{4}{\sqrt{mn}}\left(\max_{i\in[m]} L\|\|X_i^1\|\|_{\psi_1} + \max_{i\in[m]} \|\|Y_i^1\|\|_{\psi_1}\right)\left(1 + 6e\sqrt{\ln(\frac{1}{\delta})}\right).$$

Theorem 9 (Participation gap with unbounded loss)**.** *Under the same conditons as Theorem 8 and Example 1, we have*

$$\sup_{h\in\mathcal{H}} |\mathcal{L}_P(h) - \mathcal{L}_D(h)| \leq \frac{8}{\sqrt{m}}\left(L\|\|X_1\|\|_{\psi_1} + \|\|Y_1\|\|_{\psi_1}\right)\left(1 + 3e\sqrt{\ln(\frac{1}{\delta})}\right),$$

where X_1 is random vector with expectation across two-level distribution. This is, $\mathbb{E}[\|X_1\|] = \mathbb{E}_{D_i\sim P}\mathbb{E}_{X_1\sim D_i}\|X_1\|$. Similarly, $\mathbb{E}[\|Y_1\|] = \mathbb{E}_{D_i\sim P}\mathbb{E}_{Y_1\sim D_i}\|Y_1\|$.

Remark 9. *Combining the results of Theorem 8 and Theorem 9, it can be shown that upper bound of excess risk is of order $\mathcal{O}(\frac{1}{\sqrt{mn}} + \frac{1}{\sqrt{m}})$. Though this bound is derived under the unbounded assumption, its order is comparable with basic results derived in Theorem 1. Note that the upper bounds in Theorem 9 include terms such as $\|X_1\|$ and $\|Y_1\|$, whose underlying distributions are across our two-level framework. This reflects that the participation gap captures the generalization error caused by client sampling.*

Lemma 5 (Theorem 3.1 in Maurer & Pontil (2021)). *Let $f : \mathcal{X}^n \to \mathbb{R}$ and $X(X_1, \ldots, X_n)$ be a random vector whose elements are independent and take values in a space $\mathcal{X}$. Then for any $t \geq 0$*

$$\mathbb{P}\left\{f(X) - \mathbb{E}\left[f\left(X'\right)\right] > t\right\} \leq \exp\left(\frac{-t^2}{4e^2\left\|\sum_k \|f_k(X)\|_{\psi_1}^2\right\|_\infty + 2e\max_k\left\|\|f_k(X)\|_{\psi_1}\right\|_\infty t}\right)$$

PROOF OF THEOREM 8

Proof. We first define a vector space

$$\mathcal{B} = \left\{g : \mathcal{H} \to \mathbb{R} : \sup_{f\in\mathcal{F}}|g(f)| \leq \infty\right\}.$$

By definition, $\mathcal{B}$ is a normed space with norm $\|g\|_\mathcal{B} = \sup_{h\in\mathcal{H}}|g(f)|$. For each $Z_i^j \in \mathcal{Z}$, we define $\hat{Z}_i^j$ by $(mn)^{-1}(f(Z_i^j) - \mathbb{E}[f(Z_i'^j)])$. Thus, $\mathbb{E}[\hat{Z}_i^j] \equiv 0$, and

$$\left\|\sum_i\sum_j \hat{Z}_i^j\right\|_\mathcal{B} = \sup_{f\in\mathcal{F}}\left|\frac{1}{mn}\sum_i\sum_j(f(Z_i^j) - \mathbb{E}[f(Z_i'^j)])\right|.$$

From Lemma **??**, we have

$$\begin{aligned}\left\|\sum_i\sum_j \hat{Z}_i^j\right\|_\mathcal{B} - \mathbb{E}\left[\left\|\sum_i\sum_j \hat{Z}_i^j\right\|_\mathcal{B}\right] &\leq \left\|\sum_i\sum_j \hat{Z}_i^j - \mathbb{E}\left[\sum_i\sum_j \hat{Z}_i^j\right]\right\|_\mathcal{B} \\ &\leq \max_{i\in[m]} 8e\sqrt{mn}\|\|\hat{Z}_i^1\|_\mathcal{B}\|_{\psi_1}\sqrt{2\ln(1/\delta)}.\end{aligned}$$

Observe that

$$\begin{aligned}\|\|\hat{Z}_i^j\|_\mathcal{B}\|_{\psi_1} &= \frac{1}{mn}\left\|\sup_{f\in\mathcal{F}}\left|\mathbb{E}\left[f(Z_i^j) - f(Z_i'^j) \mid Z\right]\right|\right\|_{\psi_1} \leq \frac{1}{mn}\left\|\sup_{f\in\mathcal{F}}\mathbb{E}\left[\left|f(Z_i^j) - f(Z_i'^j)\right| \mid Z\right]\right\|_{\psi_1} \\ &\leq \frac{L}{mn}\left\|\mathbb{E}\left[\left\|Z_i^j - Z_i'^j\right\| \mid Z\right]\right\|_{\psi_1} \leq \frac{2L}{mn}\left\|\|Z_i^1\|\right\|_{\psi_1}.\end{aligned}$$

Therefore, we get $\max_{i\in[m]}\|\|\hat{Z}_i^1\|_\mathcal{B}\|_{\psi_1} \leq \max_{i\in[m]}\frac{2L}{mn}\|\|Z_i^1\|\|_{\psi_1}$, and

$$\left\|\sum_i\sum_j \hat{Z}_i^j\right\|_\mathcal{B} - \mathbb{E}\left[\left\|\sum_i\sum_j \hat{Z}_i^j\right\|_\mathcal{B}\right] \leq \max_{i\in[m]} 16eL\|\|Z_i^1\|\|_{\psi_1}\sqrt{\frac{2\ln(1/\delta)}{mn}}.$$

By symmetrization, we have

$$\mathbb{E}\left[\left\|\sum_i\sum_j \hat{Z}_i^j\right\|_\mathcal{B}\right] \leq 2\mathop{\mathbb{E}}_{S_1,\cdots,S_m}[\mathcal{R}_{mn}(\mathcal{F})] = 2\mathop{\mathbb{E}}_{S_1,\cdots,S_m}\left[\mathop{\mathbb{E}}_{\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{mn}\sum_{i=1}^n\sum_{j=1}^m\sigma_{ij}f(Z_i^j) \mid Z\right]\right].$$

□

PROOF OF EXAMPLE 1

Proof.

$$\begin{aligned}
\mathcal{R}_{mn}(\mathcal{F}) &= \mathop{\mathbb{E}}_{\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}f(Z_i^j)\right] = \mathop{\mathbb{E}}_{\sigma}\left[\sup_{\|w\|_H\le L}\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}\ell(\langle w, X_i^j\rangle - Y_i^j)\right] \\
&\le \mathop{\mathbb{E}}_{\sigma}\left[\sup_{\|w\|_H\le L}\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\langle w, \sigma_{ij}X_i^j\rangle\right] + \mathop{\mathbb{E}}_{\sigma}\left[\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}Y_i^j\right] \\
&= \mathop{\mathbb{E}}_{\sigma}\left[\sup_{\|w\|_H\le L}\frac{1}{mn}\langle w, \sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\rangle\right] + \mathop{\mathbb{E}}_{\sigma}\left[\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}Y_i^j\right] \\
&\le \frac{L}{mn}\mathop{\mathbb{E}}_{\sigma}\left[\|\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\|_H\right] + \mathop{\mathbb{E}}_{\sigma}\left[\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}Y_i^j\right]
\end{aligned}$$

Next, using Jensen's inequality we can see that

$$\mathop{\mathbb{E}}_{\sigma}\left[\|\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\|_H\right] = \mathop{\mathbb{E}}_{\sigma}\left[\left(\|\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\|_H^2\right)^{\frac{1}{2}}\right] \le \left(\mathop{\mathbb{E}}_{\sigma}\left[\|\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\|_H^2\right]\right)^{\frac{1}{2}}$$

By the assumption that $\sigma_{1,1},\dots,\sigma_{m,n}$ are independent, then we have

$$\begin{aligned}
\mathop{\mathbb{E}}_{\sigma}\left[\left\|\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\right\|_H^2\right] &= \mathop{\mathbb{E}}_{\sigma}\left[\sum_{(i,j)=(1,1)}^{(m,n)}\sum_{(s,k)=(1,1)}^{(m,n)}\sigma_{ij}\sigma_{sk}\langle X_i^j, X_s^k\rangle\right] \\
&= \mathop{\mathbb{E}}_{\sigma}\left[\sum_{(i,j)\neq(s,k)}\sigma_{ij}\sigma_{sk}\langle X_i^j, X_s^k\rangle\right] + \mathop{\mathbb{E}}_{\sigma}\left[\sum_{(i,j)=(1,1)}^{(m,n)}\sigma_{ij}^2\langle X_i^j, X_i^j\rangle\right] \\
&= \mathop{\mathbb{E}}_{\sigma}\left[\sum_{(i,j)=(1,1)}^{(m,n)}\sigma_{ij}^2\langle X_i^j, X_i^j\rangle\right] = \sum_{(i,j)}\|X_i^j\|_H^2.
\end{aligned}$$

Thus, we have

$$\mathop{\mathbb{E}}_{\sigma}\left[\|\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}X_i^j\|_H\right] \le \sqrt{\sum_{(i,j)}\|X_i^j\|_H^2}.$$

Similarly, we have

$$\mathop{\mathbb{E}}_{\sigma}\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}Y_i^j\right] \le \sqrt{\sum_{(i,j)}|Y_i^j|^2}$$

Therefore,

$$\mathcal{R}_{mn}(\mathcal{F}) \le \frac{L}{mn}\sqrt{\sum_{(i,j)}\|X_i^j\|_H^2} + \sqrt{\sum_{(i,j)}|Y_i^j|^2}$$

Since $X_1^1,\cdots,X_m^n$ are independent and $\|\cdot\|_2 \le 2\|\cdot\|_{\psi_1}$, we get

$$\mathbb{E}\left[\mathcal{R}_{mn}(\mathcal{H})\right] \le \max_{i\in[m]}\frac{2L}{\sqrt{mn}}\|\|X_i^1\|\|_{\psi_1} + \max_{i\in[m]}\frac{1}{\sqrt{mn}}\|\|Y_i^1\|\|_{\psi_1}$$

□

PROOF OF THEOREM 9

Proof. We first define a vector space

$$\mathcal{B} = \left\{ g : \mathcal{H} \to \mathbb{R} : \sup_{f \in \mathcal{F}} |g(f)| \leq \infty \right\}.$$

By definition, $\mathcal{B}$ is a normed space with norm $\|g\|_{\mathcal{B}} = \sup_{h \in \mathcal{H}} |g(f)|$. Let $f(D_i) = \mathbb{E}_{Z_i^1 \sim D_i}[f(Z_i^1)]$ and $\mathbb{E}[f(D_i)] = \mathbb{E}_{D_i \sim P}[\mathbb{E}_{Z_i^1 \sim D_i}[f(Z_i^1)]]$. For each $Z_i^1 \in \mathcal{Z}$, we define $\hat{Z}_i$ by $(1/m)([f(D_i)] - \mathbb{E}[f(D_i)])$. Thus, $\mathbb{E}[\hat{Z}_i] \equiv 0$, and

$$\left\| \sum_i \hat{Z}_i \right\|_{\mathcal{B}} = \sup_{f \in \mathcal{F}} \left| \frac{1}{m} \sum_i (f(D_i) - \mathbb{E}[f(D_i')]) \right|.$$

From Lemma **??**, we have

$$\begin{aligned} \left\| \sum_i \hat{Z}_i \right\|_{\mathcal{B}} - \mathbb{E}\left[\left\| \sum_i \hat{Z}_i \right\|_{\mathcal{B}} \right] &\leq \left\| \sum_i \hat{Z}_i^j - \mathbb{E}\left[\sum_i \hat{Z}_i \right] \right\|_{\mathcal{B}} \\ &\leq 8e\sqrt{m} \| \|\hat{Z}_i\|_{\mathcal{B}} \|_{\psi_1} \sqrt{2\ln(1/\delta)}. \end{aligned}$$

Observe that

$$\begin{aligned} \| \|\hat{Z}_i\|_{\mathcal{B}} \|_{\psi_1} &= \frac{1}{m} \left\| \sup_{f \in \mathcal{F}} |\mathbb{E}\left[f(D_i) - f(D_i') \mid D \right]| \right\|_{\psi_1} \leq \frac{1}{m} \left\| \sup_{f \in \mathcal{F}} \mathbb{E}\left[|f(D_i) - f(D_i')| \mid D \right] \right\|_{\psi_1} \\ &\leq \frac{1}{m} \left\| \mathbb{E}\left[\sup_{f \in \mathcal{F}} |f(D_i) - f(D_i')| \;\middle|\; D \right] \right\|_{\psi_1} \leq \frac{1}{m} \left\| \sup_{f \in \mathcal{F}} |f(D_i) - f(D_i')| \right\|_{\psi_1}. \end{aligned}$$

$$\mathbb{E}\left[\left| \sup_{f \in \mathcal{F}} |f(D_i) - f(D_i')| \right|^p \right] \leq \mathbb{E}\left[\left| \sup_{f \in \mathcal{F}} |f(Z_i) - f(Z_i')| \right|^p \right] \leq \mathbb{E}\left[|L \|Z_i - Z_i'\||^p \right].$$

Therefore $\left\| \sup_{f \in \mathcal{F}} |f(D_i) - f(D_i')| \right\|_p \leq \|L \|Z_i - Z_i'\|\|_p$ and $\left\| \sup_{f \in \mathcal{F}} |f(D_i) - f(D_i')| \right\|_{\psi_1} \leq \|L \|Z_i - Z_i'\|\|_{\psi_1}$. Then we get $\| \|\hat{Z}_i\|_{\mathcal{B}} \|_{\psi_1} \leq \frac{2L}{m} \| \|Z_i\| \|_{\psi_1}$, and

$$\left\| \sum_i \hat{Z}_i \right\|_{\mathcal{B}} - \mathbb{E}\left[\left\| \sum_i \hat{Z}_1 \right\|_{\mathcal{B}} \right] \leq 16eL \| \|Z_i\| \|_{\psi_1} \sqrt{\frac{2\ln(1/\delta)}{m}}.$$

By symmetrization, we have

$$\begin{aligned} \mathbb{E}\left[\left\| \sum_i \hat{Z}_i \right\|_{\mathcal{B}} \right] &\leq 2 \mathop{\mathbb{E}}_{D_1, \cdots, D_m} \left[\mathop{\mathbb{E}}_{\sigma} \left[\sup_{f \in \mathcal{F}} \frac{1}{mn} \sum_{i=1}^m \sigma_i f(D_i) \mid D \right] \right] \\ &\leq 2 \mathop{\mathbb{E}}_{Z_1, \cdots, Z_m} \left[\mathop{\mathbb{E}}_{\sigma} \left[\sup_{f \in \mathcal{F}} \frac{1}{mn} \sum_{i=1}^m \sigma_i f(Z_i) \mid Z \right] \right] \\ &\leq 2 \mathop{\mathbb{E}}_{Z_1, \cdots, Z_m} [\mathcal{R}_m(\mathcal{F})] \leq \frac{8}{\sqrt{m}} \left(L \| \|X_1\| \|_{\psi_1} + \| \|Y_1\| \|_{\psi_1} \right). \end{aligned}$$

□

D PROOFS OF SMALL-BALL BASED METHOD

PROOF OF SMALL-BALL PART OF THEOREM 5

Proof. Here we use a different decomposition to measure the distance between $\widehat{h}$ and $\widehat{h}^*$:

$$\begin{aligned}
\mathcal{L}_D(\widehat{h}) &= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{(X,Y)\sim D_i}\left[\widehat{h}(X)-Y\right]^2 \\
&= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{(X,Y)\sim D_i}\left[\widehat{h}(X)-\widehat{h}^*(X)+\widehat{h}^*(X)-Y\right]^2 \\
&\le \frac{2}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\widehat{h}(X)-\widehat{h}^*(X)\right]^2 + \frac{2}{m}\sum_{i=1}^{m}\mathbb{E}_{(X,Y)\sim D_i}\left[\widehat{h}^*(X)-Y\right]^2 \qquad (11)\\
&\le 2\|\widehat{h}-\widehat{h}^*\|_{L_2(D)}^2 + \frac{2}{m}\sum_{i=1}^{m}\mathbb{E}_{(X,Y)\sim D_i}\left[\widehat{h}^*(X)-Y\right]^2. \qquad (12)
\end{aligned}$$

The second term of the RHS of (12) is determined by the underline semi-empirical distribution D and the hypothesis space $\mathcal{H}$, therefore we focus on the first term in the following.

For any $h \in \mathcal{H}$ and $u > 0$,

$$|\{(i,j) : h(X_i^j) > u\}| = \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|h(X_i^j)|\ge u\}}$$

Also,

$$\begin{aligned}
\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|h(X_i^j)|\ge u\}} &= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\mathbb{1}_{\{|h(X)|\ge 2u\}}\right] \\
&\quad + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|h(X_i^j)|\ge u\}} - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\mathbb{1}_{\{|h(X)|\ge 2u\}}\right]
\end{aligned}$$

Let $\phi_u : \mathbb{R} \to [0,1]$ be the function

$$\phi_u(t) = \begin{cases} 0 & t \le u, \\ \frac{t}{u} - 1 & u \le t \le 2u, \\ 1 & t \ge 2u. \end{cases} \qquad (13)$$

Note that for every $t \in \mathbb{R}$, $\phi_u(t) \ge \mathbb{1}_{\{t\ge 2u\}}$ and $\phi_u(t) \le \mathbb{1}_{\{t\ge u\}}$. Thus,

$$\begin{aligned}
\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|h(X_i^j)|\ge u\}} &\ge \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\mathbb{1}_{\{|h(X)|\ge 2u\}}\right] \\
&\quad + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_u(|h(X_i^j)|) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\phi_u(|h(X)|)\right] \\
&\ge \inf_{h\in\mathcal{H}}\mathbb{P}(|h(X)|\ge 2u) \\
&\quad - \sup_{h\in\mathcal{H}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_u(|h(X_i^j)|) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\phi_u(|h(X)|)\right]\right|.
\end{aligned}$$

Since function $\phi_u(t)$ is bounded by 1, using Mcdiarmid's inequality, we get that, for every $\delta > 0$, with probability at least $1 - 2\exp(-2\delta^2)$,

$$\begin{aligned}
&\sup_{h\in\mathcal{H}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_u(|h(X_i^j)|) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\phi_u(|h(X)|)\right]\right| \\
&\quad \le \mathbb{E}\sup_{h\in\mathcal{H}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_u(|h(X_i^j)|) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\phi_u(|h(X)|)\right]\right| + \frac{\delta}{\sqrt{mn}}.
\end{aligned}$$

By the Lipschitz property of $\phi_u(|t|)$ and the symmetrization theorem, we have

$$\mathbb{E}\sup_{h\in\mathcal{H}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_u(|h(X_i^j)|)-\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\phi_u(|h(X)|)\right]\right|\leq\frac{4}{u}\mathbb{E}\sup_{h\in\mathcal{H}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j h(X_i^j)\right|.$$

Therefore, for every $h\in\mathcal{H}$, it follows that with probability at least $1-2\exp(-2\delta^2)$, we have

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|h(X_i^j)|\geq u\}}\geq\inf_{h\in\mathcal{H}}\mathbb{P}(|h(X)|\geq 2u)-\frac{4}{u}\mathbb{E}\sup_{h\in\mathcal{H}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j h(X_i^j)\right|-\frac{\delta}{\sqrt{mn}}. \tag{14}$$

The first term on the RHS can be bounded by small ball condtion. Let $\mathcal{H}^*=\mathcal{H}-\widehat{h}^*$. We first prove that $\mathcal{H}^*$ is star-shaped around 0. For every $h-\widehat{h}^*\in\mathcal{H}^*$ and $0\leq\lambda\leq 1$, we have $\lambda(h-\widehat{h}^*)=\lambda h+(1-\lambda)\widehat{h}^*-\widehat{h}^*$. Since $\mathcal{H}$ is convex, it follows that $\lambda h+(1-\lambda)\widehat{h}^*\in\mathcal{H}$. Then the claim follows because $\lambda(h-\widehat{h}^*)\in\mathcal{H}-\widehat{h}^*$.

Assume that these exsits $\tau>0$ for which $Q_{\mathcal{H}^*}(2\tau)>0$. The for every $s\geq\omega(\mathcal{H}^*,\tau Q_{\mathcal{H}^*}(2\tau)/16)$, we have

$$\mathbb{E}\sup_{h\in\mathcal{H}^*\cap s\mathcal{D}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j h(X_i^j)\right|\leq\frac{\tau Q_{\mathcal{H}^*}(2\tau)}{16}s.$$

Let $\mathcal{G}$ be a function class associated with $\mathcal{H}^*$

$$\mathcal{G}=\left\{\frac{h}{s}:h\in\mathcal{H}^*\cap s\mathcal{D}\right\}\subset\mathcal{D},$$

where $\mathcal{D}$ is the unit ball with respect with $L_2(D)$ and D is the semi-empirical distribution.

$$\mathbb{E}\sup_{g\in\mathcal{G}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j g(X_i^j)\right|=\mathbb{E}\sup_{h\in\mathcal{H}^*\cap s\mathcal{D}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j\frac{h(X_i^j)}{s}\right|\leq\frac{\tau Q_{\mathcal{H}^*}(2\tau)}{16}\leq\frac{\tau Q_{\mathcal{G}}(2\tau)}{16},$$

where the last inequality follows from $Q_{\mathcal{G}}(2\tau)\geq Q_{\mathcal{H}^*}(2\tau)$. By equation (14) applied to the function class G, it follows that with probability at least $1-2\exp(-2\delta^2)$

$$\begin{aligned}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|g(X_i^j)|\geq u\}}&\geq\inf_{g\in\mathcal{G}}\mathbb{P}(|g(X)|\geq 2u)-\frac{4}{u}\mathbb{E}\sup_{g\in\mathcal{G}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j g(X_i^j)\right|-\frac{\delta}{\sqrt{mn}}\\&\geq Q_G(2u)-\frac{4}{u}\mathbb{E}\sup_{g\in\mathcal{G}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j g(X_i^j)\right|-\frac{\delta}{\sqrt{mn}}\\&\geq Q_G(2u)-\frac{4}{u}\frac{\tau Q_{\mathcal{G}}(2\tau)}{16}-\frac{\delta}{\sqrt{mn}}.\end{aligned}$$

Now, setting

$$u=\tau\quad\delta=\frac{\sqrt{mn}Q_{\mathcal{G}}(2\tau)}{2},$$

it follows that with probability at least $1-2\exp(-mnQ_{\mathcal{G}}(2\tau)/2)$

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|g(X_i^j)|\geq\tau\}}\geq Q_G(2\tau)-\frac{Q_{\mathcal{G}}(2\tau)}{4}-\frac{Q_{\mathcal{G}}(2\tau)}{2}=\frac{Q_{\mathcal{G}}(2\tau)}{4}.$$

Using the condition $Q_{\mathcal{G}}(2\tau)\geq Q_{\mathcal{H}^*}(2\tau)$, for every $s\geq\omega(\mathcal{H}^*,\tau Q_{\mathcal{H}^*}(2\tau)/16)$, it follows that with probability at least $1-2\exp(-mnQ_{\mathcal{H}^*}(2\tau)/2)$, we get

$$\inf_{g\in\mathcal{G}}|\{(i,j):|g(X_i^j)|>\tau\}|=\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{1}_{\{|g(X_i^j)|\geq\tau\}}\geq\frac{mnQ_{\mathcal{G}}(2\tau)}{4}\geq\frac{mnQ_{\mathcal{H}^*}(2\tau)}{4}. \tag{15}$$

For every $h - \widehat{h}^* \in \mathcal{H}^*$ that satisfies $\|h - \widehat{h}^*\|_{L_2(D)} \geq s$, since $\mathcal{H} - \widehat{h}^*$ is star-shaped around 0, we have $\left(s/\|h - \widehat{h}^*\|_{L_2(D)}\right)(h - \widehat{h}^*) \in \mathcal{H}^* \cap s\mathcal{D}$. Thus, $(h - \widehat{h}^*)/\|h - \widehat{h}^*\|_{L_2(D)} \in \mathcal{G}$.

Combining equation (15), if $s \geq \omega(\mathcal{H}^*, \tau Q_{\mathcal{H}^*}(2\tau)/16)$, then for every $h \in \mathcal{H}$ that satisfies $\|h - \widehat{h}^*\|_{L_2(D)} \geq s$, if follows that with probability at least $1 - 2\exp(-mnQ_{\mathcal{H}^*}(2\tau)/2)$, one has

$$\left|\{(i,j) : |(h - \widehat{h}^*)(X_i^j)| > \tau\|h - \widehat{h}^*\|_{L_2(D)}\}\right| \geq \frac{mnQ_{\mathcal{H}^*}(2\tau)}{4}$$

Therefore, on that event, we get

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(h - \widehat{h}^*)^2(X_i^j) > \frac{\tau^2}{4}Q_{\mathcal{H}^*}(2\tau)\|h - \widehat{h}^*\|_{L_2(D)}^2. \tag{16}$$

Note that $\mathcal{H}^* \subset \mathcal{H} - \mathcal{H}$, the same conclusion holds with $Q_{\mathcal{H}-\mathcal{H}}(2\tau)$ replacing the larger $Q_{\mathcal{H}^*}$. That is, if $s \geq \omega(\mathcal{H}^*, \tau Q_{\mathcal{H}-\mathcal{H}}(2\tau)/16)$, then for every $h \in \mathcal{H}$ that satisfies $\|h - \widehat{h}^*\|_{L_2(D)} \geq s$, if follows that with probability at least $1 - 2\exp(-mnQ_{\mathcal{H}-\mathcal{H}}(2\tau)/2)$, one has

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(h - \widehat{h}^*)^2(X_i^j) > \frac{\tau^2}{4}Q_{\mathcal{H}-\mathcal{H}}(2\tau)\|h - \widehat{h}^*\|_{L_2(D)}^2.$$

Combining equation (2), with the same conditions we get

$$\mathcal{L}_S(h) - \mathcal{L}_S(\widehat{h}^*) \geq \frac{2}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j) + \frac{\tau^2}{4}Q_{\mathcal{H}-\mathcal{H}}(2\tau)\|h - \widehat{h}^*\|_{L_2(D)}^2. \tag{17}$$

Since $\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\xi(h - \widehat{h}^*)(X)\right] \geq 0$, then we have

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j) \geq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\xi(h - \widehat{h}^*)(X)\right].$$

We recall the definition (4) and set $\kappa_{mn}(\eta, \delta/4) \equiv \kappa_{mn}$. Therefore, with probability at least $1-\delta/4$, if $\|h - \widehat{h}^*\|_{L_2(D)} \leq \kappa$ we have

$$\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_i^j\xi_i^j(h - \widehat{h}^*)(X_i^j)\right| \leq \eta\kappa^2.$$

By the symmetrization theorem, we get

$$\sup_{h-\widehat{h}^*\in\mathcal{H}-\widehat{h}^*\cap\kappa\mathcal{D}}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\xi(h - \widehat{h}^*)(X)\right]\right| \leq 4\eta\kappa^2$$

For function $h \in \mathcal{H}$ that satisfies $\|h - \widehat{h}^*\|_{L_2(D)} \geq \kappa$, we have

$$\left(\kappa/\|h - \widehat{h}^*\|_{L_2(D)}\right)(h - \widehat{h}^*) \in \mathcal{H} - \widehat{h}^* \cap \kappa\mathcal{D}.$$

Thus, for function $h \in \mathcal{H}$ that satisfies $\|h - \widehat{h}^*\|_{L_2(D)} \geq \kappa$, we have

$$\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\xi(h - \widehat{h}^*)(X)\right]\right| \leq 4\eta\kappa^2\frac{\|h - \widehat{h}^*\|_{L_2(D)}}{\kappa}.$$

Since $\|h - \widehat{h}^*\|_{L_2(D)} \geq \kappa$, one has

$$\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_i^j(h - \widehat{h}^*)(X_i^j) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}_{X\sim D_i}\left[\xi(h - \widehat{h}^*)(X)\right]\right| \leq 4\eta\|h - \widehat{h}^*\|_{L_2(D)}^2. \tag{18}$$

Combining (2) and (18), if $\|h - \widehat{h}^*\|_{L_2(D)} \geq \max(\kappa, s)$, it follows that with probability at least $1 - \delta - 2\exp(-mnQ_{\mathcal{H}^*}(2\tau)/2)$,

$$\mathcal{L}_S(h) - \mathcal{L}_S(\widehat{h}^*) \geq \|h - \widehat{h}^*\|^2_{L_2(D)} \left(\frac{\tau^2}{4} Q_{\mathcal{H}-\mathcal{H}}(2\tau) - 8\eta \right).$$

Consider $\eta < \tau^2 Q_{\mathcal{H}-\mathcal{H}}(2\tau)/32$, we get

$$\mathcal{L}_S(h) - \mathcal{L}_S(\widehat{h}^*) \geq 0.$$

For sum of Sub-Weibull random variables $X_1, \cdots, X_n$, we have the follows inequality

$$\mathbb{P}\left(\left| \sum_{i=1}^n a_i X_i \right| \geq 2eC(\alpha)\|b\|_2 \sqrt{t} + 2eL_n^*(\alpha) t^{1/\alpha} \|b\|_{\beta(\alpha)} \right) \leq 2e^{-t} \quad \text{for all} \quad t \geq 0.$$

Thus,

$$\mathbb{P}\left(\left| \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \xi_i^j (h - \widehat{h}^*)(X_i^j) - \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{X \sim D_i} \left[\xi(h - \widehat{h}^*)(X) \right] \right| \geq \eta_0 \|h - \widehat{h}^*\|^2_{L_2(D)} \right)$$
$$\leq 2\exp\left\{ -\frac{1}{4} \left(\frac{(mn\eta_0)\|h - \widehat{h}^*\|^4_{L_2(D)}}{\sum_{i=1}^m \sum_{j=1}^n 2 \left\| V_i^j \right\|^2_{\psi_\alpha}} \wedge \frac{(mn\eta_0)\|h - \widehat{h}^*\|^2_{L_2(D)}}{\max_{(1,1) \leq (i,j) \leq (m,n)} \left\| V_i^j \right\|_{\psi_\alpha}} \right) \right\},$$

where

$$V_i^j = \xi_i^j (h - \widehat{h}^*)(X_i^j) - \mathbb{E}\left[\xi_i^j (h - \widehat{h}^*)(X_i^j) \right]$$

To make sure that the probability tends to zero as mn increase, we could chose $\|h - \widehat{h}^*\|^2_{L_2(D)} \geq \kappa_{mn} \geq (mn)^{-\frac{1}{4}+\iota}$, where $0 < \iota < \frac{1}{4}$.

□

PROOF OF THEOREM 6

Proof. For any $h \in \mathcal{H}$ and $u > 0$,

$$|\{i : |\mathbb{E}[h(X_i^1)]| > u\}| = \sum_{i=1}^m \mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq u\}}$$

Also,

$$\frac{1}{m} \sum_{i=1}^m \mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq u\}} = \mathop{\mathbb{E}}_{D_i \sim P} \left[\mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq 2u\}} \right]$$
$$+ \frac{1}{m} \sum_{i=1}^m \mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq u\}} - \mathop{\mathbb{E}}_{D_i \sim P} \left[\mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq 2u\}} \right]$$

Let $\phi_u : \mathbb{R} \to [0, 1]$ be the function

$$\phi_u(t) = \begin{cases} 0 & t \leq u, \\ \frac{t}{u} - 1 & u \leq t \leq 2u, \\ 1 & t \geq 2u. \end{cases} \tag{19}$$

Note that for every $t \in \mathbb{R}$, $\phi_u(t) \geq \mathbb{1}_{\{t \geq 2u\}}$ and $\phi_u(t) \leq \mathbb{1}_{\{t \geq u\}}$. Thus,

$$\frac{1}{m} \sum_{i=1}^m \mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq u\}} \geq \mathop{\mathbb{E}}_{D_i \sim P} \left[\mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]| \geq 2u\}} \right]$$
$$+ \frac{1}{m} \sum_{i=1}^m \phi_u(|\mathbb{E}[h(X_i^1)]|) - \mathop{\mathbb{E}}_{D_i \sim P} \left[\phi_u(|\mathbb{E}[h(X_i^1)]|) \right]$$
$$\geq \inf_{h \in \mathcal{H}} \mathbb{P}(|\mathbb{E}[h(X_i^1)]| \geq 2u)$$
$$- \sup_{h \in \mathcal{H}} \left| \frac{1}{m} \sum_{i=1}^m \phi_u(|\mathbb{E}[h(X_i^1)]|) - \mathop{\mathbb{E}}_{D_i \sim P} \left[\phi_u(|\mathbb{E}[h(X_i^1)]|) \right] \right|.$$

Since function $\phi_u(t)$ is bounded by 1, using Mcdiarmid's inequality, we get that, for every $\delta > 0$, with probability at least $1 - 2\exp(-2\delta^2)$,

$$\sup_{h\in\mathcal{H}} \left| \frac{1}{m}\sum_{i=1}^{m} \phi_u(|\mathbb{E}[h(X_i^1)]|) - \mathop{\mathbb{E}}_{D_i\sim P}\left[\phi_u(|\mathbb{E}[h(X_i^1)]|)\right] \right|$$
$$\leq \mathbb{E}\sup_{h\in\mathcal{H}} \left| \frac{1}{m}\sum_{i=1}^{m} \phi_u(|\mathbb{E}[h(X_i^1)]|) - \mathop{\mathbb{E}}_{D_i\sim P}\left[\phi_u(|\mathbb{E}[h(X_i^1)]|)\right] \right| + \frac{\delta}{\sqrt{m}}.$$

By the Lipschitz property of $\phi_u(|t|)$ and the symmetrization theorem, we have

$$\mathbb{E}\sup_{h\in\mathcal{H}} \left| \frac{1}{m}\sum_{i=1}^{m} \phi_u(|\mathbb{E}[h(X_i^1)]|) - \mathop{\mathbb{E}}_{D_i\sim P}\left[\phi_u(|\mathbb{E}[h(X_i^1)]|)\right] \right| \leq \frac{4}{u}\mathbb{E}\sup_{h\in\mathcal{H}} \left| \frac{1}{m}\sum_{i=1}^{m} \sigma_i h(X_i) \right|,$$

where X_i is the random sampled across two-level distribution framework.

Therefore, for every $h \in \mathcal{H}$, it follows that with probability at least $1 - 2\exp(-2\delta^2)$, we have

$$\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{\{|\mathbb{E}[h(X_i^1)]|\geq u\}} \geq \inf_{h\in\mathcal{H}} \mathbb{P}(|\mathbb{E}[h(X_i^1)]| \geq 2u) - \frac{4}{u}\mathbb{E}\sup_{h\in\mathcal{H}} \left| \frac{1}{m}\sum_{i=1}^{m} \sigma_i h(X_i) \right| - \frac{\delta}{\sqrt{m}}. \quad (20)$$

The first term on the RHS can be bounded by small ball condtion. Let $\mathcal{H}^* = \mathcal{H} - h^*$. We first prove that $\mathcal{H}^*$ is star-shaped around 0. For every $h - h^* \in \mathcal{H}^*$ and $0 \leq \lambda \leq 1$, we have $\lambda(h - h^*) = \lambda h + (1-\lambda)h^* - h^*$. Since $\mathcal{H}$ is convex, it follows that $\lambda h + (1-\lambda)h^* \in \mathcal{H}$. Then the claim follows because $\lambda(h - h^*) \in \mathcal{H} - h^*$.

Assume that these exsits $\tau > 0$ for which $Q_{\mathcal{H}^*}(2\tau, P) > 0$. The for every $s \geq \omega_m(\mathcal{H}^*, \tau Q_{\mathcal{H}^*}(2\tau)/16, P)$, we have

$$\mathbb{E}\sup_{h\in\mathcal{H}^*\cap sB_m} \left| \frac{1}{m}\sum_{i=1}^{m} \sigma_i h(X_i) \right| \leq \frac{\tau Q_{\mathcal{H}^*}(2\tau, P)}{16} s.$$

Let $\mathcal{G}$ be a function class associated with $\mathcal{H}^*$

$$\mathcal{G} = \left\{ \frac{h}{s} : h \in \mathcal{H}^* \cap sB_m \right\} \subset B_m,$$

where B_m is the unit ball with respect with $L_2(P)$ and P is the population distribution.

$$\mathbb{E}\sup_{g\in\mathcal{G}} \left| \frac{1}{mn}\sum_{i=1}^{m} \sigma_i g(X_i) \right| = \mathbb{E}\sup_{h\in\mathcal{H}^*\cap sB_m} \left| \frac{1}{m}\sum_{i=1}^{m} \sigma_i \frac{h(X_i)}{s} \right| \leq \frac{\tau Q_{\mathcal{H}^*}(2\tau, P)}{16} \leq \frac{\tau Q_{\mathcal{G}}(2\tau, P)}{16},$$

where the last inequality follows from $Q_{\mathcal{G}}(2\tau, P) \geq Q_{\mathcal{H}^*}(2\tau, P)$. By equation (20) applied to the function class G, it follows that with probability at least $1 - 2\exp(-2\delta^2)$

$$\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{\{|\mathbb{E}[g(X_i^1)]|\geq u\}} \geq \inf_{g\in\mathcal{G}} \mathbb{P}(|\mathbb{E}[g(X_i^1)]| \geq 2u) - \frac{4}{u}\mathbb{E}\sup_{g\in\mathcal{G}} \left| \frac{1}{m}\sum_{i=1}^{m} \sigma_i g(X_i) \right| - \frac{\delta}{\sqrt{m}}.$$
$$\geq Q_G(2u) - \frac{4}{u}\mathbb{E}\sup_{g\in\mathcal{G}} \left| \frac{1}{m}\sum_{i=1}^{m} \sigma_i g(X_i) \right| - \frac{\delta}{\sqrt{m}}$$
$$\geq Q_G(2u) - \frac{4}{u}\frac{\tau Q_{\mathcal{G}}(2\tau, P)}{16} - \frac{\delta}{\sqrt{m}}.$$

Now, setting

$$u = \tau \quad \delta = \frac{\sqrt{m} Q_{\mathcal{G}}(2\tau, P)}{2},$$

it follows that with probability at least $1 - 2\exp(-mQ_{\mathcal{G}}(2\tau, P)/2)$

$$\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{\{|\mathbb{E}[g(X_i^1)]|\geq u\}} \geq Q_G(2\tau, P) - \frac{Q_{\mathcal{G}}(2\tau, P)}{4} - \frac{Q_{\mathcal{G}}(2\tau, P)}{2} = \frac{Q_{\mathcal{G}}(2\tau, P)}{4}.$$

Using the condition $Q_{\mathcal{G}}(2\tau, P) \geq Q_{\mathcal{H}^*}(2\tau, P)$, for every $s \geq \omega_m(\mathcal{H}^*, \tau Q_{\mathcal{H}^*}(2\tau, P)/16)$, it follows that with probability at least $1 - 2\exp(-mQ_{\mathcal{H}^*}(2\tau, P)/2)$, we get

$$\inf_{g \in \mathcal{G}} |\{i : |\mathbb{E}[g(X_i^1)]| > \tau\}| = \sum_{i=1}^m \mathbb{1}_{\{|\mathbb{E}[g(X_i^1)]| \geq \tau\}} \geq \frac{mQ_{\mathcal{G}}(2\tau, P)}{4} \geq \frac{mQ_{\mathcal{H}^*}(2\tau, P)}{4}. \tag{21}$$

For every $h - h^* \in \mathcal{H}^*$ that satisfies $\|h - h^*\|_{L_2(P)} \geq s$, since $\mathcal{H} - h^*$ is star-shaped around 0, we have $\left(s/\|h - h^*\|_{L_2(P)}\right)(h - h^*) \in \mathcal{H}^* \cap sB_m$. Thus, $(h - h^*)/\|h - h^*\|_{L_2(P)} \in \mathcal{G}$.

Combining equation (21), if $s \geq \omega_m(\mathcal{H}^*, \tau Q_{\mathcal{H}^*}(2\tau, P)/16)$, then for every $h \in \mathcal{H}$ that satisfies $\|h - h^*\|_{L_2(P)} \geq s$, if follows that with probability at least $1 - 2\exp(-mQ_{\mathcal{H}^*}(2\tau, P)/2)$, one has

$$\left|\{i : |\mathbb{E}[(h - h^*)(X_i^1)]| > \tau\|h - h^*\|_{L_2(P)}\}\right| \geq \frac{mQ_{\mathcal{H}^*}(2\tau, P)}{4}$$

Therefore, on that event, we get

$$\frac{1}{m}\sum_{i=1}^m \mathbb{E}\left[(h - h^*)^2(X_i)\right] \geq \frac{1}{m}\sum_{i=1}^m \left|\mathbb{E}[(h - h^*)(X_i^1)]\right|^2 \geq \frac{\tau^2}{4} Q_{\mathcal{H}^*}(2\tau)\|h - h^*\|_{L_2(P)}^2. \tag{22}$$

Note that $\mathcal{H}^* \subset \mathcal{H} - \mathcal{H}$, the same conclusion holds with $Q_{\mathcal{H}-\mathcal{H}}(2\tau, P)$ replacing the larger $Q_{\mathcal{H}^*}(2\tau, P)$. That is, if $s \geq \omega_m(\mathcal{H}^*, \tau Q_{\mathcal{H}-\mathcal{H}}(2\tau, P)/16)$, then for every $h \in \mathcal{H}$ that satisfies $\|h - h^*\|_{L_2(P)} \geq s$, if follows that with probability at least $1 - 2\exp(-mQ_{\mathcal{H}-\mathcal{H}}(2\tau, P)/2)$, one has

$$\frac{1}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{X_i \sim D_i}\left[(h - h^*)^2(X_i)\right] > \frac{\tau^2}{4} Q_{\mathcal{H}-\mathcal{H}}(2\tau, P)\|h - h^*\|_{L_2(P)}^2.$$

Combining equation (**??**), with the same conditions we get

$$\mathcal{L}_D(h) - \mathcal{L}_D(h^*) \geq \frac{2}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{(X_i, Y_i) \sim D_i}[\xi_i(h - h^*)(X_i)] + \frac{\tau^2}{4} Q_{\mathcal{H}-\mathcal{H}}(2\tau, P)\|h - h^*\|_{L_2(P)}^2. \tag{23}$$

Since $\mathbb{E}_{D_i \sim P}\mathbb{E}_{(X_i, Y_i) \sim D_i}[\xi(h - h^*)(X_i)] \geq 0$, then we have

$$\frac{1}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{(X_i, Y_i) \sim D_i}[\xi_i(h - h^*)(X_i)] \geq \frac{1}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{(X_i, Y_i) \sim D_i}[\xi_i(h - h^*)(X_i)] - \mathbb{E}_{D_i \sim P}\mathbb{E}_{(X_i, Y_i) \sim D_i}[\xi(h - h^*)(X_i)].$$

We recall the definition (6) and set $\kappa_m(\eta, \delta/4) \equiv \kappa_m$. Therefore, with probability at least $1 - \delta/4$, if $\|h - \widehat{h}^*\|_{L_2(P)} \leq \kappa$ we have

$$\left|\frac{1}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{(X_i, Y_i) \sim D_i}[\xi_i(h - h^*)(X_i)]\right| \leq \eta\kappa^2.$$

By the symmetrization theorem, we get

$$\sup_{h - h^* \in \mathcal{H} - h^* \cap \kappa B_m}\left|\frac{1}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{(X_i, Y_i) \sim D_i}[\xi_i(h - h^*)(X_i)] - \mathbb{E}_{D_i \sim P}\mathbb{E}_{(X_i, Y_i) \sim D_i}[\xi(h - h^*)(X_i)]\right| \leq 4\eta\kappa^2$$

For function $h \in \mathcal{H}$ that satisfies $\|h - h^*\|_{L_2(P)} \geq \kappa$, we have

$$\left(\kappa/\|h - h^*\|_{L_2(P)}\right)(h - h^*) \in \mathcal{H} - h^* \cap \kappa B_m.$$

Thus, for function $h \in \mathcal{H}$ that satisfies $\|h - h^*\|_{L_2(P)} \geq \kappa$, we have

$$\left|\frac{1}{m}\sum_{i=1}^m \mathop{\mathbb{E}}_{(X_i, Y_i) \sim D_i}[\xi_i(h - h^*)(X_i)] - \mathbb{E}_{D_i \sim P}\mathbb{E}_{(X_i, Y_i) \sim D_i}[\xi(h - h^*)(X_i)]\right| \leq 4\eta\kappa^2 \frac{\|h - h^*\|_{L_2(P)}}{\kappa}.$$

Since $\|h - h^*\|_{L_2(P)} \geq \kappa$, one has

$$\left| \frac{1}{m} \sum_{i=1}^{m} \mathop{\mathbb{E}}_{(X_i,Y_i)\sim D_i} [\xi_i(h - h^*)(X_i)] - \mathbb{E}_{D_i \sim P} \mathbb{E}_{(X_i,Y_i)\sim D_i} [\xi(h - h^*)(X_i)] \right| \leq 4\eta \|h - h^*\|_{L_2(P)}^2. \tag{24}$$

Combining (**??**) and (24), if $\|h - h^*\|_{L_2(P)} \geq \max(\kappa, s)$, it follows that with probability at least $1 - \delta - 2\exp(-mQ_{\mathcal{H}^*}(2\tau, P)/2)$,

$$\mathcal{L}_D(h) - \mathcal{L}_D(h^*) \geq \|h - h^*\|_{L_2(P)}^2 \left(\frac{\tau^2}{4} Q_{\mathcal{H}-\mathcal{H}}(2\tau, P) - 8\eta \right).$$

Consider $\eta < \tau^2 Q_{\mathcal{H}-\mathcal{H}}(2\tau, P)/32$, we get

$$\mathcal{L}_D(h) - \mathcal{L}_D(h^*) \geq 0.$$

For sum of Sub-Weibull random variables $X_1, \cdots, X_n$, we have the follows inequality

$$\mathbb{P}\left(\left| \sum_{i=1}^{n} a_i X_i \right| \geq 2eC(\alpha)\|b\|_2 \sqrt{t} + 2eL_n^*(\alpha) t^{1/\alpha} \|b\|_{\beta(\alpha)} \right) \leq 2e^{-t} \quad \text{for all} \quad t \geq 0.$$

Thus,

$$\begin{aligned} &\mathbb{P}\left(\left| \frac{1}{m} \sum_{i=1}^{m} \xi_i^j (h - \widehat{h}^*)(X_i) - \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}_{X\sim D_i} \left[\xi(h - \widehat{h}^*)(X) \right] \right| \geq \eta_0 \|h - \widehat{h}^*\|_{L_2(D)}^2 \right) \\ &\quad \leq 2\exp\left\{ -\frac{1}{4} \left(\frac{(m\eta_0)\|h - \widehat{h}^*\|_{L_2(D)}^4}{\sum_{i=1}^{m}\sum_{j=1}^{n} 2 \left\| V_i^j \right\|_{\psi_\alpha}^2} \wedge \frac{(m\eta_0)\|h - \widehat{h}^*\|_{L_2(D)}^2}{\max_{(1,1)\leq(i,j)\leq(m,n)} \left\| V_i^j \right\|_{\psi_\alpha}} \right) \right\}, \end{aligned}$$

where

$$V_i^j = \xi_i^j (h - \widehat{h}^*)(X_i^j) - \mathbb{E}\left[\xi_i^j (h - \widehat{h}^*)(X_i^j) \right]$$

To make sure that the probability tends to zero as m increase, we could chose $\|h - \widehat{h}^*\|_{L_2(P)}^2 \geq \kappa_m \geq (m)^{-\frac{1}{4}+\iota}$, where $0 < \iota < \frac{1}{4}$.

□