Toward a Complete Criterion for Value of Information in Insoluble Decision Problems

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Abstract

In a decision problem, observations are said to be material if they must be taken into account, to perform optimally. Decision problems have an underlying (graphical) causal structure, which sometimes may used to evaluate certain observations as immaterial For soluble graphs — ones where important past observations are remembered — there is a complete graphical criterion; one that rules out materiality whenever this can be done on the basis of the graphical structure alone. In this work, we analyse a proposed criterion for insoluble graphs. In particular, we prove that some of the conditions used to prove immateriality are necessary; when they are not satisfied, materiality is possible. We discuss possible avenues and obstacles for proving necessity of the remaining conditions.

1 Introduction

We can view any decision problem as having an underlying causal structure — a graph consisting of chance events, decisions and outcomes, and their causal relationships. Sometimes, it is possible to establish key features of the decision problem from its causal structure alone. For example, in Figure 1a and Figure 1b, we see two such causal structures. For now, let us focus on the three endogenous vertices: the observation Z, the decision (chosen by the decision-maker) X, and the downstream outcome Y. In each graph, Z has an effect on X, which affects Y, but in Figure 1b, Z also directly influences Y, whereas in Figure 1a, it does not.

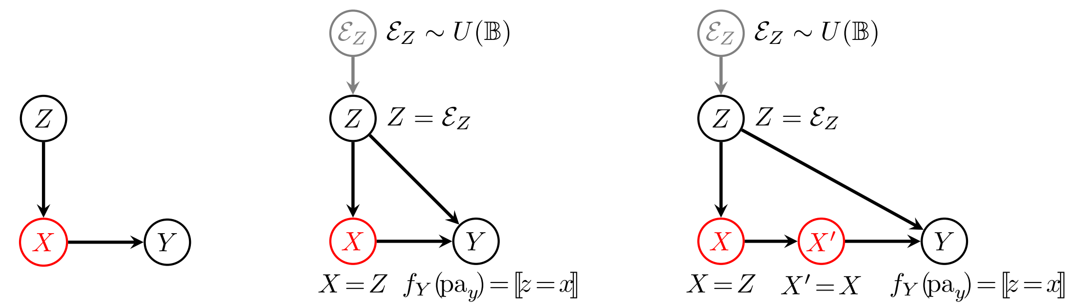
To fully describe a decision problem, we must specify probability distributions for each of the non-decision variables — distributions that must be compatible with the graphical structure. In particular, the distribution for any variable must depend only on its direct causes, i.e. its parents, a condition known as Markov compatibility. For example, in the causal structure shown in Figure 1b, one compatible decision problem is shown in the figure. The variable Z is a Bernoulli trial (i.e. a coin flip), and the decision-maker is rewarded with $Y = 1$ if they state the outcome of Z (i.e. call the outcome of the coin flip), otherwise the reward is $Y = 0$. A variable is then said to be material if the attainable reward is greater given access to an observation than without it. For example, by observing Z, the decision-maker can obtain a reward of 1, such as with the policy $Y = Z$. Without observing Z, any policy will achieve a reward of 0.5. The means that the value of information is $1 - 0.5 = 0.$, and since this quantity is strictly positive, Z is material.

For the causal structure shown in Figure 1a, we can instead make a deduction that applies to *any* decision problem compatible with the graph. In this case, for any such decision problem, there will exist an optimal decision rule that ignores the value of $Z = z$ entirely. One way to see this is that once a decision $X = x$ is chosen, the observation Z becomes independent of Y, and so there is no reason for the decision to depend on it. (This can be proved from the fact that Z is d-separated from Y given X.) So for any decision problem compatible with this graph, Z is immaterial.

There are many reasons that we may want to evaluate whether a causal structure allows an observation such as Z to be material. Firstly, for algorithmic efficiency — if an observed variable is immaterial, then the optimal policies are contained in a small subset of all available policies, that we can search exponentially more quickly. (For example, in Figure 1a, there are two choices for X, but there are four deterministic mappings from Z to X.)

Secondly, materiality can have implications regarding the fairness of a decision-making procedure. Suppose that Z designates the gender of candidates available to a recruiter, which are male $Z = 1$ or female $Z = 0$ with equal probability, while X indicates whether that person is $X = 1$ or is not $X = 0$ recruited, and Y indicates whether that person is $Y = 1$ or is not $Y = 0$ hired. If Y is correlated with Z given X, then the applicant's gender is material for the recruiter, and to maximise the hiring probability, they will have to recruit applicants at different rates based on their gender. If the causal structure is that of Figure 1a, then materiality can be ruled out, meaning that unfair behaviour is not necessary for optimal performance, whereas the causal structure of Figure 1b can incentivise unfairness. Such an analyses can be used for well-studied concepts like counterfactual fairness (Kusner et al., 2017). An arbitrary graph where Z is a sensitive variable (such as gender), counterfactual fairness can arise only when there is a path $Z \to \ldots \to O \to X$, where the observation O is material (Everitt et al., 2021).

Thirdly, materiality can have implications for AI safety — if Z represents a corrective instruction from a human overseer, and there exists no path $Z \to \ldots \to O \to X$ where O is material, then there exist optimal policies that ignore this instruction (Everitt et al., 2021).Materiality is also relevant for evaluations of agents' intent (Halpern & Kleiman-Weiner, 2018; Ward et al., 2024), and relatedly, their incentives to control parts of the environment (Everitt et al., 2021; Farquhar et al., 2022). For an agent to intentionally manipulate a variable Z to obtain an outcome $Y = y$, there must be a path $p : X \to \ldots \to Z \to \ldots \to Y$ where for each of its decisions X' lying on p, the parent O' along p is material for X'. In general, a stronger criterion for ruling out materiality will allow us to rule out unfair or unsafe behaviour for a wider range of agent-environment interactions (Everitt et al., 2021).



(a) Z is immaterial for X. **(b)** Z is material for X. **(c)** Z is material for X.

Figure 1: Three graphs, with decisions in red, and a real-valued outcome Y. We write $U(\mathbb{B})$ for a uniform distribution over $\mathbb{B}$, i.e. a Bernoulli distribution with $p = 0.5$.

Any procedure for establishing immateriality based on the causal structure may be called a *graphical criterion*. For example, if a decision X is not an ancestor of the outcome Y, then all of the variables observed at X are immaterial. An ideal graphical criterion would be proved *complete*, in that it can establish immateriality whenever this is possible from the graphical structure alone. Clearly, this criterion is not complete, because in Figure 1a, X is an ancestor of the outcome, but we still proved Z immaterial. So far, a graphical criterion from van Merwijk et al. (2022) has been proved complete, but only under some significant restrictions. The causal structure must be *soluble*, meaning that all of the important information observed from past decisions is remembered at later decision points. Also, no criteria has been proved complete for identifying immaterial decisions, i.e. past decisions that can be safely forgotten.

For insoluble graphs, there the criterion of Lee & Bareinboim (2020, Thm. 2), which can identify immaterial decisions and is (strictly) more potent in general. However, it is not yet known whether this criterion is complete. In particular, it is not yet clear whether several of its conditions are necessary. For example, one case where all existing criteria are silent is the simple graph shown in Figure 1c — we would like to know whether we can rule out X being a material observation for X'. We cannot use van Merwijk et al. (2022) because X is a decision, and because the graph is insoluble.[1] Furthermore, we cannot establish immateriality

[1]Formally, this is because $W \not\perp Y \mid X \cup X'$, and $X' \not\perp Y \mid X \cup W$, as per the definition of solubility that we will review in Section 3.

using Lee & Bareinboim (2020, Thm. 2), because it violates a property that we term LB-factorizability, which we will discuss in Section 3.3.[2]

By studying Figure 1c in a bespoke fashion, we find that there exists a decision problem with the given causal structure, where X is material for X'. As shown in Figure 1c, Z is a Bernoulli variable, and Y is equal to 1 if $Z = X'$ and to 0 otherwise. If X is observed by X', then a reward of $\mathbb{E}[Y] = 1$ can be achieved by the policy $X' = X = Z$. If X is not observed, the greatest achievable reward is lower, at $\mathbb{E}[Y] = 0.5$, implying materiality.

This raises a question: by generalising this construction, can we prove that requirement I of LB-factorizability is necessary to prove immateriality for a wide class of graphs? This work will prove that this requirement is indeed necessary, meaning that materiality cannot be excluded for a wide class of graphs including Figure 1c.

It remains an open question whether the criterion of Lee & Bareinboim (2020, thm. 2) as a whole is complete, in that its other conditions are necessary for establishing immateriality. In the case that it is complete, our work is a step toward proving this. On the other hand, we also present some graphs where materiality is difficult to establish, that — if the criterion is not complete — could bring us closer to a proof of incompleteness.

The structure of the paper is as follows. In Section 2, we will recap the formalism used by Lee & Bareinboim (2020) for modelling decision problems, based on structural causal models. In Section 3, we will review existing procedures for proving that an observation can or cannot be material. In Section 4, we will establish our main result: that requirement I of LB-factorizability is necessary to establish immateriality. In Section 5, we present some analogous results for other requirements of LB-factorizability, that could serve as a building block for proving the necessity of those requirements. We then illustrate the problems that arise in trying to prove necessity of those further requirements, and outline some possible directions for further work. Finally, in Section 6, we conclude.

2 Setup

Our analysis will follow Lee & Bareinboim (2020) by using the structural causal model (SCM) framework (Pearl, 2009, Chapter 7), although the results also apply equally to Bayesian networks and influence diagrams.

2.1 Structural causal models

A structural causal model (SCM) $\mathcal{M}$ is a tuple $\langle \boldsymbol{U}, \boldsymbol{V}, P(\boldsymbol{U}), \mathbf{F}\rangle$, where $\boldsymbol{U}$ is a set of variables determined by factors outside the model, called *endogenous* following a joint distribution $P(\boldsymbol{U})$, and $\boldsymbol{V}$ is a set of endogenous variables whose values are determined by a collection of functions $\mathbf{F} = \{f_V\}_{V \in \boldsymbol{V}}$ such that $V \leftarrow f_V(\mathrm{Pa}(V), \boldsymbol{U}_V)$ where $\mathrm{Pa}(V) \subseteq \boldsymbol{V} \setminus \{V\}$ is a set of endogenous variables and $\boldsymbol{U}_V \subseteq \boldsymbol{U}$ is a set of exogenous variables. The observational distribution $P(\boldsymbol{v})$ is defined as $\sum_{\boldsymbol{u}} \prod_{V \in \boldsymbol{V}} P(v|\mathbf{pa}_V, \boldsymbol{u}_V)P(\boldsymbol{u})$, where $\boldsymbol{u}_V$ is the assignment $\boldsymbol{u}$ restricted to variables $\boldsymbol{U}_V$. Furthermore, $\mathrm{do}(\boldsymbol{X} = \boldsymbol{x})$ represents the operation of fixing a set $\boldsymbol{X}$ to a constant $\boldsymbol{x}$ regardless of their original mechanisms. Such intervention induces a submodel $\mathcal{M}_{\boldsymbol{x}}$, which is $\mathcal{M}$ with f_X replaced by x for $X \in \boldsymbol{X}$. Then, an interventional distribution $P(\boldsymbol{v}\backslash\boldsymbol{x}|\,\mathrm{do}(\boldsymbol{x}))$ can be computed as the observational distribution in $\mathcal{M}_{\boldsymbol{x}}$. The induced graph of an SCM $\mathcal{M}$ is a DAG $\mathcal{G}$ on only the endogenous variables $\boldsymbol{V}$ where (i) $X \to Y$ if X is an argument of f_Y; and (ii) $X \leftrightarrow Y$ if $\boldsymbol{U}_X$ and $\boldsymbol{U}_Y$ are dependent, i.e. for any $\boldsymbol{u}_X, \boldsymbol{u}_Y$, $P(\boldsymbol{u}_X, \boldsymbol{u}_Y) \neq P(\boldsymbol{u}_X) \times P(\boldsymbol{u}_Y)$.

We use the notation $\mathrm{Pa}(X)$, $\mathrm{Ch}(X)$, $\mathrm{Anc}(X)$ and $\mathrm{Desc}(X)$ to represent the parents, children, ancestors and descendants of a variable X, respectively, and take ancestors and descendants to include the node X itself.[3]

We will use the notation $V_1 - V_2$ to designate an edge whose direction may be $V_1 \to V_2$ or $V_1 \leftarrow V_2$. For a path $V_1 - V_2 - \cdots - V_\ell$, we will use the shorthand $V_1 \dashrightarrow V_\ell$...

[2]Specifically, requirement I of LB-factorizability is violated because Y is d-connected to $\boldsymbol{\pi}_{X'}$ given X'.

[3]Note that $\mathrm{Pa}(X)$ is an intentional reuse of the notation used to describe the arguments of f_X in the SCM definition, because the endogenous arguments of f_X and the parents of X in the induced graph are the same variables.

the shorthand $B \overset{p}{\dashrightarrow} C$. We will use the shorthand $\boldsymbol{V}_{1:N}$ for a sequence of variables $V_1, \ldots V_N$ indexed by $1, \ldots, N$, $\boldsymbol{v}_{1:N}$ for a sequence of assignments, and $\boldsymbol{p}_{1:N}$ for a set of paths $p_1, \ldots p_N$.

There is certain notation that we will use repeatedly when constructing causal models, such as tuples, bitstrings, indexing, and Iverson brackets. We will write a tuple as $z := \langle x, y \rangle$, and this may be indexed as $z[0] = x$. A bitstring of length n, i.e. a tuple of n Booleans, may be written as $\mathbb{B}^n$, and a uniform distribution over this space, as $U(\mathbb{B}^n)$. We will denote a bitwise XOR operation by $\oplus$ so that, for example, $01 \oplus 11 = 10$. Bitstrings may also be used for indexing, for example, the y^{th} bit of x may be written as as $x[y]$, and the leftmost bits are of higher-order so that, for example, $0100[01] = 1$. Similarly, for random variables X, Y, we will write $X[Y]$ for a variable equal to $x[y]$ when $X = x$ and $Y = y$. Finally, the Iverson bracket $[\![P]\!]$ is equal to 1 if P is true, and 0 otherwise.

2.2 Modelling decision problems

To use an SCM to define a decision problem, we need to specify what policies the agent can select from and what goal the agent is trying to achieve.

We will describe the set of available policies using a Mixed Policy Scope (Lee & Bareinboim, 2020), which casts certain variables as decisions, and others as *context variables* or "observations" $\boldsymbol{C}_X$, that each decision X is allowed to depend on. Following Lee & Bareinboim (2020), we will consistently illustrate decision variables with red circles, as in Figure 1.

Definition 1 (Mixed Policy Scope (MPS))**.** Given a DAG $\mathcal{G}$ on vertices $\boldsymbol{V}$, a *mixed policy scope* $\mathcal{S} = \langle X, \boldsymbol{C}_X \rangle_{X \in \boldsymbol{X}(\mathcal{S})}$ consists of a set of decisions $\boldsymbol{X}(\mathcal{S}) \subseteq \boldsymbol{V}$ and a set of context variables $\boldsymbol{C}_X \subseteq \boldsymbol{V}$ for each decision.

For a set of decisions $\boldsymbol{X}'$, we define their contexts as $\boldsymbol{C}_{\boldsymbol{X}'} = \bigcup_{X \in \boldsymbol{X}'} \boldsymbol{C}_X$.

A policy consists of a probability distribution for each decision X, conditional on its contexts $\boldsymbol{C}_X$.

Definition 2 (Mixed Policy)**.** Given an SCM $\mathcal{M}$ and scope $\mathcal{S} = \langle X, \boldsymbol{C}_X \rangle$, a *mixed policy* $\boldsymbol{\pi}$ (or a *policy*, for short) contains for each X a decision rule $\pi_{X|\boldsymbol{C}_X}$, where $\pi_{X|\boldsymbol{C}_X} : \mathfrak{X}_X \times \mathfrak{X}_{\boldsymbol{C}_X} \mapsto [0, 1]$ is a proper probability mapping.[4]

We will say that such a policy $\boldsymbol{\pi}$ *follows* the scope $\mathcal{S}$, written $\boldsymbol{\pi} \sim \mathcal{S}$. A mixed policy is said to be *deterministic* if every decision is a deterministic function of its contexts.

Once a policy is selected, we would have a new causal structure, described by a *scoped graph*.

Definition 3 (Scoped graph)**.** The *scoped graph* $\mathcal{G}_\mathcal{S}$ is obtained by $\mathcal{G}$, by replacing, for each decision $X \in \boldsymbol{X}(\mathcal{S})$, all inbound edges to X with edges $C \to X$ for every $C \in \boldsymbol{C}_X$. We only consider scopes for which $\mathcal{G}_\mathcal{S}$ is acyclic.

We will designate one real-valued variable $Y \notin \boldsymbol{X}(\mathcal{S}) \cup \boldsymbol{C}(\mathcal{S})$ as the outcome node (also called the "utility" variable). To calculate the expected utility under a policy $\boldsymbol{\pi} \sim \mathcal{S}$, let $\boldsymbol{C}^- = \left(\bigcup_{X \in \boldsymbol{X}(\mathcal{S})} \boldsymbol{C}_X\right) \setminus \boldsymbol{X}(\mathcal{S})$ be the *non-action* contexts. Then, the expected utility is:
$\mu_{\boldsymbol{\pi},\mathcal{S}} = \sum_{y,\boldsymbol{x},\boldsymbol{c}^-} y P_{\boldsymbol{x}}(y, \boldsymbol{c}^-) \prod_{X \in \boldsymbol{X}(\mathcal{S})} \pi(x|\boldsymbol{c}_x)$. When the scope is obvious, we will simply write $\mu_{\boldsymbol{\pi}}$.

This paper is concerned with materiality — whether removing one context variable from one decision will decrease the expected utility attainable by the best policy. We define it in terms of the value of information (Howard, 1990; Everitt et al., 2021).

Definition 4 (Value of Information)**.** Given an SCM $\mathcal{M}$ and scope $\mathcal{S}$, the *maximum expected utility* (MEU) is $\mu^*_\mathcal{S} = \max_{\boldsymbol{\pi} \sim \mathcal{S}} \mu_{\boldsymbol{\pi},\mathcal{S}}$. The *value of information* (VoI) of context $Z \in \boldsymbol{C}_X$ for decision $X \in \boldsymbol{X}(\mathcal{S})$ is $\mu^*_\mathcal{S} - \mu^*_{\mathcal{S}_{Z \not\to X}}$, where $\mathcal{S}_{Z \not\to X}$ is defined as $\langle X', \boldsymbol{C}_{X'} \rangle_{X' \in \boldsymbol{X}(\mathcal{S}) \setminus \{X\}} \cup \langle X, \boldsymbol{C}_X \setminus \{Z\} \rangle$.

The context Z is *material* for X in an SCM $\mathcal{M}$ if Z has strictly positive value of information for X, otherwise it is *immaterial*.

[4]Following Lee & Bareinboim (2020), we term this a "mixed policy" due to its including mixed strategies. Note that game theory also has a distinction between "mixed" policies, where the decision rules share a source of randomness, and "behavioural" policies, where they do not, and in this sense, the "mixed" policies of Lee & Bareinboim (2020) are actually *behavioural*.

2.3 Graphical criteria for independence

Knowing when variables are independent is an important step in identifying immaterial contexts, as we will discuss in the next section. So, we will make repeated use of d-separation, a graphical criterion that establishes the independence of variables in a graph.

Definition 5 (d-separation; Verma & Pearl, 1988)**.** A path p is said to be d-separated by a set of nodes $\boldsymbol{Z}$ if and only if:

1. p contains a collider $X \to W \leftarrow Y$ such that the middle node W is not in $\boldsymbol{Z}$ and no descendants of W are in $\boldsymbol{Z}$, or
2. p contains a chain $X \to W \to Y$ or fork $X \leftarrow W \to Y$ where W is in $\boldsymbol{Z}$, or
3. one or both of the endpoints of p is in $\boldsymbol{Z}$.

A set $\boldsymbol{Z}$ is said to d-separate $\boldsymbol{X}$ from $\boldsymbol{Y}$, written $(\boldsymbol{X} \perp_{\mathcal{G}} \boldsymbol{Y} \mid \boldsymbol{Z})$, if and only if $\boldsymbol{Z}$ d-separates every path from a node in $\boldsymbol{X}$ to a node in $\boldsymbol{Y}$. Sets that are not d-separated are called d-connected, written $\boldsymbol{X} \not\perp_{\mathcal{G}} \boldsymbol{Y} \mid \boldsymbol{Z}$.

When the graph is clear from context, we will write $\perp$ in place of $\perp_{\mathcal{G}}$. When sets $\boldsymbol{X}, \boldsymbol{W}, \boldsymbol{Z}$ satsify $\boldsymbol{X} \perp \boldsymbol{W} \mid \boldsymbol{Z}$ they are conditionally independent: $P(\boldsymbol{X}, \boldsymbol{W} \mid \boldsymbol{Z}) = P(\boldsymbol{X} \mid \boldsymbol{Z})P(\boldsymbol{W} \mid \boldsymbol{Z})$ (Verma & Pearl, 1988).

If we know that a deterministic mixed policy is being followed, then we may deduce further conditional independence relations. This is because conditioning on variables $\boldsymbol{V}$ may determine some decision variables, which are called "implied" (Lee & Bareinboim, 2020), or "functionally determined" (Geiger & Pearl, 1990), making them conditionally independent of other variables in the graph.

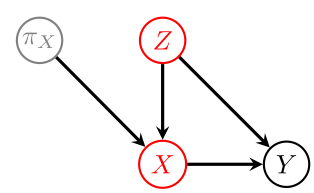


Figure 2: A graph where decisions Z, X jointly determine the outcome Y. A policy node π_X is shown, which decides the decision rule at X.

Definition 6 (Implied variables; Lee & Bareinboim, 2020)**.** To obtain the *implied variables* $\lceil \boldsymbol{Z} \rceil$ for variables $\boldsymbol{Z}$ in $\mathcal{G}$ given a mixed policy scope $\mathcal{S}$, begin with $\lceil \boldsymbol{Z} \rceil \leftarrow \boldsymbol{Z}$, then add to $\lceil \boldsymbol{Z} \rceil$ every decision X such that $C_X \subseteq \lceil \boldsymbol{Z} \rceil$, until convergence.

For example, in Figure 2, we see that $\lceil X \rceil = \{Z, X\}$, so Z is d-separated from Y given $\lceil X \rceil$. This means that under a deterministic mixed policy, Z and Y are statistically independent given X. This has implications for materiality. In particular, it means that the best deterministic mixed policy $Z = z, X = x$ does not need to observe Z at X. Moreover, the performance of the best deterministic mixed policy can never be surpassed by a stochastic policy ((Lee & Bareinboim, 2020, Proposition 1)), so Z is immaterial.

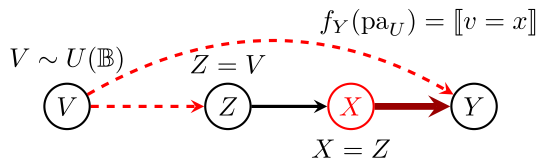
3 Review of graphical criteria for materiality

We will now review some existing techniques for proving whether or not a graph is compatible with some variable Z being material for some decision X.

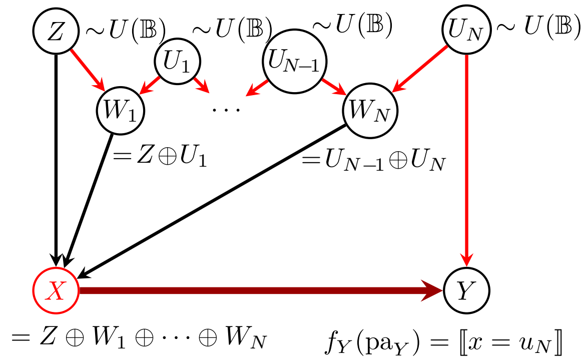
3.1 Single-decision settings

In the single-decision setting, there is a sound and complete criterion for materiality: in a scoped graph $\mathcal{G}(\mathcal{S})$, there exists an SCM where the context $Z \in \boldsymbol{C}_X$ is material if and only if $Z \not\perp Y \mid \boldsymbol{C}_X \cup \{X\} \setminus \{Z\}$ and the outcome Y is a descendant of X (Lee & Bareinboim, 2020; Everitt et al., 2021). This statement can be split into proofs for the *only if* and *if* directions, both of which are relevant to the current paper.

The argument for the *only if* is that if X is not an ancestor of the outcome Y, then its policy is completely irrelevant to the expected utility, and so all of its contexts are immaterial, and if Z is conditionally independent of the outcome Y given the decision and other observations, then it may be safely ignored without changing the outcome. These arguments are important to us because they remain equally valid as we move to a multi-decision setting — a context must be an ancestor of Y ,and must provide information about Y over and above the other contexts, in order to be material.

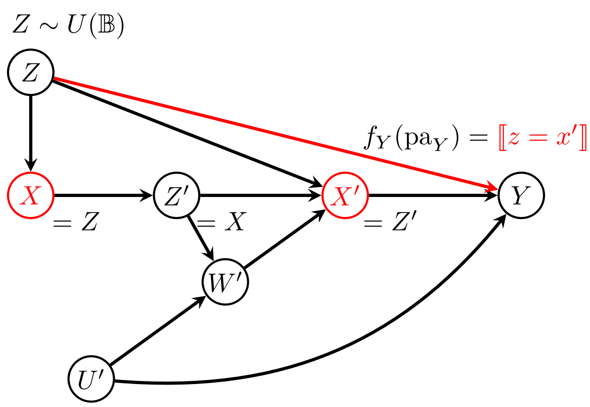


(a) The Everitt et al. (2021); Lee & Bareinboim (2020) scheme, with a red info path that lacks colliders, and the control path shown with a thick dark red arrow.

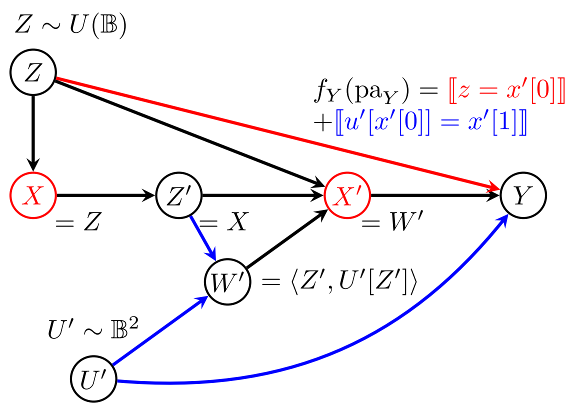


(b) The Everitt et al. (2021); Lee & Bareinboim (2020) scheme, with a red info path that contains colliders, and the control path shown in dark red.

Figure 3: Three decision problems where Z is material for X. For readability, we marginalise out exogenous variables from the SCM, so $z \sim U(\mathbb{B})$ can be understood as shorthand for $z = \varepsilon_Z$ where $\varepsilon_Z \sim U(\mathbb{B})$, and so on.



(a) The Everitt et al. (2021) scheme is applied using just the red info path; Z is immaterial for X.



(b) The van Merwijk et al. (2022) scheme is applied, using the red and blue info paths; Z is material for X.

Figure 4: Two decision problems on a soluble graph.

The *if* direction is proved by constructing a decision problem where Z is material. By assumption, there is a directed path $X \dashrightarrow Y$, called the *control path*, and a path $Z \;\text{---}\; Y$, active given $C_X \cup \{X\} \setminus \{Z\}$, called the *info path*.

In the SCM that is constructed, the variable Z will contain information about Y (due to a conditional dependency induced by the info path), and this will inform X regarding how to influence Y (using influence that is transmited along the control path).

The construction has two cases, which differ based on whether or not the info path contains colliders (Everitt et al., 2021; Lee & Bareinboim, 2020). For the case where it does not contain colliders, the graph and construction are shown in Figure 3a. (Note that when the info path is a directed path, we take this to be a special case where $V = Z$.) The functions along the info path (dashed line) are chosen to copy V to Pa_Y and to Z, and Y equals its maximum value of 1 only if X equals V, and 0 otherwise. So, X must copy Z to achieve the maximum expected utility. Without the context Z, the maximum expected utility is 0.5, proving materiality.[5]

[5]To be precise, the formalism of Lee & Bareinboim (2020) also allows the active path from Z to include one or more bidirected edges $V \leftrightarrow Y$, but to deal with these cases, we begin with the distribution that we would use for a path $V \leftarrow L \rightarrow Y$, then marginalise out L.

For the case where the info path does contains a collider, the graph and construction from Everitt et al. (2021); Lee & Bareinboim (2020) are shown in Figure 3b. Each fork U_i in the info path, along with Z, generates a random bit, while each collider W_i is assigned the XOR ($U_{i-1} \oplus U_i$) of its two parents. By observing z and the values $\boldsymbol{w}_{1:N}$, the agent has just enough information to recover u_N. In particular, hte policy that sets x equal to the XOR of z and $\boldsymbol{w}_{1:N}$, obtains $x = u_N$ and achieves the MEU, $\mathbb{E}[Y] = 1$. Without the context Z, the MEU becomes 0.5, so Z is material.

3.2 Soluble multi-decision settings

This approach has been generalised to deal with multi-decision graphs that are *soluble* (also known as graphs that respect "sufficient recall").

To recap, a graph is said to be soluble if there is an ordering $\prec = \langle X_1, \ldots, X_N \rangle$ over decisions such that at for every X_i, for every previous decision or context $V \in \{X_j \cup C_{X_j} \mid j \prec i\}$, we have $V \notin \text{Anc}(Y)$ or $V \perp Y \mid \{X_i\} \cup C_{X_i}$. That is, past decisions and contexts do not contain any information that is relevant for a later decision, and unknown at the time that this later decision is made. For example, in Figure 4a, using the ordering $X \prec X'$, the nodes Z, X are d-separated from Y by X' and its contexts $\{Z, Z', W\}$, which implies solubility.

For soluble graphs, there exists a complete criterion, for discerning whether a non-decision context Z is material for a decision X. If X lacks a *control path* (a directed path to Y), or Z lacks an info path (a path to Y, active given $\boldsymbol{C} \setminus \{Z\}$), then Z is immaterial. Conversely, if in a graph, every X decision has a control path, and each context Z has an info path, then every context is material in some decision problem with that causal structure (van Merwijk et al., 2022, Theorem 7).[6] For example, in the graph of Figure 4a, every decision is an ancestor of Y, and every context has an info path, (the info paths include $Z \to Y$, $Z' \to W' \leftarrow U' \to Y$, and $W' \leftarrow U' \to Y$), so, all contexts may be material in at least one decision problem with this causal structure.

It will be important for us to understand what obstacles can arise in proving materiality in multi-decision graphs, such as was required in proving (van Merwijk et al., 2022, Theorem 7). For example, suppose that we seek to construct a decision problem where Z is material for the graph in in Figure 4 Suppose that we apply the single-decision construction of Everitt et al. (2021) to this graph. First, we would identify the info path $Z \to Y$ and the control path $X \to Z' \to X' \to Y$. The info path has no colliders, so we will construct a decision problem using the scheme from Figure 3a, and the result is shown in Figure 4a. The idea of this construction is that X should have to copy Z in order for the value z transmitted by the info path to match the value x' transmitted by the control path. We see, however, that whatever action x is selected, the decision X' can assume the value z, thereby achieving the MEU. The MEU is then achievable whether Z is a context of X or not, so Z is immaterial in this construction.

In order to render Z material, we must adapt the construction from Figure 4a by incentivising X' to pass along the value of Z'. To this end, we will use the second info path $Z' \to W' \leftarrow U' \to Y$, shown in Figure 4b. We add a term $y_2 := [\![u'[x'[0]] = x'[1]]\!]$ to the reward, which equals 1 if X' presents one bit from U', along with its index. We then set $W' = U'[Z']$, so that X' knows only the Z'^{th} bit of U', and since the index z' is one bit, we let U' be two bits in length, i.e. $U' \sim U(\mathbb{B}^2)$. Finally, rather than requiring $z = x'$ as in Figure 4a, we now include the term $y_1 := [\![z = x'[0]]\!]$, because Z' will be the zeroth term of X'. In the resulting model, the utility is clearly $Y = 2$ in the non-intervened model, and to achieve this utility, the MEU, we must have $Y_1 = Y_2 = 1$ with probability 1. To maximise y_2, the decision X' must reproduce the only known digit from U', i.e. $x' = \langle z', u'[z'] \rangle$. To maximise y_1, we must have $Z = X'[0]$ almost surely, and since $X'[0] = X$, this requires $X = Z$ with probability 1. This can only be done if Z is a context of X, meaning that Z is material for X. There is a general principle here — if a control path for X, such as $X \to Z' \to X' \to Y$, contains decisions other than X, then we need to incentivise the downstream decision to copy information along the control path, and this will be done by choosing values for variables lying on the info path for X' (the one shown in blue in Figure 4b); we will revisit this matter in our main result.

[6] In full generality, the result allows an info path to terminate at another context, rather than at Y. This detail is not pertinent to the methods used to derive our main result in Section 4, although we do consider this scenario in Section 5.

3.3 Multi-decision settings in full generality

Once the solubility assumption is relaxed, there are some criteria for identifying immaterial variables, but it is not yet known to what extent these criteria are necessary, in that materiality is possible whenever they are not satisfied.

The simplest criteria for immateriality are those that carry over from the single-decision case:

- If a decision X is a non-ancestor of Y, then its contexts are immaterial,
- If $C \perp Y \mid \boldsymbol{C}_X \setminus \{C\}$, then the context C is immaterial.

But suppose that we have a graph where neither of these criteria is satisfied. Then on some occasions, we can still establish immateriality, using the more sophisticated criterion of Lee & Bareinboim (2020, Theorem 2). The assumptions of this criterion are split across: Lee & Bareinboim (2020, Lemma 1) and Lee & Bareinboim (2020, Theorem 2) itself. Lee & Bareinboim (2020, Lemma 1) establishes that if some target variables $\boldsymbol{Z}$, target actions $\boldsymbol{X'}$, and latent variables $\boldsymbol{U'}$ satisfy certain separation conditions, then they may be factorized in a favourable way. Lee & Bareinboim (2020, Theorem 2) then proves that under some further assumptions, the contexts $\boldsymbol{Z}$ are immaterial to the decisions $\boldsymbol{X'}$. in this paper, our focus is exclusively on the assumptions of Lee & Bareinboim (2020, Lemma 1), and we term them "LB-factorizability", after the authors' initials. Lee & Bareinboim (2020, Theorem 2) does not feature in our analysis, but for completeness sake, it is reproduced in Appendix A.

Definition 7. For a scoped graph $\mathcal{G}_{\mathcal{S}}$, we will say that target actions $\boldsymbol{X'}$, endogenous variables $\boldsymbol{Z}$ disjoint with $\boldsymbol{X'}$, contexts $\boldsymbol{C'} := \boldsymbol{C}_{\boldsymbol{X'}} \setminus (\boldsymbol{X'} \cup \boldsymbol{Z})$ and exogenous variables $\boldsymbol{U'}$ are *LB-factorizable* if there exists an ordering $\prec$ over $\boldsymbol{V'} := \boldsymbol{C'} \cup \boldsymbol{X'} \cup \boldsymbol{Z}$ such that:

I. $(Y \perp \boldsymbol{\pi}_{\boldsymbol{X'}} \mid \lceil(\boldsymbol{X'} \cup \boldsymbol{C'})\rceil)$,

II. $(C \perp \boldsymbol{\pi}_{\boldsymbol{X'}_{\prec C}}, \boldsymbol{Z}_{\prec C}, \boldsymbol{U'} \mid \lceil(\boldsymbol{X'} \cup \boldsymbol{C'})_{\prec C}\rceil)$, for every $C \in \boldsymbol{C'}$ and

III. $\boldsymbol{V'}_{\prec X}$ is disjoint with $\mathrm{Desc}(X)$ and subsumes $\mathrm{Pa}(X)$ for every $X \in \boldsymbol{X'}$,

where $\boldsymbol{\pi}_{\boldsymbol{X'}}$ consists of a new parent π_X added to each variable $X \in \boldsymbol{X'}$, and $\boldsymbol{W}_{\prec V}$, for $\boldsymbol{W} \subseteq \boldsymbol{V'}$, denotes the subset of $\boldsymbol{W}$ that is strictly prior to V in the ordering $\prec$.

For example, consider the graph Figure 2. In this case, $Y \in \mathrm{Desc}(X)$ and $Z \not\perp Y \mid X$, so the single-decision criteria cannot establish that Z is immaterial for X. However, by choosing $\boldsymbol{Z} = \{Z\}, \boldsymbol{X'} = \{X\}$, and the ordering $\prec = \langle Z, X \rangle$, we have that:

I. the outcome Y is d-separated from $\boldsymbol{\pi}_X$ by $\lceil X \rceil$, (since Z is a decision that lacks parents, we actually have $\lceil X \rceil = \{Z, X\}$),

II. the contexts $\boldsymbol{C'}$ are an empty set, so (II) is trivially true, and

III. $\boldsymbol{V'}_{\prec X} = \boldsymbol{Z}$, and $\boldsymbol{Z}$ is disjoint with $\mathrm{Desc}(X)$ and $\boldsymbol{Z} \supseteq \mathrm{Pa}(X)$

so $\boldsymbol{Z}$ and $\boldsymbol{X'}$ are LB-factorizable. As shown in Appendix A, the assumptions of Lee & Bareinboim (2020, Theorem 2) are also satisfied, enabling us to deduce that Z is immaterial for X, matching the ad hoc analysis of this graph in Section 2.

4 Main result

4.1 Theorem statement and proof overview

The goal of this paper is to prove that condition (I) of LB-factorizability is necessary to establish immateriality. More precisely, we prove that if condition (I) is unsatisfiable for all observations in the graph, then the graph is incompatible with materiality. It might initially seem unnecessarily stringent to assume that this holds for *all* observations, rather than the context Z_0 for which we are trying to prove materiality. Recall from Figure 4b, however, that proofs of materiality are recursive — to prove that Z material for X, we incentivised X to copy Z, and to do this, we had to incentivise X' has to pass on the value of Z'. To do this, we needed to assume that other contexts and decisions (such as Z' and X') have their own info paths and control paths, not just Z and X. So, in our theorem below, assumption (C) requires that (I) holds for all contexts. Assumptions (A) and (B) are also necessary for a graph to be compatible with materiality, because their negation implies immateriality, as per the single-decision criteria discussed in Section 3.1.

Theorem 8. *If, in a scoped graph $\mathcal{G}_\mathcal{S}$, for every $X \in \boldsymbol{X}(\mathcal{S})$*

A. $X \in Anc_{\mathcal{G}_\mathcal{S}}(Y)$,
B. $\forall C \in \boldsymbol{C}_X : (C \not\perp_{\mathcal{G}_\mathcal{S}} Y \mid (\{X\} \cup \boldsymbol{C}_X \setminus \{C\}))$, *and*
C. *for every decision X and context $Z \in \boldsymbol{C}_X$ in $\mathcal{G}_\mathcal{S}$, $(\pi_X \not\perp_{\mathcal{G}_\mathcal{S}} Y \mid \lceil (\boldsymbol{X}(\mathcal{S}) \cup \boldsymbol{C}_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\} \rceil)$, where π_X is a new parent of X,*

then for every $X_0 \in \boldsymbol{X}(\mathcal{S})$ and $Z_0 \in \boldsymbol{C}_{X_0}$, there exists an SCM where Z_0 is material for X_0.

We will prove this result in three stages, across the next three sections.

- In Section 4.2, we prove that for any scoped graph satisfying the assumptions of Theorem 8, for any context $Z_0 \in \boldsymbol{C}_{X_0}$, there exist certain paths, which we will call the *materiality paths*.
- In Section 4.3, we use the materiality paths to define an SCM for this scoped graph, which we will call the *materiality SCM*.
- In Section 4.4, we will prove that in the materiality SCM, Z_0 is material for X_0.

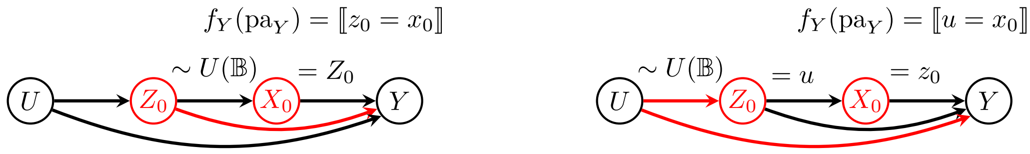
4.2 The materiality paths

To prove materiality, we will begin by selecting info paths and a control path, similar to what was described in Section 3.2 and illustrated in Figure 4b. One difference, however, is that these paths must allow for the case where we are proving the value of remembering a past decision. We will first describe how to accommodate this case in Section 4.2.1 then define a set of paths for our proof in Section 4.2.2.

4.2.1 Paths for the value of remembering a decision

One distinction between our setting and that of van Merwijk et al. (2022) is that we may need to establish the value of remembering a past decision, for example, the value of remembering Z_0 in Figure 5. In this graph, the procedures of Everitt et al. (2021) and van Merwijk et al. (2022) are silent about whether we should choose the info path $Z_0 \to Y$, and construct the graph Figure 5a, or choose the info path $Z_0 \leftarrow U \to Y$, and construct the model depicted in Figure 5b. In the first case, we have $Y = 1$ if $x_0 = z_0$, i.e. the decision X_0 is required to match the value of a past decision Figure 5a. Then, the MEU of 1 can be achieved with a deterministic policy such as $Z_0 = 1, X_0 = 1$, and Z_0 is immaterial for X_0. To understand this in terms of the paths involved, The problem is that the info path $Z_0 \to Y$ doesn't include any parents of Z_0, so Z_0 is *implied* by values outside the info path, and $Z_0 \to Y$ is rendered inactive given $\lceil U \rceil$. This means that observing Z_0 can no-longer provide useful information about how to maximise Y. In the second case, $Y = 1$ if $x_0 = u$, i.e. the decision X_0 must match the value of a random Bernoulli variable U Figure 5b. U is directly observed only by Z_0, and so in optimal policy, X_0 must observe the decision z_0, as is the case in the optimal policy $z_0 = u, x_0 = z_0$, and so Z_0 is material for X_0. The info path $Z_0 \leftarrow U \to Y$ does include a parent U of Z_0, and so Z_0 is no-longer *implied* by values outside the info path, and the path $Z_0 \leftarrow U \to Y$ remains active given $\lceil \emptyset \rceil$. Thus Z_0 may still provide useful information about Y.

For our proof, we need a general procedure for finding an info path that contains a non-decision parent for every decision. Condition (C) of Theorem 8 is useful, because it implies the presence of a path from Z to Y that is active given $\lceil (\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus Z}) \setminus Z \rceil$. Any fork or chain variables in this path will not be decisions, otherwise they would be contained in $\lceil \boldsymbol{X}(\mathcal{S}) \setminus Z \rceil$, which would make them blocked given $\lceil (\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus Z}) \setminus Z \rceil$. This deals with the possibility of decisions anywhere except for the endpoint Z. But how can we ensure that the info path contains a non-decision parent for Z, if it is a decision? We can



(a) Z_0 is immaterial for X_0.

(b) Z_0 is material for X_0

Figure 5: Two SCMs, with models constructed using different (red) info paths.

use condition (C) again, because it implies that every context that is a decision must have a non-decision parent.

Lemma 9. *If a scoped graph $\mathcal{G}(\mathcal{S})$ satisfies the condition(C) of Theorem 8, then for every context $Z \in \boldsymbol{C}_X$ where $Z, X \in \boldsymbol{X}(\mathcal{S})$ are decisions, there exists a non-decision $N \in \boldsymbol{C}_Z \setminus \lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$.*

Intuitively, this is because condition (C) states that there is an active path from Z to Y, given a superset of $\lceil \boldsymbol{X}(\mathcal{S}) \setminus \{Z\}$. If all of the parents of Z are decisions, then we would have $Z \in \lceil \boldsymbol{X}(\mathcal{S}) \setminus \{Z\}$, and every path would be blocked, and condition (C) could not be true.

Proof of Lemma 9. Assume that there is no such non-decision N, i.e. $\boldsymbol{C}_Z \subseteq \lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$, and that $\pi_X \not\perp Y \mid \lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$, (by condition (C) of Theorem 8), and we will prove a contradiction. From $\boldsymbol{C}_Z \subseteq \lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$, we deduce that $Z \in \lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$ (by the definition of $\lceil \boldsymbol{W}\rceil$), and then there can be no active path from π_X to Y given $\lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil \supseteq \boldsymbol{C}_Z \cup \{Z\}$, contradicting condition (C) of Theorem 8, and proving the result. ☐

This tells us that for any decision Z there is an edge $Z \leftarrow N$. Moreover, by condition(C) of the main result, we know that there is an info path from N to Y. By concatenating the edge and the path, we obtain a path from Z to Y, which we will prove is active given $\lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$. This is precisely the kind of info path that we are looking for: activeness given $\lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus Z}) \setminus Z\rceil$ means that forks and chains will not be decisions, and we know that the endpoint Z has a non-decision parent N.

Lemma 10. *If a scoped graph $\mathcal{G}(\mathcal{S})$ satisfies assumptions (B-C) of Theorem 8, then for every edge $Z \to X$ between decisions $Z, X \in \boldsymbol{X}(\mathcal{S})$, there exists a path $Z \leftarrow N \dashrightarrow\!\!\!- Y$, active given $\lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$, (so $N \notin \lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$).*

Some care is needed in proving that the segment N --- Y is active given $\lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{Z\}}) \setminus \{Z\}\rceil$, rather than just $\lceil(\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S})\setminus\{N\}}) \setminus \{N\}\rceil$, and the detail is presented in Lemma 10.

4.2.2 Defining the materiality paths

We will now describe how to select finitely many info paths, along with a control path, as shown in Figure 6. The assumptions of Theorem 8 allow there to be any finite number of contexts and decisions, so we will designate the target decision and context (whose materiality we are trying to establish) as $X_0 := X$ and context $Z_0 := Z$. We know from condition (A) that X_0 is an ancestor of Y, so we have a directed path $X_0 \dashrightarrow Y$. We also know that Z_0 has a chance node ancestor, because it either is a chance node, or it has a chance node parent, from Lemma 10. So we will call that chance node ancestor, A, and define a *control path* of the form $A \dashrightarrow Z_0 \to X_0 \dashrightarrow Y$, shown in black in Figure 6, where $A \dashrightarrow Z_0$ has length of either 0 or 1.

Other paths are then chosen to match this control path. We will index the decisions on the control path as $X_{i_{\min}}, \ldots, X_{i_{\max}}$, and their respective contexts are $Z_{i_{\min}}, \ldots, Z_{i_{\max}}$. where $i_{\min}$ is either 0 (if Z_0 is a chance node), or -1 (if $Z_0 = X_{-1}$). In general, we allow for the possibility that $Z_i = X_{i-1}$ for any of the decisions. We define an info path m_i for each context Z_i, which must satisfy the desirable properties established in Lemma 9. To help with our later proofs, it is also useful to define an intersection node T_i, at which the info path departs from the control path, and a truncated info path m'_i, which consists of the segment of m_i that is not in the control path. Recall from Figure 3b and Figure 4b that information from collider variables can play an important role in incentivising a decision to copy information from its context. So, for each collider $W_{i,j}$ in each info path m_i we define an *auxiliary path* $r_{i,j} : W_{i,j} \dashrightarrow Y$.

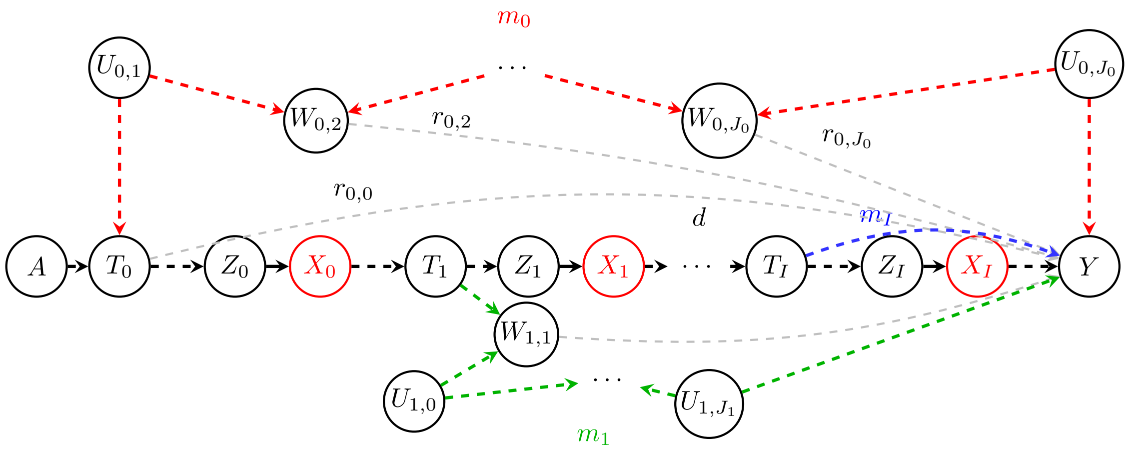


Figure 6: The set of paths proven to exist by Lemma 11 are red, green and blue. In each case, the point of departure of the active path from the (black) directed path is designated by T_i. In full generality, each path may begin either as $Z_i \dashleftarrow T_i \leftarrow \cdot$ (as in red), or as $Z_i \dashleftarrow T_i \rightarrow \cdot$ (green, blue).

Collectively, we refer to the control, info and auxiliary paths as the *materiality paths.*

Lemma 11. *Let $\mathcal{G}(\mathcal{S})$ be a scoped graph that contains a context $Z_0 \in \boldsymbol{C}_{X_0}$ and satisfies the assumptions of for Theorem 8. Then, it contains the following:*

- *A **control path**: a directed path $d : A \dashrightarrow Z_0 \rightarrow X_0 \dashrightarrow Y$, where A is a non-decision, possibly equal to Z_0, and d contains no parents of X_0 other than Z_0.*
- *We can write d as $A \dashrightarrow Z_{i_{min}} \rightarrow X_{i_{min}} \dashrightarrow \cdots Z_0 \rightarrow X_0 \dashrightarrow Z_{i_{max}} \rightarrow X_{i_{max}} \dashrightarrow Y, i_{min} \leq i \leq i_{max}$, where each Z_i is the parent of X_i along d (where $A \dashrightarrow Z_{i_{min}}$ and $X_{i-1} \dashrightarrow Z_i$ are allowed to have length 0). Then, for each i, define the **info path**: $m'_i : Z_i \text{ - - - } Y$, active given $\lceil (\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S}) \setminus Z_i}) \setminus Z_i \rceil$, that if Z_i is a decision, begins as $Z_i \leftarrow N$ (so $N \in \boldsymbol{C}_{Z_i} \setminus \lceil (\boldsymbol{X}(\mathcal{S}) \cup C_{\boldsymbol{X}(\mathcal{S}) \setminus Z_i}) \setminus Z_i \rceil$.)*
- *Let T_i be the node nearest Y in $m'_i : Z_i \text{ - - - } Y$ (and possibly equal to Z_i) such that the segment $Z_i \overset{m'_i}{\text{- - -}} T_i$ of m'_i is identical to the segment $Z_i \overset{d}{\dashleftarrow} T_i$ of d. Then, let the **truncated info path** m_i be the segment $T_i \overset{m'_i}{\text{- - -}} Y$.*
- *Write m_i as $m_i : T_i \dashrightarrow W_{i,1} \dashleftarrow U_{i,1} \dashrightarrow W_{i,2} \dashleftarrow U_{i,2} \cdots U_{i,J_i} \dashrightarrow Y$, where J_i is the number of* forks *in m_i. (We allow the possibilities that $T_i = W_{i,1}$ so that m_i begins as $T_i \dashleftarrow U_{i,1}$, or that $J_i = 0$ so that m_i is $T_i \dashrightarrow Y$.) Then, for each i and $1 \leq j \leq J_i$, let the **auxiliary path*** be any directed path *$r_{i,j} : W_{i,j} \dashrightarrow Y$ from $W_{i,j}$ to Y.*

The proof was described before the lemma statement, and is detailed in Appendix B.2.

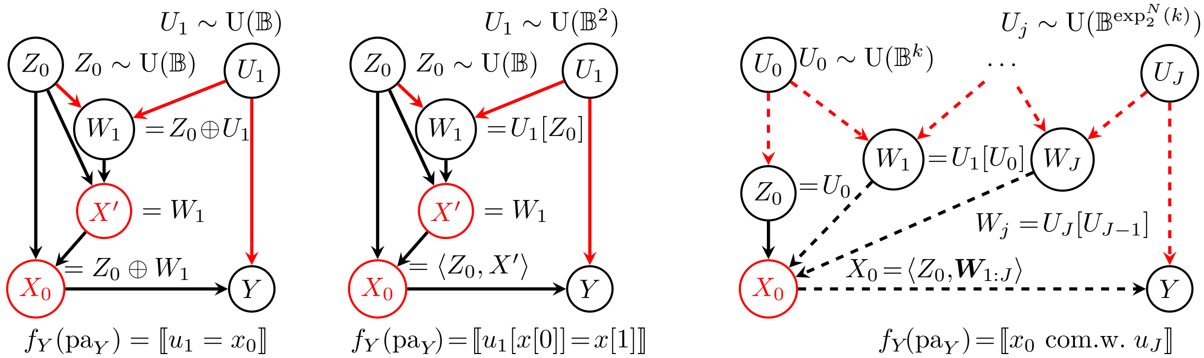
4.3 The materiality SCM

We will now show how the materiality paths can be used to define an SCM where Z_0 is material for X_0. As with the seleciton of paths, the construction of models will have to differ a little from the constructions of Sections 3.1 and 3.2, in order to better deal with insolubility. So we will first describe how we deal with insoluble graphs, in Section 4.3.1 , then define a general model in Section 4.3.2.

4.3.1 Models for insoluble graphs

Certain graphs that are allowed by Theorem 8 violate solubility, and the constructions from Everitt et al. (2021) and van Merwijk et al. (2022) will need to be altered in order to establish materiality in these graphs.

The assumption of solubility meant that upstream decisions could not contain latent, actionable information — in particular, this implied if an info path m_i contains a context C for a decision $X' \in \boldsymbol{X}(\mathcal{S}) \setminus \{X_i\}$,



(a) Z_0 is immaterial for X_0 **(b)** Z_0 is material for X_0 **(c)** Z_0 is material for X_0

Figure 7: Two SCMs (a-b), and a description of a family of SCMs, where each dashed line represents a path. The repeated exponent $\exp_2^n(k)$ is defined as k if $n = 0$, and $2^{\exp_2^{n-1}(k)}$ otherwise.

then V would have to be context of X_i, otherwise the past decision V would contain latent information that is of import to X_i (van Merwijk et al., 2022, Lemma 28). For example, in Figure 7a the red info path contains the variable W_1, which is a context for X' but not for X_0, and solubility is violated because $W_1 \perp Y \mid \{Z_0, X_0, X_1\}$ but it satisfies all the three conditions of Theorem 8.

We can nonetheless apply the construction from (van Merwijk et al., 2022) to this graph, by treating X' as through it was a non-decision. This yields the decision problem shown in Figure 7a, which is example of the construction from Figure 7c), except that there is a decision X' that observes Z_0 and W_1. In this model, the outcome Y is equal to 1 if x_0 is equal to u_1. The intended logic of this construction is that since $W_1 = Z_0 \oplus U_q$, the MEU can be achieved with the non-intervened policy $X_0 = Z_0 \oplus W_1$, which would require X_0 to depend on Z_0. In this model, however, there exists an alternative policy where $X' = U_1$ and $X_0 = X'$, which achieves the MEU of 1, without having X_0 directly depend on Z_0, and proving that Z_0 is immaterial for X_0. Essentially, the single bit of X' sufficed to transmit the value of U_1, meaning that Z_0 contained no more useful information. So long as the decision problem allows X' can do this there can be no need for X_0 to observe Z_0. So in order to exhibit materiality, we need the domain of X' to be smaller than that of U_1.

As such, we can devise a modified scheme, shown in Figure 7b. In this scheme, *two* random bits are generated at U_1. The outcome is $Y = 1$ if X_1 supplies one bit from U_1 along with its index. A random bit is sampled at Z_0, and W_1 presents the ${Z_0}^{\text{th}}$ bit from U_1, while X_1 has a domain of just one bit. Then, similar to our previous discussion of Figure 4b, the only bit from U_1 that X_0 can reliably know is the ${Z_0}^{\text{th}}$ bit. Hence the only way achieve the MEU is for X' to inform X_0 about the value of W_1, and for X_0 to equal $X_0 = \langle Z_0, X' \rangle$. Importantly, this can only be done if X_0 observes Z_0; it is material for X_0.

In Figure 7b, if x_1 produces the ${z_0}^{\text{th}}$ bit from u_1, i.e. $x_1 = \langle z_0, u_1[z_0] \rangle$, we will call it *consistent* with $\langle z_0, u_1 \rangle$. If it produces *any* bit from u_1, then we will call it *compatible* with $\langle z_0, u_1 \rangle$. For instance, either $\langle 0, 0 \rangle$ or $\langle 1, 1 \rangle$ is compatible with $z_0 = 0$ and $u_1 = 01$, but only the former is consistent with $z_0 = 0$ and $u_1 = 00$.

We can generalise these concepts to a case of multiple fork variables, rather than just Z_0 and U_1. For example, Figure 7c, we have $J + 1$ fork variables $\boldsymbol{U}_{0:J}$, which sample bitstrings of increasing length. Then, $Z_0 = W_u$, and each collider W_i has $W_i = U_j[U_{j-1}]$. The outcome Y will still check whether X_0 is compatible with U_J, but it will do so using a more general definition, as follows.

Definition 12 (Consistency and compatibility)**.** Let $\boldsymbol{w} = \langle w_0, w_1, \ldots, w_J \rangle$ where $w_0 \in \mathbb{B}^k$ and $w_n \in \mathbb{B}$ for $n \geq 1$. Then, $\boldsymbol{w}$ is *consistent with* $\boldsymbol{u} = \langle u_0, \ldots, u_J, u_i \in \mathbb{B}^{\exp_2^i(k)} \rangle$ (i.e. $\boldsymbol{w} \sim \boldsymbol{u}$) if $w_0 = u_0$ and $w_n = u_n[u_{n-1}]$ for $n \geq 1$. Moreover, $\boldsymbol{w}$ is *compatible with* $u_J \in \mathbb{B}^{\exp_2^J(k)}$ (i.e. $\boldsymbol{w} \sim u_J$) if there exists any $u_0, \ldots, u : J - 1$ such that $\boldsymbol{w}$ is consistent with $u_0, \ldots, u_J$.

In Figure 7b, if, with positive probability, the assignment of X_0 is inconsistent with $\langle z_0, u_1 \rangle$, then the decision-maker is also penalised with strictly positive probability. For instance, if the assignments $z_0 = 0$ and $u_1 = 01$ lead to the assignment $x = \langle 1, 1 \rangle$, then this policy will achieve utility of $y = 0$ given the assignments $y_0 = 0$

and $u_1 = 00$, since they cause the values $z_0 = 0$ and $w_1 = 0$, which will cause the assignment $x = \langle 1, 1\rangle$, which is not consistent with $z_0 = 0$ and $u_1 = \langle 0, 0\rangle$. We find that the same is true in the more general mode of Figure 7c. If with strictly positive probability, the assignment of X_0 is inconsistent with $\boldsymbol{u}_{0:J}$, then there will exist an alternative assignment $\boldsymbol{U}_{0:J} = \boldsymbol{u}'_{0:J}$, that produces the same assignments to the observations of X_0, but where X_0 is not compatible with $\boldsymbol{u}'_J$.

Lemma 13. *Let $\boldsymbol{w} = \langle w_0, \ldots, w_J\rangle$ and $\bar{\boldsymbol{w}} = \langle \bar{w}_0, \ldots, \bar{w}_J\rangle$ be sequences with $w_0, \bar{w}_0 \in \mathbb{B}^k$, $w_j, \bar{w}_j \in \mathbb{B}$ for $j \geq 1$, and let $J' \leq J$ be the smallest integer such that $w_{J'} \neq \bar{w}_{J'}$. Let $u_0, \ldots, u_{J'}$ be a sequence where $u_j[u_{j-1}] = w_j$ for $1 \leq j < J'$. Then, there exists some $u_{J'+1}, \ldots, u_J$ such that $\boldsymbol{w}$ is consistent with $u_0, \ldots, u_J$, but $\bar{\boldsymbol{w}}$ is incompatible with u_J.*

The proof is deferred to Appendix B.5.

This result implies that an optimal policy in Figure 7c, x_0 must be consistent with $\boldsymbol{u}_{0:J}$ with probability 1. After all, the non-intervened policy clearly achieves the MEU of 1, being that it is consistent with $\boldsymbol{u}_{0:J}$, and consistency implies compatibility. On the other hand, if x_0 is inconsistent with $\boldsymbol{u}_{0:J}$ with strictly positive probability, then there will exist an alternative assignment $\boldsymbol{u}'_{0:J}$ that produces the same assignment x_0, and since the variables $\boldsymbol{U}_{0:J}$ have full support, this will lead to $y = 0$ will strictly positive probability, and decrease the expected utility. If a policy cannot copy Z_0 without observing it, then this will make X_0 inconsistent with $\boldsymbol{u}$ with strictly positive probability, so this policy will not be optimal. One may notice that by setting U_0 to contain k bits rather than just one, this will make it very difficult for Z_0 to copy the value of Z_0 without observing it, if a sufficiently large k is chosen. We will develop a fully formal argument for materiality in Section 4.4.

4.3.2 A decision problem for any graph containing the materiality paths

We will now generalise the constructions from Figure 3a (for a truncated info path is a directed path) and Figure 7c (for a truncated info path that is not a directed path) to an arbitrary graph containing the materiality paths described in Lemma 11.

To begin with, let us note that the materiality paths may overlap. So our general approach will be to define a random variable V^p for each variable in a path p. To derive the overall materiality SCM, we will simply define V by a cartesian product over each V_p. For the outcome variable Y, we will instead take a sum over each Y^p. For any set of paths $\boldsymbol{p}$, we define $V^{\boldsymbol{p}} = \times_{p \in \boldsymbol{p}} V^p$.

Let us now discuss the control path. The initial node A will sample a bitstring that is passed along the control path, and through each intersection node T_i in particular. To describe this, we will rely on shorthand.

Definition 14 (Parents along paths)**.** When a vertex V has a unique parent $\bar{V}$ along p, $\mathrm{Pa}(V^p) = \bar{V}^p$, and for a set of paths $\boldsymbol{p}'$, let $\mathrm{Pa}(V^{\boldsymbol{p}'}) = \times_{p \in \boldsymbol{p}'} \mathrm{Pa}(V^p)$. For a collider V in a truncated info path $m_i : T_i \text{ --- } Y$, let the parent nearer T_i along m_i be $\mathrm{Pa}_L(V)$, and the parent nearer Y be $\mathrm{Pa}_R(V)$.

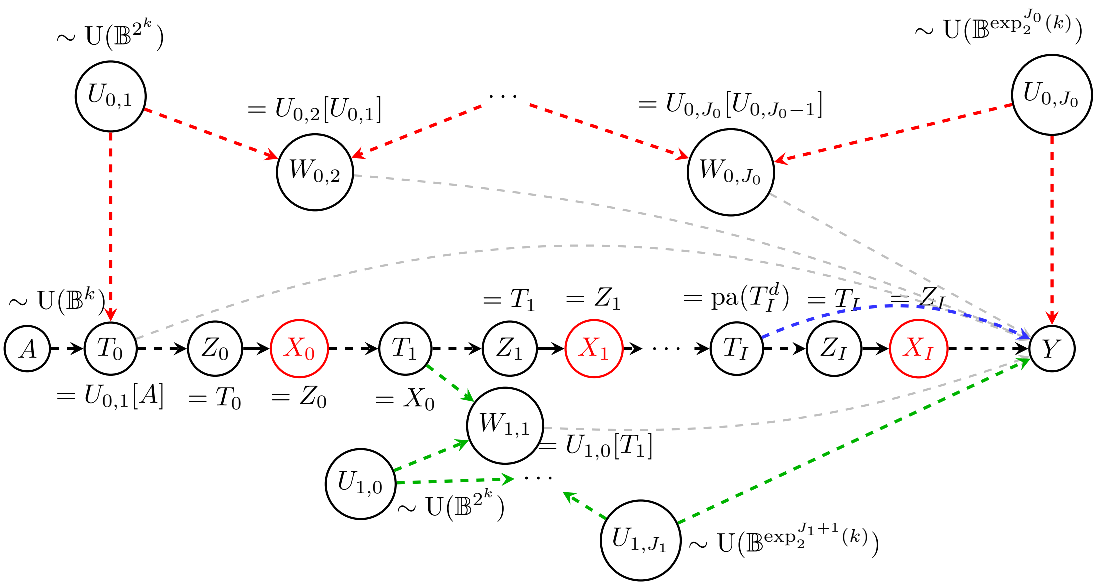
For example, a non-outcome child V of A along the control path will be assigned $V^d = \mathrm{Pa}(V^d)$.

Each info path must pass on information from upstream paths that traverse the intersection node. We therefore use the notation $\boldsymbol{p}_i$ to refer to the set of control and auxiliary paths that enter the intersection node T_i. We also devise an extended notion of parents Pa^* to include this information. Relatedly, we will define a notion of parents for the auxiliary path, which includes information from the collider $W_{i,j}$ of the info path, and a notion of parents for the paths $\boldsymbol{p}_i$, that includes the exogenous parent $\mathcal{E}_A$ of A.

Definition 15 (Extended parent relations)**.** For a truncated info path m_i, let:

$$\mathrm{Pa}^*(V^{m_i}) = \begin{cases} T_i^{\boldsymbol{p}_i} & \text{if } \mathrm{Pa}(V^{m_i}) = T_i^{m_i} \\ \mathrm{Pa}(V^{m_i}) & \text{otherwise} \end{cases}, \text{ and } \mathrm{Pa}_l^*(V) = \begin{cases} T_i^{\boldsymbol{p}_i} & \text{if } \mathrm{Pa}_L(V^{m_i}) = T_i^{m_i} \\ \mathrm{Pa}_L(V_l^{m_i}) & \text{otherwise} \end{cases}.$$

For an auxiliary path $r_{i,j}$, let $\mathrm{Pa}^*(V^{r_{i,j}}) = \begin{cases} W_{i,j}^{m_i} & \text{if } \mathrm{Pa}(V^{r_{i,j}}) = W_{i,j}^{m_i} \\ \mathrm{Pa}(V^{r_{i,j}}) & \text{otherwise} \end{cases}.$



$$f_Y(\mathrm{pa}_Y) = [\![\mathrm{pa}(Y^{\boldsymbol{p}_0,\boldsymbol{r}_{0:J^0}}) \text{ com.w. } u_{0,J_0}]\!] \;\; + [\![\mathrm{pa}(Y^{\boldsymbol{p}_1,\boldsymbol{r}_{0:J^1}}) \text{ com.w. } u_{1,J_1}]\!] \;\; + [\![x_I = t_I]\!]$$

Figure 8: The materiality SCM: a general SCM where Z_0 is material for X_0.

Finally, let: $\mathrm{Pa}^*(V^{\boldsymbol{p}_i}) = \begin{cases} \mathcal{E}_A \times \mathrm{Pa}(V^{\boldsymbol{p}_i}) & \text{if } V \text{ is } A \\ \mathrm{Pa}(V^{\boldsymbol{p}_i}) & \text{otherwise} \end{cases}$.

In other respects, the materiality SCM will behave in a similar manner to previous examples. For instance, when m_i is directed, the outcome Y^{m_i} will evaluate whether the values $\mathrm{Pa}(Y^{\boldsymbol{p}_i})$ (which mostly come from X_i) are equal to $\mathrm{Pa}(Y^{m_i})$, which come from the info path. When m_i is not directed, the outcome Y^{m_i} will evaluate whether the values from $\mathrm{Pa}(Y^{\boldsymbol{p}_i, r_{i,0:J}})$ are compatible with those from $U_{i,J}$. So let us now define the materiality SCM as follows.

Definition 16 (Materiality SCM)**.** Given a graph containing the materiality paths, we may define the following random variables.

In the control path, $d : A \dashrightarrow Y$, let:

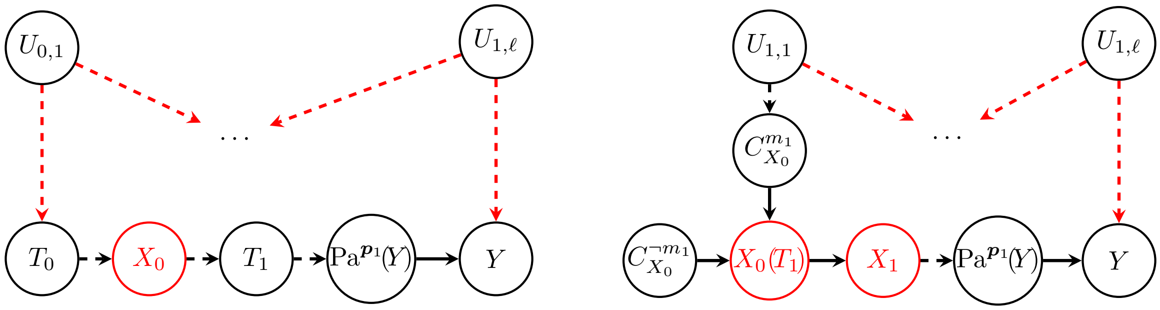
- the source be $A^d = \mathcal{E}^{A^d}$ where $\mathcal{E}^{A^d} \sim \mathrm{U}(\mathbb{B}^k)$ where k is the smallest positive integer such that $2^k > (k+c)bc$, where b is the maximum number of variables that are contexts of one decision, $b := \max_{X \in \boldsymbol{X}(\mathcal{S})} |C_X|$, and c is the maximum number of materiality paths passing through any vertex in the graph;
- every non-endpoint V have $V^d = \mathrm{Pa}(V^d)$.

In each truncated info path that is directed, $m_i : T_i \dashrightarrow Y$, let:

- the intersection node T^{m_i} have trivial domain;
- each chain node be $V^{m_i} = \mathrm{Pa}^*(V^{m_i})$
- the outcome have the function $f_{Y^{m_i}}(\mathrm{pa}_Y) = [\![\mathrm{pa}(Y^{\boldsymbol{p}_i}) = \mathrm{pa}^*(Y^{m_i})]\!]$.

In each truncated info path that is not directed, T_i - - - $\leftarrow W_{i,1} \to \cdots \leftarrow W_{i,J} \dashrightarrow Y$, let:

- each fork be $W_{i,j}^{m_i} = \mathcal{E}^{W_{i,j}^{m_i}}, \mathcal{E}^{W_{i,j}^{m_i}} \sim \mathrm{U}(\mathbb{B}^{\exp_2^j(k+|\boldsymbol{p}_i|-1)})$ where $|\boldsymbol{p}_i|$ is the number of paths in $\boldsymbol{p}_i$;
- each chain node be $V^d = \mathrm{Pa}^*(V^d)$;
- each collider be $V^{m_i} = \mathrm{Pa}_R(V^{m_i})[\mathrm{Pa}_L^*(V^{m_i})]$;



(a) The intersection node T_1 is a chance node.

(b) The intersection node T_1 is a decision. The contexts of X_0 are divided into $C^{m_1}_{X_0}$ (the parent along the info path), and $C^{\neg m_1}_{X_0}$ (the other parents).

Figure 9: The cases where the intersection node T_1 is a chance node, or a decision

- each intersection node be $T_i^{m_i} = \mathrm{Pa}(V^{m_i})[\mathrm{Pa}^*(T_i^{\boldsymbol{p}_i})]$ if the info path begins as $T_i \to \cdot$, otherwise it has empty domain;
- the outcome have the function $f_{Y^{m_i}}(\mathrm{pa}_Y) = [\![\mathrm{pa}(Y^{\boldsymbol{p}_i,\boldsymbol{r}_{i,1:J_i}})$ is compatible with $\mathrm{pa}^*(Y)]\!]$.

In each auxiliary path $r_{i,j} : W_{i,j} \to V_2 \dashrightarrow Y$, let:

- each chain node have $V^{r_{i,j}} = \mathrm{Pa}^*(v^{r_{i,j}})$.
- each source $W_{i,j}$ have trivial domain

Then, let the *materiality SCM* have outcome variable $Y = \sum_{i_{\min} \le i \le i_{\max}} Y^{m_i}$, and non-outcome variables $V = \times_{p \in \{d, m_i, r_{i,1:J_i} | i_{\min} \le i \le i_{\max}\}} V^p$.

Note that this defines an SCM because each variable is a deterministic function of only its endogenous parents and exogenous variables.

We have define the materiality SCM so that decisions behave just as non-decisions, which always do what is required to ensure that $Y^{m_i} = 1$.

Lemma 17. *In the non-intervened model, the materiality SCM has $Y = i_{max} - i_{min} + 1$, surely.*

The proof follows from the model definition, and is supplied in Appendix B.4.

We also know that each utility term Y^{m_i} is upper bounded at one, so in order to obtain the MEU, each Y^i must equal 1, almost surely.

Lemma 18. *If a policy $\boldsymbol{\pi}$ for the materiality SCM, has $P^{\boldsymbol{\pi}}(Y^{m_i} < 1) > 0$ for any i, the MEU is not achieved.*

Proof. We know that $\mathbb{E}^{\boldsymbol{\pi}}[Y] = \sum_{i_{\min} \le i \le i_{\max}} Y^{m_i}$ (Definition 16), so for all i, $Y^{m_i} \le 1$ always. So, if $P^{\boldsymbol{\pi}}(Y^{m_i} < 1) > 0$ for any i, then $\mathbb{E}^{\boldsymbol{\pi}}[Y] < i_{\max} - i_{\min} + 1$, which underperforms the policy that is followed in the non-intervened model (Lemma 17). $\square$

4.4 Proving materiality in the materiality SCM

We will now prove that in the materiality SCM, if Z_0 is removed from the contexts of X_0, then the performance for at least one of the utility variables Y^{m_i} is compromised, and so the MEU is not achieved. The proof divides into two cases, based on whether the child of X_0 along the control path is a non-decision (Section 4.4.1) or a decision (Section 4.4.2).

4.4.1 Case 1: child of X_0 along d is a non-decision.

If the child of X_0 along the control path is a non-decision and Z_0 is not a context of X_0, we will prove that $\mathbb{E}[Y^{m_0}] < 1$. In this case, either X_0 is the last decision in the control path, or otherwise there must exist an intersection node T_1, as shown in Figure 9a. If the former is true, then it is immediate that the value x_0 is transmitted to Y along the control path, based on the model definition. As such, Y_0 can directly evaluate the decision X_0. For the latter case, we want an assurance that downstream decisions will pass along the value of X, as was the case in Figure 4b. Such an assurance is provided by the following lemma, which shows that whenever an intersection node T_i is a chance node — as is T_1 — the value t_i is transmitted to Y by every optimal policy.

Lemma 19 (Chance intersection node requirement)**.** *If in the materiality SCM, where T_i is a chance node, a policy $\boldsymbol{\pi}$ has $P^{\boldsymbol{\pi}}(Pa(T_i^{\boldsymbol{p}_i}) = Pa(Y^{\boldsymbol{p}_i})) < 1$, then $P^{\boldsymbol{\pi}}(Y^{m_i} < 1) > 0$.*

First, we prove the case where m_i is a directed path. In this case, m_i copies the value $t^{\boldsymbol{p}_i}$ to Y, which Y^{m_i} checks against the value $\text{pa}(y^{\boldsymbol{p}_i})$ received via the control path. Maximising Y^{m_i} then requires them to be equal.

Proof of Lemma 19 when m_i is a directed path. We have $f_{Y^{m_i}}(\text{pa}_{Y^{m_i}}) = [\![\text{pa}(Y^{m_i}) = \text{pa}(Y^{\boldsymbol{p}_i}))]\!]$ (Definition 16). Also, $\text{Pa}(Y^{m_i}) = T_i^{\boldsymbol{p}_i} = \text{Pa}(T_i^{\boldsymbol{p}_i})$ surely, where the first equality follows from Definition 16, while the second follows from Definition 16 and T_i being a chance node. So, if $P^{\boldsymbol{\pi}}(\text{Pa}(Y^{\boldsymbol{p}_i}) = \text{Pa}(T^{\boldsymbol{p}_i})) < 1$ then $P^{\boldsymbol{\pi}}(Y^{m_i} = 1) < 1$. □

We now prove the case where m_i is a directed path. In this case, if the assignment $\text{pa}(Y^{\boldsymbol{p}_i})$ transmitted along the control path differs from the value $\text{pa}(T_i^{\boldsymbol{p}_i})$ that came in to the intersection node T_i, then just as we established for Figure 7c, there will exist an assignment $\boldsymbol{u}_{i,1:J_i}$ to the fork nodes in m_i that gives an unchanged assignment to colliders $\boldsymbol{v}_{i,1:J_i}$, but where $\text{pa}(Y^{\boldsymbol{p}_i})$ is incompatible with u_{J_i}.

Proof of Lemma 19 when m_i is not a directed path. Let us index the forks and colliders of m_i as $T_i \dashrightarrow V_{i,1} \dashleftarrow U_{i,1} \dashrightarrow W_{i,1} \dashleftarrow \cdots W_{i,J_i} \dashleftarrow U_{i,J_i} \dashrightarrow Y$. Choose any assignments $\text{pa}(T_i^{\boldsymbol{p}_i}) \neq \text{pa}(Y^{\boldsymbol{p}_i})$ that occur with strictly positive probability. Then, there must also exist assignments $\text{Pa}(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}}) = \text{pa}(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}})$, $\boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}_{1:J_i}$, and $\boldsymbol{W}_{i,1:J_i} = \boldsymbol{w}_{1:J_i}$ such that

$$P^{\boldsymbol{\pi}}(\text{pa}(T_i^{\boldsymbol{p}_i}), \text{pa}(Y^{\boldsymbol{p}_i, r_{i,1}}), t_i^{\boldsymbol{p}_i}, \boldsymbol{u}_{1:J_i}, \boldsymbol{w}_{1:J_i}) > 0.$$

By Lemma 13, there also exists an assignment $\boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i}$ such that $\text{pa}(T_i^{\boldsymbol{p}_i}), \boldsymbol{w}_{1:J_i}$ is consistent with $u'_{1:J_i}$, and $\text{pa}(Y_i^{\boldsymbol{p}}), \text{pa}(Y^{\boldsymbol{r}_{i,1:J_i}})$ is incompatible with u'_{J_i}. Now, consider the intervention $\text{do}(\boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i})$. Since T_i is a chance node, every collider in m_i is a non-decision, and is assigned the (unique) value consistent with $\text{pa}(T_i^{\boldsymbol{p}_i}), \boldsymbol{u}'_{1:J_i}$. Furthermore, $\text{pa}(T_i^{\boldsymbol{p}_i}), \boldsymbol{w}_{1:J_i}$ is consistent with $\text{pa}(T_i^{\boldsymbol{p}_i}), \boldsymbol{u}'_{1:J_i}$, so the intervention does not affect the assignments to these colliders. Moreover, from Definition 16, no variable outside of m_i is affected by assignments within m_i, except through the colliders. Therefore:

$$\begin{aligned}
&P^{\boldsymbol{\pi}}(\text{pa}(Y^{\boldsymbol{p}_i}), \text{pa}(Y^{\boldsymbol{r}_{i,1:J_i}}), \text{Pa}(Y^{m_i}) = u'_{J_i} \mid \text{do}(\boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i})) > 0 \\
\therefore\, &P^{\boldsymbol{\pi}}(Y^{m_i} = 0 \mid \text{do}(\boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i})) > 0 \\
&\qquad\qquad (\text{pa}(Y_i^{\boldsymbol{p}}), \text{pa}(Y^{\boldsymbol{r}_{i,1:J_i}}) \text{ not compatible with } u'_{J_i}) \\
\therefore\, &P^{\boldsymbol{\pi}}(Y^{m_i} = 0 \mid \boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i}) > 0 \\
&\qquad\qquad (\boldsymbol{U}_{i,1:J_i} \text{ are unconfounded, so } P^{\boldsymbol{\pi}}(\boldsymbol{V} \mid \text{do}(\boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i})) = P^{\boldsymbol{\pi}}(\boldsymbol{V} \mid \boldsymbol{U}_{i,1:J_i} = \boldsymbol{u}'_{1:J_i}) \\
\therefore\, &P^{\boldsymbol{\pi}}(Y^{m_i} = 0) > 0 \qquad\qquad (P^{\boldsymbol{\pi}}(\boldsymbol{u}_{i,1:J_i}) > 0).
\end{aligned}$$

□

If m_i is not a directed path, then this requirement extends to the values $\text{pa}(Y^{\boldsymbol{r}_{i,1:J_i}})$ passed down the auxiliary paths, not just the value $\text{pa}(Y^{\boldsymbol{p}_i})$ from the control path. Specifically, $\text{pa}(Y^{\boldsymbol{p}_i}), \text{pa}(Y^{\boldsymbol{r}_{i,1:J_i}})$ must be consistent with $\text{pa}(Y^{\boldsymbol{p}_i}), \boldsymbol{u}_{i,1:J_i}$, where $\boldsymbol{u}_{i,1:J_i}$ denotes the values of forks on the info path.

Lemma 20 (Collider path requirement)**.** *If the materiality SCM has an info path m_i that is not directed, and under the policy $\boldsymbol{\pi}$ there are assignments $Pa(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}}) = pa(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}})$ to parents of the outcome, and $\boldsymbol{U}^{m_i}_{i,1:J_i} = \boldsymbol{u}^{m_i}_{i,1:J_i}$ to the forks of m_i, with $P^{\boldsymbol{\pi}}(pa(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}}), \boldsymbol{u}^{m_i}_{i,1:J_i}) > 0$ and where $pa(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}})$ is inconsistent with $pa(Y^{\boldsymbol{p}_i}), \boldsymbol{u}^{m_i}_{i,1:J_i}$, then $P^{\boldsymbol{\pi}}(Y^{m_i} < 1) > 0$.*

The idea of the proof, similar to Lemma 19, is that whenever the bits transmitted along the auxiliary paths deviate from the values $\boldsymbol{w}_{i,1:J_i}$ of colliders in m_i, there exists an assignment $\boldsymbol{u}'_{i,1:J_i}$ to forks in m_i that will render the colliders, and hence the decision x_i unchanged, while making x_i incompatible with u_{J_i}, and thereby producing $Y^{m_i} < 0$. A detailed proof is in Appendix B.5.

In order to prove that the context Z_0 is needed, we will also need to establish that it is not deterministic, even if it is a decision. In the case where Z_0 is a decision, the idea is that random information is generated at A, which each of the decisions are required to pass along the control path. We are able to prove this as a corollary of Lemma 19.

Lemma 21 (Initial truncated info path requirements)**.** *If $\boldsymbol{\pi}$ in the materiality SCM does not satisfy: $P^{\boldsymbol{\pi}}(Pa(Y^d) = A^d) < 1$. then the MEU is not achieved.*

Proof. From Lemma 11, the control path d begins with a chance node. So, the first decision $X_{i_{\min}}$ in d must have a chance node $Z_{i_{\min}}$ as its parent along d. Furthermore, the intersection node $T_{i_{\min}}$ must be an ancestor of $Z_{i_{\min}}$ along d, so it is also a chance node. So it follows from Lemma 19, that any policy $\boldsymbol{\pi}$ must satsify $P^{\boldsymbol{\pi}}(T^{\boldsymbol{p}_{i_{\min}}}_{i_{\min}} = \mathrm{Pa}(Y^{\boldsymbol{p}_{i_{\min}}})) = 1$ if it attains the MEU. As $T_{i_{\min}}$ is in the control path, we have $d \in \boldsymbol{p}_{i_{\min}}$ (Lemma 11) so $T^d_{i_{\min}} \overset{\text{a.s.}}{=\!=} \mathrm{Pa}(Y^d)$ is also required. Moreover, all of vertices in the segment $A \dashrightarrow T_{i_{\min}}$ of d are chance nodes, because $X_{i_{\min}}$ was defined as the first decision in d, and $T_{i_{\min}}$ precedes it. And, each chance variable V^d on the control path equals its parent $\mathrm{Pa}(V^d)$ (Definition 16), so $A^d = T^d_{i_{\min}}$, and thus $A^d \overset{\text{a.s.}}{=\!=} \mathrm{Pa}(Y^d)$ is required to attain the MEU. □

We can now combine our previous results to prove that it is impossible to achieve the MEU, if Z_0 is not a context of X_0, in the case where T_1 does not exist, or is a non-decision.

Lemma 22 (Required properties unachievable if child is a non-decision)**.** *Let $\mathcal{M}$ be a materiality SCM where the child of X_0 along d is a* non-decision*. Then, the MEU for the scope $\mathcal{S}$ cannot be achieved by a deterministic policy in the scope $\mathcal{S}_{Z_0 \not\to X_0}$ (equal to $\mathcal{S}$, except that Z_0 is removed from $\boldsymbol{C}_{X_0}$).*

The logic is that if child of X_0 in the control path is a non-decision, then the value of X_0 is copied all the way to $\mathrm{Pa}(Y^d)$ (Lemma 21). Furthermore, $Z_0^d \overset{\text{a.s.}}{=\!=} \mathrm{Pa}(Y^d)$ is necessary to achieve the MEU (Lemma 19). But the materiality SCM has been constructed so that the non-Z_0 parents of X_0 do not contain enough bits to transmit all of the information about Z_0^d, so the MEU cannot be achieved. The proof is detailed in Appendix B.6.

4.4.2 Case 2: child of X_0 along d is a decision.

If the child of X_0 along d is a decision, as shown in Figure 9b, we will prove that the decision X_0 must depend on Z_0 in order to achieve $\mathbb{E}[Y_1] = 1$. This will be because without Z_0, X_0 will be limited in its ability to distinguish all of the possible values of the first fork node $U_{i,1}$ of m_1. To establish this, we will need to conceive of a possible intervention to the fork nodes in m_i, that X_i would have to respond to, and so we begin by proving that relatively few variables will be causally affected by certain interventions.

Lemma 23 (Fork information can pass in few ways)**.** *If, in the materiality SCM:*

- *the intersection node T_i is the vertex X_{i-1},*
- *π_{T_i} is a deterministic decision rule where $\pi_{T_i}(\boldsymbol{c}^{\neg m_i}(T_i, u_{i,1}) = \pi_{T_i}(\boldsymbol{c}^{\neg m_i}(T_i, u'_{i,1}))$ for assignments $u_{i,1}, u'_{i,1}$ to the first fork variable, and $\boldsymbol{c}^{\neg m_i}(T_i)$ to the contexts of T_i not on m_i, and*
- *$\boldsymbol{W}_{i,1:J_i} = \boldsymbol{w}_{i,1:J_i}$, and $\boldsymbol{U}_{i,2:J_i} = \boldsymbol{u}_{i,2:J_i}$ are assignments to forks and colliders in m_i where each $u_{i,j}$ consists of just $w_{i,j}$ repeated $\exp_2^j(k + |\boldsymbol{p}_i| - 1)$ times, then:*

$$P^{\boldsymbol{\pi}}(pa(Y^{\boldsymbol{p}_i, r_{i,1}}), \boldsymbol{c}^{\neg m_i}(T_i), \boldsymbol{w}_{i,1:J_i}, \boldsymbol{u}_{i,2:J_i} \mid \mathrm{do}(u_{i,1})) = P^{\boldsymbol{\pi}}(pa(Y^{\boldsymbol{p}_i, r_{i,1}}), \boldsymbol{c}^{\neg m_i}(T_i), \boldsymbol{w}_{i,1:J_i}, \boldsymbol{u}_{i,2:J_i} \mid \mathrm{do}(u'_{i,1})).$$

The proof follows from the definition of the materiality SCM, and it is detailed in Appendix B.7.

We can now prove that if a deterministic policy does not appropriately distinguish assignments to $U_{i,1}$, then the i^{th} component of the utility will be suboptimal $\mathbb{E}[Y^{m_i}] < 1$.

Lemma 24 (Decision must distinguish fork values)**.** *If in the materiality SCM:*

- *the intersection node T_i is the vertex X_{i-1}, and*
- *$\boldsymbol{\pi}$ is a deterministic policy that for assignments $u_{i,1}, u'_{i,1}$ to $U_{i,1}$ where $u_{i,1} \neq u'_{i,1}$, has $\pi_{T_i}(\boldsymbol{c}^{\neg m_i}(T_i), u_{i,1}) = \pi_{T_i}(\boldsymbol{c}^{\neg m_i}(T_i), u'_{i,1})$ for every $\boldsymbol{C}^{\neg m_i}_{T_i}(T_i) = \boldsymbol{c}^{\neg m_i}(T_i)$,* (†)

then $P^{\boldsymbol{\pi}}(Y^{m_i} < 1) > 0$

The idea of the proof is that if $u_{i,1}$ and $u'_{i,1}$ differ, there will be some assignment $\text{pa}(Y^{\boldsymbol{p}_i})$ such that $u_{i,1}[\text{pa}(Y^{\boldsymbol{p}_i})]$ and $u'_{i,1}[\text{pa}(Y^{\boldsymbol{p}_i})]$ differ. When $\text{Pa}(Y^{\boldsymbol{p}_i}) = \text{pa}(Y^{\boldsymbol{p}_i})$ and $u_{i,1}$, then $\text{Pa}(Y^{r_{i,1}})$ to assume one value. But if we intervene $u'_{i,1}, u_{i,2:J_i}$, then the value of $\text{Pa}(Y^{r_{i,1}})$ will be incorrect, making $\text{Pa}(Y^{\boldsymbol{p}_i, \boldsymbol{r}_{i,1:J_i}})$ inconsistent with $\text{Pa}(Y^{\boldsymbol{p}_i}, U_{i,1:J_i})$ so the maximum expected utility is impossible to achieve. The details are deferred to Appendix B.8.

This will allow us to prove that when the child of X_0 along d is a decision, the MEU cannot be achieved without Z_0 as a context of X_0.

Lemma 25 (Required properties unachievable if child is a decision)**.** *Let $\mathcal{M}$ be the materiality SCM for some scoped graph $\mathcal{G}_{\mathcal{S}}$, where $i_{max} > 0$ and T_1 is a decision. Then, there exists no deterministic policy in the scope $\mathcal{S}_{Z_0 \not\to X_0}$ that achieves the MEU.*

To prove that no deterministic policy in $\mathcal{S}_{Z_0 \not\to X_0}$ can achieve the MEU (achievable with the scope $\mathcal{S}$), we will show that if a deterministic policy $\boldsymbol{\pi}$ satisfies $P^{\boldsymbol{\pi}}(\text{Pa}(Y^d) = A^d) = 1$, as required by Lemma 21, then the domain of $X_0 \times C^{\neg m_1}_{X_0}$ is smaller than the domain of $C^{m_1}_{X_0}$, so Equation (†) will be satisfied, and thus the MEU cannot be achieved. A detailed proof is presented in Appendix C.

We now combine the lemmas for the two cases to prove the main result.

Proof of Theorem 8. Any scoped graph $\mathcal{G}(\mathcal{S})$ that satisfies assumptions (A-C) contains materiality paths for the context Z_0 of X_0 (Lemma 11), and has a materiality SCM (Definition 16) compatible with $\mathcal{G}(\mathcal{S})$. In this decision problem, whether the child of X_0 along d is or is not a decision, the MEU cannot be achieved by a deterministic policy unless X_0 is allowed to depend on Z_0 (Lemmas 22 and 25). And stochastic policies can never surpass the best deterministic policy ((Lee & Bareinboim, 2020, Proposition 1)), so no such policy can achieve the MEU, and so Z_0 is material for X_0. □

5 Toward a more general proof of materiality

So far, via Theorem 8 we have established the necessity of condition (I) of LB-factorizability for immateriality. We now outline some steps toward evaluating the necessity of conditions (II-III) of LB-factorizability, and the further condition in (Lee & Bareinboim, 2020, Thm. 2).

To begin with, condition (III) requires that we choose an ordering $\prec$, such that the parents of each decision X are somewhere before X, while the descendants are somewhere afterwards. Clearly this condition can be satisfied for any acyclic graph, so it instead

Conditions (II-III) are individually not very restrictive, but are jointly substantial. So a natural next step is to try to prove that conditions (II-III) are necessary, by defining some info paths and control paths for graphs that violate conditions (II-III), defining a materiality SCM, and proving materiality in that SCM. So far, however, we have only been able to carry out the first step — defining the paths — and difficulties have arisen in using those paths to define an SCM that exhibits materiality. In this section, we will outline what info paths and control paths can be proven to exist, and then outline the difficulties in using them to prove materiality.

5.1 A lemma for proving the existence of paths

When the variables $\boldsymbol{Z}, \boldsymbol{X}', \boldsymbol{C}', \boldsymbol{U}$ are not factorizable, we can prove the existence of info and control paths.

Lemma 26 (System Exists General)**.** *Let $\mathcal{G}_{\mathcal{S}}$ be a scoped graph that satisfies assumptions (A,B) from Theorem 8. If $\boldsymbol{Z} = \{Z_0\}$, $\boldsymbol{X}' \supseteq Ch(Z_0)$, $\boldsymbol{C}' = C_{\boldsymbol{X}'} \setminus (\boldsymbol{X}' \cup \boldsymbol{Z})$, $\boldsymbol{U} = \emptyset$ are not LB-factorizable, then there exists a pair of paths to some $C' \in \boldsymbol{C}' \cup Y$:*

- *an info path $m : Z_0 \dashrightarrow C'$, active given $\lceil \boldsymbol{X}' \cup \boldsymbol{C}' \rceil$, and*
- *a control path $d : X \dashrightarrow C'$ where $X \in \boldsymbol{X}'$.*

A proof is supplied in Appendix D.1. The intuition of this proof is that each of the conditions (I-III) implies a precedence relation between a pair of variables in $\boldsymbol{V}' \cup Y$. Each of these precedence relations can be used to build an "ordering graph" over $\boldsymbol{V}' \cup Y$. If the ordering graph is acyclic, then we can let $\prec$ be any ordering that is topological on the graph, and then $\boldsymbol{Z}, \boldsymbol{X}', \boldsymbol{C}', \boldsymbol{U}$ are LB-factorizable. Otherwise, we can use a cycle in the graph to prove the existence of an info path and a control path. By iterating through these cycles, we can obtain a series of info paths and control paths that terminate at Y.

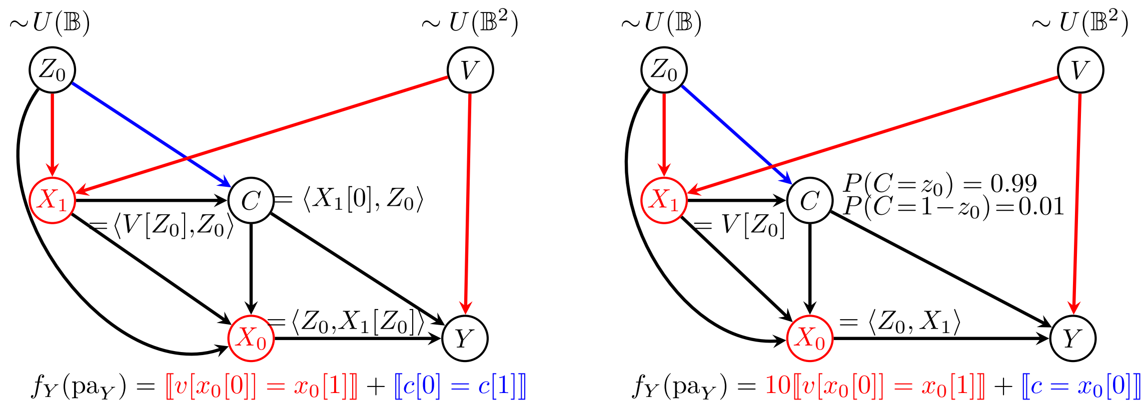
The resulting paths are in some cases, quite useful for proving materiality. For instance, we can recover the pair of info and control paths used in Figure 4b. To prove that Z is material for X, we can start by choosing $\boldsymbol{X}' = \{X, X'\}$, $\boldsymbol{C}' = \{Z'\}$, $\boldsymbol{C}' = \{Z', W\}$, and $\boldsymbol{U}' = \emptyset$. Then, Lemma 26 implies the existence of an active path from Z to some $\text{Desc}_X \cap \boldsymbol{C}'$, so we see that the first info path is the edge $Z \to Z'$. With Z' being a descendant of X, we also have the first control path, $X \to Z'$. We must then obtain some paths that exhibit why Z' is itself useful for the decision X to know about, and to influence. To do this, we can reapply Lemma 26 using the sets $\boldsymbol{X}' = \{X'\}, \boldsymbol{Z} = \{Z'\}, \boldsymbol{C}' = \{W\}$, and $\boldsymbol{U}' = \emptyset$. We then obtain the new info path $Z' \to W \leftarrow U \to Y$, and the new control path $Z' \to X' \to Y$. The SCM in Figure 4b uses these paths to prove Z material for X.

5.2 A further challenge: non-collider contexts

In some graphs, it is not clear how to use the info and control paths Lemma 26 to prove materiality, because non-collider nodes on the info path may be contexts. (In previous work, this possibility was excluded by the solubility assumption (van Merwijk et al., 2022, Lemma 28).) We will now highlight one case, in Figure 10, where it is relatively clear how this challenge can be overcome, and one case, Figure 11, where it is unclear how to make progress.

In the graph of Figure 10, we would like to prove that Z_0 is material for X_0. Using Lemma 26, we can obtain the red and blue info paths as shown, and the corresponding control paths in darker versions of the same colors. In the approach of Definition 16, shown in Figure 10a, X_0 should need to observe Z_0 in order to know which slice from V is presented at its parent X_1. Then, X_1 would play two roles, one for the red info path, and one for the dark blue control path. As a collider on the red info path, its role is to present the Z_0^{th} bit from V. As the initial endpoint of the blue control path, so its role is to copy the assignment of Z_0. The problem, however, is that X_0 then does not need to observe Z_0 in order to reproduce its value, because this value is already observed at X_1, so Z_0 is not material.

To remedy this problem, we can construct an alternative SCM, where the value of Z_0 is "concealed", i.e. it is removed from the other contexts, $C_{Z_0} \setminus Z_0$. At X_1, we directly remove Z_0, leaving this decision with a domain of only one bit. At C, we impose some random noise, so that it is not always a perfect copy of Z_0. The result is shown in Figure 10b. When this model is not intervened, an expected utility of $\mathbb{E}[Y] = 10.99$ is achieved, because the red term in Y always equals 10, while the blue term has an expectation of 0.99. (This is the MEU, because there is no way to improve the blue term to have expectation 1 without decreasing the expectation of the red term by at least 0.05.) If instead, Z_0 is removed as a context for X_0, then the expected utility can only be as high as $\mathbb{E}[Y] = 10.95$. To understand this, restrict our attention to deterministic policies, and note that in order for the red term to be better than a coin flip (with an expected value of 5), we would either need to have $X_0 = \langle C, X_1 \rangle$ — and the red term will have an expectation of 9.95, or we must have $X_1 = V[0]$ and $X_0 = \langle 0, X_1 \rangle$ — and then the blue term will have an expectation of 0.5. In either case, performance is worse than 10.99, so Z_0 is material for X_0.



(a) A model with zero VoI.

(b) Z_0 is material for X_0

Figure 10: Two alternative models that use the same two info paths, red and blue.

The problem is that concealing the value of Z_0 does not work for all graphs. To see this, let us add two decisions, X_2 and X_3, to the graph from Figure 10, to thereby obtain the graph in Figure 11. Let us retain the materiality SCM from Figure 10b, except that X_2 and X_3 copy the value from C along to Y. One might expect that Z_0 should still be material, but it is not. Now, there is a policy that achieves the new MEU of 11 by superimposing the value of Z_0 on the assignments of decisions X_2 and X_3. In this policy $\boldsymbol{\pi}$, $x_1 = v[z_0]$, $x_2 = z_0 \oplus z_0$, $x_3 = x_2 \oplus z_0$, and $x_0 = x_2 \oplus x_3 = z_0$ where $\oplus$ represents the XOR function. Under $\boldsymbol{\pi}$, the red term equals 10 always, while the blue term always equals 1, i.e. the MEU is achieved, and $\boldsymbol{\pi}$ is a valid policy even if Z_0 is not a context of X_0, meaning that Z_0 is not material for X_0.

In summary, whenever $\boldsymbol{Z} \ni Z_0, \boldsymbol{X'} \ni X_0, \boldsymbol{C'}, \boldsymbol{U}$ are not LB-factorizable, then we can find some info and control paths for Z_0 and X_0, but then X_0 can recover the value of Z_0, making it possible to achieve the MEU even when Z_0 is removed as a context of X_0. In some graphs, we can devise an alternative SCM that conceals the value of Z_0. But in others, a policy can superimpose the information from Z_0 on other decisions, such as X_2 and X_3 in Figure 11, so that X_0 can recover the value of Z_0, making Z_0 immaterial for X_0 once again.

It seems that new insights are needed to solve this superimposition problem, and that therefore that we will need new insights to establish a complete criterion for materiality in insoluble decision problems.

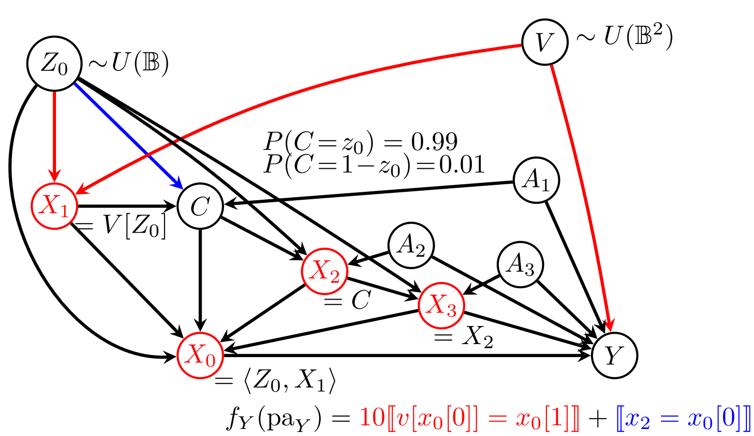


Figure 11: A model with zero VoI

6 Conclusion

We have found that in a graph whose contexts cannot satisfy condition (I) of LB-factorizability, any context can be material. We encountered some new problems for materiality proofs, and devised appropriate solutions:

- if the variable Z_i whose materiality we are trying to establish is a decision, whose value can be determined by other available contexts, — then we must choose a different info path so that non-observed variables would be needed to determine the value of Z_i
- if the info path begins with a context of multiple decisions, — then we must construct the SCM differently along the info path
- if the control path contains consecutive decisions, then we require more bits to be copied along the control path, so that not all of these bits can be copied along alternative paths.

As a next step towards establishing a complete criterion for materiality, we then considered the more general setting where no context can jointly satisfy conditions (I-III) of LB-factorizability. In this setting, it is possible to identify info paths and control paths for a target context Z_0 and decision X_0, and to apply our SCM construction to these paths. However, there may exist policies that transmit the assignment of Z_0 through alternative paths, and that achieve the MEU even when Z_0 is removed as a context of X_0. Although there exist ways of concealing the information about Z_0 from a descendant decision $X_{i'}, i < i'$, there can also be other ways that information about Z_0 may be transmitted, such as transmitting this information in other decisions, undermining materiality once again. Thus, the challenge of proving a complete criterion of materiality for insoluble graphs currently remains open.

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