Quantization of bandlimited functions using random samples

***Abstract*—We investigate the compatibility of distributed noise-shaping quantization with random samples of bandlimited functions. Let f be a bandlimited function. Suppose R is a large positive number and assume that $\{x_i\}_{i=1}^m$ is a sequence of i.i.d random variables uniformly distributed on $[-\tilde{R}, \tilde{R}]$, where $\tilde{R} > R$ is appropriately chosen. We show that on using a distributed noise-shaping quantizer to quantize the values of f at $\{x_i\}_{i=1}^m$, a function $\tilde{f}$ can be reconstructed from these quantized values such that with high probability $\|f - \tilde{f}\|_{L^2[-R,R]}$ is small.**

I. Introduction

In signal processing, one of the primary goals is to obtain a digital representation of a function in a signal space suitable for storage, transmission and recovery. This goal is usually attained through two steps, sampling and quantization. In sampling, we sample the function at appropriate data points such that the function can be stably reconstructed using those samples. For the space of the bandlimited functions, the Shannon sampling theorem, tells us that any analog time $\pi-$bandlimited signal f can be converted to a discrete time signal losslessly, by storing its sample values $\{f(n)\}_{n\in\mathbb{Z}}$. However, at this stage the samples $\{f(n)\}_{n\in\mathbb{Z}}$ still belong to the continuous $\mathbb{R}$, in the second step of quantization, we reduce these real or complex valued function samples to a discrete finite set.

More generally, given a signal x and a vector y containing linear measurements (function samples, frame measurements etc.) Ax of x. The quantization process involves replacing the measurement vector y with a vector q from a finite set $\mathcal{A}$, called the quantization alphabet, such that accurate reconstruction of x from q is possible.

One of the most intuitive approaches to quantization is memoryless scalar quantization(MSQ) [9], [18], which requires simply replacing each coefficient of $y = Ax$, with its nearest element from $\mathcal{A}$. This approach is, however not well suited for quantizing measurement vectors with redundant linear measurements, as it naively quantizes each coefficient without any regard for how the other coefficients in the vector are quantized. In order to solve this issue quantization schemes such as $\Sigma\Delta$ quantization and distributed noise-shaping quantization were introduced. These schemes follow a recursive procedure to push the quantization error $y-q$ into unoccupied portion of the signal spectrum.

$\Sigma\Delta$ quantization schemes were introduced in [14] for the quantization of oversampled bandlimited functions. Since then, they have been studied extensively [7], [8], [11]. The use of $\Sigma\Delta$ quantization in the setting of finite frames has also been explored by various authors [1], [2], [10], [15]–[17].

In their paper [3], Chou and Güntürk introduced the concept of distributed noise-shaping quantization and were able to achieve an exponentially small error bound in the quantization of Gaussian random finite frame expansions. They extended their results to the setting of classical frames in [4]. Recently, this particular scheme has was applied to fast binary embedding [13] and spectral super-resolution [12].

Although the quantization of oversampled bandlimited functions using uniform samples has been investigated significantly, to the best of our knowledge, only two papers [6], [19] are available in the literature that deal with the scenario of quantizing irregular samples of bandlimited functions. In the first [6], the authors study techniques to reconstruct a bandlimited function by estimating its uniform samples from its quantized irregular samples. In [19], the authors show that if a bandlimited function f is sampled on an interleaved collection of N uniform grids $\{kT + T_n\}_{k\in\mathbb{Z}}$ with $\{T_n\}_{n=1}^N$ chosen independently from $[0, T], T < 1$, and the samples are quantized with a first order $\Sigma\Delta$ scheme, then with high probability $\|f - \tilde{f}\|_{L^\infty(\mathbb{R})}$ is atmost of the order $\frac{\log N}{N}$.

We on the other hand do not put such tight restrictions on the sample points. Our work deals with the distributed noise-shaping quantization of bandlimited functions using random samples, where it is assumed that the samples $\{x_i\}_{i=1}^m$ are a sequence of i.i.d random variables uniformly distributed on an suitable interval, i.e. they are completely random and have no kind of structure. We state our main result, Theorem 3.1, in section III.

II. Preliminaries

A. Sampling of bandlimited functions

Let $PW_{[-\pi,\pi]}$ denote the space of π-bandlimited functions i.e. $PW_{[-\pi,\pi]} := \{f \in L^2(\mathbb{R}) : supp(\hat{f}) \subset [-\pi, \pi]\}$. Here, the Fourier transform of a function $f \in L^1(\mathbb{R})$, is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\xi} dt.$$

The celebrated Shannon sampling theorem says that for any $f \in PW_{[-\pi,\pi]}$, we have

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) sinc(t-n) \qquad \forall\ t \in \mathbb{R} \tag{1}$$

where $sinc(x) := \frac{sin\pi x}{\pi x}$. However, (1) is not useful in practice, because $sinc(x)$ decays too slowly. To circumvent this issue, it is useful to introduce oversampling. This can easily be done as shown in [6]. We review the method over here.

Let $\lambda > 1$ be a fixed real number. Choose a function g such that $\hat{g} \in C^{\infty}($ this is not a strict requirement, depending on how fast we want g to decay, it is enough to assume that $\hat{g}$ is sufficiently differentiable),

- $$\hat{g} = \begin{cases} \frac{1}{\sqrt{2\pi\lambda}} & \xi \in [-\pi, \pi] \\ 0 & |\xi| > (2\lambda - 1)\pi \end{cases}, \tag{2}$$
- $$\sum_{k \in \mathbb{Z}} \left|\hat{g}(\xi + 2k\lambda\pi)\right|^2 = \frac{1}{2\pi\lambda}. \tag{3}$$

Using [5, Theorem 9.2.5] and (3), it can be seen that $\left\{g\left(\,\cdot - \frac{k}{\lambda}\right) : k \in \mathbb{Z}\right\}$ forms a orthonormal system. Now, consider any band-limited function $f \in PW_{[-\pi,\pi]}$, then its Fourier transform can be viewed as an element in $L^2[-(2\lambda - 1)\pi, (2\lambda - 1)\pi]$. Using the Fourier series expansion of $\hat{f}$ on $[-(2\lambda - 1)\pi, (2\lambda - 1)\pi]$ and (2), it can be shown that

$$f(t) = \frac{1}{\sqrt{\lambda}} \sum_{n \in \mathbb{Z}} f\left(\frac{n}{\lambda}\right) g\left(t - \frac{n}{\lambda}\right) \;\; \forall\, t \in \mathbb{R}. \tag{4}$$

In comparison to (1), the above reconstruction formula, although requiring more samples, has the advantage that each sample is weighted in a very localized way $\left(f\left(\frac{n}{\lambda}\right)\right)$ only contributes in a small neighbourhood of $(\frac{n}{\lambda})$. This property will be exploited by us when we find an finite dimensional approximation space for $PW_{[-\pi,\pi]}$.

B. Distributed noise-shaping quantization

Let m and p be positive integers such that p divides m, then for any fixed $\beta > 1$, the block diagonal distributed noise-shaping transfer operator H_β [3] is defined as

$$H_\beta = I_p \otimes \tilde{H}_\beta \tag{5}$$

where I_p denotes the identity matrix of order p and $\tilde{H}_\beta$ is the following $\frac{m}{p} \times \frac{m}{p}$ matrix

$$(\tilde{H}_\beta)_{ij} := \begin{cases} 1 & \text{if } i = j \\ -\beta & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Definition 2.1: Given a alphabet $\mathcal{A}$, by a distributed noise-shaping quantizer we mean any map $Q : \mathbb{C}^m \to \mathcal{A}^m$ that satisfies

$$y - q = H_\beta u \tag{7}$$

where $q := Q(y)$, H_β is the $m \times m$ matrix given above and u is a vector such that $\|u\|_\infty \leq c$ for some constant c which is independent of m. The existence of such schemes is given by [13, Lemma 4.2]. Let the vector $\nu_\beta := [\beta^{-1}\beta^{-2}\cdots\beta^{-\frac{m}{p}}]$. The β-condensation operator [3] $\tilde{V}_\beta$ is defined as,

$$\tilde{V}_\beta := I_k \otimes \frac{\nu_\beta}{\|\nu_\beta\|_1}. \tag{8}$$

It can be easily seen that,

$$\|\tilde{V}_\beta H_\beta\|_{\infty \to 2} \leq \sqrt{p}\beta^{-\frac{m}{p}+1}. \tag{9}$$

III. Main result

A. Statement of the result

For a positive integer L and $\delta > 0$, let

1) $\mathcal{A}_L^\delta := \{\pm(2l - 1)\delta : 1 \leq l \leq L, l \in \mathbb{Z}\}$.
2) $\mathcal{A} := \mathcal{A}_L^\delta + i\mathcal{A}_L^\delta$.

Theorem 3.1: Let R be a positive real number. Fix $\beta \in (1, 2L - \frac{1}{\delta})$. Assume that $\{x_i\}_{i=1}^m$ is a sequence of i.i.d random variables that are uniformly distributed on $[-R - 3m^{\frac{1}{16}}, R + 3m^{\frac{1}{16}}]$. Let $\mathcal{Q}_\beta$ [13, Lemma 4.2] be the quantization scheme with alphabet $\mathcal{A}$. Consider $f \in PW_{[-\pi,\pi]}$ with $\|f\|_{L^\infty(\mathbb{R})} \leq 1$, if $m^{\frac{15}{16}}$ is an integer such that it divides m and m is sufficiently large, then with probability greater than

$$1 - 17m^{\frac{15}{16}} \exp\left(-\frac{m^{\frac{3}{8}}}{d_2 R}\right)$$

we have

$$\|f - F_{W\tilde{V}_\beta} q\|_{L^2[-R,R]} \leq \frac{d_1 R}{m^{\frac{7}{16}}} \tag{10}$$

where $F_{W\tilde{V}_\beta}$ is as defined in (14), $q := Q_\beta(y)$ and y is as defined in (15), d_1 and d_2 are explicitly known constants.

B. Sketch of proof of Theorem 3.1

Our proof combines the quantization theory of bandlimited functions with the theory of frames. It is divided into three steps. In the following we will outline the ideas of these steps, the precise proof will follow in the journal version of this manuscript, which is under preparation.

1) A suitable finite dimensional approximation space: We project the infinite-dimensional space of bandlimited functions onto a finite-dimensional space. The chosen finite-dimensional space is such that when the projection Pf of a bandlimited function f approximates f well in a neighborhood of $[-R, R]$.

Fix $\epsilon > 0$ and consider the interval $[-(1+3\epsilon)R, (1+3\epsilon)R]$. It can be partitioned into $I_{1\epsilon} := (-(1+\epsilon)R, (1+\epsilon)R)$, $I_{2\epsilon} := (-(1+2\epsilon)R, -(1+\epsilon)R] \cup [(1+\epsilon)R, (1+2\epsilon)R)$ and $I_{3\epsilon} := [-(1+3\epsilon)R, -(1+2\epsilon)R] \cup [(1+2\epsilon)R, (1+3\epsilon)R]$. Let g and λ be as defined in the subsection II-A.

Definition 3.2: For any positive real number t, let

$$[t] := \{-\lfloor t \rfloor, -(\lfloor t \rfloor - 1), \cdots, -1, 0, 1, \cdots, \lfloor t \rfloor - 1, \lfloor t \rfloor\}.$$

Define,

$$V^{(1+\frac{5}{2}\epsilon)R}(g) := span\left\{g\left(\,\cdot - \frac{k}{\lambda}\right) : k \in \left[\lambda\left(1 + \frac{5}{2}\epsilon\right)R\right]\right\}. \tag{11}$$

Let P denote the orthogonal projection from $L^2(\mathbb{R})$ onto $V^{(1+\frac{5}{2}\epsilon)R}(g)$. Then, for any $f = \frac{1}{\sqrt{\lambda}} \sum_{k \in \mathbb{Z}} f\left(\frac{k}{\lambda}\right) g\left(\,\cdot - \frac{k}{\lambda}\right) \in PW_{[-\pi,\pi]}$, we have

$$P(f) = \frac{1}{\sqrt{\lambda}} \sum_{k \in [\lambda(1+\frac{5}{2}\epsilon)R]} f\left(\frac{k}{\lambda}\right) g\left(\,\cdot - \frac{k}{\lambda}\right). \tag{12}$$

From the formula for Pf, it is clear that it has been calculated by simply replacing the samples of f outside $[-\lfloor(1+\frac{5}{2}\epsilon)R\rfloor, \lfloor(1+\frac{5}{2}\epsilon)R\rfloor]$ with 0. However, as the function samples are weighted locally because of the decay of g, this

replacement has a minimal effect on the function f in the region $I_{1\epsilon} \cup I_{2\epsilon}$, . Hence the projected function Pf will be a good approximation of f on $I_{1\epsilon}$ and $I_{2\epsilon}$, but it need not be on $I_{3\epsilon}$.

2) A random frame for $V^{(1+\frac{5}{2}\epsilon)R}(g)$: In the next step, we find a random frame for the finite-dimensional approximation space $V^{(1+\frac{5}{2}\epsilon)R}(g)$, such that the frame measurements for any $h \in V^{(1+\frac{5}{2}\epsilon)R}(g)$ can be calculated using its function samples.

If we want to work with a reasonable number of samples, then any frame satisfying the above condition will require us to sample in a region larger than $[-(1+\frac{5}{2}\epsilon)R, (1+\frac{5}{2}\epsilon)R]$. We give a simple heuristics for this. Suppose we have no sample points in an interval $[a,b] \subset [-(1+\frac{5}{2}\epsilon)R, (1+\frac{5}{2}\epsilon)R]$. Then it cannot be expected that a function f_1, which is concentrated on $[a,b]$, can be reconstructed using its frame measurements, as the frame measurements do not use the samples of f_1 on the region where it is concentrated.

Assume that $\{x_i\}_{i=1}^m$ is a sequence of i.i.d random variables that are uniformly distributed on $[-(1+3\epsilon)R, (1+3\epsilon)R]$. Since we work in the finite-dimensional space $V^{(1+\frac{5}{2}\epsilon)R}(g)$, the projected function's samples $\{Pf(x_i)\}_{i=1}^m$ are needed to compute the frame measurements of Pf. However, the samples available are of the original function f's and not the projected function Pf's.

Since by construction, Pf approximates f well in $I_{1\epsilon} \cup I_{2\epsilon}$, we use the function samples $\{f(x_i)\}_{i=1}^m$ instead of $\{Pf(x_i)\}_{i=1}^m$ to calculate frame measurements. Here, we must remember that we also have sampling points in the region $I_{3\epsilon}$, where the Pf does not approximate f well.

The goal is to devise a reconstruction process where the error generated by this approximation of samples can be controlled and minimized.

It turns out that the main issue is the completely random ordering of the sample points $\{x_i\}_{i=1}^m$, which leads to a mix-up of accurately approximated sample values $\{f(x_i)\}_{x_i \in I_{1\epsilon} \cup I_{2\epsilon}}$ and potentially inaccurately approximated sample values $\{f(x_i)\}_{x_i \in I_{3\epsilon}}$ during reconstruction. In order to solve this, we design a procedure to partition the random variables $\{x_i\}_{i=1}^m$ into three disjoint collections. As the random variables are i.i.d uniformly distributed, also the random variables in each of these three collections are i.i.d uniformly distributed, albeit on different intervals. More precisely, in any realization of the random variables $\{x_i\}_{i=1}^m$, there will be some m_1 points in $I_{1\epsilon}$, m_2 points in $I_{2\epsilon}$ and m_3 points in $I_{3\epsilon}$. These m_1, m_2 and m_3 will be random variables and can take the values $\{0, 1, \cdots, m\}$.

Definition 3.3: Let p be a positive integer that divides m. We define the following new random variables,

1) $p_i := \sum_{j=1}^{i} \lfloor \frac{m_j p}{m} \rfloor, \tilde{m}_i := \lfloor \frac{m_i p}{m} \rfloor \frac{m}{p} \;\; \forall\, i \in \{1,2,3\}$.
2) $\tilde{m} := \frac{m}{p} p_3$.
3) For $j = 1,2,3$ and for all $i \in \{1, \cdots, \tilde{m}_j\}$, define $y_i^j = x_n$ where n is the i-th number such that $x_n \in I_{j\epsilon}$. Each of the random variables $\{y_i^j\}_{i=1}^{\tilde{m}_j}$ will be i.i.d uniformly distributed on $I_{j\epsilon}$.
4) Define $\{\epsilon_i^1\}_{i=1}^{\tilde{m}_1}$,$\{\epsilon_i^2\}_{i=1}^{\tilde{m}_2}$ and $\{\epsilon_i^3\}_{i=1}^{\tilde{m}_3}$ to be sequences of ± 1 Bernoulli independent random variables that are also independent from all the above defined random variables.

Let $\tilde{V}_\beta$ be the $\beta-$condensation operator with p_3 rows and $\tilde{m}$ columns. Define the matrix

$$(W)_{ij} := \begin{cases} \sqrt{\frac{2(1+\epsilon)R}{p_1}} & i = j, i \in \{1, 2, \cdots, p_1\} \\ \sqrt{\frac{2\epsilon R}{p_2 - p_1}} & i = j, i \in \{p_1+1, \cdots, p_2\} \\ \sqrt{\frac{2\epsilon R}{p_3 - p_2}} & i = j, i \in \{p_2+1, \cdots, p_3\} \\ 0 & i \neq j \end{cases}$$

and let $h_j := \sum_{i=1}^{\tilde{m}_1} (W\tilde{V}_\beta)_{ji} \epsilon_i^1 k_{y_i^1} + \sum_{i=1}^{\tilde{m}_2} (W\tilde{V}_\beta)_{j(\tilde{m}_1+i)} \epsilon_i^2 k_{y_i^2} + \sum_{i=1}^{\tilde{m}_3} (W\tilde{V}_\beta)_{j(\tilde{m}_2+i)} \epsilon_i^3 k_{y_i^3}$ for all $j \in \{1, \cdots p_3\}$.

Lemma 3.4: Let $\gamma, t \in (0,1)$ be such that $1 - \gamma - 3t > 0$, $\{h_j\}_{j=1}^{p_3}$ be as defined above and ϵR be large enough, then

$$\frac{\|\nu_\beta\|_2^2}{\|\nu_\beta\|_1^2}(1-\gamma-3t)\|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{j=1}^{p_3} |\langle f, h_j \rangle|^2 \leq \frac{\|\nu_\beta\|_2^2}{\|\nu_\beta\|_1^2}(1+3t)\|f\|_{L^2(\mathbb{R})}^2 \quad \forall f \in V^{(1+\frac{5}{2}\epsilon)R}(g) \tag{13}$$

are true together with probability greater than $1 - 6\exp\big(-\frac{m\epsilon}{12(1+3\epsilon)}\big) - 5p\exp\big(-\frac{t^2(\beta-1)}{42(2\lambda-1)(\beta+1)R}\big(\frac{p}{2(1+3\epsilon)} - \frac{1}{\epsilon R}\big)\big)$.

3) Quantization and error bound computation: Using Lemma 3.4 , we conclude that with probability greater than $1 - 6\exp\big(-\frac{m\epsilon}{12(1+3\epsilon)}\big) - 5p\exp\big(-\frac{t^2(\beta-1)}{42(2\lambda-1)(\beta+1)R}\big(\frac{p}{2(1+3\epsilon)} - \frac{1}{\epsilon R}\big)\big)$, the following hold.

1) The collection $\{h_j\}_{j=1}^{p_3}$ forms a frame [5] for $V^{(1+\frac{5}{2}\epsilon)R}(g)$.
2) The frame operator S [5] corresponding to $\{h_j\}_{j=1}^{p_3}$, is invertible. Let $\{\tilde{h}_j\}_{j=1}^{p_3} := \{S^{-1}h_j\}_{j=1}^{p_3}$ and
$$F_{W\tilde{V}_\beta} := G_{W\tilde{V}_\beta} \cdot W\tilde{V}_\beta, \tag{14}$$
where $G_{W\tilde{V}_\beta}$ is the operator from $\mathbb{C}^{p_3}$ to $V^{(1+\frac{5}{2}\epsilon)R}(g)$ defined by
$$G_{W\tilde{V}_\beta}(\{c_j\}_{j=1}^{p_3}) = \sum_{j=1}^{p_3} c_j \tilde{h}_j \;\; \forall\, \{c_j\}_{j=1}^{p_3} \in \mathbb{C}^{p_3}.$$

Let $X = (x_1, \cdots, x_{n_1}) \in \mathbb{C}^{n_1}$ and $Y = (y_1, \cdots, y_{n_2}) \in \mathbb{C}^{n_2}$, then

$$X \frown Y := (x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}) \in \mathbb{C}^{n_1+n_2}.$$

Define,

1) $\tilde{y} := \frown_{j=1}^{3} \{\epsilon_i^j Pf(y_i^j)\}_{i=1}^{\tilde{m}_j}$,
2) $$y := \frown_{j=1}^{3} \{\epsilon_i^j f(y_i^j)\}_{i=1}^{\tilde{m}_j}, \tag{15}$$
3) $e := y - \tilde{y}$.

We quantize y using the distributed noise shaping quantizer Q_β. It satisfies $y - q = H_\beta u$. Consequently, it can be shown that

$$\|\tilde{f} - F_{W\tilde{V}_\beta} q\|_{L^2[-R,R]} \leq \|G_{W\tilde{V}_\beta}\| \|W\|_{2\to 2} \|\tilde{V}_\beta H_\beta\|_{\infty\to 2} \|u\|_\infty + \|G_{W\tilde{V}_\beta} W \tilde{V}_\beta e\|_{L^2[-R,R]}.$$

The first term $\|G_{W\tilde{V}_\beta}\| \|W\|_{2\to 2} \|\tilde{V}_\beta H_\beta\|_{\infty\to 2} \|u\|_\infty$, is bounded by using (9). It gives us exponential decay in $\frac{m}{p}$. Furthermore, the second term is bounded by using the fact that by construction, the error e is small in the interval $[-R, R]$. Next use the triangle inequality to bound $\|f - F_{W\tilde{V}_\beta} q\|_{L^2[-R,R]}$. In the end, choose $\epsilon = \frac{m^{\frac{1}{16}}}{R}, \lambda = 2, t = \frac{1}{12m^{\frac{1}{4}}}, \gamma = \frac{1}{8m^{\frac{1}{4}}}$ and $p = \frac{m}{m^{\frac{1}{16}}} = m^{\frac{15}{16}}$ to prove the result.

IV. Numerical experiments

In order to test the accuracy of Theorem 3.1, we run numerical experiments. We consider g defined via

$$\hat{g}(x) = \begin{cases} \frac{1}{\sqrt{2\lambda\pi}} & |x| \leq \pi \\ \frac{1}{\sqrt{2\lambda\pi}} \cos\left(\frac{\pi}{2}\nu\left(\frac{\frac{1}{\pi}|x|-1}{2\lambda-2}\right)\right) & \pi \leq |x| \leq (2\lambda-1)\pi \\ 0 & |(2\lambda-1)\pi| \leq x \end{cases}$$

where $\nu(x) = 35x^4 - 84x^5 + 70x^6 - 20x^7$, and $\lambda = 1.01$. Let $f(x) = sinc(x) + sinc(x+2)$, we aim to approximate f on the interval $[-3, 3]$ by collecting its function samples. We collect 600 samples $\{x_i\}_{i=1}^{600}$, where each x_i is sampled according to the uniform distribution on $[-12, 12]$. In the first graph of Fig 1, we plot f along with the reconstructed function $F_{W\tilde{V}_\beta}$. In our theory, the function g is defined in the Fourier domain to be a compactly supported smoothly decaying function satisfying (2) and (3) however if we ignore these additional restraints and choose g to be the $sinc$ function and λ as 1, we obtain a better approximation(see second graph of Fig 1) for f with the same reconstruction method and samples. An additional advantage as compared to working with g is that for the $sinc$ function, the above expressions can be efficiently computed, which is why we work with it in the remainder of our experiments. That said, it is not clear to us whether this simplified approach has the same asymptotic performance as we derived for g; we leave this problem for future investigations.

Let the function $f_1(x) := sinc^2(\frac{x}{2}) + sinc(x)$ and $\tilde{f}_1$ be the reconstructed function with $g = sinc$. In Fig 2, we plot average error $\|f_1(t_i) - \tilde{f}_1(t_i)\|_\infty$ after twenty iterations, where $\{t_i\}_{i=1}^{200}$ are evenly spaces points from $[-20, 20]$, along with the number of random samples m used to calculate $\tilde{f}_1$. Each sample x_i, is sampled according to the uniform distribution on $[-30, 30]$. We can see that the error decays initially with the increase in the number of samples, however after a certain stage it stagnates. This may be because the projection error $\|f_1 - Pf_1\|_{L^2[-30,30]}$ does not decrease with increasing samples, but also the fact that the sinc does not meet the assumptions of the theorem may play a role.

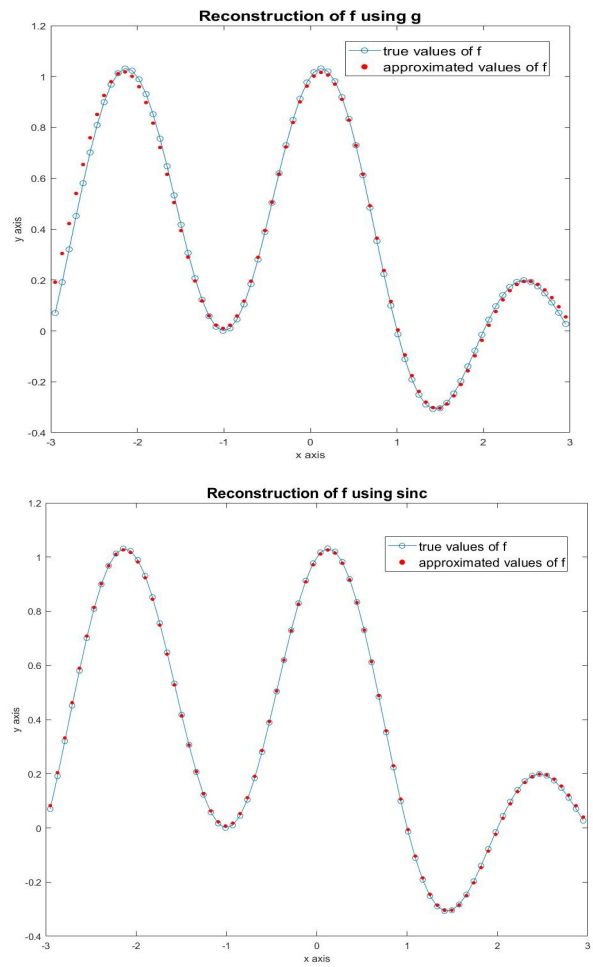

Fig. 1. We take $\beta = 6$ and the quantization alphabet as

$$\mathcal{A}_{10}^{0.1} := \left\{\pm\frac{(2l-1)}{10} : 1 \leq l \leq 10, l \in \mathbb{Z}\right\}.$$

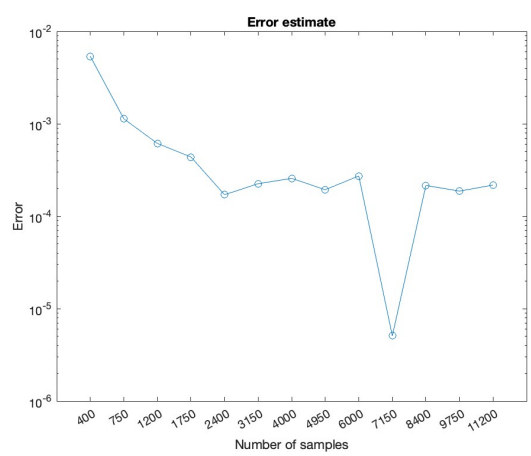

Fig. 2. We take $\beta = 3$ and the quantization alphabet as

$$\mathcal{A}_{4}^{0.25} := \left\{\pm\frac{(2l-1)}{4} : 1 \leq l \leq 4, l \in \mathbb{Z}\right\}.$$

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