Model-Based Reinforcement Learning with SINDy

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Abstract

We draw on the latest advancements in the physics community to propose a novel method for discovering the governing non-linear dynamics of physical systems in reinforcement learning (RL). We establish that this method is capable of discovering the underlying dynamics using significantly fewer trajectories (as little as one rollout with ≤ 30 time steps) than state of the art model learning algorithms. Further, the technique learns a model that is accurate enough to induce near-optimal policies given significantly fewer trajectories than those required by model-free algorithms. It brings the benefits of model-based RL without requiring a model to be developed in advance, for systems that have physics-based dynamics.

To establish the validity and applicability of this algorithm, we conduct experiments on four classic control tasks. We found that an optimal policy trained on the discovered dynamics of the underlying system can generalize well. Further, the learned policy performs well when deployed on the actual physical system, thus bridging the model to real system gap. We further compare our method to state-of-the-art model-based and model-free approaches, and show that our method requires fewer trajectories sampled on the true physical system compared other methods. Additionally, we explored approximate dynamics models and found that they also can perform well.

1. Introduction

In reinforcement learning, it is generally held that model-based approaches learn more quickly than do model-free approaches assuming the model is accurate enough (Kaelbling et al., 1996). However, it is well known that learning model is challenging in general. We show here that for systems with underlying physics dynamics, such as robotic control, one can often learn an accurate low-dimension model quickly, using very little training data from the actual system. Training with that model can result in asymptotic performance comparable to model-free approaches, using models with many fewer parameters and that are trained using less data than state of the art model-based algorithms. Thus, for systems with physics-like dynamics, we bridge the gap between model-free and model-based RL.

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2. SINDy

SINDy, which stands for Sparse Identification of Non-linear Dynamics, is an approach developed in the physics community for extracting equational models of physical systems from time series data. Specifically, SINDy can extract differential equations (ODEs, PDEs) or difference equations from data given a collection of possible equation terms (generally called *features* in the ML community) that are functions of the input values. Such terms might include polynomial and trigonometric functions of the data values. The "Sparse" of SINDy indicates that it tries to extract from a possibly large space of terms the minimum number of ones necessary for an accurate model. "Non-linear" indicates that the terms (features) may be non-linear functions of one or more input values. SINDy was introduced by Brunton et al. (Brunton et al., 2016b), who showed it is powerful enough to extract the physics even of chaotic systems such as the True Lorentz System (Brunton et al., 2016b). Similar forms apply to discrete-time and noisy systems (Brunton et al., 2016b). Brunton et al. (2016a) extends that work by extended SINDy to deal with force-driven systems (control), and (Boninsegna et al., 2018) developed a stochastic version. To demonstrate the power of the force-driven extension they solved the Lotka-Volterra predator-prey model and the Lorenz system with forcing and control.

SINDy extracts a dynamics model with applied actions by solving the equation

$$\dot{x}(t) = f(x(t); a(t)) \tag{1}$$

for f, where the vector

$$x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n \tag{2}$$

represents the observation of the system at time t, the vector

$$a(t) = [a_1(t), a_2(t), \ldots, a_k(t)]^T \in R^k \tag{3}$$

the action (typically physical forces) applied to the system at time t, the semicolon indicates vector concatenation, and the (possibly nonlinear) function $f(x(t); a(t))$ represents the dynamic constraints that define the equations of motion of the system. We write f_i for the function that defines $\dot{x}_i(t)$. SINDy uses a similar form for discrete-time difference equations. SINDy can be extended to probabilistic models, but that was not necessary in our application.

At the heart of SINDy lies a method for feature selection and sparse regression, based on the principle that only a few terms in the regression model will be important. By using intuition about the model, the user proposes a collection of feature functions, which may include polynomials, Fourier terms, etc., that the user *thinks* might govern the dynamics. Each feature is a possibly non-linear function of one of more of the input variables, that is, of the x_i and a_i. SINDy attempts to extract a model where each f_i is a *linear* function of the features.

To that end, SINDy defines

$$\hat{\Theta} = [\Theta_1, \Theta_2, \ldots, \Theta_F], \text{where } \Theta_i \in R^{n+k} \to R, \tag{4}$$

a vector of functions of the x_i and a_i that we will call *feature functions*. While the Θ_i take $n+k$ arguments, they typically depend on only a few elements of $x(t); a(t)$. F is the number of features. We further define

$$\Theta(x; a) = [\Theta_1(x; a), \Theta_2(x; a), \ldots, \Theta_F(x; a)]. \tag{5}$$

SINDY also defines an $n \times F$ matrix Ξ of real values $\xi_{i.j}$ in order to define

$$f_i(x; a) = \Theta(x; a) \cdot \Xi_i. \tag{6}$$

Thus the f_i are indeed linear functions of the possibly non-linear features, with Ξ giving the coefficients (weights) of those features for each f_i. We can now write

$$f(x; a) = \Theta(x; a) \cdot \Xi \tag{7}$$

for the overall definition of f.

We now proceed to define the optimization problem that SINDy solves. Given a sequence of observations $x(1), x(2), \ldots, x(N)$ and actions $a(1), a(2), \ldots, a(N)$, SINDy can compute actual derivatives $\dot{x}(t)$ according to several methods, and can also take user-supplied values for those derivatives or a user-supplied derivative calculating function. We used its built-in smoothed finite differencing method. We can thus form pairs for training $(\dot{x}(i), x(i); a(i))$. Note that the $x(t); a(t)$ inputs are readily extracted from RL trajectories of the typical (s, a, s') form.

We aggregate the x and a values into arrays X and A, respectively:

$$\mathbf{X} = \begin{array}{c} \xrightarrow{\text{state}} \\ \begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_n(1) \\ x_1(2) & x_2(2) & \cdots & x_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(N) & x_2(N) & \cdots & x_n(N) \end{bmatrix} \end{array} \Big\downarrow \text{time} \tag{8}$$

$$\mathbf{A} = \begin{array}{c} \xrightarrow{\text{action}} \\ \begin{bmatrix} a_1(1) & a_2(1) & \cdots & a_k(1) \\ a_1(2) & a_2(2) & \cdots & a_k(2) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(N) & a_2(N) & \cdots & a_k(N) \end{bmatrix} \end{array} \Big\downarrow \text{time} \tag{9}$$

and we write $X; A$ for the $(n+k) \times N$ array formed by appending each $a(i)$ to the corresponding $x(i)$. We extend our Θ notation to define

$$\Theta(X; A) = [\Theta(x(1); a(1)), \Theta(x(2); a(2)), \ldots, \Theta(x(N); a(N))]. \tag{10}$$

The optimization problem to be solved is then:

$$\dot{X} = \Theta(X; A) \cdot \Xi, \tag{11}$$

and we desire a solution where Ξ is sparse. Note that this is now a sparse *linear* regression problem, in terms of the (possibly non-linear) functions Θ of the input data $X; A$. SINDy can apply any of a variety of sparse regression methods. We use its Sequentially Thresholded Least Squares (STLSQ) method, which uses Ridge regression, with a threshold of 0.0009. It iteratively solves the least squares regression problem with a regularizer being the L2 norm of the weights Ξ, masking out weights below the threshold (setting them to 0).

Let us consider the very small example of a mass M moving in one dimension x under a time varying action force g. (We use g to avoid confusion with f.) Our observation are the position x and velocity v. From Newton's Law we know that $\dot{v} = g/M$, so the equations of motion are:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/M \end{bmatrix} \cdot \begin{bmatrix} x \\ v \\ g \end{bmatrix} \tag{12}$$

In this case, given suitable data, and a collection of feature functions $\hat{\Theta}$ that included $\lambda(x, v, g).v$ and $\lambda(x, v, g).g$ (or more loosely, terms v and g), SINDy should arrive at a solution Ξ whose non-zero elements are exactly the 1 and $1/M$ in the equations of motion, to within computational error. Notice that SINDy in effect discovers the mass M, that is, we knew the *form* of the equations, but not necesarily the exact values of the coefficients. This is important with real

world robots, each one of which will exhibit slight variations from a desired specification, etc.

Here we knew the exact form in advance. If we were less certain, we might include more functions in $\hat{\Theta}$, such as terms of the form 1, x, x^2, $x \cdot v$, $\sin x$, etc., and SINDy would still arrive at the same solution because of its accuracy and sparseness.

The key insights of SINDy are:

- deriving a model of physics-based dynamics using physically plausible (but possibly nonlinear) features of the observations and actions; and
- assuming a model that has a simple equational form rather than trying to learn a model via "brute force" function approximation.

Together these insights allow learning of a highly accurate model with a relatively small number of observations. Further, the models have only a small number of parameters (typically much less than the number of elements of Ξ, which itself has orders of magnitude fewer weights than a typical neural net model). One expects SINDy to do well if the problem is physics based, that is, the problem admits of solution as a relatively simple, possibly nonlinear, differential (or difference) equation. It is not a general solution for extracting a model from an arbitrary data set.

3. Dyna-Style Learning with SINDy

Algorithm 1 Dyna-Style Model-Based RL with SINDy

Hyper-parameters: Integers N_e and N
Initialize $\mathscr{D}_{env}$ and $\mathscr{D}_{SINDy}$ as empty data sets
Initialize policy π and SINDy parameters Ξ
  to random values
for N_e *rollouts* **do**
  Collect data (s_i, a, s_{i+1}) on real environment with
    random or pseudo-random policy
  $\mathscr{D}_{SINDy} \leftarrow \mathscr{D}_{SINDy} \bigcup (s_i, a, s_{i+1})$
end for
Train model Ξ on $\mathscr{D}_{SINDy}$ using SINDy
while π *is not optimal* **do**
  for N *epochs* **do**
    Collect rollout $roll_{sim}$ from model Ξ
    Train π on simulated $roll_{sim}$ using
      model-free algorithm and known reward function
  end for
  Collect a single rollout r_{real} from real environment
  Train π on r_{real} using an arbitrary model-free algorithm
  $\mathscr{D}_{SINDy} \leftarrow \mathscr{D}_{SINDy} \bigcup r_{real}$
end while

We propose a Dyna-style learning algorithm with SINDy at its base. In this algorithm SINDy learns an accurate sparse model of the non-linear dynamics of a physical RL system. We define two hyper-parameters that can be chosen depending on the complexity of the physical system we are trying to learn:

- N_e: The number of rollouts for which the algorithms collects data to train SINDy. The algorithm uses a random or pseudo-random policy for these rollouts. SINDy does not generally need much data, so N_e is typically small, and in fact a value of 1 sufficed in our experiments.
- N: The number of epochs to train using data generated by the SINDy-induced model for each epoch of training using data from the actual system, i.e., the actual system is used only one out of every $N+1$ epochs. If the SINDy model is accurate over a wide enough part of the state space, N can be set to an arbitrarily large value, which worked for a number of our experiments.

Notice that we assume $\hat{\Theta}$ has been chosen in advance and thus speak of the model as being Ξ, which strictly speaking is the coefficients of the model. The Dyna-style algorithm can readily be extended to retrain the SINDy model periodically if there were benefit to doing so, but it was not necessary in our experiments.

4. Experiments

We conducted experiments using four environments with three levels of difficulty in mind: discrete actions (Cart Pole), continuous actions (Mountain Car and Pendulum Swing Up), and realistic Mujoco physics control problems with damping and friction (Inverted Pendulum). We use the Python package PySindy open sourced by Kaptanoglu et al. (Kaptanoglu et al., 2021). We selected variants of SINDy from among its continuous-time, discrete-time, and driven models appropriate to each experiment. We use a state of the art model-free algorithm, Soft Actor-Critic (SAC), introduced by Haarnoja et al. (Haarnoja et al., 2018) and extended by Christodoulou (Christodoulou, 2019), as our basis for training on both the real system and the simulated rollouts described in Algorithm 3. We also compare our results against Model-Based Policy Optimization (MBPO), a state of the art model-based method introduced by Janner et al. (Janner et al., 2019). MBPO and our model differ primarily in that MBPO learns a model represented by a neural net while SINDy learns a model represented by a differential (or difference) equation. Our method outperforms both other methods in all experiments, giving us a 4–100×, 40×, and 200× speedup against MBPO, and 15–375×, 60×, 500×, and 5× speedup against SAC for the

Inverted Pendulum, Pendulum Swing Up, Mountain Car, and Cart Pole problems, respectively.

Figure 1 shows our experimental results. Each experiment is averaged over 10 different seeds, with further averaging performed by evaluating the agent 10 times for each seed. All results involve running the true robotic system once for a small designated number of time steps (Table 1) to obtain samples for training the SINDy model, and then performing further policy improvement steps based only on the SINDy derived model. The plots for SINDy have been shifted right by the number of time steps gathered to train SINDy based on interactions with the real environment. We now offer more details of each experiment.

Discrete Classic: Our discrete action case is the Cart Pole environment. Data from a single rollout of 30 steps was sufficient for SINDy to identify a high accuracy dynamics model. To avoid rapid termination of an episode we applied force in opposite directions at alternating time steps, with occasional random actions taken for exploration.

We can summarize the dynamics of the Cart Pole system in these equations:

$$\ddot{\theta} = \frac{(g \cdot \sin\theta - \cos\theta \cdot C)}{l \cdot (\frac{4}{3} - \frac{m_p}{m_p+m_c} \cdot (\cos\theta)^2)},$$

$$\ddot{x} = C - \frac{l}{m_p + m_c} \cdot \ddot{\theta} \cdot \cos\theta \tag{13}$$

where

$$C = (F + l \cdot \dot{\theta}^2 \sin\theta)/(m_p + m_c),$$

where l is the length of the pole, m_p its mass, x the position of the cart, m_c its mass, θ the vertical angle between the two, and F the force. Using a small θ assumption, since the pole falls and the episode terminates if the θ is not small, we can write $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, resulting in these equations:

$$\ddot{\theta} = \frac{(g \cdot \theta - C)}{l \cdot (\frac{4}{3} - \frac{m_p}{m_p+m_c})},$$

$$\ddot{x} = C - \frac{l}{m_p + m_c} \cdot \ddot{\theta} \tag{14}$$

where

$$C = (F + l \cdot \dot{\theta}^2 \cdot \theta)/(m_p + m_c),$$

Thus, substituting the right hand side of the equation for $\ddot{\theta}$ into the one for $\ddot{x}$, the right hand sides of the system of dynamics equations can be written in terms of θ, $\dot{\theta}$, x, $\dot{x}$, and constants. The $\hat{\Theta}$ we provided to SINDy was $\begin{bmatrix}1, & a, & a^2, & a \cdot b, & a^2 \cdot b\end{bmatrix}$ where a and b can be any of θ, $\dot{\theta}$, x, and $\dot{x}$. Figure 1 shows that the approximation is accurate enough that quitting when we are optimal using the SINDy model is still accurate in the real environment and needs just two further episodes of training in the real environment for fine tuning.

Continuous Classic: The equations governing the dynamics for the continuous environments of Mountain Car and Pendulum Swing Up are learned precisely by SINDy (to 4 decimal places), extracting appropriate features from among a larger set. For example, for Mountain Car, the features we used were 1, x, x^2, and $\sin kx$ and $\cos kx$ for $k = 1, 2, 3$. The results of Figure 1 show that the policy that is optimal on the SINDy dynamics is also optimal in the real environment. In fact, when we examine the equations learned by SINDy, we see that they ***match the true dynamics*** of these environments. It is worth noting that the Mountain Car domain has inelastic collisions, and our model is robust in the face of those discontinuities when learning dynamics. Furthermore SINDy learned these dynamics using a single rollout of length 100 for Mountain Car and 20 for Pendulum Swing Up.

Mujoco: We now consider the Inverted Pendulum domain of the Mujoco physics simulator. This adds the challenge of having damping on the controller and friction added to the system. The dynamics that SINDy learned generalize well to the true environment as can be seen in Figure 1 figure, and it needed ***at most one additional training episode*** for fine tuning the policy learned using the model so that it is optimal with respect to the true physical system. We used the same approximation here as for Equation 14 and used the same policy for collecting the initial samples.

Discussion: As Table 1 shows, our approach learns models that have a very small number of parameters, particularly compared with those learned by MBPO (613,036 parameters). Furthermore, since our models represent physics dynamics equations explicitly, they are highly interpretable, while function approximation neural nets generally are not. As previously discussed, the models are very accurate and can be learned with only a small amount of training data. The method works for a range of kinds of dynamics and control, both continuous and discrete. We have further seen that approximate dynamics, such as replacing $sin(\theta)$ by θ when angles tend to be small in the region of the state space that is of interest, can lead to dynamics models accurate enough for training to result in optimal behavior.

5. Conclusions and Future Work

We presented a method for learning the model of physics based RL systems. It is capable of learning either exact or high-accuracy approximate non-linear dynamics from small numbers of samples. We showed results from four environments demonstrating how training with these models results in asymptotic performance as good as that achieved by state of the art model-free methods, while converging

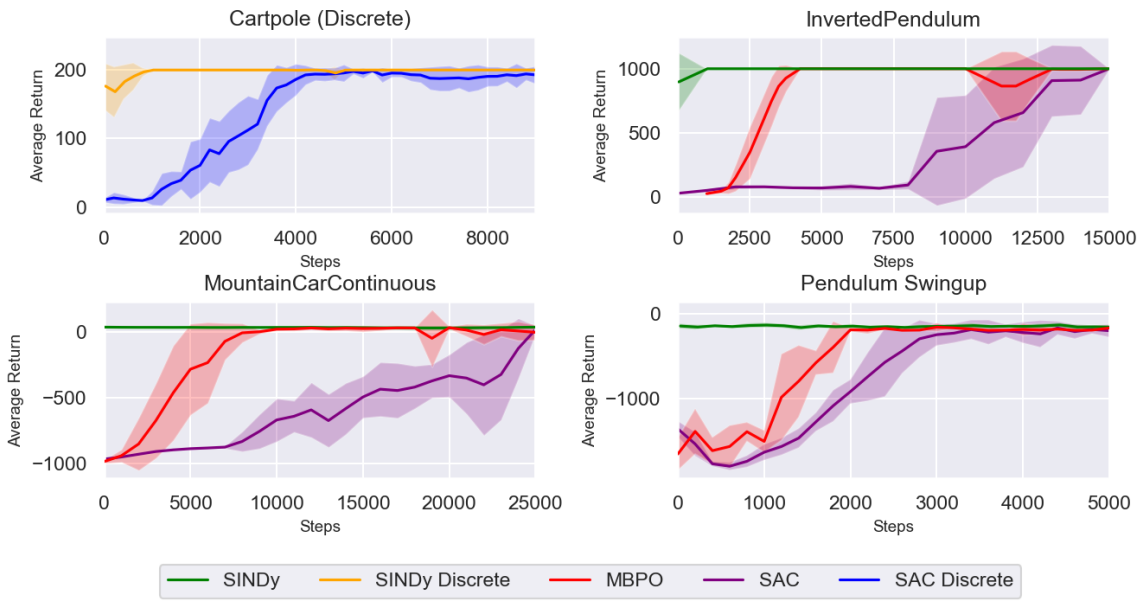

Figure 1. The results of our SINDy method compared to state of the art model-based (MBPO) and model-free (SAC) methods. Our method outperforms current methods in discrete action, continuous action, and noisy/damped environments. We found 4–100×, 40×, and 200× speedup against MBPO and 15–375×, 60×, 500×, and 5× speedup against SAC for the Inverted Pendulum, Pendulum Swing Up, Mountain Car, and Cart Pole, respectively.

significantly more rapidly and requiring less training data. Our dynamics models are of low dimension, easy to extract, and highly interpretable, advantages they have over state of the art model-based methods.

In summary, our contributions are:

1. Our algorithm matches or exceeds the asymptotic performance of existing state of the art model-based and model-free learning methods on these tasks while requiring significantly fewer time steps of interaction with the real system. In our experiments, we needed at most 50 time steps of interaction with the real system to identify high accuracy models that allow induction of near optimal policies. We reduced the time steps of interaction with the real system necessary to convergence by 4×–100×, 40×, and 200× against MBPO, and 15×–375×, 60×, 500×, and 5× against SAC for Inverted Pendulum, Pendulum Swing Up, Mountain Car, and Cart Pole, respectively.

2. Our method requires significantly fewer parameters than state of the art model-based methods. We need at most $(n+k)^2 \cdot F$ parameters, where n is the dimensionality of the state space, k the dimensions of actions, and F the number of features from which SINDy can choose. In comparison, the MBPO network used 613,036 parameters.

3. Our dynamics models are more interpretable, working from intuitively selected or approximated kernel functions, and extracting the governing physics dynamics equations and their parameters (coefficients).

A future direction for this work is exploring more complex robotic systems supported by the Mujoco physical simulation framework.

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Table 1. Hyper-parameters used for training the SINDy model and Results

Environment	N_e	R	N	Dynamics learned	Generalizes	P	P'
Cart Pole	1	30	∞	Approximate	Yes	100	$\sim$70
Mountain Car	1	50	∞	Exact	Yes	81	7
Pendulum Swing up	1	20	∞	Exact	Yes	64	10
Inverted Pendulum	1	30	∞	Approximate	Yes	100	50

N_e is the number of rollouts; R is the length of each rollout; N is the number of episodes using just the model (vs. the actual system); P is the number of parameters (size of Ξ); P' is the number of non-zero parameters (Cart Pole is stochastic so the number varied a little). The MBPO model sized was: 613,036 parameters. The SINDy model size was $(n+k)^2 \cdot F$, where n is the number of dimensions in the state, k the number of dimensions in the actions of the dynamics, and F the number of functions (features) in Ξ.

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