Understanding Accelerated Gradient Methods: Lyapunov Analyses and Hamiltonian-Assisted Interpretations

Anonymous authors
Paper under double-blind review

Abstract

We formulate two classes of first-order algorithms more general than previously studied for minimizing smooth and strongly convex or, respectively, smooth and convex functions. We establish sufficient conditions, via new discrete Lyapunov analyses, for achieving accelerated convergence rates which match Nesterov's methods in the strongly and general convex settings. Our results identify, for the first time, a simple and unified condition on gradient correction for accelerated convergence. Next, we study the convergence of limiting ordinary differential equations (ODEs), including high-resolution ODEs, and point out currently notable gaps between the convergence properties of the corresponding algorithms and ODEs, especially regarding the role of gradient correction. Finally, we propose a novel class of discrete algorithms, called the Hamiltonian-assisted gradient method, directly based on a Hamiltonian function and several interpretable operations, and then demonstrate meaningful and unified interpretations of our acceleration conditions in terms of the momentum variable updates.

1 Introduction

Optimization plays a vital role in machine learning, statistics, and many other fields. In the optimization literature, there exists a striking phenomenon where after suitable modifications of a first-order method[1], the convergence guarantee can be improved, often attaining the complexity lower bound, with a similar computational cost as before. Such acceleration has been widely studied since the seminal work of Nesterov (1983), which improves gradient descent for minimizing smooth convex functions. Examples include constrained optimization (Nesterov, 2018), mirror descent with a non-Euclidean norm (Krichene et al., 2015), composite optimization with a proximable function (Beck & Teboulle, 2009), primal-dual splitting (Chambolle & Pock, 2011), stochastic gradient methods (Zhang & Lin, 2017; Allen-Zhu, 2018), and others.

Despite extensive research, the scope and mechanism of acceleration remain to be fully understood, even including the original acceleration of gradient descent for smooth convex optimization. Typically, a class of algorithms is constructed, and their convergence proofs are provided using suitable techniques. Among them, Lyapunov analysis has been a prevalent approach for studying convergence properties of discrete algorithms, once those algorithms are defined. In fact, Nesterov (1983) used a potential function or a Lyapunov function to establish the convergence of the first accelerated algorithm, although Nesterov's later work turned to the technique of estimate sequences for studying gradient methods (Nesterov, 1988; 2018). See Bansal & Gupta (2019) and d'Aspremont et al. (2021) for overviews of Lyapunov-based proofs for gradient methods. The central step in Lyapunov analysis is to construct an appropriate Lyapunov function that is used to deduce the desired convergence. Recently, a promising strategy for constructing Lyapunov functions systematically is the linear matrix inequality (LMI) framework based on a control-theoretic perspective of optimization algorithms. The LMI framework was originally proposed in Lessard et al. (2016) and later extended by Fazlyab et al. (2018).

[1]First-order methods refer to methods using function values and gradients only, whereas second-order methods additionally rely on the Hessian matrices or their approximations.

Although being a major technical tool for establishing convergence rates, Lyapunov analysis itself may not explain conceptually when and how acceleration can be achieved. Various efforts have been made for the latter purpose. For example, geometric formulations are proposed by coupling gradient descent and mirror descent (Allen-Zhu & Orecchia, 2017), and by averaging two minimizers of an upper and a lower quadratic bound for the objective function (Bubeck et al., 2015; Drusvyatskiy et al., 2018). Another useful approach is to relate discrete algorithms to their continuous limits as ordinary differential equations (ODEs), obtained by letting the stepsize in the discrete algorithms tend to zero. ODEs are usually more tractable to study than their discrete counterparts by exploiting a rich set of analytical tools from continuous-time dynamical systems and control theory. The analyses and properties of the limiting ODEs can provide insights about those of the original discrete algorithms, for example, to facilitate the construction of Lyapunov functions for the discrete algorithms (Qian, 1999; Su et al., 2016; Yang et al., 2018; Sun et al., 2020; Shi et al., 2022). Conversely, ODEs can be directly formulated, and then their numerical discretizations are studied (Wibisono et al., 2016; Wilson et al., 2021). It has been argued that acceleration can be attributed to suitable discretizations of ODEs, such as the Runge–Kutta integrators and its variants (Zhang et al., 2018; Dobson et al., 2025), and the symplectic integrators (Shi et al., 2019; França et al., 2020; Muehlebach & Jordan, 2021).

However, the approach based solely on ODEs is insufficient to account for whether Nesterov's acceleration is achieved. For example, for minimizing strongly convex functions, both Nesterov's accelerated gradient method (NAG-SC) and Polyak's heavy-ball method (HB) admit the same limiting ODE, but only NAG-SC is known to achieve accelerated convergence. To address this limitation, Shi et al. (2022) proposed a new approach based on high-resolution ODEs. These ODEs are derived by retaining certain terms from the discrete algorithms that would otherwise vanish in the continuous-time limit as the stepsize tends to zero. In particular, the high-resolution ODE for NAG-SC includes an additional *gradient correction* term that is absent in the ODE for HB. By translating the Lyapunov analysis of high-resolution ODEs back to their discrete counterparts, Shi et al. (2022) established accelerated convergence for NAG-SC and non-accelerated convergence for HB. Their analysis suggests that gradient correction plays a crucial role for achieving acceleration. This raises two interesting questions. First, can we quantify the magnitude of gradient correction for achieving acceleration by discrete algorithms? Second, can high-resolution ODEs themselves provide insight into such a quantification?

We take a direct approach to studying the scope and mechanism of gradient correction for acceleration of discrete algorithms. There are three main contributions from our work, summarized as follows.

- We formulate two general classes of algorithms and establish sufficient conditions for when the algorithms achieve Nesterov's acceleration as well as non-accelerated convergence for the strongly and general convex settings, respectively (see Sections 2 and 3). Our results, *for the first time*, identify a simple and unified condition on the magnitude of the gradient correction term to achieve acceleration, thereby answering the first question raised above. Our proofs are based on hand-crafted Lyapunov analysis relying on neither connections with ODEs nor the LMI framework. See Section 6 for a comparison of our and existing Lyapunov analyses.
- We study high-resolution ODEs derived from our classes of algorithms and compare convergence rates of these ODEs with those of their discrete counterparts. We find that high-resolution ODEs from our classes converge at the same rate, even though the corresponding discrete algorithms exhibit different acceleration behaviors. This provides new evidence that even high-resolution ODEs may be insufficient for explaining acceleration, thereby suggesting a negative answer to the second question above. See Section 4 for a more detailed summary.
- To provide further understanding of the sufficient conditions on gradient correction, we propose a novel class of discrete algorithms called the Hamiltonian-assisted gradient method (HAG), which leads to a meaningful and unified interpretation for these conditions in terms of momentum variable updates. See Section 5.2 for a more detailed summary.

We highlight our main technical findings by the following two results, stated in a self-contained manner. Further interpretations of the technical results are provided in Section 5.2. In the strongly convex setting,

Proposition 1 can be deduced from Corollary 1 to give sufficient conditions for a simplified class of algorithms to achieve an accelerated convergence bound matching NAG-SC.

Proposition 1. *Suppose that f is L-smooth and μ-strongly convex. Let $\{x_k\}$ be the iterates from the following algorithm:*

$$x_{k+1} = x_k - s\nabla f(x_k) + (1 - c_1\sqrt{\mu s})(x_k - x_{k-1}) - \tilde{c}_2 \cdot s(\nabla f(x_k) - \nabla f(x_{k-1})),$$

with $x_1 = x_0 - s\nabla f(x_0)$. Then there exist constants C_0, $C_1 > 0$, depending only on c_1 and $\tilde{c}_2$, such that for $0 < s \leq C_0/L$, $\{x_k\}$ satisfy the bound

$$f(x_k) - f^* = O\left(L\|x_0 - x^*\|^2(1 - C_1\sqrt{\mu s})^k\right),$$

provided that $c_1 > 2$ and $\tilde{c}_2 \geq \frac{1}{2}$.

The conditions in Proposition 1 are in terms of c_1 and $\tilde{c}_2$, which controls the momentum term $x_k - x_{k-1}$ and the gradient correction term $s(\nabla f(x_k) - \nabla f(x_{k-1}))$, respectively. To our knowledge, this is the first time that simple and meaningful conditions have been established for the acceleration of a general class of algorithms in the strongly convex setting. In addition, Corollary 1 indicates that when $0 \leq \tilde{c}_2 < \frac{1}{2}$, an algorithm in the above form is proved to only have a convergence rate matching HB. Previously, such comparative results were only known for two specific algorithms, NAG-SC and HB (Shi et al., 2022).

In the general convex setting, Proposition 2 can be deduced from Theorem 4, to give sufficient conditions for a simplified class of algorithms to achieve an accelerated convergence bound matching NAG-C (the analogue of NAG-SC in the general convex setting).

Proposition 2. *Suppose that f is L-smooth and convex. Let $\{x_k\}$ be the iterates from the following algorithm*

$$x_{k+1} = x_k - s\nabla f(x_k) + \sigma_{k+1}(x_k - x_{k-1}) - \sigma_{k+1}\tilde{c}_2 \cdot s(\nabla f(x_k) - \nabla f(x_{k-1})),$$

where $x_1 = x_0 - s\nabla f(x_0)$, $\sigma_{k+1} = (\alpha_k - 1)/\alpha_{k+1}$, and $\{\alpha_k > 0\}$ with $\alpha_0 = 1$ is a scalar sequence such that $\alpha_k = \Omega(k)$, $\alpha_{k+1}(\alpha_{k+1} - 1) \leq \alpha_k^2$, $\{\alpha_{k+1}/\alpha_k\}$ is monotone (either non-increasing or non-decreasing) in k and $\lim_k \alpha_{k+1}/\alpha_k = 1$. Then there exist constants $C_0 > 0$, depending only on $\{\alpha_k\}$ and $\tilde{c}_2$, such that for $0 < s \leq C_0/L$, $\{x_k\}$ satisfy the bound

$$f(x_k) - f^* = O\left(\frac{\|x_0 - x^*\|^2}{sk^2}\right),$$

provided that $\tilde{c}_2 > \frac{1}{2}$.

Proposition 2 is more general than the related result in Shi et al. (2022) corresponding to specific choices of $\alpha_k = \frac{k+r}{r}$ and $\sigma_{k+1} = \frac{k}{k+r+1}$ with $r \geq 2$. By comparing Propositions 1 and 2, we observe a common condition requiring $\tilde{c}_2$, the coefficient for gradient correction $s(\nabla f(x_k) - \nabla f(x_{k-1}))$, to (asymptotically) surpass an explicit threshold of $\frac{1}{2}$. Such a unified condition on gradient correction is identified for the first time for Nesterov's acceleration.

In Appendix E, we present numerical results to illustrate different performances of the algorithms whose parameters either satisfy or violate the sufficient conditions for acceleration. It is found that algorithms with parameters satisfying (or lying on the boundary of) these conditions exhibit better performance. Overall, the numerical results align well with our theoretical findings. All proofs are deferred to Appendix A–D.

Notation. We largely adopt the notation in Nesterov (2018). For a smooth and convex function f, consider the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} f(x). \tag{1}$$

Denote by x^* one of the minimizers of f and $f^* = f(x^*)$. When f is strongly convex, x^* is unique. For $m \geq 1$, let $\mathcal{F}^m$ be the set of functions that are convex and m times continuously differentiable on $\mathbb{R}^n$. Moreover, we define $\mathcal{F}_L^m \subset \mathcal{F}^m$ which further requires that f is L-smooth, that is, the gradient ∇f is L-Lipschitz continuous, $\|\nabla f(y) - \nabla f(x)\| \leq L\|y - x\|$ for $x, y \in \mathbb{R}^n$, where $L > 0$ is the Lipschitz constant and

$\|\cdot\|$ denotes the Euclidean norm. Similarly, for $\mu > 0$ we define $\mathcal{S}_\mu^m \subset \mathcal{F}^m$ which further requires that f is μ-strongly convex, that is, $f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle + \frac{\mu}{2}\|y-x\|^2$ for $x, y \in \mathbb{R}^n$. Let $\mathcal{S}_{\mu,L}^m = \mathcal{F}_L^m \cap \mathcal{S}_\mu^m$. For additional properties of smoothness and strong convexity, readers are referred to, for example, Appendix A in d'Aspremont et al. (2021).

Given two sequences $\{a_k\}$ and $\{b_k\}$, we write $a_k = O(b_k)$ if there exist constants $C > 0$ and $K \geq 1$, such that $a_k \leq Cb_k$ for $k \geq K$. Similarly, we write $a_k = \Omega(b_k)$ if $a_k \geq Cb_k$ for $k \geq K$. We write $a_k = \Theta(b_k)$ if both $a_k = \Omega(b_k)$ and $a_k = O(b_k)$, and write $a_k \sim b_k$ if $\lim_k a_k/b_k = 1$. More generally, for a set of pairs (g, h), we write $g = O(h)$ or $g \lesssim h$ if there exists a constant $C > 0$ such that $g \leq Ch$ for (g, h). Similarly, we write $g = \Omega(h)$ or $g \gtrsim h$ if $g \geq Ch$ for (g, h). We write $g \asymp h$ if both $g \lesssim h$ and $g \gtrsim h$.

2 Acceleration for strongly convex functions

In this section, we first give background on several algorithms for minimizing smooth and strongly convex functions. Then we study a general class of algorithms.

2.1 Review of Nesterov's acceleration

A basic method for solving (1) is the gradient descent (GD):

$$x_{k+1} = x_k - s\nabla f(x_k), \tag{2}$$

with an initial point x_0 and a stepsize s. In this section, we study the case where $f \in \mathcal{S}_{\mu,L}^1$ is smooth and strongly convex. See Section 3 for the case where f is smooth and convex.

For $f \in \mathcal{S}_{\mu,L}^1$, GD can be exponentially convergent in k, and the convergence rate depends on the condition number $\kappa = L/\mu$. In fact, the GD iterates (2) satisfy that for $0 < s \leq 2/(\mu + L)$ (Nesterov, 2018),

$$f(x_k) - f^* \leq \frac{L\|x_0 - x^*\|^2}{2}\left(1 - \frac{2\mu s}{1 + 1/\kappa}\right)^k.$$

When $s = 1/L$, the rate is $(\frac{1-1/\kappa}{1+1/\kappa})^k \sim (1 - \frac{1}{\kappa})^{2k}$ for $\kappa \to \infty$. When $s = 2/(L + \mu)$, the rate improves to $(\frac{1-1/\kappa}{1+1/\kappa})^{2k} \sim (1 - \frac{1}{\kappa})^{4k}$. For both choices of the stepsize, the iteration complexity for $f(x_k) - f^* \leq \epsilon$ is $O(\kappa \log(\frac{1}{\epsilon}))$, which has a linear dependency on κ.

Nesterov (1988) proposed an accelerated gradient method (NAG-SC) of the following form with an additional extrapolation step:

$$y_{k+1} = x_k - s\nabla f(x_k), \tag{3a}$$
$$x_{k+1} = y_{k+1} + \sigma(y_{k+1} - y_k), \tag{3b}$$

with $x_0 = y_0$ and the momentum coefficient $\sigma = \frac{1-\sqrt{q}}{1+\sqrt{q}}$, where $q = \mu s$, a shorthand to be used throughout this paper. Equivalently, (3) can be expressed in a single-variable form:

$$x_{k+1} = \underbrace{x_k - s\nabla f(x_k)}_{\text{gradient descent}} + \underbrace{\sigma(x_k - x_{k-1})}_{\text{momentum}} - \underbrace{\sigma s(\nabla f(x_k) - \nabla f(x_{k-1}))}_{\text{gradient correction}}, \tag{4}$$

with x_0 and $x_1 = x_0 - \frac{2s\nabla f(x_0)}{1+\sqrt{q}}$. Compared with GD, the iterate (4) involves two additional terms, called the momentum and gradient correction. For $0 < s \leq 1/L$, the NAG-SC iterates satisfy $f(x_k) - f^* = O((1-\sqrt{q})^k)$. When $s = 1/L$, the bound reduces to $O((1-1/\sqrt{\kappa})^k)$, and the iteration complexity is lowered to $O(\sqrt{\kappa}\log(\frac{1}{\epsilon}))$, with a square-root dependency on κ. Drori & Taylor (2022) established that a lower bound for minimizing general $f \in \mathcal{S}_{\mu,L}^1$ is $f(x_k) - f^* = \Omega((1 - 1/\sqrt{\kappa})^{2k})$, where $\{x_k\}$ are iterates of any black-box first-order method[2]. Therefore, NAG-SC is optimal up to a constant factor of 2 in terms of the iteration complexity.

[2]Black-box means that no prior knowledge of f (e.g., f is quadratic) is available except for the class f belongs to. For $\mathcal{F}_L^1$, the available information is only the Lipschitz constant L.

There are several first-order methods *exactly* reaching the lower bound $(1-1/\sqrt{\kappa})^{2k}$, for instance, the information-theoretic exact method (ITEM) (Taylor & Drori, 2023) and triple-momentum method (TMM) (Van Scoy et al., 2017). While ITEM involves time-dependent coefficients, TMM is defined with time-independent coefficients (d'Aspremont et al., 2021):

$$y_{k+1} = x_k - s\nabla f(x_k), \tag{5a}$$

$$z_{k+1} = \sqrt{q}\left(x_k - \frac{1}{\mu}\nabla f(x_k)\right) + (1-\sqrt{q})z_k, \tag{5b}$$

$$x_{k+1} = \frac{2\sqrt{q}}{1+\sqrt{q}}z_{k+1} + \left(1 - \frac{2\sqrt{q}}{1+\sqrt{q}}\right)y_{k+1}, \tag{5c}$$

with $x_0 = z_0$. The sequence $\{z_k\}$ in TMM (5) is auxiliary. See Appendix A.1 for an equivalent form of (5) containing $\{y_{k+1}\}$ and $\{x_{k+1}\}$ only, where z_{k+1} can be recovered by

$$z_{k+1} = \frac{1+\sqrt{q}}{2\sqrt{q}}x_{k+1} + \left(1 - \frac{1+\sqrt{q}}{2\sqrt{q}}\right)y_{k+1}, \tag{6}$$

Nevertheless, $\{z_k\}$ plays a vital role in the existing analysis of TMM. Van Scoy et al. (2017) showed that $\{z_k\}$ achieves the lower bound, i.e., $f(z_k) - f^* = O((1-1/\sqrt{\kappa})^{2k})$ when $s = 1/L$. It seems to be an open question whether $\{x_k\}$ also achieves the lower bound.

In the Lyapunov analysis of NAG-SC in Bansal & Gupta (2019), a similar auxiliary sequence $\{z_k\}$ as above is introduced by defining

$$z_{k+1} = \frac{1+\sqrt{q}}{\sqrt{q}}x_{k+1} + \left(1 - \frac{1+\sqrt{q}}{\sqrt{q}}\right)y_{k+1}, \tag{7}$$

and the NAG-SC iterates in (3) together with (7) are equivalently reformulated as

$$y_{k+1} = x_k - s\nabla f(x_k), \tag{8a}$$

$$z_{k+1} = \sqrt{q}\left(x_k - \frac{1}{\mu}\nabla f(x_k)\right) + (1-\sqrt{q})z_k, \tag{8b}$$

$$x_{k+1} = \frac{\sqrt{q}}{1+\sqrt{q}}z_{k+1} + \left(1 - \frac{\sqrt{q}}{1+\sqrt{q}}\right)y_{k+1}, \tag{8c}$$

with $x_0 = z_0$. See Appendix A.1 for a derivation, provided for completeness. By comparing TMM (5) with NAG-SC (8), they differ only in how x_{k+1} is defined in (5c) and (8c). Bansal & Gupta (2019) constructed a new Lyapunov function to simplify the convergence proof for NAG-SC, and showed that $\{y_k\}$ achieves the bound $f(y_k) - f^* = O((1-1/\sqrt{\kappa})^k)$ when $s = 1/L$. By some additional arguments, similar convergence bounds can also be deduced for $\{z_k\}$ and $\{x_k\}$.

For comparison, we also mention the heavy-ball (HB) method (Polyak, 1964), defined as[3]

$$x_{k+1} = x_k - s\nabla f(x_k) + \sigma(x_k - x_{k-1}), \tag{9}$$

with $\sigma = \frac{1-\sqrt{q}}{1+\sqrt{q}}$ as in (3). Note that (9) differs from NAG-SC (4) only in the absence of gradient correction. HB may not achieve accelerated convergence for general $f \in \mathcal{S}^1_{\mu,L}$. In fact, the iterates of (9) satisfy $f(x_k) - f^* = O((1-C_1\sqrt{q})^k)$, but only provably for the stepsize $0 < s \le C_0\mu/L^2$, with some constants $C_0, C_1 > 0$ (Shi et al., 2022). Because $\mu/L^2 \ll 1/L$ by a factor of $1/\kappa$, the convergence bound for HB is still of order $O\left((1-C/\kappa)^k\right)$ for some constant $C > 0$, thus, fails to achieve acceleration.

[3]Following Shi et al. (2022), algorithm (9) is a slight modification of the original method in Polyak (1964) where $s = 4/(\sqrt{L}+\sqrt{\mu})^2$ and the momentum coefficient $\sigma = (1-\sqrt{q})^2$. If s is small, the two coefficients $(1-\sqrt{q})^2$ and $\frac{1-\sqrt{q}}{1+\sqrt{q}}$ are close. The original heavy-ball method, with the specific s, achieves the accelerated convergence rate $(\frac{1-1/\sqrt{\kappa}}{1+1/\sqrt{\kappa}})^{2k}$ for $f \in \mathcal{S}^2_{\mu,L}$ (Polyak, 1987), but it may fail to converge for some $f \in \mathcal{S}^1_{\mu,L}$ (Lessard et al., 2016).

2.2 Main results

We formulate a broad class of algorithms including NAG-SC and TMM as special cases and establish sufficient conditions for when the algorithms in the class achieve Nesterov's acceleration, which is defined as reaching an objective gap of $O((1-C/\sqrt{\kappa})^k)$ at iteration k, for a constant $C>0$. Our work does not aim to find a sharp value of C or address the question of whether these algorithms *exactly* achieve the complexity lower bound corresponding to $C=2$ (Drori & Taylor, 2022).

To unify and extend NAG-SC and TMM, we consider the following class of algorithms:

$$y_{k+1} = x_k - \eta s \nabla f(x_k), \tag{10a}$$

$$z_{k+1} = \nu\sqrt{q}\left(x_k - \frac{1}{\mu}\nabla f(x_k)\right) + (1-\nu\sqrt{q})z_k, \tag{10b}$$

$$x_{k+1} = \frac{\tau\sqrt{q}}{1+\sqrt{q}} z_{k+1} + \left(1 - \frac{\tau\sqrt{q}}{1+\sqrt{q}}\right) y_{k+1}, \tag{10c}$$

with $x_0 = z_0$, where $\eta, \nu, \tau \geq 0$ are three parameters which may depend on q. NAG-SC (3) or (8) is recovered by setting $(\eta,\nu,\tau) = (1,1,1)$, and TMM (5) is recovered by $(\eta,\nu,\tau) = (1,1,2)$. In this way, the two algorithms differ only in the choice of τ.

Equivalently, algorithm (10) can be put into a single-variable form in terms of $\{x_k\}$, similarly to (4) for NAG-SC:

$$x_{k+1} = x_k - \frac{\nu(\tau+\zeta\eta\sqrt{q})}{1+\sqrt{q}} s\nabla f(x_k) + \frac{\zeta(1-\nu\sqrt{q})}{1+\sqrt{q}}(x_k - x_{k-1}) - \frac{\zeta\eta(1-\nu\sqrt{q})}{1+\sqrt{q}} s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{11}$$

with x_0 and $x_1 = x_0 - \frac{\zeta\eta+\nu\tau}{1+\sqrt{q}} s\nabla f(x_0)$, where $\zeta = 1+(1-\tau)\sqrt{q}$, a shorthand used throughout this paper. The coefficients in the three terms for $s\nabla f(x_k)$, $x_k - x_{k-1}$, and $s(\nabla f(x_k) - \nabla f(x_{k-1}))$ are highly structured due to the translation from (10).

The following result gives sufficient conditions for the convergence of algorithm (10) in the scenario where (η,ν,τ) are constants free of $q\,(=\mu s)$.

Theorem 1. *Let $f \in \mathcal{S}^1_{\mu,L}$. Assume that $\eta = \eta_0$, $\nu = \nu_0$, and $\tau = \tau_0$ for some constants (η_0,ν_0,τ_0), free of $q\,(=\mu s)$.*

(i) There exist constants $C_0, C_1 > 0$, depending only on (η_0,ν_0,τ_0), such that for $0 < s \leq C_0\frac{\mu}{L^2}$, the iterates of (10) satisfy

$$f(x_k) - f^* = O\left(L\|x_0 - x^*\|^2(1-C_1\sqrt{\mu s})^k\right), \tag{12}$$

provided that one of the following conditions holds:

(i-a) $\nu_0, \tau_0 > 0$, $\nu_0 \neq \tau_0$, and $0 \leq \eta_0 < \nu_0\tau_0/2$;

(i-b) $\nu_0 = \tau_0 > 2$ and $\eta_0 = \tau_0^2/2$.

(ii) There exist constants $C_0, C_1 > 0$, depending only on (η_0,ν_0,τ_0), such that for $0 < s \leq \frac{C_0}{L}$, the iterates of (10) satisfy (12), provided that one of the following conditions holds:

(ii-a) $\nu_0, \tau_0 > 0$, $\nu_0 \neq \tau_0$, and $\eta_0 \geq \nu_0\tau_0/2$;

(ii-b) $\nu_0 = \tau_0 \geq 2$ and $\eta_0 > \tau_0^2/2$;

(ii-c) $1 < \nu_0 = \tau_0 < 2$ and $\eta_0 > \tau_0$;

(ii-d) $0 < \nu_0 = \tau_0 \leq 1$ and $\eta_0 \geq \tau_0$.

The convergence bound (12) exhibits two types of dependency on κ, determined by how large the stepsize s is allowed. A similar phenomenon occurs in the comparison of NAG-SC and HB. Under the conditions in Theorem 1(i), algorithm (10) achieves a usual convergence bound $O((1-C/\kappa)^k)$ for $s \asymp \frac{\mu}{L^2}$, resulting in an iteration complexity $O(\kappa \log(\frac{1}{\epsilon}))$. Under the conditions in Theorem 1(ii), algorithm (10) reaches an accelerated convergence bound $O((1-C/\sqrt{\kappa})^k)$ for $s \asymp \frac{1}{L}$, resulting in an iteration complexity $O(\sqrt{\kappa}\log(\frac{1}{\epsilon}))$. In particular, TMM with $(\eta_0, \nu_0, \tau_0) = (1, 1, 2)$ is covered by condition (ii-a), whereas NAG-SC with $(\eta_0, \nu_0, \tau_0) = (1, 1, 1)$ is covered by condition (ii-d). It is interesting that for both TMM and NAG-SC, the choice of (η_0, ν_0, τ_0) lies on boundaries of the acceleration regions identified in Theorem 1(ii).[4] In addition, although no algorithm in class (10) leads to exactly HB, there are close variations of HB which are in the class (10) with constant (η, ν, τ) and covered by the non-acceleration regions in Theorem 1(i). [5]

We next extend the constant case to a more general scenario where (η, ν, τ) are analytical functions of $\sqrt{q}$, free of negative exponents like $q^{-1/2}$, as stated in Assumption 1. In particular, (η, ν, τ) that are polynomials of $\sqrt{q}$ satisfy Assumption 1.

Assumption 1. There exist non-negative, analytic functions $\tilde{\eta}(\cdot)$, $\tilde{\nu}(\cdot)$ and $\tilde{\tau}(\cdot)$ in a neighborhood of 0 such that $\eta = \tilde{\eta}(\sqrt{q})$, $\nu = \tilde{\nu}(\sqrt{q})$ and $\tau = \tilde{\tau}(\sqrt{q})$. Then η, ν and τ admit (convergent) Taylor expansions:

$$\eta = \sum_{i=0}^{\infty} \eta_i (\sqrt{q})^i, \quad \nu = \sum_{i=0}^{\infty} \nu_i (\sqrt{q})^i, \quad \tau = \sum_{i=0}^{\infty} \tau_i (\sqrt{q})^i, \tag{13}$$

when $0 \le q \le q_0$ for some constant $q_0 > 0$.

The following result gives sufficient conditions on the convergence of the algorithm (10), in terms of only the constant coefficients, (η_0, ν_0, τ_0), in the expansions (13).

Theorem 2. *Suppose that $f \in \mathcal{S}^1_{\mu,L}$ and Assumption 1 holds.*

(i) For $0 < s \le C_0 \frac{\mu}{L^2}$ (non-accelerated convergence), the iterates of (10) satisfy (12), with $C_0, C_1 > 0$ depending only on the functions $(\tilde{\eta}, \tilde{\nu}, \tilde{\tau})$ (independent of q), provided that condition (i-a) in Theorem 1 holds.

(ii) For $0 < s \le C_0 \frac{1}{L}$ (accelerated convergence), the iterates of (10) satisfy (12), with $C_0, C_1 > 0$ depending only on the functions $(\tilde{\eta}, \tilde{\nu}, \tilde{\tau})$ (independent of q), provided that condition (ii-a) in Theorem 1 holds.

Theorem 2 serves as a generalization to Theorem 1, cases (i-a) and (ii-a). When (η, ν, τ) depend on q with leading constants $\nu_0 = \tau_0 > 0$ and $\eta_0 \ge \tau_0^2/2$, which is not addressed by Theorem 2, the following result gives sufficient conditions on convergence of algorithm (10), involving the coefficients (η_1, ν_1, τ_1) of linear terms in the expansions (13). However, in the degenerate case of constant parameters (η, ν, τ), the sufficient conditions in Theorem 3 are more restrictive than those in Theorem 1.

Theorem 3. *Suppose that $f \in \mathcal{S}^1_{\mu,L}$ and Assumption 1 holds with $\nu_0 = \tau_0 > 0$ and $\eta_0 \ge \tau_0^2/2$.*

(i) For $0 < s \le C_0 \frac{\mu}{L^2}$ (non-accelerated convergence), the iterates of (10) satisfy (12), with constants $C_0, C_1 > 0$ depending only on the functions $(\tilde{\eta}, \tilde{\nu}, \tilde{\tau})$ (independent of q), provided that the following holds:

$$\text{(i-a)} \;\; \eta_0 = \tau_0^2/2,\ \nu_1 - \tau_1 < \tau_0\left(\tfrac{\tau_0}{2} - 1\right),\ \text{and}\ 2\eta_1 < \nu_1\tau_1 + \tfrac{\tau_0^2}{2}\left(\tfrac{5}{2}\tau_0 - 2\right).$$

(ii) For $0 < s \le C_0 \frac{1}{L}$ (accelerated convergence), the iterates of (10) satisfy (12), with constants $C_0, C_1 > 0$ depending only on the functions $(\tilde{\eta}, \tilde{\nu}, \tilde{\tau})$ (independent of q), provided that one of the following conditions holds:

$$\text{(ii-a)} \;\; \eta_0 = \tau_0^2/2,\ \nu_1 - \tau_1 < \tau_0\left(\tfrac{\tau_0}{2} - 1\right),\ \text{and}\ 2\eta_1 \ge \nu_1\tau_1 + \tfrac{\tau_0^2}{2}\left(\tfrac{5}{2}\tau_0 - 2\right);$$

[4] The TMM choice, $\eta_0 = \nu_0 = 1$ and $\tau_0 = 2$, satisfies $\nu_0 \ne \tau_0$ and $\eta_0 = \nu_0\tau_0/2$, lying on the boundary of condition (ii-a) in Theorem 1. The NAG-SC choice, $\eta_0 = \nu_0 = \tau_0 = 1$, satisfies $\nu_0 = \tau_0 = 1$ and $\eta_0 = \tau_0$, lying on the boundary of condition (ii-d) in Theorem 1.

[5] For any constant $0 < \nu_0 \ne 1$, taking $(\eta, \tau, \nu) = (0, 1, \nu_0)$ in (11) yields the update $x_{k+1} = x_k - \frac{\nu_0}{1+\sqrt{q}} s\nabla f(x_k) + \frac{1-\nu_0\sqrt{q}}{1+\sqrt{q}}(x_k - x_{k-1})$, which closely resembles HB (9) as $\nu_0 \to 1$ and achieves non-accelerated convergence by Theorem 1(i-a).

(ii-b) $\eta_0 > \tau_0^2/2$ *and* $\nu_1 - \tau_1 < (\tau_0 - 1)\tau_0 - \eta_0$*;*

(ii-c) $\eta_0 > \tau_0^2/2$ *and* $(\tau_0 - 1)\tau_0 - \eta_0 < \nu_1 - \tau_1 < \eta_0 - \tau_0$*.*

All the preceding results are applicable to algorithm (10) or equivalently its single-variable form (11) with structured coefficients. To facilitate comparison and interpretation, we translate Theorem 2 in terms of a single-variable form (14) below with unstructured coefficients. It remains an open question to directly analyze (14), for example, via the LMI framework.

Corollary 1. *For* $f \in \mathcal{S}^1_{\mu,L}$*, consider the following algorithm*[6]*:*

$$x_{k+1} = x_k - (c_0 + R_1)s\nabla f(x_k) + (1 - c_1\sqrt{q} + R_2)(x_k - x_{k-1}) - (c_2\sqrt{c_0} - \frac{c_0}{2} + R_3)s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{14}$$

where $x_1 = x_0 - h_1 s\nabla f(x_0)$*;* $c_0, c_1, c_2 > 0$ *are constants independent of* q*;* $R_1 = O(\sqrt{q})$*,* $R_2 = O(q)$*,* $R_3 = O(\sqrt{q})$*, and* h_1 *are analytic functions of* $\sqrt{q}$ *around* 0*.*

(i) For $0 < s \le C_0 \frac{\mu}{L^2}$ *(non-accelerated convergence), the iterates of (14) satisfy (12), with* $C_0, C_1 > 0$ *depending only on* (c_0, c_1, c_2) *and the forms of* (R_1, R_2, R_3) *(independent of* q*), provided* $c_1^2 > 4c_0$ *and* $c_0/4 \le c_2^2 < c_0$*.*

(ii) For $0 < s \le C_0 \frac{1}{L}$ *(accelerated convergence), the iterates of (14) satisfy (12), with* $C_0, C_1 > 0$ *depending only on* (c_0, c_1, c_2) *and the forms of* (R_1, R_2, R_3) *(independent of* q*), provided* $c_1^2 > 4c_0$ *and* $c_2^2 \ge c_0$*.*

To the best of our knowledge, Corollary 1, *for the first time*, identifies simple conditions for accelerated and non-accelerated convergence in a general class of algorithms (14). Moreover, the conditions are defined in terms of c_1 and c_2, which control the momentum and gradient correction terms respectively, while the other parameter c_0 can be viewed as a non-essential rescaling parameter. We discuss these two conditions in detail.

First, the condition $c_1^2 > 4c_0$ for the momentum coefficient is due to implicit constraints in the coefficients of (14) when translated from Theorem 2 in terms of (11). We believe that $c_1^2 > 4c_0$ is technical and can be potentially relaxed to $c_1 > 0$. See Sanz Serna & Zygalakis (2021) Theorem 3.3 for related work.

Second, the condition $c_2^2 \ge c_0$ or $c_0/4 \le c_2^2 < c_0$ directly distinguishes between the accelerated or non-accelerated convergence. In other words, the leading constant $c_2\sqrt{c_0} - c_0/2$ in the coefficient of gradient correction needs to exceed a threshold $c_0/2$.

3 Acceleration for general convex functions

In this section, we first give background on several algorithms for minimizing smooth and convex functions. Then we study a general class of algorithms.

3.1 Review of Nesterov's acceleration

Consider the unconstrained minimization problem (1) with $f \in \mathcal{F}^1_L$. For stepsize $0 < s \le 1/L$, the GD iterates $\{f(x_k)\}$ is non-increasing and satisfy $f(x_k) - f^* \le \|x_0 - x^*\|^2/(2ks)$.[7] However, the $O(1/k)$ rate is not optimal. Nesterov (1988) proposed an accelerated gradient method in a similar form as NAG-SC (3):

$$\begin{aligned} y_{k+1} &= x_k - s\nabla f(x_k), \\ x_{k+1} &= y_{k+1} + \sigma_{k+1}(y_{k+1} - y_k), \end{aligned} \tag{15}$$

with $x_0 = y_0$ and $\sigma_{k+1} = k/(k+3)$. Compared with NAG-SC (3), the momentum coefficient σ_{k+1} varies with k, instead of taking a fixed value depending on the strong-convexity parameter μ. For $0 < s \le 1/L$,

[6]The seemingly unnatural parameterization of the gradient correction coefficient is motivated by the HAG algorithm in Section 5.

[7]For stepsize $1/L < s < 2/L$, the GD iterates $\{f(x_k)\}$ is still non-increasing but satisfies a slightly different bound $O\left(\frac{\|x_0 - x^*\|^2}{ks(2-Ls)}\right)$ for the objective gap (Nesterov, 2018).

the NAG-C iterates $\{f(x_k)\}$ may not be non-increasing but satisfy

$$f(x_k) - f^* = O\left(\frac{\|x_0 - x^*\|^2}{sk^2}\right). \tag{16}$$

Compared with GD, the iteration complexity for $f(x_k) - f^* \leq \epsilon$ is reduced from $O(L/\epsilon)$ to $O(\sqrt{L/\epsilon})$ when taking $s = 1/L$. Notably, the convergence bound in (16) matches with the lower bound of black-box first-order methods for minimizing functions in $\mathcal{F}_L^1$ when n (the dimension of x) is relatively large compared with k (Nesterov, 2018).

The single-variable form in terms of $\{x_k\}$ for NAG-C (15) is

$$x_{k+1} = \underbrace{x_k - s\nabla f(x_k)}_{\text{gradient descent}} + \underbrace{\sigma_{k+1}(x_k - x_{k-1})}_{\text{momentum}} - \underbrace{\sigma_{k+1} \cdot s(\nabla f(x_k) - \nabla f(x_{k-1}))}_{\text{gradient correction}}, \tag{17}$$

with x_0, $x_1 = x_0 - s\nabla f(x_0)$, and $\sigma_{k+1} = k/(k+3)$. The iterate (17) involves an additional momentum term and a gradient correction term similarly as in (4) for NAG-SC. Shi et al. (2022) studied NAG-C by relating (17) to a high-resolution ODE (see Section 4.2) and obtained a new result on the squared gradient norm: for stepsize $0 < s \leq 1/(3L)$,

$$\min_{0 \leq i \leq k} \|\nabla f(x_i)\|^2 = O\left(\frac{\|x_0 - x^*\|^2}{s^2(k+1)^3}\right). \tag{18}$$

The inverse cubic rate (18) cannot be obtained directly from (16). Moreover, Shi et al. (2022) extended the coefficients in the momentum and gradient correction terms in (17) to $\sigma_{k+1} = k/(k+r+1)$ and $\sigma_{k+1}\beta s$ respectively, and showed that the two bounds in (16) and (18) remain valid for any $r \geq 2$ and $\beta > 1/2$.

The form of momentum coefficient σ_{k+1} can be made more general. For instance, a popular scheme is to set $\sigma_{k+1} = (\alpha_k - 1)/\alpha_{k+1}$, where $\{\alpha_k\}$ is a scalar sequence to be chosen. It is known (Beck, 2017) that for any sequence $\{\alpha_k\}$ satisfying $\alpha_k = \Omega(k)$ and a recursive condition[8]

$$\alpha_{k+1}(\alpha_{k+1} - 1) \leq \alpha_k^2, \tag{19}$$

the corresponding algorithm (15) achieves the optimal bound (16). For the well-known accelerated proximal gradient method or FISTA (Beck & Teboulle, 2009), $\{\alpha_k\}$ is defined recursively as $\alpha_{k+1} = (1 + \sqrt{1 + 4\alpha_k^2})/2$ with $\alpha_0 = 1$, i.e., (19) holds as equality. It can also be verified that $\{\alpha_k = (k+r)/r\}$ satisfies (19) IFF $r \geq 2$, with corresponding $\sigma_{k+1} = k/(k+r+1)$.

3.2 Main results

We formulate a broad class of algorithms including NAG-C (15) and existing variations (Beck, 2017; Shi et al., 2022) and establish (simple and interpretable) sufficient conditions for when the algorithms in the class achieve both the optimal bound (16) for the objective gap and the inverse cubic rate (18) for the squared gradient norm, similar to NAG-C. In Appendix E, we present numerical results to illustrate different performances from specific algorithms where the algorithm parameters either satisfy or violate our sufficient conditions.

To unify and extend existing choices of the momentum and gradient correction terms related to NAG-C, we consider the following class of algorithms:

$$\begin{aligned} y_{k+1} &= x_k - \beta_k s\nabla f(x_k), \\ x_{k+1} &= x_k - \gamma_k s\nabla f(x_k) + \sigma_{k+1}(y_{k+1} - y_k), \end{aligned} \tag{20}$$

where $x_0 = y_0$, $\sigma_{k+1} = (\alpha_k - 1)/\alpha_{k+1}$, and $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$ are three scalar sequences. The equivalent single-variable form of (20) is

[8]The recursive condition (19) implies that $\alpha_{k+1} \leq (1 + \sqrt{1 + 4\alpha_k^2})/2 \leq \alpha_k + 1$, and hence $\alpha_k = \Theta(k)$.

$$x_{k+1} = x_k - (\gamma_k + \sigma_{k+1}(\beta_k - \beta_{k-1}))s\nabla f(x_k) + \sigma_{k+1}(x_k - x_{k-1}) - \sigma_{k+1}\beta_{k-1} \cdot s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{21}$$

starting from x_0 and $x_1 = x_0 - (\gamma_0 + (\alpha_0 - 1)\beta_0/\alpha_1)s\nabla f(x_0)$. The algorithms studied in Beck (2017) correspond to the case of $\beta_k = \gamma_k \equiv 1$ and α_k in a general functional form subject to the inequality constraint (19). The algorithms studied in Shi et al. (2022) correspond to $\gamma_k \equiv 1$, $\beta_k \equiv \beta$ (a constant), and $\alpha_k = (k + r)/r$ for $r \geq 2$. To highlight the main points, Proposition 2 was stated in Section 1 for a special case of (21) with constant $\gamma_k \equiv 1$ and $\beta_k \equiv \tilde{c}_2$.

Motivated by the Lyapunov analysis of NAG-C in Su et al. (2016), we introduce the following three-variable form of (20). Define $z_k = \alpha_k x_k + (1 - \alpha_k)y_k$ for $k \geq 0$, which is reminiscent of (5b) and (7) in the strongly convex setting. Then (20) can be equivalently reformulated as

$$y_{k+1} = x_k - \beta_k s\nabla f(x_k), \tag{22a}$$

$$z_{k+1} = z_k - \widetilde{\alpha}_k s\nabla f(x_k), \tag{22b}$$

$$x_{k+1} = \frac{1}{\alpha_{k+1}} z_{k+1} + \left(1 - \frac{1}{\alpha_{k+1}}\right) y_{k+1}, \tag{22c}$$

starting from $x_0 = z_0$, where $\widetilde{\alpha}_k = \beta_k\alpha_k + (\gamma_k - \beta_k)\alpha_{k+1}$. See Appendix B for a proof. By definition, given $\{\beta_k\}$ and $\{\alpha_k > 0\}$, there is a one-to-one correspondence between $\{\gamma_k\}$ and $\{\widetilde{\alpha}_k\}$: $\gamma_k = (\widetilde{\alpha}_k - \beta_k(\alpha_k - \alpha_{k+1}))/\alpha_{k+1}$. Hence, algorithm (20) or (22) can be considered to be directly parameterized by the three sequences $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\widetilde{\alpha}_k\}$. The three-variable form (22) is introduced mainly to facilitate our Lyapunov analysis, which is presented in Section 6. The z_k update (22b) appears to differ from (10b) in the strongly convex case, although (22c) resembles (10c) in the sense that the weight of z_k or y_k in the update of x_k goes to 0 or 1 respectively as $\alpha_k \to \infty$ in (22c) or $s \to 0$ in (10c).

Our main result gives sufficient conditions on when algorithm (20) or (22) achieves accelerated convergence in both the objective gap and gradient norm, similar to NAG-C. We discuss interpretations of the conditions from the perspective of HAG in Section 5.

Theorem 4. *Let $f \in \mathcal{F}_L^1$. There exist constants $C_0 > 0$, depending only on $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$, such that for $0 < s \leq C_0/L$, the iterates $\{x_k\}$ from (20) or (22) satisfy the bound (16) for the objective gap and the inverse cubic rate (18) for the squared gradient norm, provided that the following conditions jointly hold.*

(i) $\lim_k \beta_k = \beta$ *and* $\lim_k \gamma_k = \gamma$ *with* $\beta > \gamma/2 > 0$*;*

(ii) $\{\alpha_k > 0\}$ *satisfies that* $\alpha_k = \Omega(k)$, $\lim_k \alpha_{k+1}/\alpha_k = 1$, $\alpha_{k+1}(\alpha_{k+1} - 1) \leq \alpha_k^2$*;*

(iii) $\{\widetilde{\alpha}_k/\alpha_k\}$ *is monotone (either non-increasing or non-decreasing) in k.*

For Theorem 4, the existence of the limits, $\lim_k \beta_k = \beta$, $\lim_k \gamma_k = \gamma$, and $\lim_k \alpha_{k+1}/\alpha_k = 1$, are introduced mainly to simplify the sufficient conditions and may be relaxed even with the same Lyapunov function in our proofs. The monotonicity condition on $\{\widetilde{\alpha}_k/\alpha_k\}$ may also be relaxed by using other Lyapunov functions. In the setting of $\beta_k \equiv \beta$ and $\gamma_k \equiv \gamma$, we have $\widetilde{\alpha}_k/\alpha_k = \beta + (\gamma - \beta)\alpha_{k+1}/\alpha_k$. Then the monotonicity condition is equivalent to either requiring $\beta = \gamma$ without any additional constraint on $\{\alpha_k\}$ or, if $\beta \neq \gamma$, requiring that $\{\alpha_{k+1}/\alpha_k\}$ is monotone. The latter condition is satisfied by both the simple choice $\alpha_k = (k + r)/r$ and the iterative choice $\alpha_{k+1} = (1 + \sqrt{1 + 4\alpha_k^2})/2$ with $\alpha_0 = 1$ in FISTA.

We also point out that the conditions in Theorem 4 are general enough to allow the absence of a limiting ODE for (20). As described in Section 4, a necessary condition for the existence of a limiting ODE of (20) is that $\lim_k k(1 - \sigma_{k+1})$ exists. However, we provide an example where the conditions in Theorem 4 are satisfied, but $\lim_k k(1 - \sigma_{k+1})$ does not exist.

Lemma 1. *Consider the sequence $\{\alpha_k\}$ defined by alternating two rules:*

$$\alpha_k = \begin{cases} (k + r)/r, & \text{if } k \text{ is even,} \\ (1 + \sqrt{1 + 4\alpha_{k-1}^2})/2. & \text{if } k \text{ is odd,} \end{cases}$$

with $\alpha_0 = 1$. Then for any $r \geq 2$, condition (ii) in Theorem 4 holds. But $\lim_k k(1-\sigma_{k+1})$ does not exist if $r > 2$: along $\{k'\} = \{2k\}$ and $\{k''\} = \{2k+1\}$, we have

$$\lim_{k'\to\infty} k'(1-\sigma_{k'+1}) = 3r/2, \quad \lim_{k''\to\infty} k''(1-\sigma_{k''+1}) = 2 + r/2.$$

Lemma 1 shows that accelerated convergence can be achieved by algorithm (20), independent of whether a limiting ODE exists[9]. Hence, convergence of discrete algorithms may not always be explained from the ODE perspective.

4 ODE connection and comparison

As the stepsize s vanishes to 0 in a discrete algorithm, the limiting ODE (if exists) can be studied to understand the behavior of its discrete counterpart (Su et al., 2016; Shi et al., 2022). We study the convergence rates of limiting ODEs of algorithms in Sections 2 and 3, and compare them with the discrete results.

We briefly review how the limiting ODEs from NAG-SC (4) and NAG-C (17) exhibit interesting differences from that of the vanilla gradient descent (2), as discussed in Su et al. (2016). On one hand, by taking $\Delta t = s$ and $x_k = X(t_k) = X(ks)$ for a continuous-time trajectory $X_t = X(t)$, the limit of gradient descent (2) as $s \to 0$ is the gradient flow

$$\dot{X}_t + \nabla f(X_t) = 0, \tag{23}$$

with $X(0) = x_0$. It is known that when $f \in \mathcal{F}^1$, the solution X_t satisfies that $f(X_t) - f^* \leq \|x_0 - x^*\|^2/(2t)$ and when $f \in \mathcal{S}^1_\mu$, $f(X_t) - f^* \leq \mathrm{e}^{-2\mu t}(f(x_0) - f^*)$. By relating X_t to x_k (i.e., taking $t = ks$ for small s), the former $O(1/t)$ rate translates into $O(1/(ks))$, which is exactly the discrete rate of gradient descent in the general convex setting. The second $O(\mathrm{e}^{-2\mu t})$ rate resembles $O((1-2\mu s)^k)$ for $s \approx 0$, which is similar to the discrete rate $O((1 - \frac{2\mu s}{1+1/\kappa})^k)$ for gradient descent in the strongly convex setting.

On the other hand, by taking $\Delta t = \sqrt{s}$ and $x_k = X(t_k) = X(k\sqrt{s})$ for a continuous-time trajectory $X_t = X(t)$, the limit of NAG-C (17) as $s \to 0$ is

$$\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0, \tag{24}$$

with initial conditions $X(0) = x_0$ and $\dot{X}(0) = 0$. The solution X_t satisfies that for $f \in \mathcal{F}^1$, $f(X_t) - f^* \leq 2\|x_0 - x^*\|^2/t^2$. The limit of NAG-SC (4) is

$$\ddot{X}_t + 2\sqrt{\mu}\dot{X}_t + \nabla f(X_t) = 0, \tag{25}$$

with initial conditions $X(0) = x_0$ and $\dot{X}(0) = 0$. The solution X_t satisfies that when $f \in \mathcal{S}^1_\mu$, $f(X_t) - f^* \leq 2\mathrm{e}^{-\sqrt{\mu}t}(f(x_0) - f^*)$. For a fixed t, X_t can be approximated by the discrete iterate x_k with $t = k\sqrt{s}$ for a small s. Using $t = k\sqrt{s}$, the former $O(1/t^2)$ rate translates into $O(1/(sk^2))$, which is the discrete rate for NAG-C. The latter $O(\mathrm{e}^{-\sqrt{\mu}t})$ rate matches the discrete rate $O((1-\sqrt{\mu s})^k)$ for NAG-SC. In these cases, the convergence rate of the continuous-time trajectory $X_t = X(t)$ matches that of the discrete iterates $\{x_k\}$.

The overall findings of our study can be summarized as follows.

- For minimization of strongly convex functions, the ODE convergence bounds do not directly inform the range of feasible stepsizes, which are crucial in determining whether acceleration is achieved.
- For minimization of either strongly convex or general convex functions, the gradient correction term vanishes in the (low-resolution) ODEs. Although this term is explicitly retained in high-resolution ODEs, these ODEs can still converge at the same rate with or without it.

In conclusion, the convergence properties of limiting ODEs, including high-resolution ODEs, fall short of informing whether discrete algorithms achieve acceleration. Our work provides explicit convergence bounds for the low- and high-resolution ODEs through new Lyapunov analyses. These results complement and enrich the existing literature on the connections and gaps between discrete algorithms and ODEs.

[9] A similar conclusion can be found for algorithm (3) in the strongly convex case. See Sanz Serna & Zygalakis (2021) Theorem 3.3 for related work.

4.1 Strongly convex setting

For strongly convex f, we compare the convergence of algorithm (14) and related ODEs. By taking $\Delta t = \sqrt{s}$ and $x_k = X(t_k) = X(k\sqrt{s})$, the limiting ODE of (14) as $s \to 0$ is

$$\ddot{X}_t + c_1\sqrt{\mu}\dot{X}_t + c_0\nabla f(X_t) = 0, \tag{26}$$

with initial conditions $X(0) = x_0$ and $\dot{X}(0) = 0$. This can be viewed as a Newtonian equation of motion in a viscous medium in the potential field $c_0 f$ with c_0 as the rescaling constant[10]. The damping coefficient is $c_1\sqrt{\mu}$, resulting from the momentum coefficient $1-c_1\sqrt{q}$ in (14). For convenience, the three settings $c_1^2 > 4c_0$, $c_1^2 = 4c_0$, or $c_1^2 < 4c_0$ are referred to as over-damping, critical-damping, or under-damping, respectively. The parameter c_2 associated with the gradient correction vanishes. Nevertheless, for any μ-strongly convex f and any $c_0, c_1 > 0$, we show that $f(X_t)$ converges to f^* exponentially fast with a decaying rate proportional to $\sqrt{\mu}$.

Proposition 3. *Let $f \in \mathcal{S}_\mu^1$. Then for any $c_0, c_1 > 0$, the solution X_t to (26) satisfies that $f(X_t) - f^* = O(\mathrm{e}^{-C\sqrt{\mu}t}(f(x_0) - f^*))$ for a constant $C = \frac{c_1 - \sqrt{(c_1^2 - 4c_0)\vee 0}}{2} > 0$.[11] Otherwise (either $c_0 \le 0$ or $c_1 \le 0$), X_t may fail to converge to x^* as $t \to \infty$.*

The dependency on μ in the bound above is improved to $\sqrt{\mu}$, compared with the $O(\mathrm{e}^{-2\mu t})$ bound for gradient flow (23). A direct translation by $t_k = k\sqrt{s}$ suggests that a discrete bound of $O((1-C\sqrt{\mu s})^k)$ may be expected for suitable discretizations of (26), as opposed to $O((1 - C\mu s)^k)$ for gradient descent (2). However, there are notable gaps between such results suggested by ODEs and our results for discrete algorithms. First, the bound in Proposition 3 only requires $c_0, c_1 > 0$, which is much weaker than the over-damping condition $c_1^2 > 4c_0$ in Corollary 1. The condition $c_1^2 > 4c_0$ may be only technical and can be potentially relaxed. Second, the bound $O((1 - C\sqrt{\mu s})^k)$ does not directly inform whether $s \asymp 1/L$ (equivalently $q \asymp 1/\kappa$) or $s \asymp \mu/L^2$ ($q \asymp 1/\kappa^2$), corresponding to different orders of time interval in discretizing the ODE. The former leads to acceleration while the latter does not. Third, the gradient correction term from algorithm (14) vanishes in the limiting ODE (26), so that the ODE does not capture the effect of gradient correction.

To complement the preceding discussion, we study the high-resolution ODEs for NAG-SC (4) and HB (9), which are proposed to reflect the gradient correction or lack of in the two methods (Shi et al., 2022). The high-resolution ODEs are derived by retaining $O(\sqrt{s})$ terms that would otherwise vanish in the limit of $s \to 0$. By taking $\Delta t = \sqrt{s}$ and $x_k = X(t_k) = X(k\sqrt{s})$, the low-resolution ODEs for NAG-SC (4) and HB (9) are the same equation in (25). The high-resolution ODE for NAG-SC is

$$\ddot{X}_t + 2\sqrt{\mu}\dot{X}_t + \sqrt{s}\nabla^2 f(X_t)\dot{X}_t + (1 + \sqrt{\mu s})\nabla f(X_t) = 0, \tag{27}$$

with initial conditions $X(0) = x_0$ and $\dot{X}(0) = -\frac{2\sqrt{s}}{1+\sqrt{\mu s}}\nabla f(x_0)$. The high-resolution ODE for HB is

$$\ddot{X}_t + 2\sqrt{\mu}\dot{X}_t + (1 + \sqrt{\mu s})\nabla f(X_t) = 0, \tag{28}$$

with the same initial conditions. For NAG-SC, the gradient correction results in an additional Hessian term, $\sqrt{s}\nabla^2 f(X_t)\dot{X}_t$, in (27). However, the convergence rates of these two high-resolution ODEs are the same. The following result is qualitatively similar to Theorems 1 and 2 in Shi et al. (2022), but involves a sharper rate due to a new Lyapunov analysis.

Proposition 4. *Let $f \in \mathcal{S}_\mu^2$. Then the solutions X_t of both (27) and (28) satisfy that $f(X_t) - f^* \le V_0\mathrm{e}^{-\sqrt{\mu}t}$, where the constant $V_0 = (1 + \sqrt{\mu s})(f(x_0) - f^*) + \frac{1}{2}\|\sqrt{\mu}(x_0 - x^*) - \sqrt{s}\frac{1-\sqrt{\mu s}}{1+\sqrt{\mu s}}\nabla f(x_0)\|^2$ for (27) and $V_0 = (1 + \sqrt{\mu s})(f(x_0) - f^*) + \frac{1}{2}\|\sqrt{\mu}(x_0 - x^*) - \frac{2\sqrt{s}}{1+\sqrt{\mu s}}\nabla f(x_0)\|^2$ for (28).*

Unfortunately, despite explicitly incorporating the gradient correction (or not), high-resolution ODEs do not exhibit differences in convergence rates and therefore fail to capture the effect of gradient correction.

[10] It suffices to establish convergence for (26) for $c_0 = 1$, then all other cases with $c_0 > 0$ can be recovered by a time rescaling.

[11] The constant C in Proposition 3 can be improved using the LMI framework (Sanz Serna & Zygalakis, 2021). However, our analysis is based on a concise Lyapunov analysis, and our goal is not to find the sharpest constant.

4.2 General convex setting

For general convex f, we compare the convergence of algorithm (21) and related ODEs in the setting where $\gamma_k = \gamma$, $\beta_k = \beta$, and $\alpha_k = \frac{k+r}{r}$[12]. By rescaling the stepsize s to s/γ, the parameters γ and β can be reset to 1 and β/γ respectively. In this setting, (21) reduces to the sub-class in Shi et al. (2022), with two parameters r and β/γ:

$$x_{k+1} = x_k - s\nabla f(x_k) + \sigma_{k+1}(x_k - x_{k-1}) - \sigma_{k+1}\frac{\beta}{\gamma} \cdot s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{29}$$

where $\sigma_{k+1} = \frac{\alpha_k - 1}{\alpha_{k+1}} = \frac{k}{k+r+1}$. By taking $\Delta t = \sqrt{s}$ and $x_k = X(t_k) = X(k\sqrt{s})$, the limiting ODE for (29) is

$$\ddot{X}_t + \frac{r+1}{t}\dot{X}_t + \nabla f(X_t) = 0, \tag{30}$$

with initial conditions $X(0) = x_0$ and $\dot{X}(0) = 0$. This can be viewed as a Newtonian equation for a particle moving in the potential field f with friction. The damping coefficient is $\frac{r+1}{t}$, resulting from the momentum coefficient σ_{k+1}. Su et al. (2016) showed that the convergence of X_t from (30) exhibits a phase transition at $r = 2$, the choice in NAG-C (15). For $f \in \mathcal{F}^1$, if $r \geq 2$, then $f(X_t) - f^* = O(1/t^2)$. But if $r < 2$, the rate $O(1/t^2)$ may fail as illustrated by counterexamples. For $\alpha_k = \frac{k+r}{r}$, it can be directly verified that the recursive condition $\alpha_{k+1}(\alpha_{k+1} - 1) \leq \alpha_k^2$ in Theorem 4 holds if and only if $r \geq 2$. Therefore, the condition on the momentum term for acceleration of (29) is well captured by the limiting ODE.

The ODE (30), however, does not reflect the parameter β/γ associated with the gradient correction in algorithm (29), which is similarly observed in the low-resolution ODE (26) for algorithm (14) in the strongly convex setting. The gradient correction term, of order $O(\sqrt{s})$, can be incorporated in a high-resolution ODE (Shi et al., 2022). By taking $t_0 = \frac{(r+1)\sqrt{s}}{2}$ and $t_k = t_0 + k\sqrt{s}$ for algebraic convenience, the high-resolution ODE for (21) is

$$\ddot{X}_t + \frac{r+1}{t}\dot{X}_t + \frac{\beta}{\gamma}\sqrt{s}\nabla^2 f(X_t)\dot{X}_t + \left(1 + \frac{(r+1)\sqrt{s}}{2t}\right)\nabla f(X_t) = 0, \tag{31}$$

with initial conditions $X(t_0) = x_0$ and $\dot{X}(t_0) = -\sqrt{s}\nabla f(x_0)$. The following result gives convergence bounds of (31) for both $r > 2$ and $r = 2$ via a new Lyapunov analysis. Shi et al. (2022) provided similar bounds only explicitly in the case of $r > 2$.

Proposition 5. *Let $f \in \mathcal{F}^2$. If $r \geq 2$ and $\frac{\beta}{\gamma} > 0$, then there exists a time point $t_1 \geq t_0$, with $t_1/\sqrt{s}$ depending only on r and β/γ, such that the solution to (31) satisfies the bounds:*

$$f(X_t) - f^* = O\left(\frac{V_{t_1}}{t^2}\right), \quad \inf_{t_1 \leq u \leq t} \|\nabla f(X_u)\|^2 = O\left(\frac{\gamma V_{t_1}}{\beta\sqrt{s}(t^3 - t_1^3)}\right).$$

In addition, if $r \geq 2$ and $\beta = 0$, then the first bound still holds. Here V_{t_1} is the value that the continuous Lyapunov function takes at t_1: $V_t = (t + C\sqrt{s})(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s})(f(X_t) - f^*) + \frac{t + C\sqrt{s}}{t}\frac{1}{2}\|r(X_t - x^*) + t(\dot{X}_t + \frac{\beta\sqrt{s}}{\gamma}\nabla f(X_t))\|^2$, where $C = 0$ if $r > 2$ and $C = \frac{\beta}{\gamma}$ if $r = 2$.

If we replace t by $t_k = t_0 + k\sqrt{s} \sim k\sqrt{s}$, then the first bound above translates into the accelerated bound $O(1/(sk^2))$ on the objective gap as in (16) and the second becomes $O(1/(s^2k^3))$ on the squared gradient norm as in (18). These two bounds above are exactly the continuous analogs of the associated bounds for algorithm (29). However, compared with the condition $\beta/\gamma > 1/2$ in Theorem 4, the condition in the continuous ODE setting is much weaker: β/γ is allowed to be arbitrarily small, and even 0 (no Hessian term) if only the $O(1/t^2)$ rate for the objective gap is desired. The latter case corresponds to a discrete algorithm without the gradient correction, similar to HB (9). Therefore, even when a Hessian term is included as a continuous counterpart of the gradient correction, the convergence properties of the high-resolution ODE are currently not fully matched by those of algorithm (29). This remains to be further studied as an open question, as also mentioned in Shi et al. (2022), Section 5.1.

[12]For a general sequence $\{\alpha_k\}$, algorithm (21) may not admit a limiting ODE, as shown in Lemma 1.

5 Hamiltonian-assisted interpretation

As observed in Section 4, the low-resolution and high-resolution ODEs do not fully capture the convergence properties of the algorithms studied. Alternatively, we directly formulate a broad class of discrete algorithms, HAG, based on a Hamiltonian function with a position variable and a momentum variable, and demonstrate that the conditions from our convergence results in Sections 2 and 3 can be interpreted through HAG in a meaningful and unified manner in both the strongly convex and general convex settings. This development is motivated by a related formulation of Hamiltonian-assisted Metropolis sampling (HAMS) (Song & Tan, 2023; 2022).

5.1 HAG: Hamiltonian-assisted gradient method

For $f \in \mathcal{F}_L^1$, consider the following unconstrained minimization problem:

$$\min_{x,u\in\mathbb{R}^n} H(x,u) = f(x) + \frac{1}{2}\|u\|^2,$$

where $H(x,u)$ can be interpreted as a Hamiltonian function (or total energy), with x a position variable and u a momentum variable. The above problem is equivalent to the original problem (1). If x and u are updated separately, then there is no possible improvement compared with solving (1) directly. However, we show how x and u can be updated in a coupled manner to derive a rich class of first-order algorithms, which include representative algorithms from the classes (14) and (21) studied in Sections 2 and 3.

Gradient descent with a linear constraint. Given initial points (x_0, u_0), consider minimizing (or decreasing) $H(x,u)$ subject to a linear constraint $x - \delta u = x_0 - \delta u_0$ for some $\delta > 0$. By substituting $u = \frac{x - x_0 + \delta u_0}{\delta}$, the problem with respect to x becomes minimizing (or decreasing) $\tilde{f}(x; x_0, u_0) = f(x) + \frac{1}{2\delta^2}\|x - x_0 + \delta u_0\|^2$. Consider updating x_0 by gradient descent with a stepsize $s = \frac{1}{1+\frac{1}{\delta^2}} = \frac{\delta^2}{1+\delta^2}$, which, by the standard characterization of gradient descent, is equivalent to minimizing a quadratic surrogate function of $\tilde{f}$ at x_0 (with surrogate smoothness parameter being 1 for f) as follows:

$$x_1 = \arg\min_x \langle \nabla f(x_0), x - x_0\rangle + \frac{1}{2}\|x - x_0\|^2 + \frac{1}{2\delta^2}\|x - x_0 + \delta u_0\|^2.$$

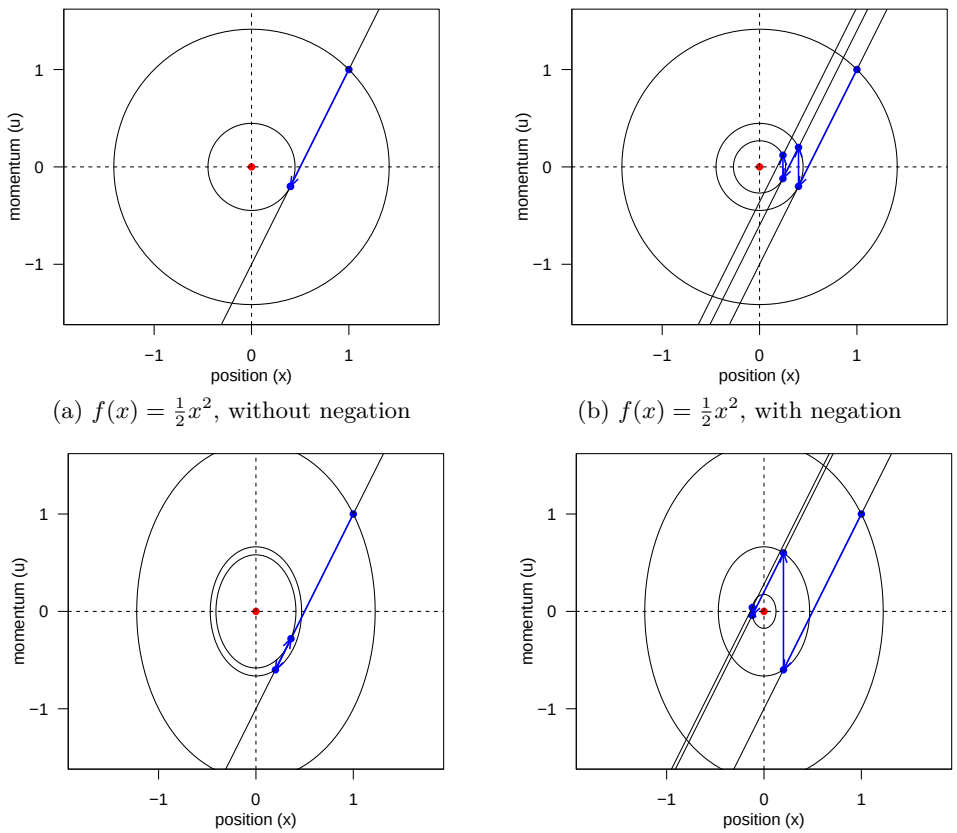
Then u_0 is updated by solving $x_1 - \delta u_1 = x_0 - \delta u_0$. The resulting update is

$$\begin{aligned} x_1 &= x_0 - \frac{\delta^2}{1+\delta^2}\nabla f(x_0) - \frac{\delta}{1+\delta^2}u_0, \\ u_1 &= u_0 - \frac{\delta}{1+\delta^2}\nabla f(x_0) - \frac{1}{1+\delta^2}u_0. \end{aligned}$$

The function $\tilde{f}(x; x_0, u_0)$ has a smoothness parameter $L_{\tilde{f}} = L + \frac{1}{\delta^2}$. If $0 < \delta^2 \le \frac{1}{L}$, then $L_{\tilde{f}} \le \frac{2}{\delta^2} < \frac{2}{s}$ and $s < \frac{2}{L_{\tilde{f}}}$, so that the above GD update is well-behaved and $\tilde{f}$ (as well as H) is non-increasing (see the footnote 7).

Momentum negation and extrapolation. The above update can be repeated on (x_1, u_1), but the constraint $x - \delta u = x_1 - \delta u_1 = x_0 - \delta u_0$ will remain the same, so that all subsequent updates also satisfy the same constraint and, in general, cannot converge to a minimizer of $H(x,u)$. To resolve this issue, we negate u at the beginning of each iteration. In other words, given (x_0, u_0), we first negate u_0 to $-u_0$, and then implement the above update. The negation of u does not change the value of H, but keeps the constraint changing from iteration to iteration.

An illustration is provided in Figure 1. For 1-dim x, the linear constraint becomes a straight line in $\mathbb{R}^2$. When $f(x) = \frac{x^2}{2}$, the level sets of $H(x,u) = \frac{x^2}{2} + \frac{u^2}{2}$ are circles, and the gradient descent update is exactly the minimizer along the line. Then the update without negation gets stuck after the first iteration. Negating the momentum changes the linear constraint, and then the updates keep moving toward the minimizer $(0,0)$. When $f(x) = x^2$, the level sets of $H(x,u) = x^2 + \frac{u^2}{2}$ are ellipses, and the gradient descent update is not exactly the minimizer along the line. Nevertheless, without negation, all the iterates will stay on the same line, hence failing to converge to the minimizer $(0,0)$. The momentum negation is also essential to the HAMS



(a) $f(x) = \frac{1}{2}x^2$, without negation

(b) $f(x) = \frac{1}{2}x^2$, with negation

(c) $f(x) = x^2$, without negation

(d) $f(x) = x^2$, with negation

Figure 1: An illustration of the gradient descent with a linear constraint and the effect of momentum negation when $x \in \mathbb{R}$. The first two iterates are plotted without or with negation, for $f(x) = x^2/2$ in the first row and $f(x) = x^2$ in the second row. The initial point is $(1, 1)$ and the global minimizer is $(0, 0)$. We fix $\delta = 1/2$. The black solid circles and ellipses correspond to different level sets of the Hamiltonian functions.

algorithm and related under-damped Langevin sampling algorithms, where a negation of the momentum is required to achieve generalized reversibility (Neal, 2011; Song & Tan, 2022).

In addition, we introduce an extrapolation step before the negation of u_0 for $\rho \geq 0$. This leads to the following update given the initial points (x_0, u_0):

$$x_1 = x_0 - \frac{(1+\rho)\delta^2}{1+\delta^2}\nabla f(x_0) + \frac{(1+\rho)\delta}{1+\delta^2}u_0,$$
$$u_1 = -u_0 - \frac{(1+\rho)\delta}{1+\delta^2}\nabla f(x_0) + \frac{1+\rho}{1+\delta^2}u_0.$$

By setting $a = \frac{(1+\rho)\delta^2}{1+\delta^2}$ and $b = \frac{1+\rho}{1+\delta^2}$, the above can be put in a simple form as

$$x_1 = x_0 - a\nabla f(x_0) + \sqrt{ab}u_0,$$
$$u_1 = -u_0 - \sqrt{ab}\nabla f(x_0) + bu_0.$$

The parameters (δ, ρ) can be determined from (a, b) as $\delta = \sqrt{a/b}$ and $\rho = a + b - 1$.

Gradient correction. The above updates of x_1 and u_1 are parallel, so that u_1 uses only the gradient information at x_0, but not at the newly updated x_1. To exploit the gradient information at x_1, we incorporate

a gradient correction into the update:

$$x_1 = x_0 - a\nabla f(x_0) + \sqrt{ab}u_0,$$
$$u_1 = -u_0 - \sqrt{ab}\nabla f(x_0) + bu_0 - \phi(\nabla f(x_1) - \nabla f(x_0)),$$

for some scalar ϕ. Lastly, we allow a, b and ϕ to vary from iteration to iteration and consider the following class of algorithms, called Hamiltonian-assisted gradient method (HAG):

$$\begin{aligned} x_{k+1} &= x_k - a_k\nabla f(x_k) + \sqrt{a_k b_k}u_k, \\ u_{k+1} &= -u_k - \sqrt{a_k b_k}\nabla f(x_k) + b_k u_k - \phi_k(\nabla f(x_{k+1}) - \nabla f(x_k)), \end{aligned} \tag{32}$$

starting from (x_0, u_0). Notably, the overall contribution from gradients remains unchanged in the update for u_{k+1}. Instead, they are re-weighted:

$$u_{k+1} = (b_k - 1)u_k - \underbrace{\left((\sqrt{a_k b_k} - \phi_k)\nabla f(x_k) + \phi_k \nabla f(x_{k+1})\right)}_{\text{re-weighted gradient}}. \tag{33}$$

The larger ϕ_k is, the larger weight is assigned to the new gradient $\nabla f(x_{k+1})$.

To facilitate comparison with algorithms in Sections 2 and 3, the HAG algorithm (32) can be put into a single-variable form involving $\{x_k\}$ only (see Appendix D):

$$\begin{aligned} x_{k+1} = x_k - \left(a_{k-1}\sqrt{\frac{a_k b_k}{a_{k-1}b_{k-1}}} + a_k\right)\nabla f(x_k) + (b_{k-1} - 1)\sqrt{\frac{a_k b_k}{a_{k-1}b_{k-1}}}(x_k - x_{k-1}) \\ - \left(\phi_{k-1}\sqrt{a_k b_k} - a_{k-1}\sqrt{\frac{a_k b_k}{a_{k-1}b_{k-1}}}\right)(\nabla f(x_k) - \nabla f(x_{k-1})), \end{aligned} \tag{34}$$

starting from x_0 and $x_1 = x_0 - a_0\nabla f(x_0) + \sqrt{a_0 b_0}u_0$. The parameter $\{\phi_k\}$ only appears in the gradient correction in (34), and in the case of $a_k b_k$ being constant in k, the parameters $\{a_k\}$ and $\{b_k\}$ fully determine the gradient descent and the momentum terms respectively. For convenience, the term of $x_k - x_{k-1}$ is still referred to as the momentum term in single-variable forms, which is not to be confused with the momentum variable u_k.

5.2 Interpretation from HAG

HAG (32) or (34) are general enough to represent various algorithms studied in Sections 2 and 3, by choosing the parameters $\{a_k\}$, $\{b_k\}$ and $\{\phi_k\}$ accordingly. We examine our convergence results from the HAG perspective and obtain meaningful and unified interpretations of the conditions for acceleration in both strongly convex and general convex settings. Our findings are summarized as follows.

- The parameter a_k acts as a re-scaled stepsize s. The parameter b_k, which controls the momentum term in the single-variable form, has a leading constant 2, and the gap $2 - b_k$ is $\Omega(\sqrt{q})$ with $q = \mu s$ or $\Omega(\frac{1}{k})$ in the strongly or general convex setting respectively. [13] In terms of (δ_k, ρ_k) in the HAG derivation, this indicates that $\delta_k \sim \sqrt{\frac{a_k}{2}}$ and $1-\rho_k \sim 2-b_k$. Therefore, δ_k is of order $\sqrt{s}$ (comparable to Δt in deriving ODEs for NAG-SC and NAG-C), ρ_k has a leading constant 1 (indicating symmetric extrapolation), and the gap $1 - \rho_k$ is $\Omega(\sqrt{q})$ or $\Omega(\frac{1}{k})$ in the strongly or general convex setting.
- The parameter ϕ_k is of order $\sqrt{s}$ and controls the gradient correction, such that in the re-weighted gradient for the momentum update (33), the new gradient $\nabla f(x_{k+1})$ *fully dominates* with its weight greater than the total weight, whereas the old gradient $\nabla f(x_k)$ has a negative weight. The boundary case of a zero weight for $\nabla f(x_k)$ is also allowed in the strongly convex setting. Such heuristic interpretations are not feasible from the single-variable forms.

[13] The two orders, $\sqrt{q}$ and $\frac{1}{k}$, can be seen to match each other, because the accelerated algorithms and their continuous limits are linked via $t = k\sqrt{s}$ in Section 4 and hence $\sqrt{s}$ can be viewed to be in the order of $\frac{1}{k}$, as $s \to 0$ or $k \to \infty$.

It is remarkable that such meaningful interpretations are based on convergence results for (14) and (22). It remains an interesting question to analyze HAG directly and study further implications.

Strongly convex setting. For $f \in \mathcal{S}^1_{\mu,L}$, we set

$$a_k \equiv a = (\frac{c_0}{2} + O(\sqrt{q}))s, \quad b_k \equiv b = 2 - c_1\sqrt{q} + O(q), \quad \phi_k \equiv \phi = (c_2 + O(\sqrt{q}))\sqrt{s},$$

where $c_0, c_1, c_2 > 0$. Then HAG (34) reduces to (14) exactly:

$$\begin{aligned} x_{k+1} &= x_k - 2a\nabla f(x_k) + (b-1)(x_k - x_{k-1}) - (\phi\sqrt{ab} - a)(\nabla f(x_k) - \nabla f(x_{k-1})) \\ &= x_k - (c_0 + O(\sqrt{q}))s\nabla f(x_k) + (1 - c_1\sqrt{q} + O(q))(x_k - x_{k-1}) \\ &\quad - (c_2\sqrt{c_0} - \frac{c_0}{2} + O(\sqrt{q}))s(\nabla f(x_k) - \nabla f(x_{k-1})). \end{aligned} \tag{35}$$

In the following, we interpret the conditions in Corollary 1 through HAG.

The parameter $a = (\frac{c_0}{2} + O(\sqrt{q}))s$ plays the role of a re-scaled stepsize s. The parameter $b = 2 - c_1\sqrt{q} + O(q)$ controls the momentum term $(x_k - x_{k-1})$ in (35). The leading constant of b is 2, which ensures the existence of a limiting ODE as $s \to 0$. In terms of c_1, Corollary 1 requires the condition $c_1^2 > 4c_0$, which as mentioned earlier may be potentially relaxed, for example, to $c_1 > 0$. This would lead to a simple condition $2 - b = \Omega(\sqrt{q})$ as $s \to 0$. In terms of (δ, ρ) in the HAG derivation, the preceding discussion also gives

$$\delta = \sqrt{a/b} = \frac{1}{2}\sqrt{c_0 s} + O(s), \quad 1 - \rho = 2 - b - a = \Omega(\sqrt{q}). \tag{36}$$

We observe that δ is in the order of $\sqrt{s}$, similarly as Δt in deriving the ODEs (25) and (26), and ρ is close to 1 (symmetric extrapolation) but with a gap being $\Omega(\sqrt{\mu s})$.

The parameter $\phi = (c_2 + O(\sqrt{q}))\sqrt{s}$ controls the gradient correction $(\nabla f(x_{k+1}) - \nabla f(x_k))$ in (35). The re-weighted gradient in the momentum update (33) for u_{k+1} becomes

$$(\sqrt{c_0} - c_2 + O(\sqrt{q}))\sqrt{s}\nabla f(x_k) + (c_2 + O(\sqrt{q}))\sqrt{s}\nabla f(x_{k+1}). \tag{37}$$

For accelerated convergence in Corollary 1, the condition $c_2^2 \geq c_0$ indicates that the new gradient $\nabla f(x_{k+1})$ fully dominates with its weight, $c_2\sqrt{s}$, no smaller than the total weight $\sqrt{c_0 s}$, whereas the old gradient $\nabla f(x_k)$ has a zero or negative weight. For non-accelerated (sub-optimal) convergence in Corollary 1, the condition $c_0/4 \leq c_2^2 < c_0$ indicates that both the new and old gradients have positive weights, but $\nabla f(x_{k+1})$ contributes no less than $\nabla f(x_k)$. Interestingly, such a meaningful interpretation cannot be obtained from the single-variable form (14) or (35), even though a similar re-weighted gradient can also be identified there[14].

General convex setting. For $f \in \mathcal{F}^1_L$, we set

$$b_k = 1 + \sigma_{k+2}, \quad a_k = \frac{c_0}{b_k}s, \quad \phi_k \equiv c_2\sqrt{s}, \tag{38}$$

where $c_0, c_2 > 0$ and, as before, $\sigma_{k+1} = \frac{\alpha_k - 1}{\alpha_{k+1}}$ for some sequence $\{\alpha_k\}$. Then HAG (34) reduces to

$$x_{k+1} = x_k - c_0\left(\frac{1}{b_{k-1}} + \frac{1}{b_k}\right)s\nabla f(x_k) + \sigma_{k+1}(x_k - x_{k-1}) - \left(c_2\sqrt{c_0} - \frac{c_0}{b_{k-1}}\right)s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{39}$$

[14]In (14) or (35), it seems reasonable to consider the re-weighted gradient as

$$\begin{aligned} &(c_0 + O(\sqrt{q}))s\nabla f(x_k) + (c_2\sqrt{c_0} - \frac{c_0}{2} + O(\sqrt{q}))s(\nabla f(x_k) - \nabla f(x_{k-1})) \\ &\qquad = (c_2\sqrt{c_0} + \frac{c_0}{2} + O(\sqrt{q}))s\nabla f(x_k) + (-c_2\sqrt{c_0} + \frac{c_0}{2} + O(\sqrt{q}))s\nabla f(x_{k-1}). \end{aligned}$$

However, the condition $c_2^2 \geq c_0$ indicates that the weight of $\nabla f(x_k)$ is no smaller than $\frac{3}{2}c_0$, whereas the weight of $\nabla f(x_{k-1})$ is no larger than $-\frac{1}{2}c_0$, with a total weight $c_0 s$. Therefore, the conditions bear no meaningful interpretations.

which falls in the class (21) with γ_k and β_k varying in k, whose convergence property can be readily deduced from Theorem 4. We choose a_k varying in k such that $a_k b_k$ is constant in k, mainly to simplify the coefficients from (34).

Given any $\{\alpha_k\}$ satisfying $\alpha_k = \Omega(k)$ and $\frac{\alpha_{k+1}}{\alpha_k} = 1 + O(\frac{1}{k})$, we have $\sigma_{k+1} = \frac{\alpha_k - 1}{\alpha_{k+1}} = 1 - O(\frac{1}{k})$, $b_k = 1 + \sigma_{k+2} = 2 - O(\frac{1}{k})$, $a_k = \frac{c_0}{b_k} s = (c_0 + O(\frac{1}{k}))s/2$, and the preceding HAG algorithm (39) is further simplified to

$$x_{k+1} = x_k - \left(c_0 + O(\frac{1}{k})\right) s\nabla f(x_k) + \sigma_{k+1}(x_k - x_{k-1}) - \left(c_2\sqrt{c_0} - \frac{c_0}{2} - O(\frac{1}{k})\right) s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{40}$$

which resembles algorithm (35) in the strongly convex setting, in all the leading constants involved. Algorithm (40) can be put into (21) with $\gamma_k \to \gamma = c_0$ and $\beta_k \to \beta = c_2\sqrt{c_0} - \frac{c_0}{2}$. Note that (39) or (40) is more complex than (29) with constant γ_k and β_k studied in Shi et al. (2022). In the following, we interpret conditions (i) and (ii) in Theorem 4 through HAG for the choice $\alpha_k = \frac{k+r}{r}$. The technical monotonicity condition (iii) can be verified for this specific choice (see Appendix D for a proof).

Similarly to the strongly convex setting, the parameter $a_k = (c_0 + O(\frac{1}{k}))s/2$ plays the role of a re-scaled stepsize s. Moreover, the parameter $b_k = 1 + \sigma_{k+2}$ controls the momentum term $(x_k - x_{k-1})$ in (40), and has its leading constant being 2, which is necessary (but not sufficient by Lemma 1) for a limiting ODE to exist as $s \to 0$. For the choice $\alpha_k = \frac{k+r}{r}$ and $\sigma_{k+1} = \frac{k}{k+r+1}$, the recursive condition $\alpha_{k+1}(\alpha_{k+1} - 1) \le \alpha_k^2$ is equivalent to $r \ge 2$. Then it implies a simple condition $2 - b_k = (r+1)/k - O(1/k^2) = \Omega(1/k)$ as $k \to \infty$ (or $s \sim 1/k^2 \to 0$). In terms of the time-dependent (δ_k, ρ_k) in HAG, the preceding discussion gives

$$\delta_k = \sqrt{a_k/b_k} = \frac{1}{2}\sqrt{c_0 s} + O(s), \quad 1 - \rho_k = 2 - b_k - a_k = \Omega(\frac{1}{k}). \tag{41}$$

We observe that δ_k is in the order of $\sqrt{s}$, similarly as Δt in deriving the ODE (24), and ρ_k is close to 1 (symmetric extrapolation) but with a gap being $\Omega(\frac{1}{k})$. The conditions, (36) and (41), for the strongly convex and general convex settings respectively, are of similar forms, with $\sqrt{\mu s}\,(= \sqrt{q})$ and $\frac{1}{k}$ exchanged with each other.

The parameter $\phi_k = c_2\sqrt{s}$ plays a similar role of controlling the gradient correction term $(\nabla f(x_k) - \nabla f(x_{k-1}))$ in (40), as does $\phi = (c_2 + O(\sqrt{q}))\sqrt{s}$ in the strongly convex setting. The re-weighted gradient in the momentum update (33) for u_{k+1} is similar to (37):

$$(\sqrt{c_0} - c_2)\sqrt{s}\nabla f(x_k) + c_2\sqrt{s}\nabla f(x_{k+1}).$$

With $\gamma = c_0$ and $\beta = c_2\sqrt{c_0} - \frac{c_0}{2}$, the condition $\beta > \gamma/2$ in Theorem 4 reduces to $c_2^2 > c_0$, which indicates that the new gradient $\nabla f(x_{k+1})$ fully dominates with its weight, $c_2\sqrt{s}$, greater than the total weight $\sqrt{c_0 s}$, whereas the old gradient $\nabla f(x_k)$ has a negative weight. This is the same as the condition, $c_2^2 \ge c_0$, for accelerated convergence in the strongly convex setting except that the boundary case $c_2^2 = c_0$ is excluded here. As discussed earlier, the single-variable form (40) does not admit a meaningful interpretation for the re-weighted gradient.

6 Outlines of Lyapunov analyses

We outline our Lyapunov analyses to prove the convergence results for the discrete algorithms in Sections 2 and 3. See Appendix C for our Lyapunov analyses for the convergence of ODEs in Section 4. Compared with existing ones, our Lyapunov functions are constructed to handle more general algorithms and ODEs or to achieve simpler analysis and sometimes sharper results. Before giving the outlines of our analyses, we summarize the comparison of Lyapunov analyses. The comparison is restricted among analytic Lyapunov functions, excluding those constructed by the LMI framework, which require additional numerical solvers (Lessard et al., 2016; Sanz Serna & Zygalakis, 2021; Dobson et al., 2025).

Strongly convex setting. We construct the discrete Lyapunov (43) to establish the convergence of algorithm (10) including NAG-SC and TMM as special cases. The auxiliary-energy term $\mu\|z_{k+1} - x^*\|^2/2$ in (43)

is also used in Bansal & Gupta (2019) for NAG-SC and in d'Aspremont et al. (2021) for TMM. However, the potential-energy term in our Lyapunov is in $\{x_k\}$ whereas the one in Bansal & Gupta (2019) is in $\{y_k\}$. The potential-energy term of the Lyapunov in d'Aspremont et al. (2021) is in $\{x_k\}$ like ours, but is not lower-bounded by $f(x_k) - f^*$, so that their analysis only establishes the convergence for $\{z_k\}$. In addition, compared with the analysis of NAG-SC in Shi et al. (2022), our Lyapunov function (43) has fewer terms, and our analysis is much simpler.

Our continuous Lyapunov function to analyze the class of low-resolution ODEs (26) for Proposition 3 is extended from the one proposed in Wilson et al. (2021) for $(c_0, c_1) = (1, 2)$ (i.e., the low-resolution ODE of NAG-SC). Furthermore, we construct suitable Lyapunov functions to analyze the high-resolution ODEs of NAG-SC and HB. Compared with the ones used in Shi et al. (2022), our Lyapunov functions lead to a sharper convergence bound for NAG-SC and HB (Proposition 4).

General convex setting. We construct the discrete Lyapunov (46) to establish the convergence of (20) or equivalently (22), including those in Beck (2017) and Shi et al. (2022) as special cases. The auxiliary-energy term $\|z_{k+1} - x^*\|^2/2$ in (46) is motivated by Su et al. (2016) for NAG-C and Shi et al. (2022) for (29), a sub-class of (20) with $\gamma_k = 1$, $\beta_k = \beta/\gamma$, and $\alpha_k = (k + r)/r$ for $r \geq 2$. However, the potential-energy term in our Lyapunov (46) differs from the related ones in Su et al. (2016) and Shi et al. (2022). Moreover, Shi et al. (2022) analyzed the three cases, $r = 2$ and $\beta \leq 1$, $r = 2$ and $\beta > 1$, and $r > 2$, separately. It seems difficult to extend their case-by-case analysis to cover the more general results in our Theorem 4.

The continuous Lyapunov function for the class of high-resolution ODEs (31) is the same as in Shi et al. (2022) when $r > 2$, but involves a technical modification when $r = 2$. The modification helps to establish the convergence bounds in Proposition 5 for both $r > 2$ and $r = 2$, whereas similar bounds are provided in Shi et al. (2022) only explicitly for $r > 2$.

6.1 Strongly convex setting

We provide a unified Lyapunov analysis to establish Theorems 1–3 for the class of algorithms (10). Proof details are presented in Appendix A. Although our Lyapunov function, like most existing ones, is manually designed, our analysis proceeds in several structured steps.

Step 1. Bounding the differencing of an auxiliary energy. The sequence $\{z_k\}$ plays a key role in our formulation of (10) in a three-variable form. We identify $\frac{\mu}{2}\|z_k - x^*\|^2$ as an auxiliary-energy term and bound its differencing, $\frac{1}{1-\nu\sqrt{q}}\frac{\mu}{2}\|z_{k+1} - x^*\| - \frac{\mu}{2}\|z_k - x^*\|^2$.

Lemma 2. *Let $f \in \mathcal{S}^1_{\mu,L}$. For any $s > 0$ such that $0 \leq \eta s \leq 1/L$, $\nu \geq 0$, $1 - \nu\sqrt{q} > 0$, $\tau > 0$ and $\zeta = 1 + (1 - \tau)\sqrt{q} \geq 0$, the iterates of (10) satisfy that for $k \geq 1$,*

$$\left(\frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q}) + \frac{\nu\sqrt{q}}{1 - \nu\sqrt{q}}\right)(f(x_k) - f^*) - \frac{\nu s}{2}\left(\frac{\nu}{1 - \nu\sqrt{q}} - \frac{\zeta\eta}{\tau}\right)\|\nabla f(x_k)\|^2 + \frac{1}{1 - \nu\sqrt{q}}\frac{\mu}{2}\|z_{k+1} - x^*\|^2$$
$$\leq \frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q})\left(f(x_{k-1}) - f^* - \frac{\eta s}{2}\|\nabla f(x_{k-1})\|^2\right) + \frac{\mu}{2}\|z_k - x^*\|^2. \quad (42)$$

Step 2. Constructing a discrete Lyapunov function. We define a Lyapunov function simply from the right-hand-side of (42): for $k \geq 0$,

$$V_{k+1} = \frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q})\left(f(x_k) - f^* - \frac{\eta s}{2}\|\nabla f(x_k)\|^2\right) + \frac{\mu}{2}\|z_{k+1} - x^*\|^2. \quad (43)$$

We refer to the first term above as a potential energy, involving both $f(x_k) - f^*$ and $\|\nabla f(x_k)\|^2$, and the second term $\frac{\mu}{2}\|z_k - x^*\|^2$ as an auxiliary energy. The negative term $-\frac{\eta s}{2}\|\nabla f(x_k)\|^2$ in (43) is a technical adjustment and can be traced to (42). A similar term can be found in our Lyapunov (46) in the general convex setting. By Assumption 1 for smoothness of $\tilde{\eta}(\cdot)$ near 0, a sufficiently small $C_0 \in (0, \frac{1}{4\eta_0}]$ can be picked with $\eta_0 > 0$ such that when $0 < s \leq \frac{C_0}{L}$, we have $0 < q = \mu s \leq C_0$, $\eta = \tilde{\eta}(\sqrt{q}) \leq 2\eta_0$, and $\eta L s \leq 2\eta_0 C_0 \leq \frac{1}{2}$. Then for $0 < s \leq \frac{C_0}{L}$, by the L-smoothness of f, we have $f(x_k) - f^* - \frac{\eta s}{2}\|\nabla f(x_k)\|^2 \geq$

$(1-\eta Ls)(f(x_k)-f^*) \geq \frac{1}{2}(f(x_k)-f^*)$. More details are given in Appendix A.3. The negative term in (46) is handled similarly in Appendix B.3.

Step 3. Identifying sufficient conditions for Lyapunov contraction. As expected from (42), we further bound $V_{k+1}-(1-\nu\sqrt{q})V_k$ and identify conditions such that a contraction inequality holds for the Lyapunov function: $V_{k+1}-(1-\nu\sqrt{q})V_k \leq 0$.

Lemma 3. *Define* $\mathbf{I}$ *and* $\mathbf{II}$ *as polynomials of* $\sqrt{q}$*,* η*,* ν *and* τ *(hence functions of* $\sqrt{q}$*) taking the following forms: (recall that* $\zeta = 1+(1-\tau)\sqrt{q}$*)*

$$\begin{aligned}\mathbf{I} &= \zeta\nu(\tau+\zeta\eta\sqrt{q})-\tau^2 \\ &= \tau(\nu-\tau)+\nu(\eta-\tau(\tau-1))\sqrt{q}-2\eta\nu(\tau-1)q+\eta\nu(\tau-1)^2q^{\frac{3}{2}}, \\ \mathbf{II} &= \tau(\nu\tau-2\zeta\eta)+\zeta\eta(\nu\tau-\zeta\eta)\sqrt{q} \\ &= \tau(\nu\tau-2\eta)+\eta(-\eta+\nu\tau+2\tau(\tau-1))\sqrt{q}+\eta(\tau-1)(2\eta-\nu\tau)q-\eta^2(\tau-1)^2q^{\frac{3}{2}}.\end{aligned} \tag{44}$$

Under the condition in Lemma 2, if one of the following (mutually exclusive) conditions holds: (i) $\mathbf{I}>0$ *and* $\mathbf{I}+\sqrt{q}\mathbf{II}\leq 0$*, or (ii)* $\mathbf{II}>0$ *and* $(1/\kappa)\mathbf{I}+\sqrt{q}\mathbf{II}\leq 0$*, or (iii)* $\mathbf{I}\leq 0$ *and* $\mathbf{II}\leq 0$*, then we have the contraction inequality* $V_{k+1}\leq(1-\nu\sqrt{q})V_k$ *for* $k\geq 1$*.*

Step 4. Verifying the contraction conditions and completing the analysis. For completing the analysis, the final step is to show that the sufficient conditions for Lyapunov contraction in Lemma 3 are satisfied under the conditions included in Theorems 1–3. This step can be algebraically tedious, and the details are left to Appendix A.

The contraction inequality, if verified, directly leads to a convergence bound $O((1-C_1\sqrt{\mu s})^k)$ as stated in (12). However, to fulfil the conditions in Lemma 3, we find that two ranges of stepsize s (or q) are allowed: $0<s\lesssim\mu/L^2$ (or $0<q\lesssim 1/\kappa^2$) and $0<s\lesssim 1/L$ (or $0<q\lesssim 1/\kappa$). As discussed after Theorem 1, the upper bound of feasible s determines whether the vanilla or accelerated convergence is achieved, in terms of the dependency on κ.

6.2 General convex setting

We provide a unified Lyapunov analysis to establish Theorem 4 for the class of algorithms (20) or equivalently (22). Proof details are presented in Appendix B. Our analysis proceeds in several structured steps, similarly as in the strongly convex setting (Section 6.1).

Step 1. Bounding the differencing of an auxiliary energy. From our formulation of the three-variable form (22), we identify $\frac{1}{2}\|z_k-x^*\|^2$ as an auxiliary-energy term and bound its differencing, $\frac{1}{2}\|z_{k+1}-x^*\|^2-\frac{1}{2}\|z_k-x^*\|^2$.

Lemma 4. *Let* $f\in\mathcal{F}_L^1$*. Then for each* $k\geq 1$ *such that* $\alpha_k\geq 1$ *and* $\widetilde{\alpha}_k\geq 0$*, the iterates of (22) satisfy*

$$\begin{aligned}&\alpha_k\widetilde{\alpha}_k s(f(x_k)-f^*)-\frac{\widetilde{\alpha}_k^2 s^2}{2}\|\nabla f(x_k)\|^2+\frac{1}{2}\|z_{k+1}-x^*\|^2 \\ &\qquad\leq \widetilde{\alpha}_k(\alpha_k-1)s\left((f(x_{k-1})-f^*)-\frac{(2-\beta_{k-1}Ls)\beta_{k-1}s}{2}\|\nabla f(x_{k-1})\|^2\right)+\frac{1}{2}\|z_k-x^*\|^2.\end{aligned} \tag{45}$$

Step 2. Constructing a discrete Lyapunov function. We define a Lyapunov function simply as the left-hand-side of (45) up to a scalar sequence $\{\omega_k\}$: for $k\geq 0$,

$$V_{k+1}=\omega_{k+1}\left(\alpha_k\widetilde{\alpha}_k s(f(x_k)-f^*)-\frac{\widetilde{\alpha}_k^2 s^2}{2}\|\nabla f(x_k)\|^2+\frac{1}{2}\|z_{k+1}-x^*\|^2\right). \tag{46}$$

The sequence $\{\omega_k\}$ is introduced to later deal with the mismatching of coefficients on two sides of (45). Similarly as in Section 6.1, we refer to the term involving $f(x_k)-f^*$ and $\|\nabla f(x_k)\|^2$ as a potential energy, and the term $\frac{1}{2}\|z_k-x^*\|^2$ as an auxiliary energy.

Step 3. Identifying sufficient conditions for Lyapunov contraction. As expected from (45), We further bound $V_{k+1}-V_k$ and identify conditions such that a contraction inequality holds for the Lyapunov function: $V_{k+1}-V_k\leq 0$ or $V_{k+1}-V_k\lesssim -k^2s^2\|\nabla f(x_{k-1})\|^2$.

Lemma 5. *Define* **I** *and* **II** *as follows:*

$$\mathbf{I} = \omega_k \alpha_{k-1} \widetilde{\alpha}_{k-1} - \omega_{k+1} \widetilde{\alpha}_k (\alpha_k - 1),$$
$$\mathbf{II} = \omega_{k+1} \widetilde{\alpha}_k (\alpha_k - 1) \beta_{k-1} (2 - \beta_{k-1} L s) - \omega_k \widetilde{\alpha}_{k-1}^2.$$

For any $k \geq 1$ such that $\alpha_k \geq 1$, $\widetilde{\alpha}_k \geq 0$, $\omega_k \geq \omega_{k+1}$, $\mathbf{I} \geq 0$, and $\mathbf{II} \geq 0$, the Lyapunov function (46) satisfies $V_{k+1} \leq V_k$. If further $\mathbf{II} \geq Ck^2$ for a constant $C > 0$, then $V_{k+1} - V_k \leq -\frac{C}{2} k^2 s^2 \|\nabla f(x_{k-1})\|^2$.

Step 4. Verifying the contraction conditions and completing the analysis. The final step is to show that the sufficient conditions for Lyapunov contraction in Lemma 5 are satisfied under the conditions included in Theorem 4. The details are left to Appendix B.

The contraction inequality $V_{k+1} - V_k \leq 0$ leads to the convergence bound (16) for the objective gap, whereas the contraction inequality $V_{k+1} - V_k \lesssim -k^2 s^2 \|\nabla f(x_{k-1})\|^2$ leads to the inverse cubic rate (18) for the squared gradient norm. Unlike in the strongly convex setting, the stepsize s can be simply set in the range $0 < s \lesssim 1/L$ to fulfill the conditions in Lemma 5.

7 Conclusion

Our work contributes to understanding the acceleration of first-order algorithms for convex optimization. We directly formulate discrete algorithms as general as we can and establish sufficient conditions for accelerated convergence using discrete Lyapunov functions. We point out currently notable gaps between the convergence properties of the corresponding algorithms and ODEs. We propose the Hamiltonian-assisted gradient method, HAG, and demonstrate meaningful and unified interpretations of our acceleration conditions, especially those for gradient correction. Future work is needed to address various open questions, including the extent to which our sufficient conditions are necessary, improving analyses using the LMI framework, and further understanding the gradient correction and its relationship with existing explanations, such as symplectic integrations.

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A Technical details in Section 2

A.1 Reformulation of NAG-SC and TMM

First, we show that NAG-SC (3) with (7) can be reformulated as (8), i.e., (10) with $\eta = \nu = \tau = 1$. Apparently (3a) is the same as (8a). It suffices to derive (8b) and (8c) from (3). Because $x_0 = y_0 = z_0$, (7) also holds for z_0. Therefore, for $k \geq 0$ we have

$$z_k = \frac{1+\sqrt{q}}{\sqrt{q}} x_k + \left(1 - \frac{1+\sqrt{q}}{\sqrt{q}}\right) y_k,$$

which implies

$$x_k = \frac{\sqrt{q}}{1+\sqrt{q}} z_k + \frac{1}{1+\sqrt{q}} y_k, \tag{47}$$

and

$$y_k = (1+\sqrt{q}) x_k - \sqrt{q} z_k. \tag{48}$$

Eq. (47) with k replaced by $k+1$ becomes (8c). Substituting x_{k+1} from (47) into (3b), we have

$$\frac{\sqrt{q}}{1+\sqrt{q}} z_{k+1} + \frac{1}{1+\sqrt{q}} y_{k+1} = y_{k+1} + \frac{1-\sqrt{q}}{1+\sqrt{q}} (y_{k+1} - y_k),$$

which yields

$$z_{k+1} = \frac{1}{\sqrt{q}} y_{k+1} - \frac{1-\sqrt{q}}{\sqrt{q}} y_k.$$

Substituting y_{k+1} from (3a) into the above display, we have

$$z_{k+1} = \frac{1}{\sqrt{q}} (x_k - s\nabla f(x_k)) - \frac{1-\sqrt{q}}{\sqrt{q}} y_k,$$

which together with (48) gives

$$z_{k+1} = \frac{1}{\sqrt{q}}(x_k - s\nabla f(x_k)) - \frac{1-\sqrt{q}}{\sqrt{q}}\big((1+\sqrt{q})x_k - \sqrt{q}z_k\big) = \sqrt{q}\left(x_k - \frac{1}{\mu}\nabla f(x_k)\right) + (1-\sqrt{q})z_k,$$

which is (8b).

Second, we show that TMM (5), i.e., (10) with $\eta = \nu = 1$ and $\tau = 2$, can be reformulated as follows, with (49c) same as (6) for z_{k+1}:

$$y_{k+1} = x_k - s\nabla f(x_k), \tag{49a}$$

$$x_{k+1} = x_k - (2-\sqrt{q})s\nabla f(x_k) + \frac{(1-\sqrt{q})^2}{1+\sqrt{q}}(y_{k+1} - y_k), \tag{49b}$$

$$z_{k+1} = \frac{1+\sqrt{q}}{2\sqrt{q}}x_{k+1} + \left(1 - \frac{1+\sqrt{q}}{2\sqrt{q}}\right)y_{k+1}. \tag{49c}$$

Apparently (5a) is the same as (49a). It suffices to derive (49b) and (49c). Solving (5c) for z_{k+1} gives (49c). Because $x_0 = y_0 = z_0$, (49c) also holds for z_0. That is, for all $k \geq 0$,

$$z_k = \frac{1+\sqrt{q}}{2\sqrt{q}}x_k + \left(1 - \frac{1+\sqrt{q}}{2\sqrt{q}}\right)y_k,$$

which together with (5b) yields

$$\begin{aligned} z_{k+1} &= \sqrt{q}\left(x_k - \frac{1}{\mu}\nabla f(x_k)\right) + (1-\sqrt{q})\left[\frac{1+\sqrt{q}}{2\sqrt{q}}x_k + \left(1 - \frac{1+\sqrt{q}}{2\sqrt{q}}\right)y_k\right] \\ &= \frac{1+q}{2\sqrt{q}}x_k - \frac{\sqrt{q}}{\mu}\nabla f(x_k) - \frac{(1-\sqrt{q})^2}{2\sqrt{q}}y_k. \end{aligned}$$

To derive (49b), substituting the above display into (5c) and using (5a) yields

$$\begin{aligned} x_{k+1} &= \frac{2\sqrt{q}}{1+\sqrt{q}}\left(\frac{1+q}{2\sqrt{q}}x_k - \frac{\sqrt{q}}{\mu}\nabla f(x_k) - \frac{(1-\sqrt{q})^2}{2\sqrt{q}}y_k\right) + \frac{1-\sqrt{q}}{1+\sqrt{q}}y_{k+1} \\ &= \frac{1+q}{1+\sqrt{q}}x_k - \frac{2s}{1+\sqrt{q}}\nabla f(x_k) + \frac{(1-\sqrt{q})^2}{1+\sqrt{q}}(y_{k+1} - y_k) + \frac{\sqrt{q}(1-\sqrt{q})}{1+\sqrt{q}}y_{k+1} \\ &= \frac{1+q}{1+\sqrt{q}}x_k - \frac{2s}{1+\sqrt{q}}\nabla f(x_k) + \frac{(1-\sqrt{q})^2}{1+\sqrt{q}}(y_{k+1} - y_k) + \frac{\sqrt{q}(1-\sqrt{q})}{1+\sqrt{q}}(x_k - s\nabla f(x_k)) \\ &= x_k - (2-\sqrt{q})s\nabla f(x_k) + \frac{(1-\sqrt{q})^2}{1+\sqrt{q}}(y_{k+1} - y_k), \end{aligned}$$

which is (49b).

A.2 Proofs of Lemmas 2–3 in Section 6.1

Proof of Lemma 2. By (10b), we have

$$\begin{aligned} &\frac{1}{1-\nu\sqrt{q}}\frac{\mu}{2}\|z_{k+1} - x^*\|^2 - \frac{\mu}{2}\|z_k - x^*\|^2 \\ &= \frac{1}{1-\nu\sqrt{q}}\frac{\mu}{2}\left\|\nu\sqrt{q}(x_k - \frac{1}{\mu}\nabla f(x_k) - x^*) + (1-\nu\sqrt{q})(z_k - x^*)\right\|^2 - \frac{\mu}{2}\|z_k - x^*\|^2 \\ &= -\frac{\mu\nu\sqrt{q}}{2}\|z_k - x^*\|^2 + \mu\nu\sqrt{q}\langle z_k - x^*, x_k - \frac{1}{\mu}\nabla f(x_k) - x^*\rangle + \frac{\nu^2 q}{1-\nu\sqrt{q}}\frac{\mu}{2}\left\|x_k - \frac{1}{\mu}\nabla f(x_k) - x^*\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mu\nu\sqrt{q}}{2}\left\|(z_k - x^*) - (x_k - \frac{1}{\mu}\nabla f(x_k) - x^*)\right\|^2 + \frac{\nu\sqrt{q}}{1-\nu\sqrt{q}}\frac{\mu}{2}\left\|x_k - \frac{1}{\mu}\nabla f(x_k) - x^*\right\|^2 \\
&= -\frac{\mu\nu\sqrt{q}}{2}\left\|z_k - x_k + \frac{1}{\mu}\nabla f(x_k)\right\|^2 + \frac{\nu\sqrt{q}}{1-\nu\sqrt{q}}\frac{\mu}{2}\left\|x_k - \frac{1}{\mu}\nabla f(x_k) - x^*\right\|^2 .
\end{aligned} \tag{50}
$$

Next, we bound the two terms in the above display separately.

For the second term, by the μ-strong convexity of f,

$$
f^* - f(x_k) \geq \langle \nabla f(x_k), x^* - x_k\rangle + \frac{\mu}{2}\|x_k - x^*\|^2,
$$

from which we have

$$
\begin{aligned}
&\frac{\nu\sqrt{q}}{1-\nu\sqrt{q}}\frac{\mu}{2}\left\|x_k - \frac{1}{\mu}\nabla f(x_k) - x^*\right\|^2 \\
&= \frac{\nu\sqrt{q}}{1-\nu\sqrt{q}}\left(\frac{\mu}{2}\|x_k - x^*\|^2 - \langle x_k - x^*, \nabla f(x_k)\rangle + \frac{1}{2\mu}\|\nabla f(x_k)\|^2\right) \\
&\leq \frac{\nu\sqrt{q}}{1-\nu\sqrt{q}}\left(f^* - f(x_k) + \frac{1}{2\mu}\|\nabla f(x_k)\|^2\right).
\end{aligned} \tag{51}
$$

For the first term, solving (10c) for z_{k+1} we obtain

$$
z_{k+1} = x_{k+1} + \frac{1+(1-\tau)\sqrt{q}}{\tau\sqrt{q}}(x_{k+1} - y_{k+1}) = x_{k+1} + \frac{\zeta}{\tau\sqrt{q}}(x_{k+1} - y_{k+1}),
$$

which together with (10a) yields

$$
z_{k+1} = x_{k+1} + \frac{\zeta}{\tau\sqrt{q}}\left(x_{k+1} - x_k + \eta s\nabla f(x_k)\right).
$$

Hence for $k \geq 1$ we have

$$
\begin{aligned}
&-\frac{\mu\nu\sqrt{q}}{2}\left\|z_k - x_k + \frac{1}{\mu}\nabla f(x_k)\right\|^2 \\
&= -\frac{\mu\nu\sqrt{q}}{2}\left\|\frac{\zeta}{\tau\sqrt{q}}(x_k - x_{k-1}) + \frac{\zeta\eta s}{\tau\sqrt{q}}\nabla f(x_{k-1}) + \frac{1}{\mu}\nabla f(x_k)\right\|^2 \\
&= -\frac{\zeta^2\nu}{\tau^2\sqrt{q}}\frac{\mu}{2}\|x_k - x_{k-1}\|^2 - \frac{\zeta^2\eta^2\nu\sqrt{q}}{\tau^2}\frac{s}{2}\|\nabla f(x_{k-1})\|^2 - \frac{\nu\sqrt{q}}{2\mu}\|\nabla f(x_k)\|^2 \\
&\quad - \frac{\zeta^2\eta\nu\sqrt{q}}{\tau^2}\langle\nabla f(x_{k-1}), x_k - x_{k-1}\rangle - \frac{\zeta\nu}{\tau}\langle\nabla f(x_k), x_k - x_{k-1}\rangle - \frac{\zeta\eta\nu}{\tau}s\langle\nabla f(x_{k-1}), \nabla f(x_k)\rangle.
\end{aligned} \tag{52}
$$

By the L-smoothness of f, we have

$$
\begin{cases}
f(x_{k-1}) \geq f(x_k) + \langle\nabla f(x_k), x_{k-1} - x_k\rangle + \frac{1}{2L}\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2, \\
f(x_k) \leq f(x_{k-1}) + \langle\nabla f(x_{k-1}), x_k - x_{k-1}\rangle + \frac{L}{2}\|x_k - x_{k-1}\|^2.
\end{cases}
$$

Therefore,

$$
\begin{aligned}
&-\frac{\zeta^2\eta\nu\sqrt{q}}{\tau^2}\langle\nabla f(x_{k-1}), x_k - x_{k-1}\rangle - \frac{\zeta\nu}{\tau}\langle\nabla f(x_k), x_k - x_{k-1}\rangle \\
&\leq -\frac{\zeta^2\eta\nu\sqrt{q}}{\tau^2}\left(f(x_k) - f(x_{k-1}) - \frac{L}{2}\|x_k - x_{k-1}\|^2\right) - \frac{\zeta\nu}{\tau}\left(f(x_k) - f(x_{k-1}) + \frac{1}{2L}\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2\right) \\
&= \frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q})(f(x_{k-1}) - f(x_k)) + \frac{\zeta^2\eta\nu\sqrt{q}}{\tau^2}\frac{L}{2}\|x_k - x_{k-1}\|^2 - \frac{\zeta\nu}{\tau}\frac{1}{2L}\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2.
\end{aligned} \tag{53}
$$

By the cosine rule,

$$-\frac{\zeta\eta\nu s}{\tau}\langle\nabla f(x_{k-1}),\nabla f(x_k)\rangle = -\frac{\zeta\eta\nu s}{2\tau}\Big(\|\nabla f(x_{k-1})\|^2 + \|\nabla f(x_k)\|^2 - \|\nabla f(x_k) - \nabla f(x_{k-1})\|^2\Big). \quad (54)$$

Combining (52), (53) and (54) and noting $0 \le \eta s \le 1/L$ by assumption, we obtain

$$\begin{aligned}
&-\frac{\mu\nu\sqrt{q}}{2}\left\|z_k - x_k + \frac{1}{\mu}\nabla f(x_k)\right\|^2 \\
&\le -\frac{\zeta^2\mu\nu}{2\tau^2\sqrt{q}}(1-\eta Ls)\|x_k - x_{k-1}\|^2 - \frac{\zeta\nu}{2\tau L}(1-\eta Ls)\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 \\
&\quad + \frac{\zeta\nu}{\tau^2}(\tau+\zeta\eta\sqrt{q})(f(x_{k-1}) - f(x_k)) - \frac{\zeta\eta\nu s}{2\tau^2}(\tau+\zeta\eta\sqrt{q})\|\nabla f(x_{k-1})\|^2 - \frac{\nu\sqrt{q}}{2\mu\tau}(\tau+\zeta\eta\sqrt{q})\|\nabla f(x_k)\|^2 \\
&\le \frac{\zeta\nu}{\tau^2}(\tau+\zeta\eta\sqrt{q})(f(x_{k-1}) - f(x_k)) - \frac{\zeta\eta\nu s}{2\tau^2}(\tau+\zeta\eta\sqrt{q})\|\nabla f(x_{k-1})\|^2 - \frac{\nu\sqrt{q}}{2\mu\tau}(\tau+\zeta\eta\sqrt{q})\|\nabla f(x_k)\|^2.
\end{aligned} \quad (55)$$

Collecting (50), (51) and (55) completes the proof.

Proof of Lemma 3. The three sets of conditions are mutually exclusive because if $\mathbf{I} > 0$ and $\mathbf{II} > 0$, then $\mathbf{I} + \sqrt{q}\mathbf{II} > 0$ and $\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II} > 0$. Applying (42) to the Lyapunov function (43) and using the fact that $\frac{1}{2L}\|\nabla f(x_k)\|^2 \le f(x_k) - f^* \le \frac{1}{2\mu}\|\nabla f(x_k)\|^2$, we have

$$\begin{aligned}
&V_{k+1} - (1-\nu\sqrt{q})V_k \\
&\le \frac{\nu}{2\tau^2}\Big(2\sqrt{q}(\mathbf{I})(f(x_k) - f^*) + s(\mathbf{II})\|\nabla f(x_k)\|^2\Big) \\
&\le \begin{cases}
\frac{\nu}{2\tau^2}\frac{\sqrt{q}}{\mu}\Big(\mathbf{I} + \sqrt{q}\mathbf{II}\Big)\|\nabla f(x_k)\|^2, & \text{if } \mathbf{I} \ge 0; \\
\frac{\nu}{2\tau^2}\frac{\sqrt{q}}{\mu}\Big(\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II}\Big)\|\nabla f(x_k)\|^2, & \text{if } \mathbf{I} \le 0; \\
\frac{\nu}{\tau^2}\frac{\sqrt{q}L}{\mu}\Big(\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II}\Big)(f(x_k) - f^*), & \text{if } \mathbf{II} \ge 0; \\
\frac{\nu}{\tau^2}\sqrt{q}\Big(\mathbf{I} + \sqrt{q}\mathbf{II}\Big)(f(x_k) - f^*), & \text{if } \mathbf{II} \le 0.
\end{cases}
\end{aligned}$$

The rest is straightforward.

A.3 Proofs of Theorems 1–3

To prepare for the proofs of Theorems 1–3, we show that under Assumption 1, the leading coefficients in the Taylor expansions of $\mathbf{I}$ and $\mathbf{II}$ in $\sqrt{q}$ can be used to verify the conditions in Lemma 3. Throughout, the range $0 < q \lesssim \mu^2/L^2$ or $0 < q \lesssim \mu/L$ is interpreted as, respectively, $0 < q \le C_0\mu^2/L^2$ or $0 < q \le C_0\mu/L$ for a constant $C_0 > 0$.

Lemma 6. *Under Assumption 1, denote the Taylor expansions of* $\mathbf{I}$ *and* $\mathbf{II}$ *as*

$$\mathbf{I} = \sum_{n=0}^{\infty} a_n(\sqrt{q})^n, \quad \mathbf{II} = \sum_{m=0}^{\infty} b_m(\sqrt{q})^m,$$

where $\{a_n\}_{n\ge 0}$ *and* $\{b_m\}_{m\ge 0}$ *are real sequences. When* $\mathbf{I}$ *is not constant* 0*, define* N *as the minimal of* n *such that* $a_n \ne 0$*. Define* M *in a similar manner for* $\mathbf{II}$*. Then*
(i-a) If $a_N > 0$*,* $b_M < 0$ *and* $M \le N-2$*, then* $\mathbf{I} > 0$ *and* $\mathbf{I} + \sqrt{q}\mathbf{II} \le 0$ *for* $0 < q \lesssim \mu/L$*.*
(i-b) If $a_N > 0$*,* $b_M < 0$*,* $M = N-1$*, and the first nonzero element of* $\{a_n + b_{n-1}\}_{n\ge N}$ *(i.e., the first nonzero coefficient in the expansion of* $\mathbf{I} + \sqrt{q}\mathbf{II}$*) is negative or the entire sequence is* 0 *(i.e.,* $\mathbf{I} + \sqrt{q}\mathbf{II} \equiv 0$*), then*

$\mathbf{I} > 0$ *and* $\mathbf{I} + \sqrt{q}\mathbf{II} \leq 0$ *for* $0 < q \lesssim \mu/L$.
(ii-a) If $a_N < 0$, $b_M > 0$ *and* $M = N$, *then* $\mathbf{II} > 0$ *and* $\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II} \leq 0$ *for* $0 < q \lesssim \mu^2/L^2$.
(ii-b) If $a_N < 0$, $b_M > 0$ *and* $M \geq N+1$, *then* $\mathbf{II} > 0$ *and* $\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II} \leq 0$ *for* $0 < q \lesssim \mu/L$.
(iii) If $a_N < 0$ *(or* N *does not exist) and* $b_M < 0$ *(or* M *does not exist), then* $\mathbf{I} \leq 0$ *and* $\mathbf{II} \leq 0$ *for* $0 < q \lesssim \mu/L$.

Proof. With $0 < \mu/L \leq 1$, q can be made sufficiently small by picking C_0 in the range $0 < q \leq C_0\mu^2/L^2$ or $0 < q \leq C_0\mu/L$. Hence it suffices to study the leading terms of $\mathbf{I}$ and $\mathbf{II}$.

For (i-a) and (i-b), $a_N > 0$ ensures that $\mathbf{I} \sim a_N(\sqrt{q})^N > 0$. For (i-a), with $M+1 < N$ and $b_M < 0$, we have $\mathbf{I} + \sqrt{q}\mathbf{II} \sim b_M(\sqrt{q})^{M+1} < 0$. For (i-b), with $M+1 = N$, we have $\mathbf{I} + \sqrt{q}\mathbf{II} = \sum_{n=N}(a_n + b_{n-1})(\sqrt{q})^n \leq 0$ if the first nonzero element of $\{a_n + b_{n-1}\}_{n\geq N}$ is negative or the entire sequence is 0.

For (ii-a) and (ii-b), $b_M > 0$ ensures that $\mathbf{II} \sim b_M(\sqrt{q})^M > 0$. With $a_N < 0$ and $b_M > 0$, we have that for sufficiently small $q > 0$, $\mathbf{I} < \frac{a_N}{2}(\sqrt{q})^N$ and $\mathbf{II} < 2b_M(\sqrt{q})^M$. Hence, $\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II} < \frac{\mu a_N}{2L}(\sqrt{q})^N + 2b_M(\sqrt{q})^{M+1} = (\sqrt{q})^N(\frac{\mu a_N}{2L} + 2b_M(\sqrt{q})^{M+1-N})$. For (ii-a), $M = N$ and $0 < q \leq C_0^2\mu^2/L^2$ imply that $\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II} < (\sqrt{q})^N\frac{\mu}{2L}(a_N + 4C_0b_M) < 0$ by picking sufficiently small C_0 with $a_N < 0$. For (ii-b), $M \geq N+1$ and $0 < q \leq C_0\mu/L$ imply $\frac{\mu}{L}\mathbf{I} + \sqrt{q}\mathbf{II} < (\sqrt{q})^N(\frac{\mu a_N}{2L} + 2b_M(\sqrt{q})^2) \leq (\sqrt{q})^N\frac{\mu}{2L}(a_N + 4C_0b_M) < 0$ again by picking sufficiently small C_0 with $a_N < 0$.

The case (iii) is straightforward to verify. If N (or M) does not exist, then $\mathbf{I} \equiv 0$ (or $\mathbf{II} \equiv 0$).

We notice that the constant C_0 in the range of q is picked, depending only on $\{a_n\}$ and $\{b_m\}$, which are determined by the algorithm parameters $\tilde{\eta}$, $\tilde{\nu}$ and $\tilde{\tau}$. □

Next, we show Theorems 1*, 2*, and 3, where Theorems 1* and 2* are the same as Theorems 1 and 2 except with conditions (i-a) and (ii-a) replaced by (i-a*) and (ii-a*) as follows:

(i-a*) $0 < \nu_0 < \tau_0$, and $0 \leq \eta_0 < \nu_0\tau_0/2$;

(ii-a*) $0 < \nu_0 < \tau_0$, and $\eta_0 \geq \nu_0\tau_0/2$.

Conditions (i-a) and (ii-a) are the symmetrized (hence weaker) versions of (i-a*) and (ii-a*), by allowing either $0 < \nu_0 < \tau_0$ or $0 < \tau_0 < \nu_0$. To show Theorems 1*, 2*, and 3, it suffices to verify that the conditions in Lemma 3 are satisfied. In fact, the contraction inequality in Lemma 3 directly implies that that for $k \geq 1$,

$$C(f(x_k) - f^*) \;\leq\; \frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q})(1 - \eta Ls)(f(x_k) - f^*) \;\leq\; V_{k+1} \;\leq\; (1 - \nu\sqrt{q})^k V_1 \;\leq\; (1 - \frac{\nu_0}{2}\sqrt{q})^k V_1,$$

for some constant $C > 0$, where the first inequality holds by noting $\nu_0, \tau_0 > 0$ in each condition of Theorems 1*, 2*, and 3 and picking sufficiently small C_0 in $0 < q \leq C_0\mu/L$ such that, for example, $\nu \geq \nu_0/2$, $\tau_0/2 \leq \tau \leq 2\tau_0$, $\zeta \geq 1/2$, $0 \leq \eta s \leq 1/(2L)$ (i.e., $0 \leq \eta q \leq \mu/(2L)$) with $C = \nu_0/(16\tau_0)$. Moreover, by the definition of V_1,

$$\begin{aligned}
V_1 &\leq \frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q})(f(x_0) - f^*) + \frac{\mu}{2}\|z_1 - x^*\|^2 \\
&= \frac{\zeta\nu}{\tau^2}(\tau + \zeta\eta\sqrt{q})(f(x_0) - f^*) + \frac{\mu}{2}\|x_0 - x^* - \frac{\nu\sqrt{q}}{\mu}\nabla f(x_0)\|^2 \\
&\lesssim L\|x_0 - x^*\|^2 + \mu\|x_0 - x^*\|^2 + s\|\nabla f(x_0)\|^2 \\
&\leq (L + \mu + sL^2)\|x_0 - x^*\|^2 \\
&\lesssim L\|x_0 - x^*\|^2.
\end{aligned} \tag{56}$$

Then $f(x_k) - f^* \lesssim L(1 - \frac{\nu_0}{2}\sqrt{q})^k\|x_0 - x^*\|^2$, which is (12). The conditions in Lemma 2, which are required in Lemma 3, can be easily verified by noting $\nu_0, \tau_0 > 0$ and picking sufficiently small C_0. Therefore, it remains to verify the conditions involving $\mathbf{I}$ and $\mathbf{II}$ in Lemma 3.

Proof of Theorems 2* and 3. We apply Lemma 6 to verify the conditions involving **I** and **II** in Lemma 3. The Taylor expansions of **I** and **II** up to $\sqrt{q}$-terms are

$$\begin{aligned}\mathbf{I} &= \underbrace{\tau_0(\nu_0-\tau_0)}_{a_0} + \big[\tau_0(\nu_1-\tau_1)+\tau_1(\nu_0-\tau_0)+\nu_0(\eta_0-\tau_0(\tau_0-1))\big]\sqrt{q}+O(q),\\ \mathbf{II} &= \underbrace{\tau_0(\nu_0\tau_0-2\eta_0)}_{b_0} + \big[\tau_0(\nu_1\tau_1-2\eta_1)+\tau_1(\nu_0\tau_0-2\eta_0)+\eta_0\big(2\tau_0(\tau_0-1)+\nu_0\tau_0-\eta_0\big)\big]\sqrt{q}+O(q).\end{aligned} \tag{57}$$

Consider the following scenarios.

Scenario 1: $\boldsymbol{\eta_0 = 0,\ 0 < \nu_0, \tau_0}$. Then $b_M = b_0 = \nu_0\tau_0^2 > 0$. Because $M = 0$, only case (ii-a) in Lemma 6 is feasible, which holds when $N = 0$ and $a_N = a_0 = \tau_0(\nu_0 - \tau_0) < 0$, i.e., $\nu_0 < \tau_0$. To conclude, if $\eta_0 = 0$, $0 < \nu_0 < \tau_0$, then Lemma 3 holds for $0 < q \lesssim \mu^2/L^2$.

Assume η_0, ν_0 and τ_0 are all positive. We notice that $0 < \nu_0 \le \tau_0$ is necessary. Otherwise, $a_0 = \tau_0(\nu_0-\tau_0) > 0$ and $N = 0$. Then only cases (i-a) and (i-b) in Lemma 6 are feasible, which require $M \le N-1$, contradicting $N = 0$. To proceed, we further split $0 < \nu_0 \le \tau_0$ into Scenario 2 ($0 < \nu_0 < \tau_0$) and Scenario 3 ($0 < \nu_0 = \tau_0$) as below.

Scenario 2: $\boldsymbol{\eta_0 > 0,\ 0 < \nu_0 < \tau_0}$. Then $a_0 = \tau_0(\nu_0 - \tau_0) < 0$ and $N = 0$. We notice that either M does not exist (i.e., $\mathbf{II} \equiv 0$), or M exists and has $b_M > 0$ or $b_M < 0$ for some $M \ge N = 0$. Therefore, one of case (ii-a), case (ii-b), and case (iii) in Lemma 6 is valid. Case (ii-a) holds if and only if $M = N = 0$ and $b_M = b_0 = \tau_0(\nu_0\tau_0 - 2\eta_0) > 0$, i.e., $\eta_0 < \frac{\nu_0\tau_0}{2}$. Therefore, if $\eta_0 \ge \frac{\nu_0\tau_0}{2}$, case (ii-b) or case (iii) holds. To conclude, if $0 < \eta_0 < \frac{\nu_0\tau_0}{2}$ and $0 < \nu_0 < \tau_0$, then Lemma 3 holds for $0 < q \lesssim \mu^2/L^2$. If $\eta_0 \ge \frac{\nu_0\tau_0}{2}$ and $0 < \nu_0 < \tau_0$, then the range of q is relaxed to $0 < q \lesssim \mu/L$.

Scenario 3: $\boldsymbol{\eta_0 > 0,\ 0 < \nu_0 = \tau_0}$. In this scenario, (57) reduces to

$$\begin{aligned}\mathbf{I} &= \underbrace{\tau_0\big[\nu_1-\tau_1+\eta_0-\tau_0(\tau_0-1)\big]}_{a_1}\sqrt{q}+O(q),\\ \mathbf{II} &= \underbrace{\tau_0(\tau_0^2-2\eta_0)}_{b_0} + \underbrace{\big[\tau_0(\nu_1\tau_1-2\eta_1)+\tau_1(\tau_0^2-2\eta_0)+\eta_0\big(\tau_0(3\tau_0-2)-\eta_0\big)\big]}_{b_1}\sqrt{q}+O(q).\end{aligned}$$

Then $a_0 = 0$ and hence $N \ge 1$. We point out that $\eta_0 \ge \frac{\tau_0^2}{2}$ is necessary. Otherwise, $b_0 = \tau_0(\tau_0^2 - 2\eta_0) > 0$ and hence $M = 0$. Then only case (ii-a) or (ii-b) in Lemma 6 is possible, which requires $M \ge N$. But this contradicts the fact that $N \ge 1$ and $M = 0$. To proceed, we split $\eta_0 \ge \frac{\tau_0^2}{2}$ into Scenario 3.1 ($\eta_0 > \frac{\tau_0^2}{2}$) and 3.2 ($\eta_0 = \frac{\tau_0^2}{2}$) based on η_0.

Scenario 3.1: $\boldsymbol{\eta_0 > \frac{\tau_0^2}{2},\ 0 < \nu_0 = \tau_0}$. Then $b_0 = \tau_0(\tau_0^2 - 2\eta_0) < 0$ and $M = 0$. Case (i-a), case (i-b) and case (iii) in Lemma 6 are each possible. We consider several special cases involving only η_1, ν_1 and τ_1.

- For case (i-a) to hold, even higher-order coefficients are needed, and we skip this case.
- For case (i-b) to hold, let $N = M + 1 = 1$ and $a_N = a_1 = \tau_0\big[\nu_1 - \tau_1 + \eta_0 - \tau_0(\tau_0 - 1)\big] > 0$, which is equivalent to $\nu_1 - \tau_1 + \eta_0 - \tau_0(\tau_0 - 1) > 0$. Moreover, let $a_N + b_{N-1} = a_1 + b_0 = \tau_0\big[\nu_1 - \tau_1 + \eta_0 - (\tau_0 - 1)\tau_0\big] + \tau_0(\tau_0^2 - 2\eta_0) = \tau_0[\nu_1 - \tau_1 + \tau_0 - \eta_0] < 0$, which is equivalent to $\nu_1 - \tau_1 < \eta_0 - \tau_0$.
- For case (iii) to hold, let $a_1 = \tau_0\big[\nu_1 - \tau_1 + \eta_0 - \tau_0(\tau_0 - 1)\big] < 0$, which is equivalent to $\nu_1 - \tau_1 < \tau_0(\tau_0 - 1) - \eta_0$.

To conclude, if $\eta_0 > \frac{\tau_0^2}{2}$, $0 < \nu_0 = \tau_0$, and either $\tau_0(\tau_0-1)-\eta_0 < \nu_1-\tau_1 < \eta_0-\tau_0$ or $\nu_1-\tau_1 < \tau_0(\tau_0-1)-\eta_0$, then Lemma 3 holds for $0 < q \lesssim \mu/L$.

Scenario 3.2: $\boldsymbol{\eta_0 = \frac{\tau_0^2}{2}}$, $\boldsymbol{0 < \nu_0 = \tau_0}$. Then (57) further reduces to

$$\mathbf{I} = \underbrace{\tau_0\big(\nu_1 - \tau_1 + \tau_0(1 - \frac{\tau_0}{2})\big)}_{a_1} \sqrt{q} + O(q),$$

$$\mathbf{II} = \underbrace{\tau_0\big[\nu_1\tau_1 - 2\eta_1 + \frac{\tau_0^2}{2}(\frac{5}{2}\tau_0 - 2)\big]}_{b_1} \sqrt{q} + O(q).$$

Then $a_0 = b_0 = 0$. We consider several special cases where (η_1, ν_1, τ_1) are enough to determine the convergence. Let $a_1 = \tau_0\left(\nu_1 - \tau_1 + \tau_0(1 - \frac{\tau_0}{2})\right) < 0$, which is equivalent to $\nu_1 - \tau_1 + \tau_0(1 - \frac{\tau_0}{2}) < 0$. Then $N = 1$. We distinguish three cases by the sign of $b_1 = \tau_0\big[\nu_1\tau_1 - 2\eta_1 + \frac{\tau_0^2}{2}(\frac{5}{2}\tau_0 - 2)\big]$.

- If $b_1 < 0$, then $M = 1$ and $b_M < 0$, and hence case (iii) holds.
- If $b_1 > 0$, then $M = N = 1$ and $b_M > 0$, and hence case (ii-a) holds. Note that the range for q is $0 < q \lesssim \mu^2/L^2$.
- If $b_1 = 0$, then $M \geq 2 = N + 1$ or M does not exist. If M does not exist, case (iii) is valid. If M exists, then either $b_M > 0$ or $b_M < 0$. The former satisfies case (ii-b) and the latter satisfies case (iii).

Collecting the results of Scenario 1, 2 and 3 concludes the proof for Theorem 2* and 3.

Proof of Theorems 1*. When (η, ν, τ) are constants, the above analysis still holds, but we can unfold Scenario 3 without imposing strong conditions on (η_1, ν_1, τ_1). We continue with Scenario 3, that is, assume $\eta \geq \frac{\tau^2}{2}$ and $0 < \nu = \tau$. Then we have the Taylor expansions in finite terms:

$$\begin{aligned}
\mathbf{I} &= \tau(\eta - (\tau - 1)\tau)\sqrt{q} - 2\eta\tau(\tau - 1)q + \eta\tau(\tau - 1)^2 q^{\frac{3}{2}}, \\
\mathbf{II} &= \tau(\tau^2 - 2\eta) + \eta[\tau(3\tau - 2) - \eta]\sqrt{q} + \eta(\tau - 1)(2\eta - \tau^2)q - \eta^2(\tau - 1)^2 q^{\frac{3}{2}}, \\
\mathbf{I} + \sqrt{q}\mathbf{II} &= \tau(\tau - \eta)\sqrt{q} + \eta(\tau^2 - \eta)q + \eta(\tau - 1)(2\eta - \tau)q^{\frac{3}{2}} - \eta^2(\tau - 1)^2 q^2.
\end{aligned} \tag{58}$$

To proceed, we split Scenario 3 into Scenario 3.1* and 3.2* based on η.

Scenario 3.1*: $\boldsymbol{\eta > \frac{\tau^2}{2}}$, $\boldsymbol{0 < \nu = \tau}$. Then $b_0 = \tau(\tau^2 - 2\eta) < 0$, and $M = 0$.

- $\tau > 2$ (hence $\tau < \frac{\tau^2}{2} < \tau(\tau - 1)$).
  - If $\eta > \tau(\tau - 1)$, then $a_1 = \tau(\eta - \tau(\tau - 1)) > 0$ and $N = M + 1 = 1$. Moreover, $a_1 + b_0 = \tau(\tau - \eta) < 0$, and hence case (i-b) in Lemma 6 holds.
  - If $\eta = \tau(\tau - 1)$, then $a_1 = \tau(\eta - \tau(\tau - 1)) = 0$, and $a_2 = -2\eta\tau(\tau - 1) = -2\tau^2(\tau - 1)^2 < 0$. Hence case (iii) in Lemma 6 holds.
  - If $\frac{\tau^2}{2} < \eta < \tau(\tau - 1)$, then $a_1 = \tau(\eta - \tau(\tau - 1)) < 0$. Hence case (iii) in Lemma 6 holds.
- $\tau = 2$ (hence $\tau = \frac{\tau^2}{2} = \tau(\tau - 1) = 2$). Then $\mathbf{I} = 2(\eta - 2)\sqrt{q} + O(q)$ and $\mathbf{I} + \sqrt{q}\mathbf{II} = 2(2 - \eta)\sqrt{q} + O(q)$. By $\eta > \frac{\tau^2}{2} = 2$, we have $a_1 = 2(\eta - 2) > 0$ and $a_1 + b_0 = 2(2 - \eta) < 0$. Case (i-b) holds.
- $0 < \tau < 2$ (hence $\tau > \frac{\tau^2}{2} > \tau(\tau - 1)$). Then $\eta > \frac{\tau^2}{2}$ implies that $\eta > \tau(\tau - 1)$. Hence $a_1 = \tau(\eta - \tau(\tau - 1)) > 0$ and $N = M + 1 = 1$. Only case (i-b) in Lemma 6 is possible, which requires the first non-zero coefficient of $\mathbf{I} + \sqrt{q}\mathbf{II}$ in (58) to be negative, or $\mathbf{I} + \sqrt{q}\mathbf{II} \equiv 0$.
  - If $\eta > \tau$, then $\mathbf{I} + \sqrt{q}\mathbf{II} = \tau(\tau - \eta)\sqrt{q} + O(q) < 0$.
  - If $\eta = \tau$, then $\mathbf{I} + \sqrt{q}\mathbf{II} = \tau^2(\tau - 1)q + \tau^2(\tau - 1)q^{\frac{3}{2}} - \tau^2(\tau - 1)^2 q^2$. Hence for $0 < \tau \leq 1$, $\mathbf{I} + \sqrt{q}\mathbf{II} \leq 0$.

– If $\frac{\tau^2}{2} < \eta < \tau$, then $\mathbf{I} + \sqrt{q}\mathbf{II} = \tau(\tau - \eta)\sqrt{q} + O(q) > 0$, which violates case (i-b).

Scenario 3.2*: $\boldsymbol{\eta = \frac{\tau^2}{2}}$, $\mathbf{0 < \nu = \tau}$. Then (58) only involves τ: $\mathbf{I} = \tau^2(1-\frac{\tau}{2})\sqrt{q} - \tau^3(\tau-1)q + \frac{1}{2}\tau^3(\tau-1)^2 q^{\frac{3}{2}}$, $\mathbf{II} = \frac{1}{2}\tau^3(\frac{5}{2}\tau - 2)\sqrt{q} - \frac{1}{4}\tau^4(\tau-1)^2 q^{\frac{3}{2}}$, and $\mathbf{I} + \sqrt{q}\mathbf{II} = \tau^2(1-\frac{\tau}{2})\sqrt{q} + O(q)$.

- If $\tau > 2$, then $a_N = a_1 = \tau^2(1-\frac{\tau}{2}) < 0$, and $b_M = b_1 = \frac{1}{2}\tau^3(\frac{5}{2}\tau - 2) > 0$. Hence case (ii-a) in Lemma 6 holds. Note that the range of q is $0 < q \lesssim \mu^2/L^2$.
- If $\tau = 2$, then $\mathbf{I} = -8q + 4q^{\frac{3}{2}}$ and $\mathbf{II} = 12\sqrt{q} - 4q^{\frac{3}{2}}$. Hence $a_N = a_2 = -8 < 0$ and $b_M = b_1 = 12 > 0$, with $1 = M < N = 2$, which contradicts case (ii-a) and (ii-b) requiring $M \geq N$. This case is invalid.
- If $0 < \tau < 2$, then $a_N = a_1 = \tau^2(1-\frac{\tau}{2}) > 0$. Only case (i-a) or (i-b) in Lemma 6 is possible. But then $\mathbf{I} + \sqrt{q}\mathbf{II} = \tau^2(1-\frac{\tau}{2})\sqrt{q} + O(q) > 0$, contradicting the conclusion in (i-a) and (i-b). This case is invalid.

Collecting the results in Scenario 3.1* and 3.2* concludes the proof for Theorem 1*.

Finally, we derive Theorems 1 and 2 from Theorems 1* and 2* by exploiting symmetrization in the case of $\nu_0 \neq \tau_0$ due to Lemma 7 in Appendix A.4.

Proof of Theorems 1 and 2. It suffices to only deal with conditions (i-a) and (ii-a) in Theorem 2, which directly implies the conclusions from conditions (i-a) and (ii-a) in Theorem 1.

For $\eta \sim \eta_0 \geq 0$, $\nu \sim \nu_0 > 0$, $\tau \sim \tau_0 > 0$, and $\nu_0 \neq \tau_0$, by Lemma 7, algorithm (10) can be first put into (59), with R_1, R_2, R_3 and $h_1 = \frac{\zeta\eta + \nu\tau}{1+\sqrt{q}}$. Next, we keep η_0 and the remainder terms, but exchange the role of ν_0 and τ_0 by setting $\bar{\nu}_0 = \tau_0$ and $\bar{\tau}_0 = \nu_0$ and then translate (59) back to (10) with new parameters $\bar{\eta} \sim \eta_0$, $\bar{\nu} \sim \bar{\nu}_0 = \tau_0$ and $\bar{\tau} \sim \bar{\tau}_0 = \nu_0$ (ζ is also translated to the new $\bar{\zeta} = 1 + (1-\bar{\tau})\sqrt{q}$), and a possibly nonzero $h_2 = \frac{\bar{\zeta}\bar{\eta} + \bar{\nu}\bar{\tau} - (\zeta\eta + \nu\tau)}{\bar{\tau}(1-\bar{\nu}\sqrt{q})}$ in z_0. In other words, the original algorithm (10) can be reformulated such that the leading constants in ν and τ are exchanged and the algorithm now starts with x_0 and possibly $z_0 \neq x_0$.

For the reformulated algorithm, the proof of Theorem 2* remains valid except for the bound of V_1 in (56) with the new z_0. Nevertheless, an inspection of (56) reveals that $V_1 \lesssim L\|x_0 - x^*\|^2$ still holds because h_2 can be easily shown to be bounded. Hence, the desired result follows by symmetrizing the conditions (i-a*) and (ii-a*) in Theorem 2*.

A.4 Proof of Corollary 1

To facilitate interpretation of (10) and prepare for the proof of Corollary 1, we study the single-variable form of (10). The following lemma shows that the two forms can be transformed into each other, provided that the leading constants in ν and τ differ from each other. The initial points need to be aligned because (10) starts from x_0 and z_0 while (59) starts from x_0 and x_1.

Lemma 7. *Let $\zeta = 1 + (1-\tau)\sqrt{q}$ as in Lemma 2. (i) Algorithm (10) with tuning parameters η, ν and τ under Assumption 1 admits the single-variable form (11), which can be expressed as*

$$x_{k+1} = x_k - (\nu_0\tau_0 + R_1)s\nabla f(x_k) + (1 - (\nu_0 + \tau_0)\sqrt{q} + R_2)(x_k - x_{k-1}) - (\eta_0 + R_3)s(\nabla f(x_k) - \nabla f(x_{k-1})), \tag{59}$$

with x_0, $x_1 = x_0 - h_1 s\nabla f(x_0)$, where $R_1 = O(\sqrt{q})$, $R_2 = O(q)$, $R_3 = O(\sqrt{q})$ and $h_1 = \frac{\zeta\eta + \nu\tau}{1+\sqrt{q}}$ are analytic functions of $\sqrt{q}$ around 0. (ii) Conversely, given any analytic functions of $\sqrt{q}$: $R_1 = O(\sqrt{q})$, $R_2 = O(q)$, $R_3 = O(\sqrt{q})$, h_1, and three scalars $\eta_0 \geq 0$, $\nu_0, \tau_0 > 0$, $\nu_0 \neq \tau_0$, there exist $\eta \sim \eta_0$, $\nu \sim \nu_0$ and $\tau \sim \tau_0$ satisfying Assumption 1 such that (59) starting from x_0 and $x_1 = x_0 - h_1 s\nabla f(x_0)$ is equivalent to (10) starting from x_0 and $z_0 = x_0 + h_2\frac{\sqrt{q}}{\mu}\nabla f(x_0)$ where

$$h_2 = \frac{\zeta\eta + \nu\tau - (1+\sqrt{q})h_1}{\tau(1-\nu\sqrt{q})}.$$

Proof. First, we show that (10) admits the single-variable form (11), i.e.,

$$x_{k+1} = x_k - \frac{\nu(\tau+\zeta\eta\sqrt{q})}{1+\sqrt{q}}s\nabla f(x_k) + \frac{\zeta(1-\nu\sqrt{q})}{1+\sqrt{q}}(x_k - x_{k-1}) - \frac{\zeta\eta(1-\nu\sqrt{q})}{1+\sqrt{q}}s(\nabla f(x_k) - \nabla f(x_{k-1})),$$

for $k \geq 1$ starting from x_0 and $x_1 = x_0 - \frac{\zeta\eta+\nu\tau}{1+\sqrt{q}}s\nabla f(x_0)$. The calculation for x_1 is straightforward and hence omitted. To show (11), from (10c) and (10a) we have for $k \geq 0$,

$$z_{k+1} = \frac{1+\sqrt{q}}{\tau\sqrt{q}}x_{k+1} - \frac{\zeta}{\tau\sqrt{q}}y_{k+1} = \frac{1+\sqrt{q}}{\tau\sqrt{q}}x_{k+1} - \frac{\zeta}{\tau\sqrt{q}}(x_k - \eta s\nabla f(x_k)).$$

Substituting the above display with subscript $k+1$ and k for $k \geq 1$ into (10b), we have

$$\begin{aligned}&\frac{1+\sqrt{q}}{\tau\sqrt{q}}x_{k+1} - \frac{\zeta}{\tau\sqrt{q}}(x_k - \eta s\nabla f(x_k))\\ &\qquad = \nu\sqrt{q}(x_k - \frac{1}{\mu}\nabla f(x_k)) + (1-\nu\sqrt{q})\left(\frac{1+\sqrt{q}}{\tau\sqrt{q}}x_k - \frac{\zeta}{\tau\sqrt{q}}(x_{k-1} - \eta s\nabla f(x_{k-1}))\right).\end{aligned}$$

After rearrangement we obtain

$$\begin{aligned}\frac{1+\sqrt{q}}{\tau\sqrt{q}}x_{k+1} = \left(\frac{1+\sqrt{q}}{\tau\sqrt{q}} + \frac{\zeta(1-\nu\sqrt{q})}{\tau\sqrt{q}}\right)x_k - \frac{\zeta(1-\nu\sqrt{q})}{\tau\sqrt{q}}x_{k-1}\\ - \frac{\zeta\eta+\nu\tau}{\tau\sqrt{q}}s\nabla f(x_k) + \frac{\zeta\eta(1-\nu\sqrt{q})}{\tau\sqrt{q}}s\nabla f(x_{k-1}).\end{aligned}$$

Solving for x_{k+1} yields (11). Expanding the coefficients in (11) yields (59).

Second, we prove the reverse statement. Given $R_1 = O(\sqrt{q})$, $R_2 = O(q)$, $R_3 = O(\sqrt{q})$ and the leading constants η_0, ν_0 and τ_0, we determine η, ν and τ by solving the following equations

$$\begin{aligned}&\nu(\tau+\zeta\eta\sqrt{q}) = (\nu_0\tau_0 + R_1)(1+\sqrt{q}) = g_1 \sim \nu_0\tau_0 + O(\sqrt{q}),\\ &\zeta(1-\nu\sqrt{q}) = (1-(\nu_0+\tau_0)\sqrt{q} + R_2)(1+\sqrt{q}) = g_2 \sim 1 - (\nu_0+\tau_0-1)\sqrt{q} + O(q),\\ &\zeta\eta(1-\nu\sqrt{q}) = (\eta_0 + R_3)(1+\sqrt{q}).\end{aligned}$$

Note that g_1 and g_2 above are known. Solving the equations we obtain

$$\eta = \frac{\eta_0 + R_3}{1-(\nu_0+\tau_0)\sqrt{q} + R_2} \sim \eta_0,$$

and ν depending on τ as

$$\nu = \frac{g_1}{\tau+\zeta\eta\sqrt{q}} \sim \frac{\nu_0\tau_0}{\tau_0} = \nu_0,$$

and τ as a root for the quadratic equation

$$\alpha_2 \cdot \tau^2 + \alpha_1 \cdot \tau + \alpha_0 = 0,$$

where

$$\begin{aligned}&\alpha_2 = -\sqrt{q}(1-\eta q),\\ &\alpha_1 = 1 - g_2 + \sqrt{q} + (g_1 - 2\eta + \eta g_2)q - 2\eta q^{\frac{3}{2}} = \sqrt{q}(\nu_0 + \tau_0 + O(\sqrt{q})),\\ &\alpha_0 = (\eta - g_1 - \eta g_2)\sqrt{q} + (2\eta - g_1 - \eta g_2)q + \eta q^{\frac{3}{2}} = \sqrt{q}(-\nu_0\tau_0 + O(\sqrt{q})).\end{aligned}$$

The discriminant is $\Delta = \alpha_1^2 - 4\alpha_2 \cdot \alpha_0 = q[(\nu_0-\tau_0)^2 + O(\sqrt{q})]$. For $\nu_0 \neq \tau_0 > 0$, the root

$$\tau = \frac{-\alpha_1 \pm \sqrt{\Delta}}{2\cdot\alpha_2} = \frac{\alpha_1/\sqrt{q} \pm \sqrt{\Delta/q}}{-2\cdot\alpha_2/\sqrt{q}} = \frac{\nu_0+\tau_0+O(\sqrt{q}) \pm \sqrt{(\nu_0-\tau_0)^2 + O(\sqrt{q})}}{2(1-\eta q)} \sim \frac{\nu_0+\tau_0 \pm |\nu_0-\tau_0|}{2}$$

is well-defined for small q. The sign is determined by the sign of $\nu_0 - \tau_0$ to make $\tau \sim \tau_0$. An inspection of the expressions suggests that η, ν and τ are all analytic functions of $\sqrt{q}$.

Lastly, from the above calculation, the updating formula for x_k has been matched between (10) and (59), starting from x_2. The initial points can be aligned by picking suitable z_0 such that x_1 from (10) is exactly $x_0 - h_1 s\nabla f(x_0)$. The calculation is straightforward and hence omitted. □

Interestingly, the leading coefficients in the single-variable form (59) for (10) are symmetric with regard to ν_0 and τ_0. This suggests that the convergence properties of (10) should also be symmetric in ν_0 and τ_0. Indeed, such symmetrization in the case of $\nu_0 \neq \tau_0$ is exploited in our derivation of Theorems 1–2 from Theorems 1*–2* in Appendix A.3.

Corollary 1 is derived by translating the conclusions from Theorem 2 in the case of $\nu_0 \neq \tau_0$ in terms of the leading coefficients in a single-variable form.

Proof of Corollary 1. Let $c_1 = \nu_0 + \tau_0$, $c_0 = \nu_0\tau_0$, $c_2\sqrt{c_0} - c_0/2 = \eta_0$ and solve for η_0, ν_0 and τ_0. The conditions $c_1^2 > 4c_0$ and $c_0, c_1 > 0$ ensure that ν_0 and τ_0 exist, satisfying $\nu_0, \tau_0 > 0$ and $\nu_0 \neq \tau_0$. The conditions on c_2 can be directly translated into the conditions on η_0. The desired result follows by applying Lemma 7 and Theorem 2.

B Technical details in Section 3

B.1 Derivation of three-variable form (22)

By comparing (22) with (20), the update of y_{k+1} is the same. Rearranging $z_{k+1} = \alpha_{k+1}x_{k+1} + (1-\alpha_{k+1})y_{k+1}$, we obtain the update of x_{k+1} in (22c). It suffices to prove the update of z_{k+1} in (22b). With (20) in place, we have

$$\begin{aligned}
z_{k+1} &= \alpha_{k+1}x_{k+1} + (1-\alpha_{k+1})y_{k+1} \\
&= \alpha_{k+1}[x_k - \gamma_k s\nabla f(x_k) + \sigma_{k+1}(y_{k+1} - y_k)] + (1-\alpha_{k+1})y_{k+1} \\
&= \alpha_{k+1}(x_k - \gamma_k s\nabla f(x_k)) + (\alpha_k - 1)(y_{k+1} - y_k) + (1-\alpha_{k+1})y_{k+1} \\
&= \alpha_{k+1}(x_k - \gamma_k s\nabla f(x_k)) + (\alpha_k - \alpha_{k+1})y_{k+1} + (1-\alpha_k)y_k \\
&= \alpha_{k+1}(x_k - \gamma_k s\nabla f(x_k)) + (\alpha_k - \alpha_{k+1})(x_k - \beta_k s\nabla f(x_k)) + (1-\alpha_k)y_k \\
&= \alpha_k x_k + (1-\alpha_k)y_k - (\gamma_k\alpha_{k+1} + \beta_k(\alpha_k - \alpha_{k+1}))s\nabla f(x_k) \\
&= z_k - \widetilde{\alpha}_k s\nabla f(x_k).
\end{aligned}$$

B.2 Proofs of Lemmas 4 and 5 in Section 6.2

Proof of Lemma 4. Using (22b), we have

$$\begin{aligned}
\frac{1}{2}\|z_{k+1} - x^*\|^2 - \frac{1}{2}\|z_k - x^*\|^2 &= \frac{1}{2}\|z_k - x^* - \widetilde{\alpha}_k s\nabla f(x_k)\|^2 - \frac{1}{2}\|z_k - x^*\|^2 \\
&= -\widetilde{\alpha}_k s\langle z_k - x^*, \nabla f(x_k)\rangle + \frac{\widetilde{\alpha}_k^2 s^2}{2}\|\nabla f(x_k)\|^2.
\end{aligned}$$

Substituting $z_k = x_k + (\alpha_k - 1)(x_k - y_k)$ into the above display, we have

$$\begin{aligned}
&\frac{1}{2}\|z_{k+1} - x^*\|^2 - \frac{1}{2}\|z_k - x^*\|^2 \qquad (60) \\
&= -\widetilde{\alpha}_k s\langle x_k - x^*, \nabla f(x_k)\rangle - \widetilde{\alpha}_k(\alpha_k - 1)s\langle x_k - y_k, \nabla f(x_k)\rangle + \frac{\widetilde{\alpha}_k^2 s^2}{2}\|\nabla f(x_k)\|^2 \\
&\leq -\widetilde{\alpha}_k s(f(x_k) - f^*) - \widetilde{\alpha}_k(\alpha_k - 1)s(f(x_k) - f(y_k)) + \frac{\widetilde{\alpha}_k^2 s^2}{2}\|\nabla f(x_k)\|^2
\end{aligned}$$

$$= -\alpha_k \widetilde{\alpha}_k s(f(x_k) - f^*) + \widetilde{\alpha}_k(\alpha_k - 1)s(f(y_k) - f^*) + \frac{\widetilde{\alpha}_k^2 s^2}{2}\|\nabla f(x_k)\|^2,$$

where the inequality holds because by the convexity of f and the assumption that $\widetilde{\alpha}_k \geq 0$ and $\alpha_k \geq 1$, we have $\langle x_k - x^*, \nabla f(x_k)\rangle \geq f(x_k) - f^*$ and $\langle x_k - y_k, \nabla f(x_k)\rangle \geq f(x_k) - f(y_k)$. When $k \geq 1$, by the L-smoothness of f and (22a), we have

$$\begin{aligned} f(y_k) - f^* &\leq f(x_{k-1}) - f^* + \langle \nabla f(x_{k-1}), y_k - x_{k-1}\rangle + \frac{L}{2}\|y_k - x_{k-1}\|^2 \\ &= f(x_{k-1}) - f^* - (2 - \beta_{k-1}Ls)\frac{\beta_{k-1}s}{2}\|\nabla f(x_{k-1})\|^2. \end{aligned} \tag{61}$$

Combining (60) and (61) yields (45), which completes the proof.

Proof of Lemma 5. By Lemma 4, for any $k \geq 1$ such that $\alpha_k \geq 1$, $\widetilde{\alpha}_k \geq 0$, we have

$$\begin{aligned} V_{k+1} - V_k \leq &-\frac{1}{2}(\omega_k - \omega_{k+1})\|z_k - x^*\|^2 \\ &- \underbrace{(\omega_k \alpha_{k-1}\widetilde{\alpha}_{k-1} - \omega_{k+1}\widetilde{\alpha}_k(\alpha_k - 1))}_{\mathrm{I}} s(f(x_{k-1}) - f^*) \\ &- \underbrace{\left(\omega_{k+1}\widetilde{\alpha}_k(\alpha_k - 1)\beta_{k-1}(2 - \beta_{k-1}Ls) - \omega_k \widetilde{\alpha}_{k-1}^2\right)}_{\mathrm{II}} \frac{s^2}{2}\|\nabla f(x_{k-1})\|^2. \end{aligned} \tag{62}$$

If $\omega_k \geq \omega_{k+1}$, then $-(\omega_k - \omega_{k+1})\|z_k - x^*\|^2/2 \leq 0$. The rest is straightforward.

B.3 Proof of Theorem 4

To prepare for the proof for Theorem 4, we provide a simple bound which will be used in the last step of the proof.

Lemma 8. *Let $f \in \mathcal{F}_L^1$. When $0 < s \leq C_0/L$ for some constant $C_0 > 0$, the iterates of (22) satisfy that for any fixed K, there exists a constant C such that $s(f(x_K) - f^*)$, $s^2\|\nabla f(x_K)\|^2$ and $\|z_{K+1} - x^*\|^2$ are upper-bounded by $C\|x_0 - x^*\|^2$. The constant C depends only on C_0, K and the algorithm parameters $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$.*

Proof. For notational simplicity, we assume $C_0 = 1$ so that $0 < Ls \leq 1$. First, by convexity of f, we have

$$s(f(x_K) - f^*) \leq \frac{Ls}{2}\|x_K - x^*\|^2 \leq \frac{1}{2}\|x_K - x^*\|^2.$$

Second, by the L-smoothness, we have

$$s^2\|\nabla f(x_K)\|^2 = s^2\|\nabla f(x_K) - \nabla f(x^*)\|^2 \leq (Ls)^2\|x_K - x^*\|^2 \leq \|x_K - x^*\|^2.$$

Third, by (22b) and the Cauchy–Schwartz inequality, we have

$$\frac{1}{2}\|z_{K+1} - x^*\|^2 = \frac{1}{2}\|z_K - x^* - \widetilde{\alpha}_K s\nabla f(x_K)\|^2 \leq \|z_K - x^*\|^2 + \widetilde{\alpha}_K^2 s^2\|\nabla f(x_K)\|^2,$$

which together with the preceding bound on $s^2\|\nabla f(x_K)\|^2$ yields

$$\frac{1}{2}\|z_{K+1} - x^*\|^2 \leq \|z_K - x^*\|^2 + \widetilde{\alpha}_K^2\|x_K - x^*\|^2.$$

Because $z_0 = x_0$, it suffices to bound $\|x_k - x^*\|^2$ by $\|x_0 - x^*\|^2$ up to a constant for general $k \geq 1$. The fact that (22) is a first-order method implies that for each $k \geq 1$, there exist scalars $\{c_{k,i}\}$ (depending only on the algorithm parameters) for $i = 0, \ldots, k-1$ such that

$$x_k = x_{k-1} + \sum_{i=0}^{k-1} c_{k,i} s\nabla f(x_i).$$

Then by the Cauchy–Schwartz inequality,

$$\begin{aligned}\|x_k - x^*\|^2 &= \|x_{k-1} - x^* + \sum_{i=0}^{k-1} c_{k,i} s \nabla f(x_i)\|^2 \\ &\le k\left(\|x_{k-1} - x^*\|^2 + \sum_{i=0}^{k-1} c_{k,i}^2 s^2 \|\nabla f(x_i)\|^2\right) \\ &\le k\left(\|x_{k-1} - x^*\|^2 + \sum_{i=0}^{k-1} c_{k,i}^2 \|x_i - x^*\|^2\right).\end{aligned}$$

Because K is fixed, the proof is completed by applying the preceding bound for $1 \le k \le K$. □

With Lemma 5 and Lemma 8 in place, we are ready to prove Theorem 4.

Proof of Theorem 4. By the definition of $\widetilde{\alpha}_k$ and condition (i)-(ii), we have

$$\lim_{k\to\infty} \frac{\widetilde{\alpha}_k}{\alpha_k} = \lim_{k\to\infty} \beta_k + (\gamma_k - \beta_k)\frac{\alpha_{k+1}}{\alpha_k} = \gamma > 0.$$

Then $\alpha_k = \Omega(k)$ implies that $\widetilde{\alpha}_k = \Omega(k)$. By Lemma 5, the key is to bound **I** and **II** from below for all k sufficiently large. The beginning V_k can be dealt with by the simple bound in Lemma 8 so they do not affect the convergence. To proceed, we consider two choices of $\{\omega_k\}$ depending on the monotonicity of $\{\widetilde{\alpha}_k/\alpha_k\}$. Note that $\{\omega_k\}$ needs to be non-increasing, to ensure that the first term $-(\omega_k - \omega_{k+1})\|z_k - x^*\|^2/2$ on the right-hand-side of (62) is always non-positive.

Choice 1: $\boldsymbol{\omega_k \equiv 1}$. We pick w_k simply as constant 1 when $\{\widetilde{\alpha}_k/\alpha_k\}$ is non-increasing. As for **I**, by condition (ii) we have

$$\mathbf{I} = \alpha_{k-1}\widetilde{\alpha}_{k-1} - \widetilde{\alpha}_k(\alpha_k - 1) = \alpha_{k-1}^2\left(\frac{\widetilde{\alpha}_{k-1}}{\alpha_{k-1}} - \frac{\alpha_k(\alpha_k-1)}{\alpha_{k-1}^2}\cdot\frac{\widetilde{\alpha}_k}{\alpha_k}\right) \ge \alpha_{k-1}^2\left(\frac{\widetilde{\alpha}_{k-1}}{\alpha_{k-1}} - \frac{\widetilde{\alpha}_k}{\alpha_k}\right) \ge 0.$$

As for **II**, we have

$$\mathbf{II} = \widetilde{\alpha}_k(\alpha_k - 1)\beta_{k-1}(2 - \beta_{k-1}Ls) - \widetilde{\alpha}_{k-1}^2 = \alpha_k(\alpha_k-1)\left(\frac{\widetilde{\alpha}_k}{\alpha_k}\beta_{k-1}(2-\beta_{k-1}Ls) - \frac{\widetilde{\alpha}_{k-1}^2}{\alpha_k(\alpha_k-1)}\right).$$

For $0 < s \le (2-\gamma/\beta)/(2\beta L)$ with $\beta > \gamma/2 > 0$,

$$\mathbf{II} \ge \alpha_k(\alpha_k-1)\left[\frac{\widetilde{\alpha}_k}{\alpha_k}\beta_{k-1}\left(2 - \frac{\beta_{k-1}}{2\beta}(2-\frac{\gamma}{\beta})\right) - \frac{\widetilde{\alpha}_{k-1}^2}{\alpha_k(\alpha_k-1)}\right],$$

where the limit of the term in the square brackets is

$$\lim_{k\to\infty}\frac{\widetilde{\alpha}_k}{\alpha_k}\beta_{k-1}\left(2 - \frac{\beta_{k-1}}{2\beta}(2-\frac{\gamma}{\beta})\right) - \frac{\widetilde{\alpha}_{k-1}^2}{\alpha_k(\alpha_k-1)} = \gamma(\beta - \frac{\gamma}{2}) > 0.$$

Combining the preceding two displays and $\alpha_k = \Omega(k)$ shows that there exist constants K and $C > 0$ depending only on algorithm parameters such that for $0 < s \le (2-\gamma/\beta)/(2\beta L)$ and $k \ge K$,

$$\mathbf{II} \ge \frac{\gamma}{2}(\beta - \frac{\gamma}{2})\alpha_k(\alpha_k - 1) \ge Ck^2.$$

Choice 2: $\boldsymbol{\omega_{k+1} = \alpha_k/\widetilde{\alpha}_k}$. When $\{\widetilde{\alpha}_k/\alpha_k\}$ is non-decreasing, $\{\omega_k\}$ is non-increasing. In this case we have $\mathbf{I} = \alpha_{k-1}^2 - \alpha_k(\alpha_k-1) \ge 0$. For $0 < s \le (2-\gamma/\beta)/(2\beta L)$,

$$\begin{aligned}\mathbf{II} &= \alpha_k(\alpha_k-1)\beta_{k-1}(2-\beta_{k-1}Ls) - \alpha_{k-1}\widetilde{\alpha}_{k-1} \\ &= \alpha_k(\alpha_k-1)\left(\beta_{k-1}(2-\beta_{k-1}Ls) - \frac{\alpha_{k-1}^2}{\alpha_k(\alpha_k-1)}\cdot\frac{\widetilde{\alpha}_{k-1}}{\alpha_{k-1}}\right) \\ &\ge \alpha_k(\alpha_k-1)\left[\beta_{k-1}\left(2 - \frac{\beta_{k-1}}{2\beta}(2-\frac{\gamma}{\beta})\right) - \frac{\alpha_{k-1}^2}{\alpha_k(\alpha_k-1)}\cdot\frac{\widetilde{\alpha}_{k-1}}{\alpha_{k-1}}\right],\end{aligned}$$

where the limit of the term in the square brackets is

$$\lim_{k\to\infty} \beta_{k-1}\left(2-\frac{\beta_{k-1}}{2\beta}(2-\frac{\gamma}{\beta})\right)-\frac{\alpha_{k-1}^2}{\alpha_k(\alpha_k-1)}\cdot\frac{\widetilde{\alpha}_{k-1}}{\alpha_{k-1}}=\beta-\frac{\gamma}{2}>0.$$

Similarly as in the first choice, there exist constants K and $C>0$ depending only on algorithm parameters such that for $0<s\le(2-\gamma/\beta)/(2\beta L)$ and $k\ge K$,

$$\mathbf{II}\ge(\beta-\frac{\gamma}{2})\alpha_k(\alpha_k-1)\ge Ck^2.$$

Combining the two choices above, we see that when $\{\widetilde{\alpha}_k/\alpha_k\}$ is either non-increasing or non-decreasing in k, there exist constants K and $C>0$ such that for $k\ge K$, we have $\mathbf{I}\ge 0$ and $\inf_{0<s\le C_0/L}\mathbf{II}\ge Ck^2$, where $C_0=(2-\gamma/\beta)/(2\beta)$. By Lemma 5,

$$V_{k+1}-V_k\le-\frac{C}{2}k^2s^2\|\nabla f(x_{k-1})\|^2\le 0.$$

To complete the proof of (16) for the objective gap, it suffices to bound V_k from below.

By convexity of f, we have $\|\nabla f(x_k)\|^2\le(2L)(f(x_k)-f^*)$ which together with the definition of V_{k+1} in (46) yields that for $k\ge K$ and $0<s\le C_0/L$,

$$V_K\ge V_{k+1}\ge\omega_{k+1}(\alpha_k\widetilde{\alpha}_k-\widetilde{\alpha}_k^2Ls)s(f(x_k)-f^*)=\omega_{k+1}\alpha_k\widetilde{\alpha}_k\left(1-\frac{\widetilde{\alpha}_k}{\alpha_k}Ls\right)s(f(x_k)-f^*).$$

Because $\lim_k\omega_k$ is 1 or $1/\gamma>0$ in the Choice 1 or 2 above, and $\lim_k\widetilde{\alpha}_k/\alpha_k=\gamma>0$, we reset K large enough such that the above is further bounded from below as

$$V_K\ge V_{k+1}\ge\frac{1}{2}(\frac{1}{\gamma}\vee 1)\alpha_k\widetilde{\alpha}_k(1-2\gamma Ls)s(f(x_k)-f^*).$$

By resetting $C_0=\frac{2-\gamma/\beta}{2\beta}\wedge\frac{1}{4\gamma}$, we have that for $k\ge K$ and $0<s\le C_0/L$,

$$V_K\ge V_{k+1}\ge\frac{1}{4}(\frac{1}{\gamma}\vee 1)\alpha_k\widetilde{\alpha}_ks(f(x_k)-f^*).$$

Using Lemma 8, we have $f(x_k)-f^*=O(V_K/(\alpha_k\widetilde{\alpha}_ks))=O(V_K/(sk^2))=O(\|x_0-x^*\|^2/(sk^2))$, which is the optimal rate (16).

Finally, we complete the proof of (18) for the squared gradient norm. For $k\ge K$ and $0<s\le C_0/L$, using $V_{k+1}-V_k\le-\frac{C}{2}k^2s^2\|\nabla f(x_{k-1})\|^2$, we have

$$\frac{C}{2}\left(\sum_{i=K}^k i^2\right)s^2\min_{0\le i\le k-1}\|\nabla f(x_i)\|^2\le\frac{C}{2}\sum_{i=K}^k i^2s^2\|\nabla f(x_{i-1})\|^2\le\sum_{i=K}^k(V_i-V_{i+1})=V_K-V_{k+1}\le V_K,$$

which together with Lemma 8 implies that for $k\ge 2K$,

$$\min_{0\le i\le k-1}\|\nabla f(x_i)\|^2\le\frac{2V_K}{Cs^2\sum_{i=K}^k i^2}\le\frac{4V_K}{Cs^2\sum_{i=1}^k i^2}=\frac{24V_K}{Cs^2k(k+1)(2k+1)}=O\left(\frac{\|x_0-x^*\|^2}{s^2k^3}\right),$$

which is the desired bound (18).

B.4 Proof of Lemma 1

First, we show $\alpha_k=\Omega(k)$. When k is odd, $\alpha_k=(1+\sqrt{1+4\alpha_{k-1}^2})/2>(1+2\alpha_{k-1})/2=(1+2(k+r-1)/r)/2=(k+r-1)/r+1/2$. So for any $r>0$, $\alpha_k=\Omega(k)$.

Second, we show (19), $\alpha_k(\alpha_k - 1) \leq \alpha_{k-1}^2$ for $k \geq 1$. When k is odd, by construction (19) holds. When $k \geq 2$ and k is even, $k-1$ is odd and $k-2$ is even. Then $\alpha_k = (k+r)/r$ and $\alpha_{k-1} = (1+\sqrt{1+4\alpha_{k-2}^2})/2 = (1+\sqrt{r^2+4(k+r-2)^2}/r)/2$. Simple algebra yields

$$\alpha_k(\alpha_k - 1) - \alpha_{k-1}^2 = \frac{4-r}{r^2}k - \frac{\sqrt{(k+r-2)^2 + r^2/4}}{r} - \frac{1}{2} - \left(\frac{r-2}{r}\right)^2. \tag{63}$$

When $r \geq 4$, (63) is negative for all k. For $0 < r < 4$, we further rearrange (63) as

$$\begin{aligned}
&\frac{(4-r)^2k^2 - r^2\left((k+r-2)^2 + \frac{r^2}{4}\right)}{r^2\left((4-r)k + r\sqrt{(k+r-2)^2 + \frac{r^2}{4}}\right)} - \frac{1}{2} - \left(\frac{r-2}{r}\right)^2 \\
&= -\frac{8(r-2)k^2 + 2r^2(r-2)k + r^2\left((r-2)^2 + \frac{r^2}{4}\right)}{r^2\left((4-r)k + r\sqrt{(k+r-2)^2 + \frac{r^2}{4}}\right)} - \frac{1}{2} - \left(\frac{r-2}{r}\right)^2,
\end{aligned}$$

which remains negative for $2 \leq r < 4$ but blows up to ∞ as $k \to \infty$ if $0 < r < 2$.

Third, we calculate $\lim_k k(1-\sigma_{k+1})$ along $\{k'\} = \{2k\}$ and $\{k''\} = \{2k+1\}$. By the definition of σ_{k+1} we have

$$\lim_{k\to\infty} k(1-\sigma_{k+1}) = \lim_{k\to\infty} k\left(1 - \frac{\alpha_k - 1}{\alpha_{k+1}}\right) = \lim_{k\to\infty} \frac{k}{\alpha_{k+1}}\left(\alpha_{k+1} - \alpha_k + 1\right). \tag{64}$$

Along $\{k'\} = \{2k\}$,

$$\lim_{k\to\infty} \frac{2k}{\alpha_{2k+1}} = \lim_{k\to\infty} \frac{4k}{1+\sqrt{1+4\alpha_{2k}^2}} = \lim_{k\to\infty} \frac{4k}{2\alpha_{2k}} = \lim_{k\to\infty} \frac{4k}{2(2k+r)/r} = r,$$

and

$$\begin{aligned}
\lim_{k\to\infty} \alpha_{2k+1} - \alpha_{2k} &= \lim_{k\to\infty} \frac{1+\sqrt{1+4\alpha_{2k}^2}}{2} - \alpha_{2k} \\
&= \lim_{k\to\infty} \frac{1}{2}\left(1 + 2\alpha_{2k}\sqrt{1+\frac{1}{4\alpha_{2k}^2}}\right) - \alpha_{2k} \\
&= \lim_{k\to\infty} \frac{1}{2} + \alpha_{2k}\left(1 + O(\frac{1}{\alpha_{2k}^2})\right) - \alpha_{2k} \\
&= \frac{1}{2}.
\end{aligned}$$

Using (64) we have

$$\lim_{k'\to\infty} k'(1-\sigma_{k'+1}) = r(1+1/2) = 3r/2. \tag{65}$$

Along $\{k''\} = \{2k+1\}$,

$$\lim_{k\to\infty} \frac{2k+1}{\alpha_{2k+2}} = \lim_{k\to\infty} \frac{2k+1}{(2k+2+r)/r} = r,$$

and

$$\begin{aligned}\lim_{k\to\infty}\alpha_{2k+2}-\alpha_{2k+1} &= \lim_{k\to\infty}\alpha_{2k+2}-\frac{1+\sqrt{1+4\alpha_{2k}^2}}{2}\\ &= \lim_{k\to\infty}\alpha_{2k+2}-\frac{1}{2}\left(1+2\alpha_{2k}\sqrt{1+\frac{1}{4\alpha_{2k}^2}}\right)\\ &= \lim_{k\to\infty}\alpha_{2k+2}-\frac{1}{2}-\alpha_{2k}\left(1+O(\frac{1}{\alpha_{2k}^2})\right)\\ &= \lim_{k\to\infty}\frac{2k+2+r}{r}-\frac{2k+r}{r}-\frac{1}{2}\\ &= \frac{2}{r}-\frac{1}{2}.\end{aligned}$$

Using (64) we have

$$\lim_{k''\to\infty}k''(1-\sigma_{k''+1}) = r\left(\frac{2}{r}-\frac{1}{2}+1\right) = 2+\frac{r}{2}. \tag{66}$$

Lastly, combining the limits (65) and (66) gives $\lim_k \sigma_{k+1} = 1$, which implies that $\lim_k \alpha_{k+1}/\alpha_k = 1$. Therefore, condition (ii) holds. The proof of Lemma 1 is completed.

C Technical details in Section 4

C.1 Convergence of ODEs (23) and (25)

In the general convex setting where $f \in \mathcal{F}^1$, the convergence rates of gradient flow (23) and the ODE (24) for NAG-C are proved in Su et al. (2016). For completeness, we present the proofs for the convergence rates stated in the strongly convex setting where $f \in \mathcal{S}_\mu^1$.

For gradient flow (23), consider the Lyapunov function $V_t = f(X_t) - f^*$, i.e., the potential gap itself. Then by (23), we have $\dot{V}_t = \langle \nabla f(X_t), \dot{X}_t\rangle = -\|\nabla f(X_t)\|^2$. By strong convexity, $f(X_t) - f^* \le \frac{1}{2\mu}\|\nabla f(X_t)\|^2$. Therefore, $\dot{V}_t \le -2\mu(f(X_t)-f^*) = -2\mu V_t$, which implies that $f(X_t) - f^* = V_t \le V_0 \cdot \mathrm{e}^{-2\mu t} = \mathrm{e}^{-2\mu t}(f(x_0) - f^*)$.

For the ODE (25) corresponding to NAG-SC (4), consider the Lyapunov function $V_t = f(X_t) - f^* + \frac{1}{2}\|\dot{X}_t + \sqrt{\mu}(X_t - x^*)\|^2$. Then $V_0 = f(x_0) - f^* + \frac{\mu}{2}\|x_0 - x^*\|^2 \le 2(f(x_0) - f^*)$ by strong convexity. Using the ODE (25), we have

$$\begin{aligned}\frac{\mathrm{d}V_t}{\mathrm{d}t} &= \langle \nabla f(X_t), \dot{X}_t\rangle + \langle \dot{X}_t + \sqrt{\mu}(X_t - x^*), \ddot{X}_t + \sqrt{\mu}\dot{X}_t\rangle\\ &= \langle \nabla f(X_t), \dot{X}_t\rangle - \langle \dot{X}_t + \sqrt{\mu}(X_t - x^*), \sqrt{\mu}\dot{X}_t + \nabla f(X_t)\rangle\\ &= -\sqrt{\mu}\|\dot{X}_t\|^2 - \sqrt{\mu}\langle \dot{X}_t, \sqrt{\mu}(X_t - x^*)\rangle - \sqrt{\mu}\langle X_t - x^*, \nabla f(X_t)\rangle,\end{aligned}$$

which together with the inequality $\langle X_t - x^*, \nabla f(X_t)\rangle \ge f(X_t) - f^* + \frac{\mu}{2}\|X_t - x^*\|^2$ by μ-strong convexity suggests that

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} \le -\frac{\sqrt{\mu}}{2}\|\dot{X}_t\|^2 - \sqrt{\mu}\left(f(X_t) - f^* + \frac{1}{2}\|\dot{X}_t + \sqrt{\mu}(X_t - x^*)\|^2\right) \le -\sqrt{\mu}V_t.$$

Hence, $f(X_t) - f^* \le V_t \le V_0 \cdot \mathrm{e}^{-\sqrt{\mu}t} \le 2\mathrm{e}^{-\sqrt{\mu}t}(f(x_0) - f^*)$.

C.2 Proof of Proposition 3

Consider the Lyapunov function

$$V_t = c_0(f(X_t) - f^*) + \frac{1}{2}\|\dot{X}_t + \lambda\sqrt{\mu}(X_t - x^*)\|^2,$$

where λ is to be chosen later. Using the ODE (26), we have

$$\begin{aligned}
\frac{\mathrm{d}V_t}{\mathrm{d}t} &= c_0\langle \nabla f(X_t), \dot{X}_t\rangle + \langle \dot{X}_t + \lambda\sqrt{\mu}(X_t - x^*), \ddot{X}_t + \lambda\sqrt{\mu}\dot{X}_t\rangle \\
&= c_0\langle \nabla f(X_t), \dot{X}_t\rangle + \langle \dot{X}_t + \lambda\sqrt{\mu}(X_t - x^*), -(c_1-\lambda)\sqrt{\mu}\dot{X}_t - c_0\nabla f(X_t)\rangle \\
&= -(c_1-\lambda)\sqrt{\mu}\|\dot{X}_t\|^2 - (c_1-\lambda)\sqrt{\mu}\langle \dot{X}_t, \lambda\sqrt{\mu}(X_t - x^*)\rangle - c_0\lambda\sqrt{\mu}\langle X_t - x^*, \nabla f(X_t)\rangle \\
&= -\frac{c_1-\lambda}{2}\sqrt{\mu}\|\dot{X}_t\|^2 - \frac{c_1-\lambda}{2}\sqrt{\mu}\|\dot{X}_t + \lambda\sqrt{\mu}(X_t - x^*)\|^2 + \frac{c_1-\lambda}{2}\lambda^2\mu^{\frac{3}{2}}\|X_t - x^*\|^2 \\
&\quad - c_0\lambda\sqrt{\mu}\langle X_t - x^*, \nabla f(X_t)\rangle \\
&= -2C\sqrt{\mu}V_t - \frac{c_1-\lambda}{2}\sqrt{\mu}\|\dot{X}_t\|^2 - \left(\frac{c_1-\lambda}{2} - C\right)\sqrt{\mu}\|\dot{X}_t + \lambda\sqrt{\mu}(X_t - x^*)\|^2 \\
&\quad + 2Cc_0\sqrt{\mu}(f(X_t) - f^*) + \frac{c_1-\lambda}{2}\lambda^2\mu^{\frac{3}{2}}\|X_t - x^*\|^2 - c_0\lambda\sqrt{\mu}\langle X_t - x^*, \nabla f(X_t)\rangle,
\end{aligned}$$

which together with the inequality $\langle X_t - x^*, \nabla f(X_t)\rangle \geq f(X_t) - f^* + \frac{\mu}{2}\|X_t - x^*\|^2$ by μ-strong convexity suggests that

$$\begin{aligned}
\frac{\mathrm{d}V_t}{\mathrm{d}t} &\leq -2C\sqrt{\mu}V_t - \frac{c_1-\lambda}{2}\sqrt{\mu}\|\dot{X}_t\|^2 - \left(\frac{c_1-\lambda}{2} - C\right)\sqrt{\mu}\|\dot{X}_t + \lambda\sqrt{\mu}(X_t - x^*)\|^2 \\
&\quad - (\lambda - 2C)c_0\sqrt{\mu}(f(X_t) - f^*) - \frac{\lambda\mu^{\frac{3}{2}}}{2}(c_0 - (c_1-\lambda)\lambda)\|X_t - x^*\|^2.
\end{aligned}$$

Because $c_0 > 0$ and $c_1 > 0$, we pick λ such that $0 < \lambda < c_1$ and $(c_1 - \lambda)\lambda \leq c_0$. Moreover, we pick C such that $0 < C \leq \frac{c_1-\lambda}{2} \wedge \frac{\lambda}{2}$. Then

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq -2C\sqrt{\mu}V_t \implies V_t \leq V_0 \cdot \mathrm{e}^{-2C\sqrt{\mu}t}.$$

In particular, if $c_1^2 \leq 4c_0$, we pick $\lambda = c_1/2$ and $C = c_1/4$. If $c_1^2 > 4c_0$, we pick $\lambda = \frac{c_1+\sqrt{c_1^2-4c_0}}{2}$ and $C = \frac{c_1-\sqrt{c_1^2-4c_0}}{4}$. The conclusion then follows by bounding $V_0 = c_0(f(x_0) - f^*) + \frac{\lambda^2\mu}{2}\|x_0 - x^*\|^2 \leq (c_0 + \lambda^2)(f(x_0) - f^*)$ because $\frac{\mu}{2}\|x_0 - x^*\|^2 \leq f(x_0) - f^*$.

When $c_0 \leq 0$ or $c_1 \leq 0$, $f(X_t) - f^*$ is not guaranteed to converge. A simple counterexample is the harmonic oscillator, with $f(x) = \mu x^2/2$ for $x \in \mathbb{R}$,

$$\ddot{X}_t + c_1\sqrt{\mu}\dot{X}_t + c_0\mu X_t = 0, \tag{67}$$

starting from the initial position $X(0) = x_0$ with velocity $\dot{X}(0) = 0$. Since (67) is a second-order linear ODE with constant coefficients, its general solutions admit closed forms. Consider the characteristic equation $w^2 + c_1\sqrt{\mu}w + c_0\mu = 0$ with discriminant $\Delta = (c_1^2 - 4c_0)\mu$.

If $c_1^2 = 4c_0$, then there are two identical real roots $w = w_1 = w_2 = -c_1\sqrt{\mu}/2$, and $X_t = (\alpha_0 + \alpha_1 t)\mathrm{e}^{wt}$ for some real numbers α_0 and α_1. By the initial condition $\dot{X}(0) = 0$, we find $\alpha_1 = -\alpha_0 w$. Then $X_t = \alpha_0(1 - wt)\mathrm{e}^{wt}$. To achieve $X_t \to 0$, we need $w < 0$, which means $c_1 > 0$ and $c_0 = c_1^2/4 > 0$ too.

If $c_1^2 > 4c_0$, then there are two distinct real roots $w_1 = \frac{-c_1+\sqrt{c_1^2-4c_0}}{2}\sqrt{\mu}$ and $w_2 = \frac{-c_1-\sqrt{c_1^2-4c_0}}{2}\sqrt{\mu}$, with $w_1 > w_2$, and $X_t = \alpha_0\mathrm{e}^{w_1 t} + \alpha_1\mathrm{e}^{w_2 t}$. By the initial condition $\dot{X}(0) = 0$, we find $\alpha_0 w_1 + \alpha_1 w_2 = 0$. If $w_2 = 0$, then $w_1 > 0$ and hence $\alpha_0 = 0$ and $X_t \equiv \alpha_1 = x_0$, which contradicts $X_t \to 0$ for any $x_0 \neq 0$. If $w_2 \neq 0$, then $\alpha_1 = -\alpha_0 w_1/w_2$, and $X_t = \alpha_0(\mathrm{e}^{w_1 t} - \frac{w_1}{w_2}\mathrm{e}^{w_2 t})$ with $w_1 \neq w_2$. To achieve $X_t \to 0$, we also need $w_1 < 0$. Then $-c_1 + \sqrt{c_1^2 - 4c_0} < 0$, which implies that $c_0 > 0$ and $c_1 > 0$.

If $c_1^2 < 4c_0$, then there are two complex roots $w_{1,2} = \frac{-c_1 \pm \sqrt{4c_0 - c_1^2}i}{2}\sqrt{\mu}$, and X_t is a linear combination of $\mathrm{e}^{-c_1\sqrt{\mu}t/2} \times \sin(\sqrt{\mu(4c_0 - c_1^2)}t/2)$ and $\mathrm{e}^{-c_1\sqrt{\mu}t/2} \times \cos(\sqrt{\mu(4c_0 - c_1^2)}t/2)$. We need $c_1 > 0$ to make $X_t \to 0$. Then $c_0 > c_1^2/4 > 0$ too.

C.3 Proof of Proposition 4

For (27), we consider the Lyapunov function

$$V_t = (1+\sqrt{\mu s})(f(X_t)-f^*) + \frac{1}{2}\|\dot{X}_t + \sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t)\|^2.$$

Using the ODE (27), we have

$$\begin{aligned}
\frac{\mathrm{d}V_t}{\mathrm{d}t} &= (1+\sqrt{\mu s})\langle \nabla f(X_t), \dot{X}_t\rangle + \langle \dot{X}_t + \sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t), \ddot{X}_t + \sqrt{\mu}\dot{X}_t + \sqrt{s}\nabla^2 f(X_t)\dot{X}_t\rangle \\
&= (1+\sqrt{\mu s})\langle \nabla f(X_t), \dot{X}_t\rangle + \langle \dot{X}_t + \sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t), -\sqrt{\mu}\dot{X}_t - (1+\sqrt{\mu s})\nabla f(X_t)\rangle \\
&= -\sqrt{\mu}\|\dot{X}_t\|^2 - \sqrt{\mu}\langle \dot{X}_t, \sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t)\rangle - (1+\sqrt{\mu s})\langle \nabla f(X_t), \sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t)\rangle \\
&= -\frac{\sqrt{\mu}}{2}\|\dot{X}_t\|^2 - \frac{\sqrt{\mu}}{2}\|\dot{X}_t + \sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t)\|^2 + \frac{\sqrt{\mu}}{2}\|\sqrt{\mu}(X_t - x^*) + \sqrt{s}\nabla f(X_t)\|^2 \\
&\quad - (1+\sqrt{\mu s})\sqrt{\mu}\langle X_t - x^*, \nabla f(X_t)\rangle - (1+\sqrt{\mu s})\sqrt{s}\|\nabla f(X_t)\|^2 \\
&= -\sqrt{\mu}V_t + \sqrt{\mu}(1+\sqrt{\mu s})(f(X_t)-f^*) - \frac{\sqrt{\mu}}{2}\|\dot{X}_t\|^2 + \frac{\mu^{\frac{3}{2}}}{2}\|X_t - x^*\|^2 - \sqrt{\mu}\langle X_t - x^*, \nabla f(X_t)\rangle \\
&\quad - \sqrt{s}(1+\frac{\sqrt{\mu s}}{2})\|\nabla f(X_t)\|^2.
\end{aligned}$$

By the strong convexity, $\langle X_t - x^*, \nabla f(X_t)\rangle \geq f(X_t) - f^* + \frac{\mu}{2}\|X_t - x^*\|^2$. Then the above display yields

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq -\sqrt{\mu}V_t + \sqrt{\mu}\sqrt{\mu s}(f(X_t)-f^*) - \sqrt{s}(1+\frac{\sqrt{\mu s}}{2})\|\nabla f(X_t)\|^2.$$

By the strong convexity again, we have $f(X_t) - f^* \leq \frac{1}{2\mu}\|\nabla f(X_t)\|^2$. Hence

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq -\sqrt{\mu}V_t - \frac{\sqrt{s}}{2}(1+\sqrt{\mu s})\|\nabla f(X_t)\|^2 \leq -\sqrt{\mu}V_t.$$

The conclusion follows by integrating with t.

For (28), we consider the Lyapunov function

$$V_t = (1+\sqrt{\mu s})(f(X_t)-f^*) + \frac{1}{2}\|\dot{X}_t + \sqrt{\mu}(X_t - x^*)\|^2.$$

Using the ODE (28), we have

$$\begin{aligned}
\frac{\mathrm{d}V_t}{\mathrm{d}t} &= (1+\sqrt{\mu s})\langle \nabla f(X_t), \dot{X}_t\rangle + \langle \dot{X}_t + \sqrt{\mu}(X_t - x^*), \ddot{X}_t + \sqrt{\mu}\dot{X}_t\rangle \\
&= (1+\sqrt{\mu s})\langle \nabla f(X_t), \dot{X}_t\rangle - \langle \dot{X}_t + \sqrt{\mu}(X_t - x^*), \sqrt{\mu}\dot{X}_t + (1+\sqrt{\mu s})\nabla f(X_t)\rangle \\
&= -\sqrt{\mu}\|\dot{X}_t\|^2 - \sqrt{\mu}\langle \dot{X}_t, \sqrt{\mu}(X_t - x^*)\rangle - \sqrt{\mu}(1+\sqrt{\mu s})\langle X_t - x^*, \nabla f(X_t)\rangle \\
&= -\frac{\sqrt{\mu}}{2}\|\dot{X}_t\|^2 - \frac{\sqrt{\mu}}{2}\|\dot{X}_t + \sqrt{\mu}(X_t - x^*)\|^2 + \frac{\mu^{\frac{3}{2}}}{2}\|X_t - x^*\|^2 - \sqrt{\mu}(1+\sqrt{\mu s})\langle X_t - x^*, \nabla f(X_t)\rangle \\
&\leq -\sqrt{\mu}V_t + \sqrt{\mu}(1+\sqrt{\mu s})(f(X_t)-f^*) + \frac{\mu^{\frac{3}{2}}}{2}\|X_t - x^*\|^2 - \sqrt{\mu}(1+\sqrt{\mu s})\langle X_t - x^*, \nabla f(X_t)\rangle.
\end{aligned}$$

By the strong convexity, $\langle X_t - x^*, \nabla f(X_t)\rangle \geq f(X_t) - f^* + \frac{\mu}{2}\|X_t - x^*\|^2$. Then the above display yields

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq -\sqrt{\mu}V_t - \frac{\mu^2\sqrt{s}}{2}\|X_t - x^*\|^2 \leq -\sqrt{\mu}V_t.$$

The conclusion follows by integrating with t.

C.4 Proof of Proposition 5

The construction of our continuous-time Lyapunov function is motivated by Su et al. (2016) and Shi et al. (2022). We consider the auxiliary-energy term defined as

$$V_t^A = \frac{1}{2}\left\| r(X_t - x^*) + t\left(\dot{X}_t + \frac{\beta\sqrt{s}}{\gamma}\nabla f(X_t)\right)\right\|^2.$$

Using the ODE (31), for $t \geq t_0$ we have

$$\begin{aligned}
\frac{\mathrm{d}V_t^A}{\mathrm{d}t} &= \big\langle r(X_t - x^*) + t\big(\dot{X}_t + \frac{\beta\sqrt{s}}{\gamma}\nabla f(X_t)\big), r\dot{X}_t + \dot{X}_t + \frac{\beta\sqrt{s}}{\gamma}\nabla f(X_t) + t\big(\ddot{X}_t + \frac{\beta\sqrt{s}}{\gamma}\nabla^2 f(X_t)\dot{X}_t\big)\big\rangle \\
&= \big\langle r(X_t - x^*) + t\big(\dot{X}_t + \frac{\beta\sqrt{s}}{\gamma}\nabla f(X_t)\big), -\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\nabla f(X_t)\big\rangle \\
&= -r\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\langle X_t - x^*, \nabla f(X_t)\rangle - t\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\langle \dot{X}_t, \nabla f(X_t)\rangle \\
&\quad - \frac{\beta\sqrt{s}}{\gamma}t\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\|\nabla f(X_t)\|^2.
\end{aligned}$$

Let $C \geq 0$ be a constant to be chosen later, and introduce a factor of $\frac{t+C\sqrt{s}}{t}$ for technical adjustment. Because $\frac{t+C\sqrt{s}}{t} = 1 + \frac{C\sqrt{s}}{t}$ is decreasing in t, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t+C\sqrt{s}}{t}V_t^A\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t+C\sqrt{s}}{t}\right)\cdot V_t^A + \frac{t+C\sqrt{s}}{t}\cdot\frac{\mathrm{d}V_t^A}{\mathrm{d}t} \leq \frac{t+C\sqrt{s}}{t}\cdot\frac{\mathrm{d}V_t^A}{\mathrm{d}t} \\
&= -r\frac{t+C\sqrt{s}}{t}\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\langle X_t - x^*, \nabla f(X_t)\rangle - (t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\langle \dot{X}_t, \nabla f(X_t)\rangle \\
&\quad - \frac{\beta\sqrt{s}}{\gamma}(t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\|\nabla f(X_t)\|^2.
\end{aligned} \tag{68}$$

To eliminate the term of $\langle \dot{X}_t, \nabla f(X_t)\rangle$, we define the potential-energy term as

$$V_t^P = (t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)(f(X_t) - f^*).$$

Then

$$\frac{\mathrm{d}V_t^P}{\mathrm{d}t} = \big(2t + (\frac{r+1}{2} - \frac{\beta}{\gamma} + C)\sqrt{s}\big)(f(X_t) - f^*) + (t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\langle \nabla f(X_t), \dot{X}_t\rangle. \tag{69}$$

Define the continuous Lyapunov function as $V_t = V_t^P + \frac{t+C\sqrt{s}}{t}V_t^A$. Set $t_0^* = t_0 \vee \{2|C|\sqrt{s}\} \vee \{2|\frac{r+1}{2} - \frac{\beta}{\gamma}|\sqrt{s}\} > 0$. Then for $t \geq t_0^*$, we have $V_t \geq 0$ and

$$(t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big) \geq \frac{t}{2}\cdot\frac{t}{2} = \frac{t^2}{4}. \tag{70}$$

Combining (68) and (69), we obtain

$$\begin{aligned}
\frac{\mathrm{d}V_t}{\mathrm{d}t} &\leq -r\frac{t+C\sqrt{s}}{t}\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\langle X_t - x^*, \nabla f(X_t)\rangle + \big(2t + (\frac{r+1}{2} - \frac{\beta}{\gamma} + C)\sqrt{s}\big)(f(X_t) - f^*) \\
&\quad - \frac{\beta\sqrt{s}}{\gamma}(t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)\|\nabla f(X_t)\|^2,
\end{aligned}$$

which together with the convexity inequality $\langle X_t - x^*, \nabla f(X_t)\rangle \geq f(X_t) - f^*$ suggests that for $r \geq 0$ and $t \geq t_0^*$,

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq -\underbrace{\left((r-2)t + (r-1)C\sqrt{s} + (r-1+\frac{rC\sqrt{s}}{t})(\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\right)}_{\mathbf{I}_t} \cdot (f(X_t) - f^*)$$
$$- \underbrace{\frac{\beta}{\gamma}(t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)}_{\mathbf{II}_t} \cdot \sqrt{s}\|\nabla f(X_t)\|^2.$$

From (70), if $\frac{\beta}{\gamma} \geq 0$, then $\mathbf{II}_t \geq \frac{\beta}{4\gamma}t^2 \geq 0$ for $t \geq t_0^*$. It remains to deal with $\mathbf{I}_t$.

When $r > 2$, we set $C = 0$. Then $\mathbf{I}_t = (r-2)t + (r-1)(\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s} \geq 0$ for $t \geq -\frac{r-1}{r-2}(\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}$. In this case, we pick $t_1 = t_0^* \vee \{-\frac{r-1}{r-2}(\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\}$.

When $r = 2$, $\mathbf{I}_t = C\sqrt{s} + (1 + \frac{2C\sqrt{s}}{t})(\frac{3}{2} - \frac{\beta}{\gamma})\sqrt{s}$. We set the constant $C > \frac{\beta}{\gamma} - \frac{3}{2}$ (e.g., $\frac{\beta}{\gamma}$). Then when t is large (e.g., $t \geq \frac{4\beta}{3\gamma}(\frac{\beta}{\gamma} - \frac{3}{2})\sqrt{s}$), $\mathbf{I}_t \geq 0$ always holds. In this case, we pick $t_1 = t_0^* \vee \{\frac{4\beta}{3\gamma}(\frac{\beta}{\gamma} - \frac{3}{2})\sqrt{s}\}$.

To summarize, when $r \geq 2$ and $\frac{\beta}{\gamma} \geq 0$, there exist $C \geq 0$ and t_1 such that when $t \geq t_1$, $\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq 0$. If further $\frac{\beta}{\gamma} > 0$, then $-\frac{\mathrm{d}V_t}{\mathrm{d}t} \geq \frac{\beta\sqrt{s}}{4\gamma}t^2\|\nabla f(X_t)\|^2$. The remaining proof is similar to the proof of Theorem 4, but in a continuous way. We sketch the reasoning below.

If $\frac{\mathrm{d}V_t}{\mathrm{d}t} \leq 0$, then for $t \geq t_1$,

$$V_{t_1} \geq V_t \geq V_t^P = (t + C\sqrt{s})\big(t + (\frac{r+1}{2} - \frac{\beta}{\gamma})\sqrt{s}\big)(f(X_t) - f^*) \geq \frac{t^2}{4}(f(X_t) - f^*),$$

which implies that $f(X_t) - f^* = O\left(\frac{V_{t_1}}{t^2}\right)$.

If further $-\frac{\mathrm{d}V_t}{\mathrm{d}t} \geq \frac{\beta\sqrt{s}}{4\gamma}t^2\|\nabla f(X_t)\|^2$ with $\frac{\beta}{\gamma} > 0$, then for $t > t_1$, we have $V_t \geq 0$, and

$$V_{t_1} \geq V_{t_1} - V_t = (-V_t) - (-V_{t_1}) = \int_{t_1}^{t} \frac{\mathrm{d}}{\mathrm{d}u}(-V_u)\mathrm{d}u \geq \frac{\beta\sqrt{s}}{4\gamma}\int_{t_1}^{t} u^2\|\nabla f(X_u)\|^2\mathrm{d}u$$
$$\geq \frac{\beta\sqrt{s}}{4\gamma}\inf_{t_1 \leq u \leq t}\|\nabla f(X_u)\|^2 \int_{t_1}^{t} u^2\mathrm{d}u = \frac{\beta\sqrt{s}(t^3 - t_1^3)}{12\gamma}\inf_{t_1 \leq u \leq t}\|\nabla f(X_u)\|^2,$$

which implies that

$$\inf_{t_1 \leq u \leq t}\|\nabla f(X_u)\|^2 \leq \frac{12\gamma V_{t_1}}{\beta\sqrt{s}(t^3 - t_1^3)} = O\left(\frac{\gamma V_{t_1}}{\beta\sqrt{s}(t^3 - t_1^3)}\right).$$

D Technical details in Section 5

D.1 Single-variable form for HAG (32)

We derive the single-variable form (34) for HAG. From (32) solving the first equation for u_k, and plugging the expressions of u_k and u_{k+1} into the second equation, we obtain

$$\frac{x_{k+2} - x_{k+1} + a_{k+1}\nabla f(x_{k+1})}{\sqrt{a_{k+1}b_{k+1}}} = (b_k - 1)\frac{x_{k+1} - x_k + a_k\nabla f(x_k)}{\sqrt{a_k b_k}} - \sqrt{a_k b_k}\nabla f(x_k) - \phi_k(\nabla f(x_{k+1}) - \nabla f(x_k)).$$

Rearrange the above display to conclude.

D.2 Verification of monotonicity condition for (39)

We verify that the monotonicity condition in Theorem 4 is satisfied by (39), i.e., HAG under the configuration (38). First, we notice that (39) can be put into (21) with

$$\beta_{k-1} = \frac{1}{\sigma_{k+1}}\left(c_2\sqrt{c_0} - \frac{c_0}{b_{k-1}}\right), \quad \gamma_k = c_0\left(\frac{1}{b_{k-1}} + \frac{1}{b_k}\right) - \sigma_{k+1}(\beta_k - \beta_{k-1}).$$

When $\alpha_k = \frac{k+r}{r}$, we have $\sigma_{k+1} = \frac{\alpha_k - 1}{\alpha_{k+1}} = \frac{k}{k+r+1}$, $b_{k-1} = 1 + \sigma_{k+1} = \frac{2k+r+1}{k+r+1}$, and $\frac{\alpha_{k+1}}{\alpha_k} = \frac{k+r+1}{k+r}$. Without loss of generality, take $c_0 = 1$. Substituting σ_{k+1}, b_{k-1} and b_k into the display above and after some algebra, we obtain

$$\begin{aligned}\frac{\widetilde{\alpha}_k}{\alpha_k} &= \gamma_k\frac{\alpha_{k+1}}{\alpha_k} - \beta_k\left(\frac{\alpha_{k+1}}{\alpha_k} - 1\right) = \gamma_k\frac{k+r+1}{k+r} - \beta_k\frac{1}{k+r} \\ &= \frac{2k^2 + (7 - 2c_2 + 4r)k + 6 - 3c_2 + 7r - c_2 r + 2r^2}{(k+r)(2k+r+3)},\end{aligned}$$

which is monotone in k when $k \geq K$ for some K depending only on c_2 and r.

E Numerical experiments

We apply algorithms (14) and (29) to various objective functions (strongly or general convex) under different parameters and step sizes. It is particularly of interest to compare the performances from the parameter choices falling inside (or on the boundaries) versus outside the sufficient conditions for achieving accelerated convergence in our theoretical results.

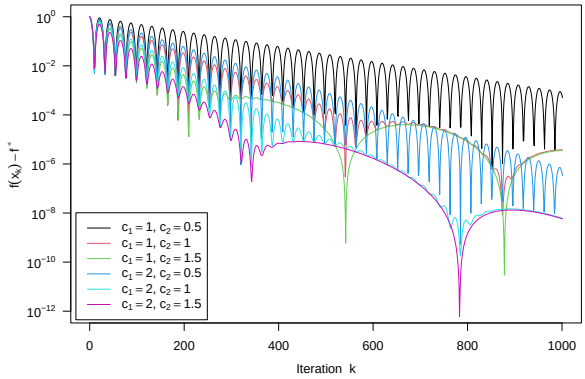
The numerical results below are overall consistent with our theoretical results in both strongly and general convex settings. Better performances are observed from algorithms inside (or on the boundaries) our sufficient conditions for accelerated convergence, that is, with more friction (over- or critical-damping, $c_1^2 \geq 4c_0$, vs under-damping, $c_1^2 < 4c_0$, in (14) and $r \geq 2$ vs $r < 1$ in (29)) and larger gradient correction ($c_2^2 \geq c_0$ vs $c_2^2 < c_0$ in (14) and $\beta/\gamma \geq 1/2$ vs $\beta/\gamma < 1/2$ in (29)). The effect of gradient correction becomes more significant as the step size increases.

Strongly convex setting. Figure 2 presents several numerical results of (14) applied to two quadratic functions on $\mathbb{R}^2$ similarly to Su et al. (2016), with c_0 fixed at 1, $c_1 = 1$, 2 (boundary) and $c_2 = 1/2, 1$ (boundary), 3/2. All remainder terms R_1, R_2, and R_3 are taken to be 0. In particular, $c_1 = 1, 2$ corresponds to under-damping ($c_1^2 < 4c_0$) and critical-damping ($c_1^2 = 4c_0$) respectively, and $c_2 = 1/2, 1, 3/2$ corresponds to the leading constant of the gradient correction coefficient in (14) being $0, 1/2, 1$ respectively. For NAG-SC (4), $c_1 = 2$ and $c_2 = 3/2$, and for heavy-ball (9), $c_1 = 2$ and $c_2 = 1/2$, but both with nonzero remainder terms R_2 and R_3. Hence the algorithms studied are not exactly NAG-SC or heavy-ball.

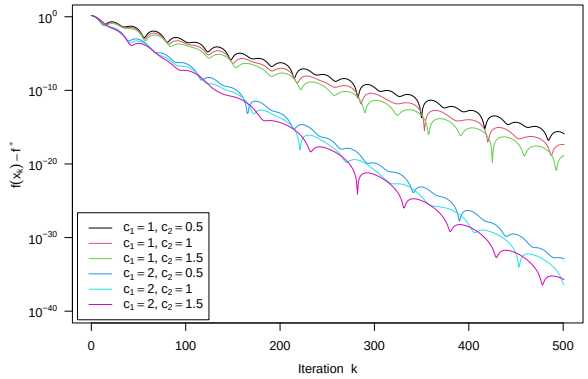
From Figure 2 we observe that critical-damping ($c_1 = 2$) results in a faster convergence than under-damping ($c_1 = 1$). Increasing c_2 (gradient correction) also tends to improve the performance, but to a small extent in the ill-conditioned case once $c_2 \geq 1$. The error plots when $c_2 = 1/2$ in the ill-conditioned case appear to be a sum of two oscillations.

General convex setting. Following Su et al. (2016), we consider a quadratic objective $f(x) = \frac{1}{2}x^{\mathrm{T}}Ax + b^{\mathrm{T}}x$ and the log-sum-exp objective $f(x) = \rho \log \sum_{i=1}^{200} \mathrm{e}^{\frac{a_i^{\mathrm{T}}x - b_i}{\rho}}$. The quadratic function is strongly convex but with $\mu \approx 0$, and the log-sum-exp function is not strongly convex. Figures 3 and 4 present the numerical results of applying (29) to these two objectives respectively. We set $\gamma \equiv 1$, $\sigma_{k+1} = \frac{k}{k+r+1}$ for $r = 1, 2$ (boundary) and $\beta = 0, 0.5$ (boundary), 1. In particular, NAG-C corresponds to $r = 2$ and $\beta = 1$. In the log-sum-exp case, the minimizer has no closed form. We obtain an approximation of f^* by running NAG-C, which converges fast in hundreds of iterations using a relatively large step size. See Figure 4 (e).

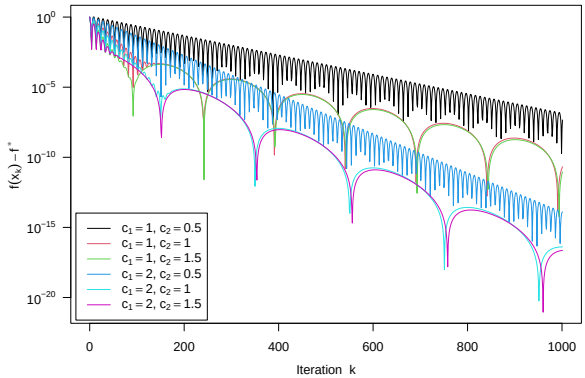
In both Figures 3 and Figure 4, for fixed β, increasing $r = 1$ to $r = 2$ significantly accelerates the convergence of $f(x_k) - f^*$ and $\min_{0 \leq i \leq k} \|\nabla f(x_i)\|^2$. In particular, it can be observed in both figures that the bound $\min_{0 \leq i \leq k} \|\nabla f(x_i)\|^2 = O(1/k^3)$ may fail when $r = 1$. When r is fixed, in Figure 4 increasing



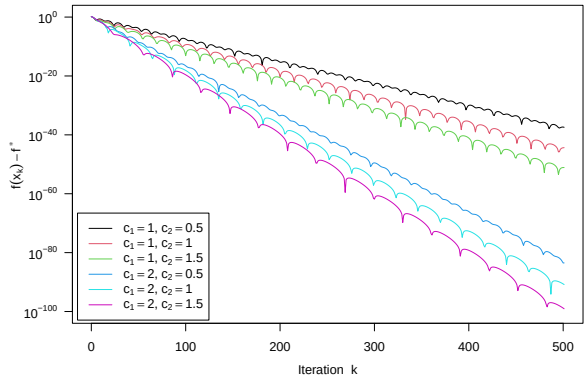
(a) Errors for ill-conditioned f ($s = 0.01$).



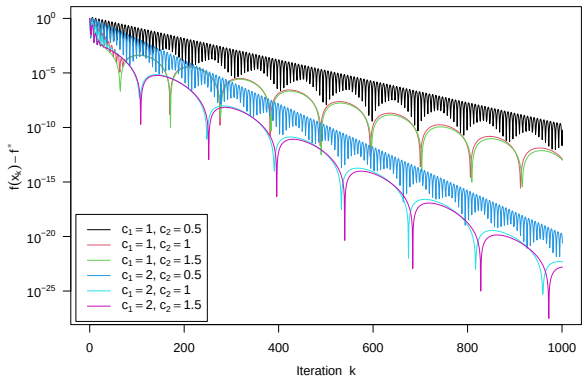
(b) Errors for well-conditioned f ($s = 0.01$).



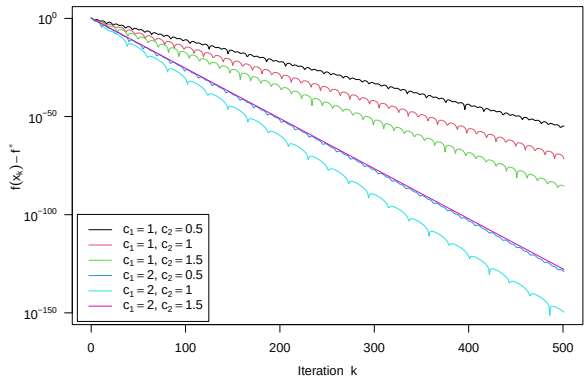
(c) Errors for ill-conditioned f ($s = 0.05$)



(d) Errors for well-conditioned f ($s = 0.05$)



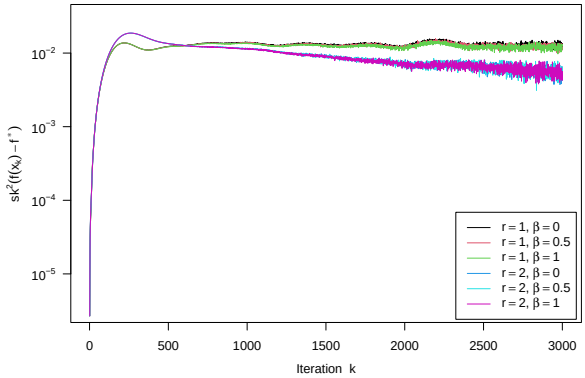
(e) Errors for ill-conditioned f ($s = 0.1$)



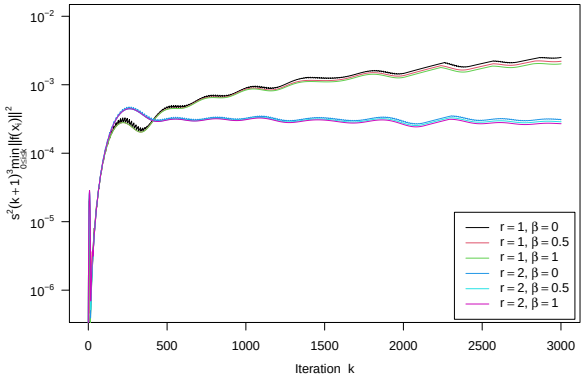
(f) Errors for well-conditioned f ($s = 0.1$)

Figure 2: Minimizing strongly convex functions by (14). The left column is for ill-conditioned $f(x_1, x_2) = 5 \times 10^{-3} x_1^2 + x_2^2$, and the right column is for well-conditioned $f(x_1, x_2) = 5 \times 10^{-1} x_1^2 + x_2^2$. Fix $c_0 = 1$. The initial iterates are $x_0 = (1, 1)$ and $x_1 = x_0 - 2s\nabla f(x_0)/(1 + \sqrt{\mu s})$.

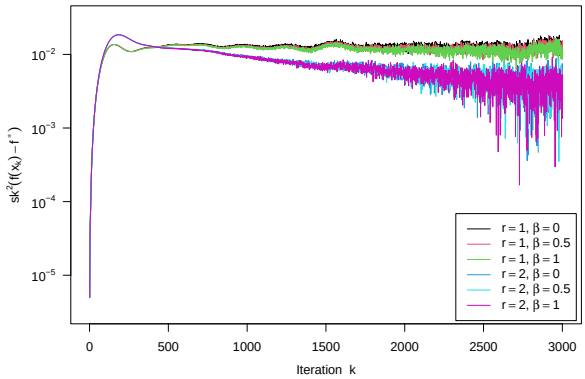
β markedly improves the performance in decreasing $f(x_k) - f^*$ and $\min_{0 \le i \le k} \|\nabla f(x_i)\|^2$, especially when the step size is large. A similar beneficial effect of a larger β can be observed from Figure 3 in decreasing $\min_{0 \le i \le k} \|\nabla f(x_i)\|^2$, but the effect there is less obvious in decreasing $f(x_k) - f^*$, where all algorithms exhibit oscillations during iterations, and the oscillations become stronger as the step size increases.



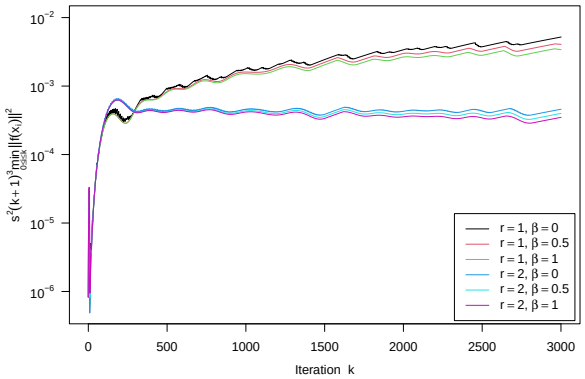
(a) Scaled errors ($s = 0.05/\|A\|$)



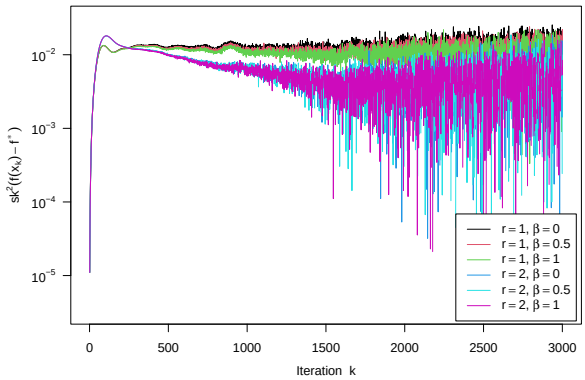
(b) Scaled gradient norms ($s = 0.05/\|A\|$)



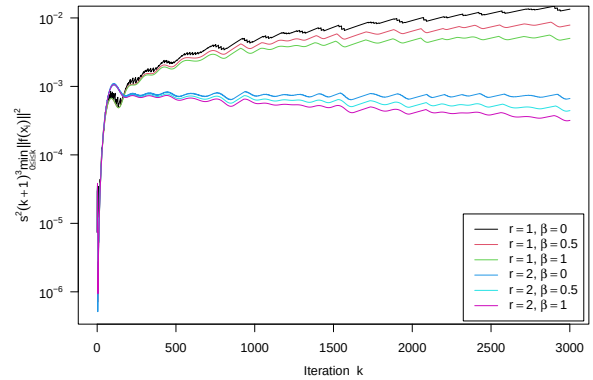
(c) Scaled errors ($s = 0.1/\|A\|$)



(d) Scaled gradient norms ($s = 0.1/\|A\|$)

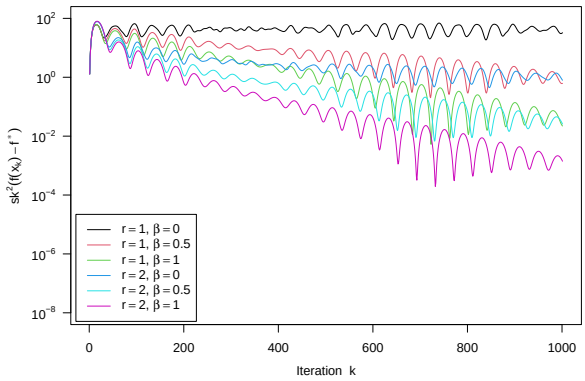


(e) Scaled errors ($s = 0.3/\|A\|$)

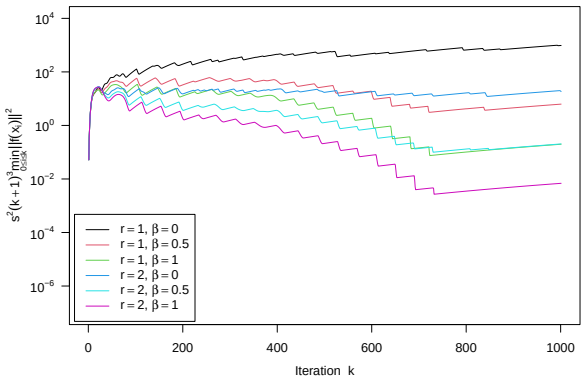


(f) Scaled gradient norms ($s = 0.3/\|A\|$)

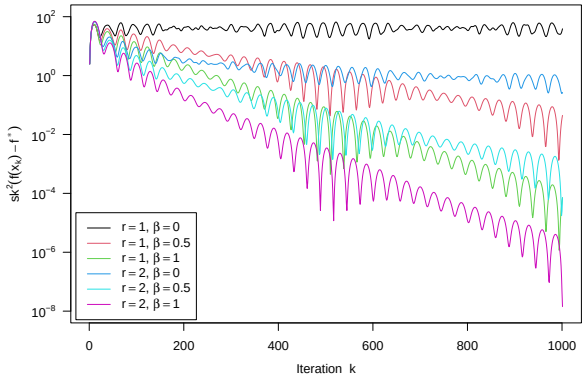
Figure 3: Scaled errors and squared gradient norms in minimizing $f(x) = \frac{1}{2}x^{\mathrm{T}}Ax + b^{\mathrm{T}}x$ by (29) under different step sizes, where $A = B^{\mathrm{T}}B$ for $B \in \mathbb{R}^{500\times 500}$, $b \in \mathbb{R}^{500}$. All entries in B and b are i.i.d. draws from $U(0,1)$, and $\|A\|$ is the spectral norm of A. We take $\gamma = 1$, $\sigma_{k+1} = \frac{k}{k+r+1}$ for $r = 1, 2$ and $\beta = 0, 0.5, 1$ in (29).



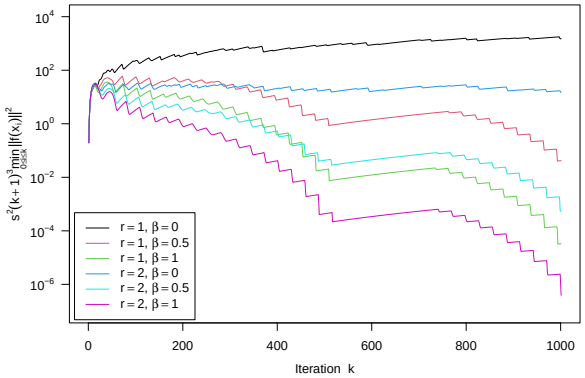
(a) Scaled errors ($s = 0.5$)



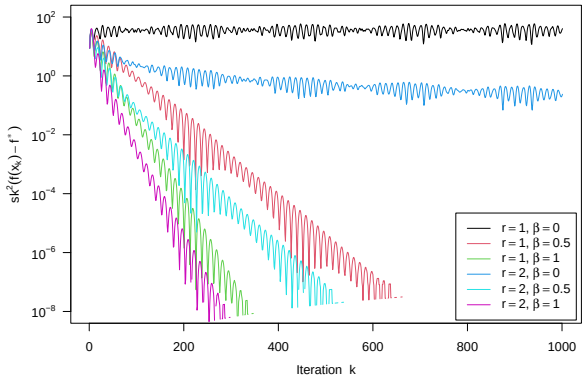
(b) Scaled gradient norms ($s = 0.5$)



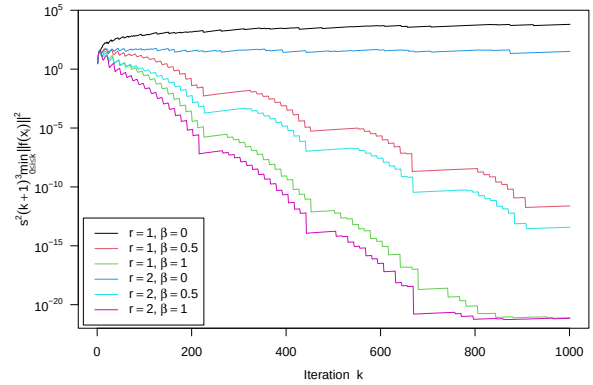
(c) Scaled errors ($s = 1$)



(d) Scaled gradient norms ($s = 1$)



(e) Scaled errors ($s = 5$)



(f) Scaled gradient norms ($s = 5$)

Figure 4: Scaled errors and squared gradient norms in minimizing $f(x) = \rho \log \sum_{i=1}^{200} \mathrm{e}^{\frac{a_i^{\mathrm{T}} x - b_i}{\rho}}$ by (29) under different step sizes, where $A = [a_1, \ldots, a_{200}] \in \mathbb{R}^{50\times 200}$, $b \in \mathbb{R}^{200}$, and $\rho = 20$. All entries in A and b are i.i.d. draws from $\mathrm{N}(0,1)$. We take $\gamma = 1$, $\sigma_{k+1} = \frac{k}{k+r+1}$ for $r = 1, 2$ and $\beta = 0, 0.5, 1$ in (29).