The non-periodic Littlewood conjecture

***Abstract*—In the 1920's, Littlewood conjectured that, among all trigonometric polynomials having only 0 or 1 coefficients, the Dirichlet kernel is the one that has minimal L^1-norm. This has been essentially solved independently by McGehee, Pigno, Smith [5] and by Konyagyn [4] in 1981. A few years later, Nazarov [6] extended the result to non-harmonic trigonometric polynomials. We here further extend these results by replacing the L^1-norms by $\mathcal{B}_1$-Besikovich norms.**[1]

***Index Terms*—Hardy's Inequality, Littlewood conjecture, Besicovich norm**

I. Introduction

The aim of this note is to give a control of coefficients of a non-harmonic (non-periodic) trigonometric polynomial $\sum c_j e^{2i\pi\lambda_j t}$ in terms of L^1-norms of this polynomial.

Let us start with some facts about L^2-theory where those questions have a long ranging history. First, note that the problem is trivial for harmonic trigonometric polynomials ($\lambda_j = j$) for $L^2([-1/2, 1/2])$-norms, thanks to Parseval's relation and orthogonoality. The same is true for non-harmonic trigonometric polynomial in the $\mathcal{B}^2$-Besikovitch norm. The question becomes more interresting for $L^2([-T/2, T/2])$-norms (whith T large enough) and has been answered in a celevrated paper by Ingham [3]: *if* $|\lambda_j - \lambda_k| \geq 1$ *then for* $T > 1$,

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \geq \frac{\pi^2(T^2-1)}{8T^2}\sum_{k=1}^{N}|a_k|^2.$$

Note that the somewhat dual question of whether the non-harmonic Fourier coefficients $\left\langle f, e^{2i\pi\lambda_k t}\right\rangle_{L^2([-T/2,T/2])}$ are controlled and control f can be seen as the origin of frame theory can be traced back to the work of Duffin and Schaeffer [**?**].

When considering L^1-type norms the situation is much more delicate and the (almost) optimal lower bound is due independently to McGehee, Pigno, Smith [5], Konyagyn [4] in their celebrated solution of the Littlewood conjecture:

Theorem I.1. *There exists a constant C such that, for every N, if $\lambda_1, \dots, \lambda_N$ are distinct integers and $(a_j)_{j=1,\dots,N} \subset \mathbb{C}$ then*

$$\sum_{k=1}^{N}\frac{|a_k|}{k} \leq C\int_{-1/2}^{1/2}\left|\sum_{j=1}^{N} a_k e^{2i\pi\lambda_j t}\right|\mathrm{d}t.$$

The constant C is explicit ($C = 30$) and the best possible constant being $C = \frac{\pi^2}{4}$ (in view of standard estimates of the Dirichlet kernel) is still conjectured. The result was then extended by Nazarov [6] to the non-periodic case, *i.e.* real frequencies $(\lambda_j)_{j=1,\dots,N} \subset \mathbb{R}$ with $|\lambda_j - \lambda_k| \geq 1$ with the cost of slightly enlarging the interval $[-1/2, 1/2]$ to $[-\Delta, \Delta]$, $\Delta > 1/2$. However, the constant is not explicit. Further, Hudson, Leckband [2] have shown how to deduce a version of this result for the Besikovitch norm from Theorem 1 (the converse direction being obvious):

Theorem I.2. *For every N, if $\lambda_1, \dots, \lambda_N$ are distinct* real *numbers and $(a_j)_{j=1,\dots,N} \subset \mathbb{C}$ then*

$$\sum_{k=1}^{N}\frac{|a_k|}{k} \leq C\lim_{T\to+\infty}\fint_{-[T/2,T/2]}\left|\sum_{j=1}^{N} a_k e^{2i\pi\lambda_j t}\right|\mathrm{d}t.$$

In a forthcomming papper, we have shown how a modification of Nazarov's argument allows to obtain this result directly with the best constants today, thanks to the introduction of several parameters that we optimize in the last step. The aim of this note is to present a simplification of the argument that avoids the parameter optimisation, but at the cost of not computing constants explicitely.

II. The proof

We write $A \lesssim B$ to mean $A \leq CB$ for some C.

Next observe that in Theorem I.2, we may scale the sequence λ_j and thus assume that $|\lambda_j - \lambda_k| \geq 1$.

We write $|a_j| = a_j u_j$, $|u_j| = 1$. We then write

$$\Phi(t) = \sum_{j=1}^{N} a_j e^{2i\pi\lambda_j t},\ U(t) = \sum_{j=1}^{N}\frac{u_j}{j+7}e^{2i\pi\lambda_j t}$$

[1] The Besikovich $\mathcal{B}_p$-norms are defined for $1 \leq p < +\infty$ by $\|\phi\|_{\mathcal{B}_p}^p = \lim_{T\to+\infty}\fint_{[-T,T]}|\phi(x)|^p\,\mathrm{d}x$ where $\fint_I$ is the mean over I.

and[2] $S = \sum_{j=1}^{N} \frac{|a_j|}{j+7}$. We are going to prove the following: take $T = 2K$ a large enough even integer, set $I_K = [-K, K]$, then $S \lesssim \fint_{I_K} |\phi(t)| \, \mathrm{d}t$.

A. First step: an auxilary function

We now define $\varphi = 2\mathbf{1}_{[-K/2,K/2]} * \bigstar_k \mathbf{1}_{[-1/2,1/2]}$ where $\bigstar_k \psi$ denotes the k-fold convolution of ψ by itself. The Fourier transform of φ is

$$\mathcal{F}[\varphi](\lambda) = 2\frac{\sin K\pi\lambda}{\pi\lambda}\left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^K$$

and one easily checks that φ is even, non negative with $\|\varphi\|_\infty \leq 2$ and $\mathcal{F}[\varphi](0) = \|\varphi\|_1 = 2K$ while $|\mathcal{F}[\varphi](\lambda)| \leq \frac{2K}{(\pi\lambda)^K}$. Then

Lemma II.1. *For $1 \leq k \leq N$,*

$$\sum_{\substack{1\leq j\leq N\\ j\neq k}} \frac{|\mathcal{F}[\varphi](\lambda_j - \lambda_k)|}{j+7} \leq \frac{1}{2}\frac{1}{k+7}. \tag{II.1}$$

Proof. We split the sum into two part, E_1 where we sum of the $j's$ for which $j + 7 \geq (k+7)/2$ and E_2 for the remaining terms. To estimate E_1, we note that $|\lambda_j - \lambda_k| \geq |j-k|$ and use the bound on $\mathcal{F}[\varphi]$ to get

$$\begin{aligned} E_1 &\leq \frac{2K}{\pi^K(k+7)} \sum_{\substack{1\leq j\leq N\\ j\neq k}} \frac{1}{|j-k|^K} \\ &\leq \frac{4K}{\pi^K(k+7)} \sum_{\ell=1}^{\infty} \frac{1}{\ell^K} \leq \frac{1}{4(k+7)} \end{aligned}$$

if K is large enough.

For the second sum, note that $\lambda_k - \lambda_j \geq k - j \geq (k+7)/2$. The bound on $\mathcal{F}[\varphi]$ then gives

$$\begin{aligned} E_2 &\leq \sum_{j+7<(k+7)/2} |\mathcal{F}[\varphi](\lambda_j - \lambda_k)| \\ &\leq k\frac{K2^K}{\big(\pi(k+7)\big)^K} \end{aligned}$$

since there are at most k terms in this sum. It remains to notice that

$$E_2 \leq \frac{k}{k+7}\frac{K2^{K+1}}{\pi^K 7^{K-2}}\frac{1}{k+7} \leq \frac{1}{4(k+7)}$$

if K is large enough. □

From these, we deduce the following:

Corollary II.2. *For $k = 1, \dots, N$*

$$\left|\fint_{I_K} U(t)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t - \frac{u_k}{k+7}\right| \leq \frac{1}{4K}\frac{1}{k+7}$$

[2]The +7 is harmless and only changes numerical constants.

Proof. Indeed, we write the integral as

$$\begin{aligned} &= \sum_{j=1}^{N} \frac{u_j}{j+7}\frac{1}{2K}\int_{I_K} e^{-2i\pi\lambda_j t}e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t \\ &= \sum_{j=1}^{N} \frac{u_j}{j+7}\frac{1}{2K}\mathcal{F}[\varphi](\lambda_j - \lambda_k). \end{aligned}$$

Separating the term $j = k$ and estimating the remaing ones with the previous lemma gives the result. □

It remains to multiply by a_k and to sum over k to obtain

Proposition II.3. *If K is large enough*

$$S \leq \frac{5}{4}\left|\fint_{I_K} U(t)\phi(t)\varphi(t)\,\mathrm{d}t\right|.$$

Here we need an explicit constant ($5/4$ is an upper bound of $4K/(4K-1)$ when K is large enough).

B. Modification of U into a uniformly bounded V

First, up to adding a few zeros terms to the sums, we can assume that $N + 7 = 2^{n+1} - 1$. We will now decompose U into dyadic blocs: write $\mathcal{D}_j = \{k \in \mathbb{N} : 2^j \leq k < 2^{j+1}\}$ and

$$f_j(t) = \sum_{r+7\in\mathcal{D}_j} \frac{u_r}{r+7}e^{-2i\pi\lambda_r t}$$

so that $U = \sum_{j=3}^{n} f_j$. We will now modify U into an other sum V that is near to U but uniformly bounded. We start by estimating the norms of the f_j's:

Lemma II.4. *With the above notation, we have*

1) $\|f_j\|_{L^2(I_K)} \leq \sqrt{2K+12}\,2^{-\frac{j}{2}}$;
2) $\|f_j\|_\infty \leq 1$.

Proof. This is a standard fact. Set $v_r = \frac{u_r}{r+7}$, then

$$\|f_j\|^2_{L^2(I_K)} = \sum_{r,s\in\mathcal{D}_j-7} v_r\overline{v_s}\int_{I_K} e^{2i\pi(\lambda_s-\lambda_r)t}\,\mathrm{d}t$$

One then isolates the term $r = s$ in the sum and estimate the remaining ones using Hilbert's Inequality [7]. □

For a function $F \in L^2(I_K)$ and $s \in \mathbb{Z}$, we write its Fourier coefficients $c_s^K(F) = \fint_{I_K} F(t)e^{-i\pi\frac{st}{K}}\,\mathrm{d}t$.

To each $|f_j| \in L^2(I_K)$ we associate $h_j \in L^2(I_K)$ defined via its Fourier series as

$$h_j(t) = c_0^K(|f_j|) + 2\sum_{s=1}^{\infty} c_s^K(|f_j|)e^{-i\pi\frac{st}{K}}.$$

Lemma II.5. *For $3 \le j \le n$, the following two properties hold*

1) $\mathrm{Re}(h_j) = |f_j| \ge 0$ *and* $\Re(h_j) \in L^\infty(I_K)$.
2) $\|h_j\|_{L^2(I_K)} \lesssim \sqrt{2K+1}2^{-\frac{j}{2}}$.

Proof. First, as $|f_j|$ is real valued, $c_0^K(|f_j|)$ is also real, and $\overline{c_s^K(|f_j|)} = c_{-s}^K(|f_j|)$ for every $s \ge 1$. A direct computation then shows that $\mathrm{Re}(h_j) = |f_j|$ while Parseval shows that $\|h_j\|_2 \leqslant \sqrt{2}\|f_j\|_2$.

Finally, as f_j is a trigonometric polynomial, it is a bounded function thus so is $\mathrm{Re}(h_j)$. □

We now take a parameter $0 < \eta \le 1$ that will be a small number ($\eta = 10^{-3}$ would do) and define a sequence $(F_j)_{j=3,\dots,n}$ inductively through

$$F_3 = f_3 \quad \text{and} \quad F_{j+1} = F_j e^{-\eta h_{j+1}} + f_{j+1}.$$

Lemma II.6. *For $3 \le j \le n$, $\|F_j\|_\infty \lesssim \eta^{-1}$.*

Proof. Set $E := \sup_{0<x\le 1} \dfrac{x}{1-e^{-\eta x}} = \dfrac{1}{1-e^{-\eta}}$ and note that $\eta^{-1} \le E \le 2\eta^{-1}$.

By definition of E, if $C \le E$ and $0 \le x \le 1$, then $Ce^{-\eta x} + x \le Ee^{-\eta x} + x \le E$.

We can now prove by induction over j that $|F_j| \le E$ from which the lemma follows. First, when $j = m$, from Lemma II.4 we get $\|F_3\|_\infty = \|f_3\|_\infty \le 1 \le E$.

Assume now that $\|F_j\|_\infty \le E$, then

$$\begin{aligned} |F_{j+1}(t)| &= |F_j(t)e^{-\eta h_{j+1}(t)} + f_{j+1}(t)| \\ &\le |F_j(t)|e^{-\eta\Re\left(h_{j+1}(t)\right)} + |f_{j+1}(t)| \\ &= |F_j(t)|e^{-\eta|f_{j+1}(t)|} + |f_{j+1}(t)|. \end{aligned}$$

As $|f_{j+1}(t)| \le 1$ and $|F_j(t)| \le E$, we get $|F_{j+1}(t)| \le E$ as claimed. □

Lemma II.7. *For $3 \le k \le n$ and $j = 3, \dots, k$ let $g_{j,k} = e^{-\eta H_{j,k}}$ with*

$$H_{j,k} = \begin{cases} h_{j+1} + \dots + h_k & \text{when } j < k \\ 0 & \text{when } j = k \end{cases}.$$

Then $F_k = \sum_{j=3}^{k} f_j g_{j,k}$. Moreover

$$\|H_{j,k}\|_{L^2(I_K)} \lesssim \sqrt{2K+1}2^{-\frac{j}{2}}.$$

Proof. The first part is easily obtained by induction on k For the second, the triangular inequality and Lemma II.5 reduces this to the sum of a geometric series. □

The next Lemma is a direct consequence of [5, Lemma, p614]:

Lemma II.8. *Let $3 \le k \le n$ and $3 \le j \le k$, then*

1) *the negative Fourier coefficients of $g_{j,k}(t) - 1$ vanish so that its Fourier series writes*

$$g_{j,k}(t) - 1 = \sum_{s\geqslant 0} c_s^K(g_{j,k} - 1)e^{i\pi\frac{st}{K}};$$

2) $\|g_{j,k} - 1\|_{L^2(I_K)} \lesssim \eta\sqrt{2K+1}2^{-\frac{j}{2}}$ *so that* $|c_s^k(g_{j,k} - 1)| \lesssim \sqrt{2K+1}2^{-\frac{j}{2}}$.

We now recall the definition of U and define V:

$$U(t) = \sum_{j=3}^{n} f_j \text{ and } V = F_n = \sum_{j=3}^{n} f_j g_{j,n}.$$

Proposition II.9. *There is an η such that, for K large enough and for $1 \le k \le N$ we have*

$$\left| \fint_{I_K} \big(U(t) - V(t)\big)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t \right| \le \frac{1}{2}\frac{1}{k+7}.$$

Proof. To simplify notation, we will simply write $g_j = g_{j,n}$. We fix $k \in [1, N]$ and denote by ℓ the unique integer such that $k + 7 \in \mathcal{N}_\ell$. We want to bound $R_1 + R_2$ where

$$R_1 = \fint_{I_K} \sum_{3\le j\le \ell-2} f_j(g_j - 1)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t,$$

$$R_2 = \fint_{I_K} \sum_{\ell-1\le j\le n} f_j(g_j - 1)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t.$$

Let us first bound R_1. For this, notice that if $s \in \mathbb{Z}$, then $\displaystyle\int_{I_K} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}e^{-2i\pi\frac{s}{K}t}\,\mathrm{d}t =$

$$\begin{aligned} &= \int_{I_K} \sum_{r+7\in\mathcal{N}_j} \frac{u_r}{r+7}\varphi(t)e^{-2i\pi(\lambda_r - \lambda_k + \frac{s}{K})}\,\mathrm{d}t \\ &= \sum_{r+7\in\mathcal{N}_j} \frac{u_r}{r+7}\mathcal{F}[\varphi]\left(\lambda_r - \lambda_k + \frac{s}{K}\right). \end{aligned}$$

It follows that $\int_{I_K} f_j(g_j-1)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t =$

$$\begin{aligned}
&= \int_{I_K} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}\sum_{s\geq 0} c_s^K(g_j-1)e^{2i\pi\frac{st}{K}}\,\mathrm{d}t\\
&= \sum_{s=0}^{+\infty} c_s^K(g_j-1)\int_{I_K} f_j(t)\varphi(t)e^{2i\pi\lambda_k t+\frac{2i\pi st}{K}}\,\mathrm{d}t\\
&= \sum_{s=0}^{+\infty} c_s^K(g_j-1)\times\\
&\qquad\times\sum_{r+M\in\mathcal{N}_j}\frac{u_r}{r+7}\mathcal{F}[\varphi]\left(\lambda_r-\lambda_k-\frac{s}{K}\right)
\end{aligned}$$

Inverting both sums and injecting the result in the definition of R_1 we obtain

$$\begin{aligned}
R_1 = \frac{1}{2K}\sum_{3\leq j\leq\ell-2}\sum_{r+7\in\mathcal{N}_j}\frac{u_r}{r+M}\times\\
\times\sum_{s=0}^{\infty} c_s^K(g_j-1)\mathcal{F}[\varphi]\left(\lambda_r-\lambda_k-\frac{s}{K}\right).
\end{aligned}$$

Now set $c_s(r)=c_s^K(g_j-1)$ when $r+7\in\mathcal{N}_j$ and use parity of $\mathcal{F}[\varphi]$ to rewrite

$$\begin{aligned}
R_1 &= \frac{1}{2K}\sum_{8\leq r+7<2^{\ell-1}}\frac{u_r}{r+7}\\
&\qquad\times\sum_{s=0}^{\infty} c_s(r)\mathcal{F}[\varphi]\left(\lambda_k-\lambda_r+\frac{s}{K}\right).
\end{aligned}$$

Using $|c_s(r)|\lesssim\sqrt{2K+1}$ we get

$$|R_1|\lesssim\sqrt{2K+1}\sum_{8\leq r+7<2^{\ell-1}}\frac{1}{r+7}E_r \qquad \text{(II.2)}$$

with

$$E_r=\sum_{s=0}^{\infty}\frac{1}{2K}\left|\mathcal{F}[\varphi]\left(\lambda_k-\lambda_r+\frac{s}{K}\right)\right|.$$

To bound E_r, one first notices that the choice of ℓ and of the cutoff in the sum implies that

$$\lambda_k-\lambda_r-1\geq k-r-1\geq\frac{1}{8}(k+7).$$

One then uses the bound on $\mathcal{F}[\varphi]$ and compare the sum in E_r with an integral to get $E_r\lesssim\frac{8^K}{\pi^K}\frac{1}{(k+7)^{K-1}}$. Then $|R_1|$ is

$$\begin{aligned}
&\lesssim \sqrt{2K+1}\frac{8^K}{\pi^K}\frac{1}{(k+7)^{K-1}}\sum_{8\leq r+7\leq 2^{\ell-1}}\frac{1}{r+7}\\
&\lesssim \sqrt{2K+1}\frac{8^K}{\pi^K}\frac{1}{(k+7)^{K-2}}
\end{aligned}$$

since the sum has at most $2^{\ell-1}\lesssim k+M$ terms ≤ 1. We thus obtain $R_1=o\left(\frac{1}{k+7}\right)$.

We can now estimate R_2. Using the fact that φ is bounded and Cauchy-Schwarz we get

$$\begin{aligned}
|R_2| &\lesssim \sum_{\ell-1\leq j\leq n}\frac{1}{2K}\|f_j\|_{L^2(I_K)}\|g_j-1\|_{L^2(I_K)}\\
&\lesssim \frac{2K+1}{2K}\eta\sum_{\ell-1\leqslant j\leqslant n}2^{-j}\lesssim\eta\frac{1}{k+7}
\end{aligned}$$

since we have chosen ℓ so that $2^{-\ell-1}\leq\frac{1}{k+7}$.

It follows that $|R_1|+|R_2|\lesssim\big(\eta+o(1)\big)\frac{1}{k+7}$ is $\leq\frac{1}{2}\frac{1}{k+7}$ if η is a small enough number (that depends only on the constant in the bound of R_2) and K is large enough. □

Corollary II.10.

$$\left|\fint_{I_K}\big(U(t)-V(t)\big)\phi(t)\varphi(t)\,\mathrm{d}t\right|\leq\frac{1}{2}S$$

Proof. As $\phi(t)=\sum_{k=1}^N a_k e^{2i\pi\lambda_k t}$, it suffices to use the triangular inequality and Proposition II.9. □

C. End of the proof

The end of the proof consists in applying first Corollary II.3 $S\leq\frac{5}{4}\left|\fint_{I_K}U(t)\phi(t)\varphi(t)\,\mathrm{d}t\right|$. Then, applying (II.10) we get

$$\begin{aligned}
S &\leq \frac{5}{4}\left|\fint_{I_K}\big(U(t)-V(t)\big)\phi(t)\varphi(t)\,\mathrm{d}t\right|\\
&\quad+\left|\fint_{I_K}V(t)\phi(t)\varphi(t)\,\mathrm{d}t\right|\\
&\leq \frac{5}{8}S+\|V\|_\infty\|\varphi\|_\infty\fint_{I_K}|\Phi(t)|\,\mathrm{d}t
\end{aligned}$$

This gives the desired bound $\frac{3}{8}S\lesssim\fint_{I_K}|\Phi(t)|\,\mathrm{d}t$.

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