Online Optimal Tracking of Linear Systems with Adversarial Disturbances

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Abstract

This paper presents a memory-augmented control solution to the optimal reference tracking problem for linear systems subject to adversarial disturbances. We assume that the dynamics of the linear system are known and that the reference signal is generated by a linear system with unknown dynamics. Under these assumptions, finding the optimal tracking controller is formalized as an online convex optimization problem that leverages memory of past disturbance and reference values to capture their temporal effects on the performance. That is, a (disturbance,reference)-action control policy is formalized, which selects the control actions as a linear map of the past disturbance and reference values. The online convex optimization is then formulated over the parameters of the policy on its past disturbance and reference values to optimize general convex costs. It is shown that our approach outperforms robust control methods and achieves a tight regret bound $\mathcal{O}(\sqrt{T})$.

1 Introduction

Reference tracking is one of the key concepts in control theory (Isidori, 1989; Huang, 2004). In the reference tracking problem, the aim is to design a controller such that the state of the system tracks a desired reference trajectory. There are typically two common approaches for the reference tracking problem (Isidori, 1989; Huang, 2004). The first approach is called the "feedforward design". In this approach, the controller is a summation of i) a feedforward term depending on the reference signal, which is derived from the dynamics of the system and reference generator and ii) an internal state feedback to stabilize the system in the absence of disturbances. The second approach is called the "internal model". In this approach, a dynamic controller contains an internal model of the reference signal and the control signal is a feedback from the internal state of the controller and the state of the system. Both approaches require the full knowledge of the system dynamics and reference signal generator dynamics.

Asymptotic reference following is the bare minimum requirement for the tracking control problem. To account for the transient response and the overall performance of the control design, an optimal reference tracking control problem is typically formalized and solved. One major factor that can adversely affect the performance of the tracking controllers is the presence of disturbances, which is typically ignored in the optimal reference tracking problem (Zhang et al., 2011; 2008; Huang & Liu, 2014; Kamalapurkar et al., 2015; Adib Yaghmaie et al., 2019; Modares & Lewis, 2014; Kiumarsi & Lewis, 2015; Kiumarsi et al., 2014; 2015). The account for the effect of the disturbance, it is common to assume one of the following: 1- the disturbance is generated by a dynamical system (Isidori, 1989; Huang, 2004), 2- the disturbance is Gaussian (Bertsekas, 2012), and 3- the disturbance is energy bounded and the effect of its worst-case realization is attenuated on the control performance in the robust control terminology (Khalil, 2002). In most cases, however, the disturbance is neither Gaussian nor generated by a dynamic system. Besides, the robust control approach yields conservative results because the disturbance is most likely far away from its worst-case realization (Khalil, 2002). In the related works, we discuss each case in detail.

In this paper, we design optimal tracking controllers for linear systems subject to adversarial disturbances. The adversarial disturbances are arbitrary and thus are not limited to Gaussian or those that are generated by a dynamical system. The reference signal to be tracked is assumed to be generated by a linear system

with unknown dynamics. We assume that only the output of the reference is measurable. We design (disturbance,reference)-action control policies where a fixed-size history of disturbance and reference values are used to parameterize the proposed policy. This is partially inspired by Agarwal et al. (2019); Zhao et al. (2022), which are designed for solving the optimal regulation control problems. In contrast, we leverage the past values of both of disturbances and reference values. Using the past history of reference values is motivated by a classical result giving necessary and sufficient conditions for tracking in control theory. Our approach results in a neat parameterization of the control policy from which any general convex cost function can be optimized using online convex optimization. Indeed, we prove that the cost function is convex with respect to the parameters of the presented controller.

The resulting algorithm is online in contrast to rollout or batch-wise reinforcement learning algorithms where it is required to collect enough samples before updating controller (Abbasi-Yadkori et al., 2014). In sharp contrast to the robust control design approach, a history of fixed-size past disturbance and references values are calculated, stored, and used by the control policy to avoid hedging against the worst-case disturbances that rarely occur in reality (Khalil, 2002; Modares et al., 2015). We show that our proposed algorithm achieves a tight regret bound. Simulation results compare the presented approach against the H_∞ control as well as the LQR control to show its superiority.

2 Related works

In this section, we summarize related works to the problem of optimal tracking in presence of disturbance.

2.1 Tracking in the presence of disturbance

Depending on the nature of the disturbance, different strategies can be followed.

Output regulation theory The output regulation theory (Isidori, 1989; Huang, 2004) can be leveraged to attenuate the effect of disturbances that are generated by a dynamic system (Chen et al., 2019; Jiang et al., 2020; Gao & Jiang, 2016; 2015). The disturbance, however, is rarely generated by a dynamic system, which limits the applicability of the output regulation theory.

Gaussian disturbance: For linear systems with Gaussian disturbance (noise), linear quadratic regulator (LQR) control can be used to design an optimal controller for the regulation problem by minimizing a quadratic cost (Bertsekas, 2012). The feedforward gain is then calculated using the full knowledge of the dynamic of the system and reference. However, there are many control systems for which the distribution of the disturbance is not Gaussian.

Robust control design: For general but limited-energy disturbances, one can use the H_∞-control theory to guarantee an $\mathcal{L}_2$-gain performance bound (Khalil, 2002; Modares et al., 2015). The H_∞ approach is typically overly conservative as the resulting robust controller hedges against the worst-case disturbance sequence, which rarely occurs in reality. A daunting challenge is to design non-conservative optimal tracking controllers for systems with arbitrary adversarial disturbances that do not follow assumptions such as being generated by an i.i.d. Gaussian noise sequence or by a system dynamic.

2.2 Notion of Optimality

To account for the transient response and the overall performance, one can introduce optimal control design to the tracking controller problem. This is usually done by designing (some part of) the controller by optimizing a performance index using the optimal control theory or reinforcement learning.

Average or discounted costs: In Zhang et al. (2011; 2008); Huang & Liu (2014); Kamalapurkar et al. (2015); Dierks & Jagannathan (2010) a feedforward approach is used to solve the tracking problem. The feedback part of the controller is designed by minimizing an average or discounted cost in reinforcement learning frameworks, and the feedforward part of the controller is found by dynamic inversion, assuming that the dynamics is known.

Similarly, Adib Yaghmaie et al. (2019); Modares & Lewis (2014); Kiumarsi & Lewis (2015); Kiumarsi et al. (2014; 2015) consider a feedforward approach to solve the tracking problem. But this time, both feedback and feedforward parts of the controller are designed optimally by minimizing average cost (Adib Yaghmaie et al., 2019) or discounted cost (Modares & Lewis, 2014; Kiumarsi & Lewis, 2015; Kiumarsi et al., 2014; 2015).

Regret: The regret compares the performance of an online control algorithm with a fixed (usually the optimal) policy in hindsight. In the context of control theory, the regret analysis is *usually* given in the regulation problem where there is no reference signal to be tracked, and the aim is to make the state vector converges to zero (Agarwal et al., 2019; Zhao et al., 2022).

In Abbasi-Yadkori et al. (2014), tracking adversarial reference trajectories with quadratic costs are considered. There are no disturbances in the problem formulation, and the regret grows as $\mathcal{O}(\log^2 N)$, where N is the number of rollouts. Tracking adversarial references with convex costs is considered in Zhang et al. (2022) where an algorithm is given to estimate the state of the system based on observed data. The control signal is then generated by an algorithm, and it is not parameterized as it usually is in control theory. Since the control signal is not parameterized, it is difficult to perform analysis in the context of control theory. It has been shown that the regret is $\mathcal{O}(\sqrt{|\mathcal{I}|})$ for time interval $\mathcal{I}$ in the time horizon $[1, T]$.

3 Optimal Reference Tracking Problem

Notations and preliminaries: Let I denote an identity matrix with appropriate dimension. Let $\mathbf{1}$ and $\mathbf{0}$ denote one and zero matrices with appropriate dimensions respectively. Let $\nabla_x f$ denote the gradient of function $f(x)$ with respect to x. The $\mathcal{L}_2$-norm of x is denoted by $\|x\|_{\mathcal{L}_2} = (\sum_{k=0}^{+\infty} |x_k|^2)^{\frac{1}{2}}$ where $|x_k|$ is the instantaneous Euclidean norm of x_k. The Frobenius norm of matrix A is denoted by $\|A\|_F$. Let $\mathbb{I}_E$ be an indicator function on set E. For a time-dependent variable x_k, the notation $x_{i:j}$, $j \geq i$ is defined as $x_{i,j} = \{x_i, x_{i+1}, .., x_j\}$. The notation $\mathcal{O}(\mathring{u})$ is leveraged throughout the paper to express the regret upper bound as a function of T.

Definition 1 *(Agarwal et al., 2019) Consider*

$$x_{k+1} = Ax_k + Bu_k$$

and $\gamma \in [0, 1)$, $\kappa > 1$. A linear controller K is (κ, γ)-stable if $\|K\| \leq \kappa$ and $\|\tilde{A}_K^t\|_2 \leq \kappa^2(1-\gamma)^t \ \forall\, t \geq 0$ where $\tilde{A}_K = A + BK$.

3.1 Dynamical System and Reference Signal

Consider the following linear dynamical system

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ denote the state and the control input of the system, respectively. $w_k \in \mathbb{R}^n$ denotes the adversarial input, which is captured by a general (i.e., arbitrary and unknown) disturbance. The only assumption on the disturbance is that it is bounded. We can assume without loss of generality that $x_0 = \mathbf{0}$ and push the initial condition into w_0.

Assumption 1 (dynamical system) *The pair (A, B) is known and stabilizable. Moreover, the system matrices are bounded, i.e., $\|A\| \leq \kappa_a$ and $\|B\| \leq \kappa_b$.*

Assumption 2 (disturbance) *The disturbance sequence w_k is bounded, i.e., $\|w_k\| \leq \kappa_w$ for some $\kappa_w > 0$. Moreover, $w_k = \boldsymbol{0}$ for $k < 0$.*

Since the system dynamics are assumed to be known, at each time k, $w_{1:k-1}$ are known. This is because $w_{k-1} = x_k - Ax_{k-1} - Bu_{k-1}$ and the state x_k is assumed measurable.

Remark 1 *Assumption 1 is a standard one. If the dynamics are not known, one can use Algorithm 2 in Hazan et al. (2020) and identify the dynamics by injecting random input to equation 1. In Assumption 2, we make a standard assumption that the disturbance is bounded.*

Our aim in this paper is to select the input u_k such that the state of the system x_k tracks an unknown linear reference signal r_k generated by

$$\begin{aligned} z_{k+1} &= Sz_k, \\ r_k &= Fz_k, \end{aligned} \tag{2}$$

where $z_k \in \mathbb{R}^p$ and $r_k \in \mathbb{R}^n$ denote the state and output of the reference signal, respectively.

Assumption 3 (reference signal) *The following assumptions are made on the reference signal*

- *The pair (S, F) is unknown, but observable.*
- *The state of the reference signal z_k is not measurable but the output r_k is measurable.*
- *The reference signal r_k is bounded, i.e., $\|r_k\| \leq \kappa_r$.*

Remark 2 *Assumption 3 is general in the sense that it allows the reference signal to have arbitrary linear dynamics. The only limiting point in Assumption 3 is that the reference signal needs to be bounded. Removing this limiting assumption is a direction of our future work.*

We first bring a classical result in Theorem 1 specifying the necessary and sufficient condition for the existence of a linear feedback policy to solve the classical state tracking problem, i.e., to ensure that $x_k \to r_k$, in the absence of disturbances. A linear feedback policy is defined as follows

$$u_k^{\text{lin}}(K_f) = K_{fb}x_k + K_{ff}z_k. \tag{3}$$

where $K_f = [K_{fb}\ K_{ff}] \in \mathcal{K}$ and $\mathcal{K} = \{K_f : A + BK_{fb} \text{ is } (\kappa, \gamma) - \text{stable}\}$.

Theorem 1 *(Isidori, 1989) Consider the dynamical system in equation 1 and the reference signal in equation 2. Assume that $w_k \equiv \boldsymbol{0}$, (A, B) is stabilizable and (S, F) is detectable. Select K_{fb} such that $A + BK_{fb}$ is strictly stable. Then, the controller*

$$u_k = K_{fb}x_k + (\Gamma - K_{fb}\Pi)z_k \tag{4}$$

solves the classical state tracking problem $x_k \to r_k$ if and only if there exist matrices $\Pi \in \mathbb{R}^{n\times p}$ and $\Gamma \in \mathbb{R}^{m\times p}$ such that

$$\Pi S = A\Pi + B\Gamma, \quad \Pi - F = \boldsymbol{0}. \tag{5}$$

We show in the next lemma that even though z_k is not measurable, it can be extractable from the current and past outputs of the reference if the dynamics of the reference are known.

Lemma 1 *Assume that (S, F) is observable. Let l denote the observability index of equation 2; i.e., the smallest positive integer $l \geq 1$ such that*

$$\mathcal{O}_l = \begin{bmatrix} F \\ \vdots \\ FS^{l-1} \end{bmatrix} \in \mathbb{R}^{nl\times p} \tag{6}$$

has full column rank. That is, rank$(\mathcal{O}_l) = p$*. Let*

$$\begin{aligned} &\mathcal{O}_l^+ = (\mathcal{O}_l^T \mathcal{O}_l)^{-1} \mathcal{O}_l^T, \\ &N = \begin{bmatrix} N^{[1]} & \dots & N^{[l]} \end{bmatrix} = S^{l-1} \mathcal{O}_l^+, \\ &N^{[s]} \in \mathbb{R}^{p \times n}, s = 1, ..., l. \end{aligned} \tag{7}$$

Then, the state of the reference signal can be expressed as a linear function of the current and $l-1$ *past outputs of the reference*

$$z_k = \sum_{q=0}^{l-1} N^{[l-q]} r_{k-q}. \tag{8}$$

Proof: See Appendix A.

The following corollary uses the results of this lemma to formalize the controller as a memory-augmented controller, which depends on the past values of the reference outputs.

Corollary 1 *Consider the dynamical system in equation 1, the reference signal in equation 2 and* $w_k \equiv \mathbf{0}$*. Assume that there exist matrices* $\Pi \in \mathbb{R}^{n \times p}$ *and* $\Gamma \in \mathbb{R}^{m \times p}$ *such that equation 5 holds. Select* K_{fb} *such that* $A + BK_{fb}$ *is strictly stable. Then*

$$u_k^{lin} = K_{fb} x_k + \sum_{s=0}^{l-1} (\Gamma - K_{fb} \Pi) N^{[l-s]} r_{k-s} \tag{9}$$

solves the classical state-tracking problem $x_k \to r_k$*, where* l *is the observability index of equation 2 and* N *is given in equation 7.*

Proof: The proof is based on Lemma 1 and Theorem 1.

The controller in equation 9 only guarantees asymptotic convergence of the system's state to the reference trajectory. To account for the performance, an optimal state tracking controller is typically designed by optimizing a cost function with respect to the control gains. However, the controller in equation 4 (equation 9) requires the knowledge of Π (Π and N), which is found by solving equation 5 (equation 5 and equation 7), which in turns requires the complete knowledge of the reference dynamics. As stated in Assumption 3, this knowledge is typically not available. Besides, the disturbance is either ignored in this control design approach or attenuated using overly-conservative robust control design methods. Therefore, to account for the unknown dynamics of the reference generator and to design non-conservative controllers against adversarial disturbances, a new controller is designed in the subsequent sections that leverages the past disturbances and reference values to capture their temporal effects on the performance.

3.2 Optimal (Disturbance,Reference)-Action Policy Design

The overall objective of this paper is to design a control policy $\pi : (x_{1:k}, w_{1:k}, r_{1:k}) \to u_k$ that optimizes a cost function that captures the intention of the designer. The average cost associated with a policy π is defined as follows

$$J_T(\pi) = \frac{1}{T} \sum_{k=1}^{T} c_k(e_k, u_k), \tag{10}$$

where c_k is the rolling cost, and

$$e_k = x_k - r_k. \tag{11}$$

is the state tracking error.

Assumption 4 (cost function) *The cost $c_k(e_k, u_k)$ is convex in e_k, u_k. Moreover, when $\|e\|$, $\|u\| \leq D$, it holds that $|c_k(e_k, u_k)| \leq \beta D^2$ and $\|\nabla_e c_k(e, u)\|, \|\nabla_u c_k(e, u)\| \leq G_c D$ for some $\beta > 0$ and $G_c > 0$.*

Assumption 4 limits the cost function to general convex functions, which is more general than typical quadratic cost functions. To optimize this cost function, the following parameterization of the control policy is leveraged.

Definition 2 *(Memory-augmented Control Policy). A (disturbance-reference)-action policy $\pi(K, M, P)$ with memory is specified by parameters $M = [M^{[0]}, ..., M^{[m_w-1]}]$, $P = [P^{[0]}, ..., P^{[m_r-1]}]$, and a fixed matrix K. At every time k, this policy chooses the action u_k at a state x_k using the following parameterized controller*

$$u_k^{\pi}(K, M, P) = Kx_k + \sum_{t=1}^{m_w} M^{[t-1]} w_{k-t} + \sum_{s=0}^{m_r-1} P^{[s]} r_{k-s}, \tag{12}$$

Since the policy parameters will be learned, and thus are changing over time, we refer to $M_k = [M_k^{[0]}, ..., M_k^{[m_w-1]}]$ and $P_k = [P_k^{[0]}, ..., P_k^{[m_r-1]}]$ as the policy parameters at time k.

The memory-augmented controller in equation 12 is linear in the state x_k, the history of the reference signal of length m_r and the history of disturbance of length m_w. We call this controller a *linear history-based policy or memory-augmented control policy.*

Inspired by Agarwal et al. (2019); Zhao et al. (2022), the following assumption on the control parameters is included on the memory-augmented controller for technical simplicity and without losing generality.

Assumption 5 *For the control policy $\pi(K, M, P)$ in Definition 2,*

- *The control gain K makes $A + BK$ (κ, γ)-strongly stable.*
- $\|M^{[t]}\|$, $\|P^{[t]}\| \leq \kappa_b \, \kappa^3 (1 - \gamma)^t$.

Remark 3 *In the absence of the reference signal $r_k \equiv \mathbf{0}$, $\forall k$, the controller in equation 12 is simplified to the disturbance-action policy in Agarwal et al. (2019); Zhao et al. (2022). The optimal reference tracking problem leads to a challenge of designing the controllers parameters such that the output regulator equations in equation 5 are implicitly solved by solving the optimal control problem. Note that m_r is the smallest integer for which rank$(\mathcal{O}_l) = p$, where $\mathcal{O}_l$ is defined in equation 6. Therefore, if l is known, one can select m_r accordingly. Otherwise, one can select m_r large enough based on the knowledge of the dimensions of the reference generator dynamics. However, as shown later, for the past disturbances part, there is a tradeoff between the performance and the computational tractability in selecting m_w.*

Problem 1 (Optimal Tracking Against Adversarial Disturbances): Consider the system in equation 1 under Assumptions 1 and 2. Let the reference signal be generated by equation 2 under Assumption 3. Design an algorithm or control policy that generates the control actions in the form of equation 12 to optimize the convex cost function in equation 10.

4 Properties of Memory-augmented Control Policies

To solve Problem 1, we select the controller gain K in the controller in equation 12 to stabilize the dynamics in the absence of disturbance w_k and reference r_k. We keep K unchanged and aim to learn M and P to achieve optimality. The learning procedure is given and discussed in details in Section 5. Before presenting the learning algorithm and proving its regret analysis, the following results are needed.

For the standard linear controller in the form of equation 3, in the presence of an adversarial or arbitrary disturbance, the H_∞ control design finds the gains K_{fb}, K_{ff} to attenuate the effect of the disturbance on the

cost function. However, besides its conservativeness, as shown next, the cost function $c_k(e_k, u_k)$ is not convex in K_{fb}, K_{ff}, which makes the online control design intractable. To circumstance this difficulty and to avoid the design of an overly-conservative controller, a controller in the form of equation 12 is designed. We show next that, first, the cost function $c_k(e_k, u_k)$ is convex in M, P (see Lemma 2), and, second, equation 12 can approximate any linear feedback policy in the form of equation 3 (see Theorem 2). Therefore, the presented history-based or (disturbance,reference)-action controller is favorable over the linear feedback policy u_k^{lin}.

Lemma 2 *Consider the dynamical system and reference signal in equation 1-equation 2. Then, the cost function $c_k(e_k, u_k)$ is convex in M, P for the memory-augmented controller in the form of equation 12, but is not convex in K_{fb}, K_{ff} for the memoryless controller in the form of equation 3.*

Proof: See Appendix B.

Since the policy parameters will be learned, they are changing over time. In this case, the following Lemma shows that the state at time k depends on the entire past memory of the control parameters.

Lemma 3 *Let x_k^π be the state attained upon execution of the policy $\pi(K, M_{0:k-1}, P_{0:k-1})$ that generates the control input in equation 12 at time k. Then, assuming $x_0 = 0$, one has*

$$x_k^\pi x_k^K(M_{0:k-1}, P_{0:k-1}) = \sum_{y=0}^{k-1} \Psi_{k,y}^{K,k}(M_{0:k-1}) w_{k-y-1} + \sum_{z=0}^{k-1} \psi_{k,z}^{K,k}(P_{0:k-1}) r_{k-z} \tag{13}$$

where

$$\Psi_{k,y}^{K,h}(M_{k-h-1:k-1}) = \tilde{A}_K^y \mathbb{I}_{y \le h-1} + \sum_{j=0}^{h-1} \tilde{A}_K^j B M_{k-j-1}^{[y-j-1]} \mathbb{I}_{1 \le y-j \le m_w}, \tag{14}$$

$$\psi_{k,z}^{K,h}(P_{k-h-1:k-1}) = \sum_{j=0}^{h-1} \tilde{A}_K^j B P_{k-j-1}^{[z-j-1]} \mathbb{I}_{1 \le z-j \le m_r}, \tag{15}$$

with $\tilde{A}_K = A + BK$.

Proof: See Appendix C.

The results of Lemma 3 shows that the memory length grows with time, which is not feasible for developing an online gradient descent based algorithm for learning the policy parameters. Inspired by Agarwal et al. (2019), we present a truncated method that truncates the state with fixed memories lengths for both the disturbance and the reference. We also define a truncated loss accordingly.

More specifically, we truncate the state with a fixed memory length H. Let $\tilde{x}_k^\pi, \tilde{u}_k^\pi$, $\tilde{c}_k$ denote the truncated state, input and loss if the system would have started at $\tilde{x}_{k-H}^\pi = \mathbf{0}$. By setting $H = h$ in equation 32 and using equation 12, $\tilde{x}_k^\pi$, $\tilde{u}_k^\pi$ read

$$\tilde{x}_k^\pi(M_{k-H-1:k-1}, P_{k-H-1:k-1}) = \tag{16}$$
$$\sum_{y=0}^{m_w+H-1} \Psi_{k,y}^{K,H}(M_{k-H-1:k-1}) w_{k-y-1} + \sum_{z=0}^{m_r+H-1} \psi_{k,z}^{K,H}(P_{k-H-1:k-1}) r_{k-z},$$
$$\tilde{u}_k^\pi(M_{k-H-1:k}, P_{k-H-1:k}) = \tag{17}$$
$$K\tilde{x}_k^K(M_{k-H-1:k-1}, P_{k-H-1:k-1}) + \sum_{t=1}^{m_w} M_k^{[t-1]} w_{k-t} + \sum_{s=0}^{m_r-1} P_k^{[s]} r_{k-s},$$

and the truncated loss reads

$$\tilde{c}_k(\tilde{x}_k^\pi - r_k, \tilde{u}_k^\pi). \tag{18}$$

In Appendix D, we bring several lemmas which will be used to prove the main results in Theorems 2-4. Specifically,

- Lemma 4 gives bounds for $\Psi_{k,y}^{K,h}, \psi_{k,z}^{K,h}$ in equation 14-equation 15.
- Lemma 5 gives bounds on the states and inputs.
- Lemma 6 defines the Lipschitz condition on the truncated cost.
- Lemma 7 gives a bound on the gradient of the truncated cost.

Theorem 2 *Consider the dynamical system and reference signal in equation 1-equation 2. Let l denote the observability index of equation 2. Let u_k^{lin} in equation 3 be a linear feedback policy with K_{fb} being (κ,γ)-strongly stable. Then, for any (κ,γ)-strongly stable K, there exists a linear history-based policy of form equation 12, with $l \leq m_r$ and*

$$\begin{aligned} M^{[t]} =&(K_{fb}-K)(A+BK_{fb})^t, \quad 0 \leq t < m_w \\ P^{[0]} =&K_{ff}N^{[l]}, \\ P^{[s]} =& \sum_{q=0}^{\min(s-1,l-1)} (K_{fb}-K)(A+BK_{fb})^{s-q-1}BK_{ff}N^{[l-q]} \\ &+ \mathbb{I}_{0<s<l}K_{ff}N^{[l-s]}, \qquad\qquad 0 < s < m_r, \end{aligned} \tag{19}$$

such that u_k^π in equation 12 approximates u_k^{lin} in equation 3. Moreover, for $k > \max(m_w, m_r)$

$$\|u_k^{lin} - u_k^\pi\| \leq (k-m_w)\kappa_b\kappa^3(1-\gamma)^{m_w}\kappa_w + (k-m_r+1)\kappa_b\kappa^3(1-\gamma)^{m_r}\kappa_r.$$

Proof. See Appendix E.

Remark 4 *In Theorem 2, we proved that equation 12 can approximate equation 3. According to equation 2, since $\|u_k^{lin} - u_k^\pi\|$ is a function of $(1-\gamma)^{m_w}$ and $(1-\gamma)^{m_r}$, the approximation error decreases for longer history lengths m_w, m_r.*

We have seen in Theorem 2 that a linear history-based policy u_k^π can approximate the linear feedback policy in equation 3. In the next theorem, we show that if we use u_k^{lin} and u_k^π for the same system in equation 1 with the same sequences of disturbance, the corresponding trajectories x_k^{lin} and x_k^π are close.

Theorem 3 *Consider the dynamical system and reference signal in equation 1-equation 2. Let l denote the observability index of equation 2. Assume that K_{fb} and K are (κ,γ)-strongly stable. Let x_k^π denote the trajectory of the system using the linear history-based policy u_k^π in equation 12 with the parameters in equation 19 and x_k^{lin} denote the trajectory of the system using the linear policy u_k^{lin} in equation 3. Assume that $w_{k-i}, r_{k-i}, i < k$ are the same in both cases. Then x_k^π is close to x_k^{lin}. More specifically for $k > \max(m_w, m_r)$*

$$\|x_k^{lin} - x_k^\pi\| \leq \gamma^{-1}(k-1-m_w)\kappa^5\kappa_b^2(1-\gamma)^{m_w}\kappa_w + \gamma^{-1}(k-1-m_r)\kappa^5\kappa_b^2(1-\gamma)^{m_r}\kappa_r. \tag{20}$$

Proof. See Appendix F.

5 Memory-augmented online state-tracking algorithm

In this section, we will give an algorithm to tune the parameters of the linear history-based policy u_k^π in equation 12, namely M, P to provide optimality in terms of minimization of the cost in equation 10. Note that we consider optimizing over the class of linear history-based policy u_k^π in equation 12 *not* the class of linear feedback policy u_k^{lin} in equation 3. The reason is that c_k is convex with respect to M, P appearing in u_k^π but is not convex in K_{fb}, K_{ff} in u_k^{lin}, see Lemma 2. The following Algorithm 1 optimizes the truncated cost $\tilde{c}_k$ using the gradient descent method.

5.1 The memory-augmented algorithm

Algorithm 1 summarizes the online state tracking procedure. In **Line 1**, the algorithm is initialized by selecting a stabilizing controller gain K and setting M, P arbitrarily. One way to select K is by solving a Linear Quadratic Regulator (LQR) problem. After initiating the algorithm, the online procedure starts. In **Line 3**, the current output of the reference signal r_k is recorded. Then, u_k^π in equation 12 is calculated and applied to the system. Then, in **Line 4**, the next state of the system x_{k+1} is observed and the disturbance w_k is recorded. In **Line 5**, the algorithm suffers the cost $c_k(e_k, u_k)$ and the truncated cost $\tilde{c}_k$ is computed according to equation 16-equation 18. In **Line 6**, the weights M, P are updated using gradient descent on the truncated cost $\tilde{c}_k$, see equation 21 where Π_M, Π_P specify the set of matrices with appropriate dimensions and bounded norms, and η is the learning rate.

Algorithm 1 Online state tracking algorithm

1: **Initialize:** Select a stabilizing K and set M, P arbitrarily.
2: **for** $k = 1, .., T$ **do**
3: Record r_k and execute u_k^π in equation 12.
4: Observe x_{k+1} and record $w_k = k_{k+1} - Ax_k - Bu_k$.
5: Suffer $c_k(e_k, u_k)$ and compute $\tilde{c}_k$ in equation 16-equation 18.
6: Update M, P

$$\begin{aligned} M &= \Pi_M(M - \eta \nabla_M \tilde{c}_k), \\ P &= \Pi_P(P - \eta \nabla_P \tilde{c}_k). \end{aligned} \tag{21}$$

Remark 5 *Algorithm 1 is online: it updates M, P in each time step to minimize the cost c_k. Since u_k^π approximates u_k^{lin} (see Theorem 2), Algorithm 1 tries to approximate a linear feedback policy minimizing the cost c_k. There are two important characteristics for Algorithm 1. 1) The cost function c_k does not need to be quadratic. We can have any convex c_k. Note that in the classical approaches the cost is quadratic (Bertsekas, 2012; Khalil, 2002). 2) Parameterization of the controller u_k^π based on the recent values of w_k and r_k, and online tuning help us to approximate the best linear feedback policy for the recent values of w_k and r_k. Clearly, it is less conservative than selecting the linear feedback policy for the worse case disturbance in the H_∞ method. This results in a lower average cost; see the simulation results in Section 6.*

Note that at each time step k, $w_{1:k-1}, r_{1:k}$ are known (see **Lines 3-4** in Algorithm 1) and $w_k, r_k \equiv \mathbf{0}$ for $k < 0$ (see Assumptions 3 and 2). As such, equation 16-equation 18 are computable.

5.2 Regret Analysis

The standard measure for online control based on the gradient descent is the policy regret (Agarwal et al., 2019), which is defined here as the difference between cumulative loss of the designed parameterized control policy π learned by Algorithm 1 and that of the optimal linear control policy in the form of equation 3. The following definition is needed before defining the regret.

Definition 3 *Consider the system in equation 1. Let the control policy be deigned to generate the control action u_k in equation 12 at time k. Let Algorithm 1 be used to update the parameters of u_k. Then, its regret is defined as*

$$Regret = \sum_{k=1}^{T} c_k(e_k, u_k) - T \min_{K_f \in \mathcal{K}} J_T(K_f) \tag{22}$$

where $J_T(K_f)$ is the cost of the linear feedback controller in equation 3.

Theorem 4 *Suppose Algorithm 1 is executed under Assumptions 1-5. Let $H = m_w = m_r$. Select the learning rate η and the memory size H to satisfy $\eta = \mathcal{O}(\frac{1}{\sqrt{T}})$ and $H = \mathcal{O}(log\, T)$ to solve Problem 1. Then,*

$$Regret = \mathcal{O}(\sqrt{T}) \tag{23}$$

Proof: See Appendix G.

6 Simulation results

In this section, we give our simulation results.

6.1 The dynamical system, reference and cost function

We consider the dynamical system as

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_k + w_k.$$

The reference signal is generated by

$$\begin{aligned} z_{k+1} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} z_k,\ z_0 = [1, -2, 0.5]^T, \\ r_k &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z_k. \end{aligned} \tag{24}$$

We consider a quadratic cost with $Q = 20I_2,\ R = I_2$; that is

$$c_k = e_k^T Q e_k + u_k^T R u_k.$$

Note that the algorithm can handle any convex cost function. The choice of a quadratic cost is to enable comparison with the classical control approaches like Linear Quadratic Regulator (LQR) and H_∞ controllers.

6.2 Disturbances

For the simulation, we consider 6 cases for the disturbance; all of them are bounded and Assumption 2 is satisfied. In each case, we generate the disturbance in the beginning of the simulation so the disturbance sequence is the same for all algorithms. In cases 1-3, the disturbance is discontinuous and noisy. They are useful for comparing algorithms for stochastic disturbances. In cases 4-6, we consider continuous disturbances; they are useful to study the performance of the algorithms when the disturbance is not stochastic.

6.2.1 Gaussian disturbance

In the fourth case, we consider a Gaussian disturbance $w_{1k} = w_{2k} \sim 0.1\mathcal{N}(0, 1)$. It is well known that the LQR is the optimal controller for the stabilization of the system in equation 1 (Bertsekas, 2012).

6.2.2 Random walk disturbance

In the fifth case, we assume that the disturbance is a random walk and generated by $w_k = 0.999w_{k-1} + \eta_{k-1},\ \eta_{k-1} \sim 0.1\mathcal{N}(\mathbf{0}, \mathbf{1})$. Note that we have considered the internal dynamics for the random walk as 0.999 instead of 1 to ensue boundedness of the disturbance. When the noise is a random walk, the optimal controller is an LQR. To see this point, we replace the random walk disturbance in equation 1

$$x_{k+1} = Ax_k + Bu_k + 0.999w_{k-1} + \eta_{k-1}.$$

Not that in each time step k, the state x_k is measured and according to Assumption 1, w_{k-1} is known. Introducing a new state variable $\bar{x}_k = [x_k^T, w_{k-1}^T]$, we have

$$\bar{x}_{k+1} = \begin{bmatrix} A & 0.999I \\ \mathbf{0} & 0.999I \end{bmatrix} \bar{x}_k + \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \eta_{k-1}. \tag{25}$$

Hence, equation 1 with a random walk disturbance can be seen as the extended system in equation 25 where the noise η_{k-1} is Gaussian. As a result, the optimal controller is the LQR for the extended system in equation 25.

6.2.3 Uniformly sampled disturbance

In the sixth case, we assume that the disturbance is uniformly sampled from the interval $[0, 1]$.

6.2.4 Constant disturbance

The first case is when the disturbance is constant. We consider $w_{1k} = w_{2k} = 1$.

6.2.5 Amplitude modulation disturbance

In the second case, we consider the disturbance as $w_{1k} = w_{2k} = \sin(6\pi k/500)\sin(8\pi k/500)$.

6.2.6 Sinusoidal disturbance

In the third case, a sinusoidal disturbance is considered $w_{1k} = w_{2k} = \sin(8\pi k/100)$.

6.3 The compared control approaches

We compare our online tracking algorithm with the LQR and H_∞ control approaches: both of them optimize a quadratic performance index and they are also optimal for the Gaussian and worst-case disturbances. As such, they present the best possible performance for an algorithm for example when the disturbance is Gaussian or worst-case. Other adaptive algorithms which do not optimize a performance index or do not consider disturbance in the dynamics are not included.

In our simulation results, we study the following algorithms:

- **Online state tracking in Algorithm 1:** Let P_r be the solution to the Algebraic Riccati Equation $ARE(A, B, Q, R)$. We select K in equation 12 as

  $$K = -(R + B^T P_r B)^{-1} B^T P_r A.$$

  We keep K unchanged during running the algorithm. We set $H = 5$, $m_r = 5$, $m_w = 5$, $\eta = 0.0001$ and initialize $M = \mathbf{0}$, $P = \mathbf{0}$. We do not use any information about the dynamics of the reference signal; we only use measured outputs of the reference signal r_k in this algorithm. We also do not use any information about the disturbance (except the fact that the disturbance is bounded in Assumption 2) in this algorithm.

- **LQR and LQR for random walk:** We apply the controller in equation 4. We select $K_{fb} = -(R + B^T P_r B)^{-1} B^T P_r A$ where $P_r = ARE(A, B, Q, R)$. We assume that we *know* the dynamics of the reference in equation 24. We then compute K_{ff} according to equation 4-equation 5. To apply equation 4, we need to know the state of the reference z_k. We use the dynamics of the reference in equation 24 and build z_k from r_k according to Lemma 1.Then, we apply $u_k = K_{fb} x_k + K_{ff} z_k$.

  Note that K_{fb} is the optimal feedback controller gain for stabilizing the system in equation 1 when the disturbance w_k is Gaussian. No other controller can beat the LQR controller in the average cost when the Gaussian disturbance in Subsection 6.2.1 is considered. This is because, firstly, the LQR is the optimal stabilizing controller for the system in equation 1 with a Gaussian disturbance w_k and a quadratic cost, and secondly, we use full information about the dynamics of the reference signal.

We saw in Subsection 6.2.2 that when the disturbance is a random walk, one can extend the dynamics according to equation 25. The extended dynamics has a Gaussian disturbance and as a result, LQR for the extended dynamics is the optimal controller. In this case, we call the algorithm "*LQR for random walk*".

- **H_∞-control:** In the H_∞, the controller is defined to have a finite $\mathcal{L}_2$-gain with respect to the worst-case disturbance. The H_∞ controller is of the form in equation 4. We design K_{fb} for the system in equation 1 such that $\frac{||\sqrt{Q}x||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} \leq 1.5$. Note that 1.5 is the best achievable $\mathcal{L}_2$-gain for this system. We assume that we *know* the dynamics of the reference in equation 24. We then compute K_{ff} according to equation 4-equation 5. To apply equation 4, we need to know the state of the reference z_k. We use the dynamics of the reference in equation 24 and build z_k from r_k according to Lemma 1.Then, we apply $u_k = K_{fb}x_k + K_{ff}z_k$.

  The H_∞-control results in a conservative controller as it guarantees a finite $\mathcal{L}_2$-gain for the worst-case disturbance.

6.4 Performance during learning

In this subsection, we discuss the performance of the algorithms in Subsection 6.3 for the 6 cases of the disturbance in Subsection 6.2. In Table 1, we summarize the simulation time T and the final average cost suffered by the algorithms. We use **bold** to refer to the algorithm with the lowest average cost in each case of the disturbance.

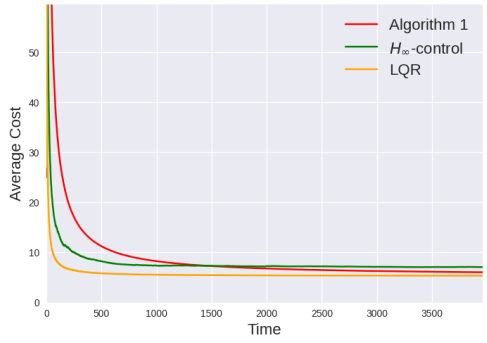
The evolution of the average cost for different cases of the disturbance is shown in Fig. 1. As the simulation goes on, Algorithm 1 learns M, P by minimizing the counterfactual cost as described in Section 4. As a result, the average cost for Algorithm 1 decreases with time.

To understand the performance of Algorithm 1, it is worthy to remind that Algorithm 1 learns a linear history-based policy in form of equation 12 by minimizing the average cost. By Theorem 2, the linear history-based policy in equation 12 is an approximation of the linear feedback policy in equation 3. In other words, Algorithm 1, tries to minimize the average cost by approximating the best linear feedback policy. That is, Algorithm 1 converges to an approximation of the best linear feedback policy minimizing the average cost.

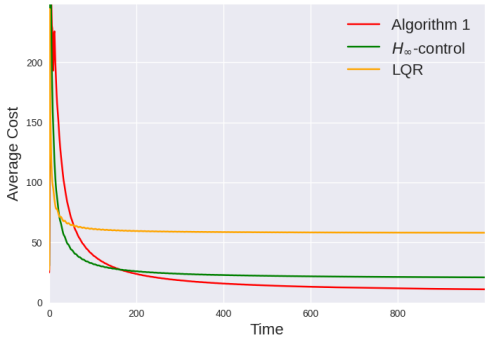
When the disturbance is Gaussian, the best linear feedback policy can be found by first selecting $K_{fb} = -(R + B^T P_r B)^{-1} B^T P_r A$ where $P_r = ARE(A, B, Q, R)$ and then computing K_{ff} according to equation 4-equation 5 using the dynamics of the reference. This is the best linear feedback policy in case of Gaussian disturbance with full information about the dynamics of the reference: no algorithm can beat this controller in terms of the average cost. The price for having this best linear feedback policy is firstly to know that the disturbance is Gaussian and secondly to know the dynamics of the reference signal. From Fig. 1a, we can see that the average cost by Algorithm 1 approaches the average cost with the best linear feedback policy, without knowing the nature of the disturbance and the dynamics of the reference. If we run Algorithm 1 for longer time, for example for $T = 8000$ steps, we will see that the average cost becomes closer to the average cost by the LQR $J_{8000}(\text{Algorithm 1}) = 5.66$, but we cannot get exactly the performance of the LQR because the linear history-based policy in equation 12 is an approximation of the linear feedback policy in equation 3.

A similar discussion is also valid for the case of random-walk disturbance, see Subsection 6.2.2, where we showed that the optimal controller for the system when the disturbance is random walk, is obtained by solving an LQR problem for the extended system. This controller is called "LQR for random walk".

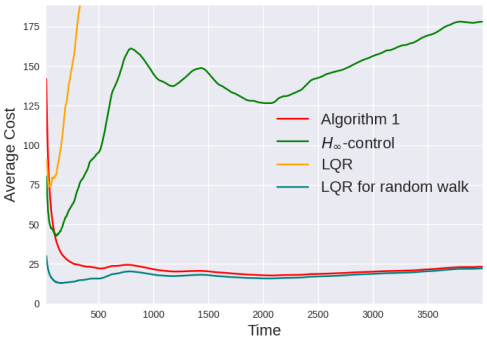
When the disturbance is not Gaussian or random walk, there is no analytical way to determine the best linear feedback policy. In such cases, usually the H_∞-controller is used to design a linear feedback policy to guarantee a finite $\mathcal{L}_2$-gain for the worst-case disturbance and as such it is conservative. Indeed, if the disturbance is not the worst-case, the H_∞-controller does not have the best performance. As one can see in Table 1, Algorithm 1 has lower average costs for uniformly sampled, constant, amplitude modulation, and sinusoidal disturbances.



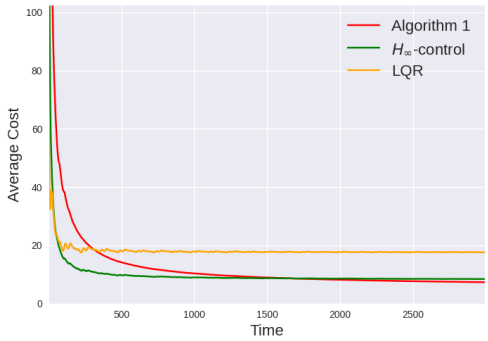
(a) Gaussian



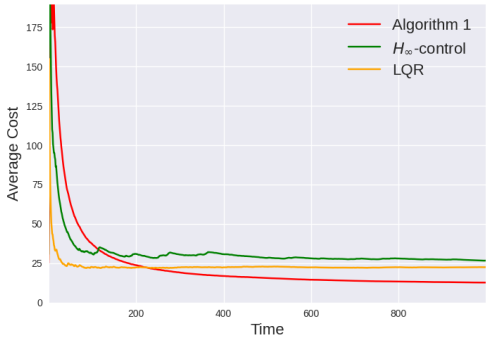
(b) Constant



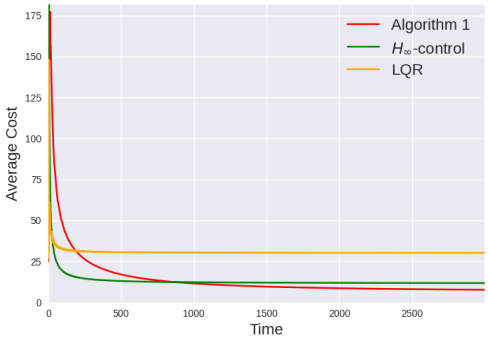
(c) Random walk



(d) Amplitude modulation



(e) Uniformly sampled



(f) Sinusoidal

Figure 1: The average cost for six disturbances

Table 1: The average costs suffered by the algorithms for running each algorithm for T steps. Bold values show the lowest average cost for each case of disturbance. The algorithm LQR for random walk is the optimal controller in the case of random walk disturbance and thus it is only evaluated in this case.

Disturbance	T	Algorithm 1	LQR	H_∞	LQR for random walk
Gaussian	4000	6.03	**5.34**	7.06	N.A.
Random walk	4000	23.24	415.65	178.03	**22.16**
Uniformly sam.	1000	**12.62**	22.44	26.62	N.A.
Constant	1000	**10.92**	58.08	20.89	N.A.
Amplitude mod.	3000	**7.30**	17.62	8.40	N.A.
Sinusoidal	3000	**7.71**	30.24	11.80	N.A.

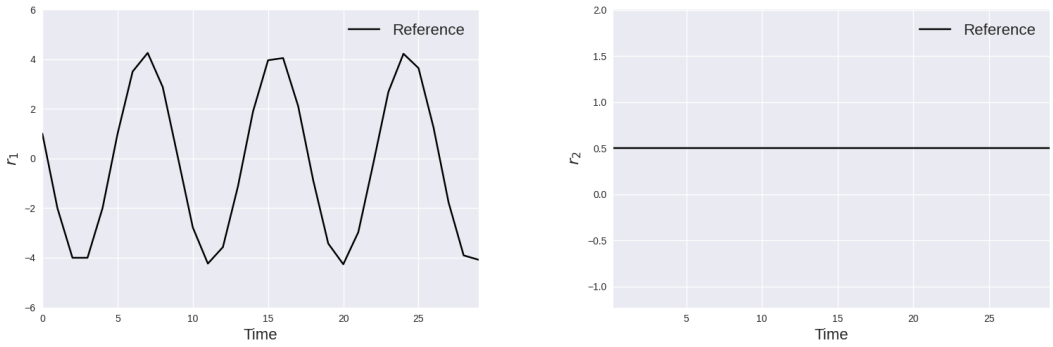


Figure 2: The reference

6.5 Evaluation after learning

We further evaluate the performance of the learned controllers by Algorithm 1 by using them to control the system in equation 1 for $T_{\text{eval}} = 30$ steps. The reference trajectory for the evaluation is shown in Fig. 2. We generate the disturbance in the beginning of the simulation so the disturbance sequence is the same for all algorithms during the evaluation. In Fig. 3-8, one can see the state tracking error for different cases of the disturbances using the methods in Subsection 6.3. Figures 3-8 also confirm the results in Table 1. For non-stochastic disturbances, namely the constant, amplitude modulation and sinusoidal disturbances, the tracking error by Algorithm 1 is near to zero while other approaches have significant nonzero errors. This observation is align with the results in Table 1. For stochastic disturbances, it is more difficult to understand the behavior from the tracking error plots and the performance is better understood by the numbers in Table 1 but we have kept the figures for completeness of the results. For the Gaussian disturbance, the performance of Algorithm 1 is almost identical to the LQR controller which is the optimal controller in this case. For the random-walk and uniformly-sampled disturbances, the behaviors of the compared algorithms are not completely understandable from Fig. 4-5 but according to Table 1, one can see that Algorithm 1 has the closest performance to the optimal performance.

7 Conclusion

In this paper, we have considered the problem of state tracking in presence of general disturbances. We have proposed a linear history-based controller and given an online algorithm to tune the parameters of the controller. Our proposed algorithm tunes the parameters of the controller online to achieve state tracking and disturbance rejection while minimizing general convex costs. We have proved that the algorithm attains

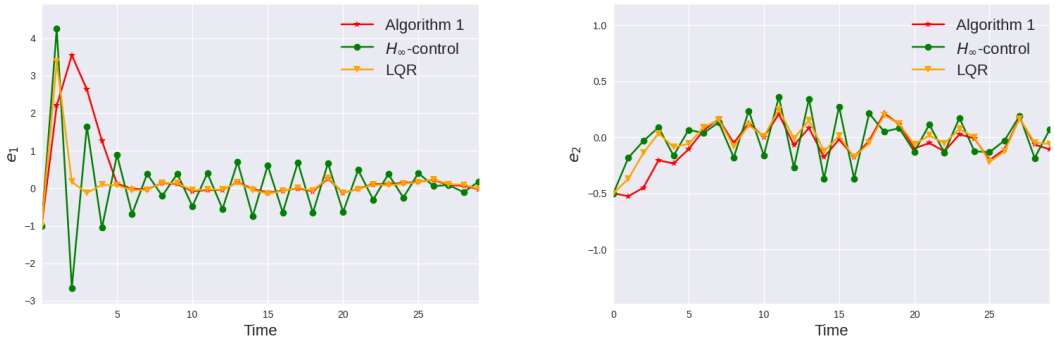


Figure 3: Tracking error the Gaussian disturbance

Figure 4: Tracking error the random walk disturbance

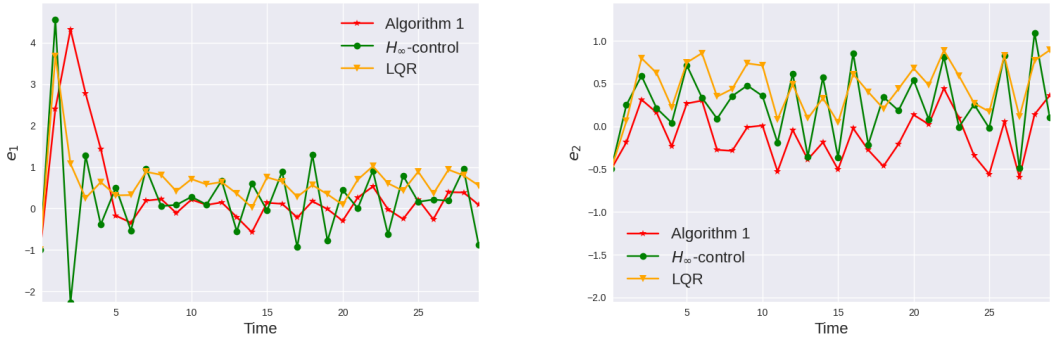


Figure 5: Tracking error the uniformly sampled disturbance

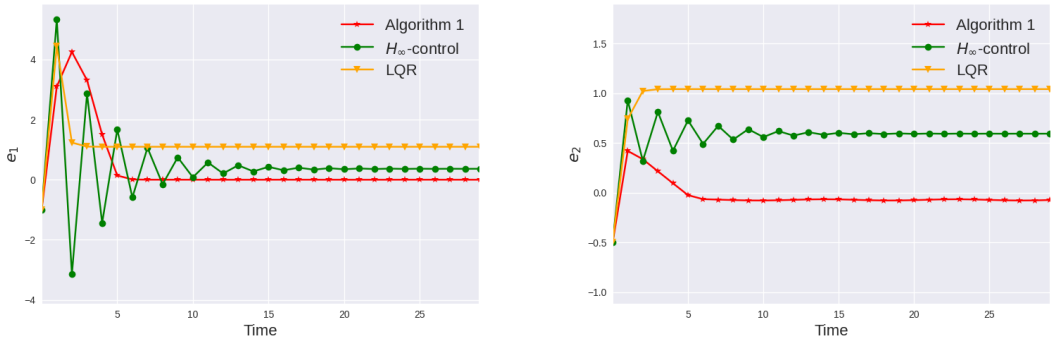


Figure 6: Tracking error for the constant disturbance

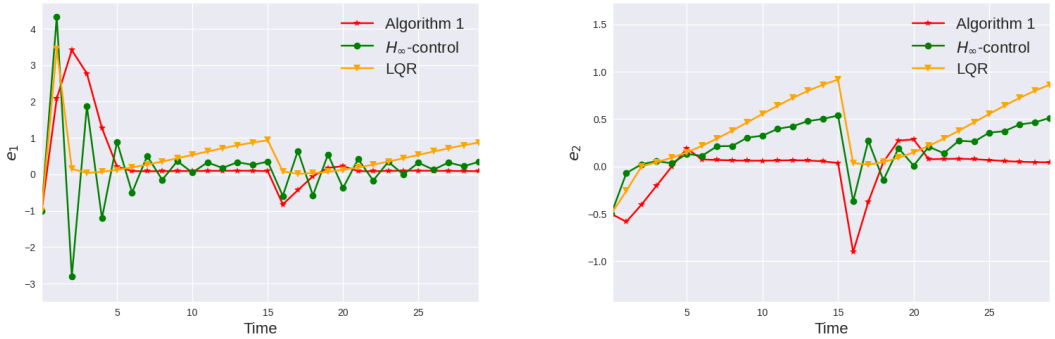


Figure 7: Tracking error for the amplitude mod. disturbance

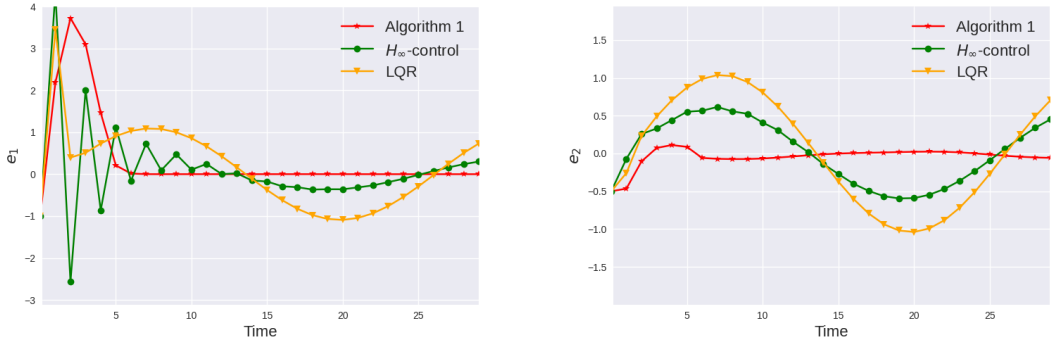


Figure 8: Tracking error the sinusoidal disturbance

$\mathcal{O}(\sqrt{\mathcal{T}})$-policy regret. In our future works, we will consider partially observable dynamical systems and aim to remove the bounded assumption on the reference signal.

References

Yasin Abbasi-Yadkori, Peter Bartlett, and Varun Kanade. Tracking adversarial targets. In *International Conference on Machine Learning*, pp. 369–377. PMLR, 2014.

Farnaz Adib Yaghmaie, Svante Gunnarsson, and Frank L. Lewis. Output regulation of unknown linear systems using average cost reinforcement learning. *Automatica*, 110:108549, 2019. ISSN 00051098. doi: 10.1016/j.automatica.2019.108549. URL `https://doi.org/10.1016/j.automatica.2019.108549`.

Naman Agarwal, Brian Bullins, Elad Hazan, Sham M. Kakade, and Karan Singh. Online control with adversarial disturbances. *36th International Conference on Machine Learning, ICML 2019*, 2019-June: 154–165, 2019.

Dimitri Bertsekas. *Dynamic programming and optimal control: Volume I*. Athena scientific, 2012.

Ci Chen, Hamidreza Modares, Kan Xie, Frank L. Lewis, Yan Wan, and Shengli Xie. Reinforcement Learning-Based Adaptive Optimal Exponential Tracking Control of Linear Systems with Unknown Dynamics. *IEEE Transactions on Automatic Control*, 64(11):4423–4438, 2019. ISSN 15582523. doi: 10.1109/TAC.2019.2905215.

T. Dierks and S. Jagannathan. Optimal control of affine nonlinear continuous-time systems. *Proceedings of the 2010 American Control Conference, ACC 2010*, pp. 1568–1573, 2010. doi: 10.1109/acc.2010.5531586.

Weinan Gao and Zhong Ping Jiang. Global Optimal Output Regulation of Partially Linear Systems via Robust Adaptive Dynamic Programming. *IFAC-PapersOnLine*, 48(11 11):742–747, 2015. ISSN 24058963. doi: 10.1016/j.ifacol.2015.09.278. URL `http://dx.doi.org/10.1016/j.ifacol.2015.09.278`.

Weinan Gao and Zhong Ping Jiang. Adaptive Dynamic Programming and Adaptive Optimal Output Regulation of Linear Systems. *IEEE Transactions on Automatic Control*, 61(12):4164–4169, 2016. ISSN 00189286. doi: 10.1109/TAC.2016.2548662.

Elad Hazan, Sham M. Kakade, and Karan Singh. The Nonstochastic Control Problem. In *Algorithmic Learning Theory*, pp. 408—-421, 2020. URL `http://arxiv.org/abs/1911.12178`.

Jie Huang. *Nonlinear output regulation: theory and applications*. SIAM, 2004.

Yuzhu Huang and Derong Liu. Neural-network-based optimal tracking control scheme for a class of unknown discrete-time nonlinear systems using iterative ADP algorithm. *Neurocomputing*, 125:46–56, 2014. ISSN 09252312. doi: 10.1016/j.neucom.2012.07.047. URL `http://dx.doi.org/10.1016/j.neucom.2012.07.047`.

Alberto Isidori. *Nonlinear control systems*. 1989.

Yi Jiang, Bahare Kiumarsi, Jialu Fan, Tianyou Chai, Jinna Li, and Frank L. Lewis. Optimal Output Regulation of Linear Discrete-Time Systems with Unknown Dynamics Using Reinforcement Learning. *IEEE Transactions on Cybernetics*, 50(7):3147–3156, 2020. ISSN 21682275. doi: 10.1109/TCYB.2018.2890046.

Rushikesh Kamalapurkar, Huyen Dinh, Shubhendu Bhasin, and Warren E. Dixon. Approximate optimal trajectory tracking for continuous-time nonlinear systems. *Automatica*, 51:40–48, 2015. ISSN 00051098. doi: 10.1016/j.automatica.2014.10.103. URL `http://dx.doi.org/10.1016/j.automatica.2014.10.103`.

Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, second edition, 2002.

Bahare Kiumarsi and Frank L. Lewis. Actor-critic-based optimal tracking for partially unknown nonlinear discrete-time systems. *IEEE Transactions on Neural Networks and Learning Systems*, 26(1):140–151, 2015. ISSN 21622388. doi: 10.1109/TNNLS.2014.2358227.

Bahare Kiumarsi, Frank L. Lewis, Hamidreza Modares, Ali Karimpour, and Mohammad Bagher Naghibi-Sistani. Reinforcement Q-learning for optimal tracking control of linear discrete-time systems with unknown dynamics. *Automatica*, 50(4):1167–1175, 2014. ISSN 00051098. doi: 10.1016/j.automatica.2014.02.015. URL http://dx.doi.org/10.1016/j.automatica.2014.02.015.

Bahare Kiumarsi, Frank L. Lewis, Mohammad Bagher Naghibi-Sistani, and Ali Karimpour. Optimal tracking control of unknown discrete-time linear systems using input-output measured data. *IEEE Transactions on Cybernetics*, 45(12):2770–2779, 2015. ISSN 21682267. doi: 10.1109/TCYB.2014.2384016.

Hamidreza Modares and Frank L. Lewis. Optimal tracking control of nonlinear partially-unknown constrained-input systems using integral reinforcement learning. *Automatica*, 50(7):1780–1792, 2014. ISSN 00051098. doi: 10.1016/j.automatica.2014.05.011.

Hamidreza Modares, Frank L. Lewis, and Zhong Ping Jiang. H_∞ Tracking Control of Completely Unknown Continuous-Time Systems via Off-Policy Reinforcement Learning. *IEEE Transactions on Neural Networks and Learning Systems*, 26(10):2550–2562, 2015. ISSN 21622388. doi: 10.1109/TNNLS.2015.2441749.

Huaguang Zhang, Lili Cui, Xin Zhang, and Yanhong Luo. Data-driven robust approximate optimal tracking control for unknown general nonlinear systems using adaptive dynamic programming method. *IEEE Transactions on Neural Networks*, 22(12 PART 2):2226–2236, 2011. ISSN 10459227. doi: 10.1109/TNN.2011.2168538.

Huanguang Zhang, Qinglai Wei, and Yanhong Luo. A novel infinite-time optimal tracking control scheme for a class of discrete-time nonlinear systems via the greedy HDP iteration algorithm. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 38(4):937–942, 2008. ISSN 10834419. doi: 10.1109/TSMCB.2008.920269.

Zhiyu Zhang, Ashok Cutkosky, and Ioannis Paschalidis. Adversarial tracking control via strongly adaptive online learning with memory. In *International Conference on Artificial Intelligence and Statistics*, pp. 8458–8492. PMLR, 2022.

Peng Zhao, Yu-Xiang Wang, and Zhi-Hua Zhou. Non-stationary online learning with memory and non-stochastic control. In *International Conference on Artificial Intelligence and Statistics*, pp. 2101–2133. PMLR, 2022.

A Proof of Lemma 1

Consider the current and the $l-1$ past outputs of the system

$$\begin{aligned} r_{k-l+1} &= Fz_{k-l+1}, \\ z_{k-l} &= Sz_{k-l+1}, \\ r_{k-l} &= Fz_{k-l} = FSz_{k-l+1}, \\ &\vdots \end{aligned}$$

Continuing like this for l steps, we get in the end

$$\begin{aligned} z_k &= S^{l-1}z_{k-l+1}, \\ r_k &= FS^{l-1}z_{k-l+1}. \end{aligned} \tag{26}$$

Let $\bar{r}_k = \begin{bmatrix} r_{k-l+1} \\ \vdots \\ r_k \end{bmatrix}$ be the concatenation of the outputs from r_{k-l+1} to r_k. We have $\bar{r}_k = \mathcal{O}_l z_{k-l+1}$. Since the matrix $\mathcal{O}_l$ has full column rank $\text{rank}(\mathcal{O}_l) = p$, any $p \times p$ matrix can be spans by columns of $\mathcal{O}_l$. In particular, there exists a matrix N such that $S^{l-1} = N\mathcal{O}_{l-1}$ and $N = S^{l-1}\mathcal{O}_l^+$. Using this result in the first equation in equation 26

$$z_k = S^{l-1}z_{k-l+1} = N\mathcal{O}_{l-1}z_{k-l+1} = N\bar{r}_k.$$

B Proof of Lemma 2

Let $x_k^{\text{lin}}(K_{fb}, K_{ff})$ denote solution to the system in equation 1 using the linear feedback policy $u_k^{\text{lin}}(K_{fb}, K_{ff})$ in equation 3 and $x_k^{\pi}(K, M, P)$ denote the solution to the system in equation 1 using the linear history-based policy $u_k^{\pi}(K, M, P)$ in equation 12. We drop the arguments in x_k^{lin}, u_k^{lin}, x_k^{π}, u_k^{π} for clarity in the proof. Using u_k^{lin}, the closed-loop system reads

$$\begin{aligned} x_{k+1}^{\text{lin}} &= (A + BK_{fb})x_k^{\text{lin}} + BK_{ff}z_k + w_k \\ &= (A + BK_{fb})x_k^{\text{lin}} + BK_{ff}\sum_{q=0}^{l-1} N^{[l-q]}r_{k-q} + w_k, \end{aligned}$$

where we have used equation 8 in Lemma 1 to replace z_k in the second line. x_k^{lin} reads

$$\begin{aligned} x_k^{\text{lin}} =& \sum_{i=1}^{k}(A + BK_{fb})^{i-1}BK_{ff}\sum_{q=0}^{l-1} N^{[l-q]}r_{k-i-q} + \sum_{i=1}^{k}(A + BK_{fb})^{i-1}w_{k-i} \\ =& \sum_{i=1}^{k}\sum_{q=0}^{l-1}(A + BK_{fb})^{i-1}BK_{ff}N^{[l-q]}r_{k-i-q} + \sum_{i=1}^{k}(A + BK_{fb})^{i-1}w_{k-i}. \end{aligned} \tag{27}$$

Now, we change the index in the first summation; it will simplify our derivations in other proofs. Introduce $j = i + q$ and let $1 \leq j \leq k$. We have two bounds for i which give two bounds for q after the change of the variable $j = i + q$.

- The lower bound on $i = j - q \geq 1$. From this, one should have that $q \leq j - 1$. Since $q \leq l - 1$ also in the summation, we will have that $q \leq \min(l - 1, j - 1)$.
- The upper bound on $i = j - q \leq k$. Since $j \leq k$, we have $q \geq$ (a negative number). Since $q \geq 0$ in the summation, we will have that $q \geq 0$.

Combining these two boundaries for q, we have $0 \leq q \leq \min(l - 1, j - 1)$. As a result, the above equation can be written as

$$x_k^{\text{lin}} = \sum_{j=1}^{k}\sum_{q=0}^{\min(l-1,j-1)}(A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]}r_{k-j} + \sum_{i=1}^{k}(A + BK_{fb})^{i-1}w_{k-i}. \tag{28}$$

Note that c_k is convex in x_k^{lin}, but based on equation 28, x_k^{lin} is not convex in K_{fb}, K_{ff}. As a result, $c_k(e_k, u_k)$ is not convex in K_{fb}, K_{ff}.

Next, we study the solution to the system in equation 1 using the linear history-based policy u_k^{π}. Using u_k^{π}, the closed-loop system reads

$$x_{k+1}^{\pi} = \tilde{A}_K x_k^{\pi} + B\sum_{t=1}^{m_w} M^{[t-1]}w_{k-t} + w_k + B\sum_{s=0}^{m_r-1} P^{[s]}r_{k-s}.$$

As a result, x_k^{π} reads

$$\begin{aligned} x_k^{\pi} =& \sum_{i=1}^{k}\tilde{A}_K^{i-1}B\sum_{t=1}^{m_w} M^{[t-1]}w_{k-i-t} + \sum_{i=1}^{k}\tilde{A}_K^{i-1}w_{k-i} \\ &+ \sum_{i=1}^{k}\tilde{A}_K^{i-1}B\sum_{s=0}^{m_r-1} P^{[s]}r_{k-i-s} = \sum_{i=1}^{k}\sum_{t=1}^{m_w}\tilde{A}_K^{i-1}BM^{[t-1]}w_{k-i-t} \\ &+ \sum_{i=1}^{k}\tilde{A}_K^{i-1}w_{k-i} + \sum_{i=1}^{k}\sum_{s=0}^{m_r-1}\tilde{A}_K^{i-1}BP^{[s]}r_{k-i-s}. \end{aligned} \tag{29}$$

Again, we change the indices in the summations. The reasoning is similar to what we discussed earlier. For the first summation, introduce $j = i + t$ and let $1 \leq j \leq k$. We have two bounds for i which give two bounds for t after the change of the variable $j = i + t$.

- The lower bound on $i = j - t \geq 1$. From this, one should have that $t \leq j - 1$. Since $t \leq m_w$ also in the summation, we will have that $t \leq \min(m_w, j - 1)$.
- The upper bound on $i = j - t \leq k$. Since $j \leq k$, we have $t \geq$ (a negative number). Since $t \geq 1$ in the summation, we will have that $t \geq 1$.

Combining these two boundaries for t, we have $1 \leq t \leq \min(m_w, j - 1)$. For the third summation, introduce $j = i + s$ and let $1 \leq j \leq k$. We can follow a similar reasoning and show that $0 \leq s \leq \min(m_r - 1, j - 1)$.

$$x_k^\pi = \sum_{j=1}^{k} \sum_{t=1}^{\min(m_w, j-1)} \tilde{A}_K^{j-t-1} B M^{[t-1]} w_{k-j} + \sum_{i=1}^{k} \tilde{A}_K^{i-1} w_{k-i} \tag{30}$$
$$+ \sum_{j=1}^{k} \sum_{s=0}^{\min(m_r - 1, j-1)} \tilde{A}_K^{j-s-1} B P^{[s]} r_{k-j}.$$

Note that c_k is convex in x_k^π and x_k^π is linear in M, P. As a result, $c_k(e_k, u_k)$ is convex in M, P.

C Proof of Lemma 3

Based on Lemma 2, the expression of the state at time k becomes,

$$x_k^\pi = \sum_{i=1}^{k} \sum_{t=1}^{m_w} \tilde{A}_K^{i-1} B M_{k-i}^{[t-1]} w_{k-i-t} + \sum_{i=1}^{k} \tilde{A}_K^{i-1} w_{k-i} + \sum_{i=1}^{k} \sum_{s=0}^{m_r - 1} \tilde{A}_K^{i-1} B P_{k-i}^{[s]} r_{k-i-s}. \tag{31}$$

Forming x_{k-h}^π for any $h \geq 0$, multiplying it by $\tilde{A}_K^h$, and subtracting x_k^π from $\tilde{A}_K^h x_{k-h}^\pi$ yields

$$\begin{aligned}
x_k^\pi - \tilde{A}_K^h x_{k-h}^\pi &= \sum_{j=1}^{k} \tilde{A}_K^{j-1} w_{k-j} - \sum_{j=h+1}^{k} \tilde{A}_K^{j-1} w_{k-i} + \sum_{j=1}^{k} \sum_{t=1}^{m_w} \tilde{A}_K^{j-1} B M_{k-j}^{[t-1]} w_{k-j-t} \\
&- \sum_{j=h+1}^{k} \sum_{t=1}^{m_w} \tilde{A}_K^{j-1} B M_{k-j}^{[t-1]} w_{k-j-t} + \sum_{j=1}^{k} \sum_{s=0}^{m_r - 1} \tilde{A}_K^{j-1} B P_{k-j}^{[s]} r_{k-j-s} - \sum_{j=h+1}^{k} \sum_{s=0}^{m_r - 1} \tilde{A}_K^{j-1} B P_{k-j}^{[s]} r_{k-j-s} \\
&= \sum_{j=1}^{h} \tilde{A}_K^{j-1} w_{k-j} + \sum_{j=1}^{h} \sum_{t=1}^{m_w} \tilde{A}_K^{j-1} B M_{k-j}^{[t-1]} w_{k-j-t} + \sum_{j=1}^{h} \sum_{s=0}^{m_r - 1} \tilde{A}_K^{j-1} B P_{k-j}^{[s]} r_{k-j-s}
\end{aligned}$$

In the second summation, introduce $y = j + t - 1$ and change the order of summations, so $y = 0, ..., h + m_w - 1$. Similarly, introduce $z = j + s$ so $z = 0, ..., h + m_r - 1$ and then $j - 1 \to j$

$$\begin{aligned}
x_k^\pi - \tilde{A}_K^h x_{k-h}^\pi &= \sum_{j=0}^{h-1} \tilde{A}_K^j w_{k-j-1} + \sum_{y=0}^{m_w + h - 1} \sum_{j=1}^{h} \tilde{A}_K^{j-1} B M_{k-j}^{[y-j]} w_{k-y-1} + \sum_{z=0}^{h+m_r-1} \sum_{j=1}^{h} \tilde{A}_K^{j-1} B P_{k-j}^{[z-j]} r_{k-z} \\
&= \sum_{a=0}^{h-1} \tilde{A}_K^a w_{k-a-1} + \sum_{y=0}^{m_w+h-1} \sum_{j=0}^{h-1} \tilde{A}_K^j B M_{k-j-1}^{[y-j-1]} w_{k-y-1} + \sum_{z=0}^{h+m_r-1} \sum_{j=0}^{h-1} \tilde{A}_K^j B P_{k-j-1}^{[z-j-1]} r_{k-z}
\end{aligned}$$

Then, based on equation 14, we have

$$\begin{aligned}
x_k^\pi = x_k^K(M_{0:k-1}, P_{0:k-1}) &= \tilde{A}_K^h x_{k-h}^\pi + \sum_{y=0}^{m_w+h-1} \Psi_{k,y}^{K,h}(M_{k-h-1:k-1}) w_{k-y-1} \\
&+ \sum_{z=0}^{m_r+h-1} \psi_{k,z}^{K,h}(P_{k-h-1:k-1}) r_{k-z}.
\end{aligned} \tag{32}$$

Set $h = k$ and note that $y \leq k-1$ and $z \leq k-1$. So equation 13 is concluded.

D Supporting Lemmas

Lemma 4 *Let Assumptions 1-5 hold. Suppose that K is (κ, γ)-strongly stable. Then,*

$$\begin{aligned} \|\Psi_{k,y}^{K,h}\| &\leq \kappa^2(1-\gamma)^y \mathbb{I}_{y\leq h-1} + m_w \kappa^5 \kappa_b^2 (1-\gamma)^{y-1}, \\ \|\psi_{k,z}^{K,h}\| &\leq m_r \kappa^5 \kappa_b^2 (1-\gamma)^{z-1}. \end{aligned} \tag{33}$$

Proof: To prove the first statement in equation 33 note that

$$\begin{aligned} \|\Psi_{k,y}^{K,h}\| \leq & \|\tilde{A}_K^y \mathbb{I}_{y\leq h-1}\| + \|\sum_{j=0}^{h-1} \tilde{A}_K^j B M_{k-j-1}^{[y-j-1]} \mathbb{I}_{1\leq y-j\leq m_w}\| \\ \leq & \kappa^2(1-\gamma)^y \mathbb{I}_{y\leq h-1} + \sum_{j=0}^{h-1} \kappa^2 \kappa_b^2 \kappa^3 (1-\gamma)^{y-1} \mathbb{I}_{1\leq y-j\leq m_w} \\ \leq & \kappa^2(1-\gamma)^y \mathbb{I}_{y\leq h-1} + m_w \kappa^5 \kappa_b^2 (1-\gamma)^{y-1}, \end{aligned}$$

where the last inequality follows from (κ, γ)-stability of the controller gain K and Assumption 5. The proof of the second statement follows similarly.

Lemma 5 *Let Assumptions 1-5 hold. Define*

$$\begin{aligned} Y_{0:k} &:= [M_{0:k}, P_{0:k}], \\ Y_{H,k} &:= [M_{k-H:k}, P_{k-H:k}]. \end{aligned} \tag{34}$$

$$D\gamma^{-1} \frac{\kappa_w \kappa^3 + (\kappa_r m_r + \kappa_w m_w)(1-\gamma)^{-1} \kappa^6 \kappa_b^2}{1-\kappa^2(1-\gamma)^H} + \frac{(\kappa_w + \kappa_r)\kappa_b \kappa^3}{\gamma}.$$

Suppose that K and K_{fb}^* are (κ, γ)-strongly stable. Define $x_k^{lin}(K_{fb}^*, K_{ff}^*)$ as the system state corresponding to an optimal linear feedback controller. Then, one has

$$\begin{aligned} &\max(\|x_k^\pi(Y_{0:k-1})\|, \|\tilde{x}_k^\pi(Y_{H,k-1})\|, \\ &\qquad \|x_k^{lin}(K_{fb}^*, K_{ff}^*)\|) \leq D, && (35) \\ &\max(\|u_k^\pi(Y_{0:k})\|, \|\tilde{u}_k^\pi(Y_{H,k})\|) \leq D, && (36) \\ &\|x_k^\pi(Y_{0:k-1}) - \tilde{x}_k^\pi(Y_{H,k-1})\| \leq \kappa^2(1-\gamma)^H D, && (37) \\ &\|u_k^\pi(Y_{0:k}) - \tilde{u}_k^\pi(Y_{H,k})\| \leq \kappa^3(1-\gamma)^H D. && (38) \end{aligned}$$

Proof: Using equation 32, we have

$$\begin{aligned} \|x_k^\pi\| \leq & \|\tilde{A}_K^H\| x_{k-H}^\pi + \kappa_w \sum_{y=0}^{m_w+H-1} \|\Psi_{k,y}^{K,H}(M_{k-H-1:k-1})\| + \kappa_r \sum_{z=0}^{m_r+H-1} \psi_{k,z}^{K,H}(P_{k-H-1:k-1})\| \\ \leq & \kappa^2(1-\gamma)^H \|x_{k-H}^\pi\| + \kappa_w \gamma^{-1}(\kappa^2 + m_w \kappa^5 \kappa_b^2 (1-\gamma)^{-1}) + \kappa_r \gamma^{-1}(m_r \kappa^5 \kappa_b^2)(1-\gamma)^{-1}. \end{aligned}$$

The above recursion satisfies

$$\|x_k^\pi\| \leq \gamma^{-1} \frac{\kappa_w \kappa^2 + (\kappa_r m_r + \kappa_w m_w)(1-\gamma)^{-1} \kappa^5 \kappa_b^2}{1-\kappa^2(1-\gamma)^H}.$$

Similarly, from equation 16, one has

$$\|\tilde{x}_k^\pi(Y_{H,k-1})\| \leq \sum_{y=0}^{m_w+H-1} \|\Psi_{k,y}^{K,H}(M_{k-H-1:k-1})w_{k-y-1}\| + \sum_{z=0}^{m_r+H-1} \|\psi_{k,z}^{K,H}(P_{k-H-1:k-1})r_{k-z}\| \leq$$
$$\gamma^{-1}\kappa_w\kappa^2 + \gamma^{-1}(\kappa_w m_w + \kappa_r m_r)\kappa^5\kappa_b^2(1-\gamma)^{-1} \leq D.$$

where the last inequality is obtained because $0 \leq 1-\kappa^2(1-\gamma)^H \leq 1$. Moreover,

$$\|x_k^{lin}(K_{fb}^*, K_{ff}^*)\| = \|\sum_{y=0}^{k-1} \tilde{A}_{K_{fb}^*}^i w_{k-i-1} + \sum_{z=0}^{k-1} \tilde{A}_{K_{fb}^*}^i r_{k-i}\| \leq (\kappa_w+\kappa_r)\kappa^2\gamma^{-1} \leq D.$$

Beside, one has

$$\|u_k^\pi(Y_{0:k})\| = \|Kx_k^\pi(Y_{0:k-1}) + \sum_{t=1}^{m_w} M^{[t-1]}w_{k-t} + \sum_{s=0}^{m_r-1} P^{[s]}r_{k-s}\| \leq$$
$$\kappa\|x_k^\pi(Y_{0:k-1})\| + \kappa_w\sum_{t=1}^{m_w}\kappa_b\kappa^3(1-\gamma)^{(t-1)} + \kappa_r\sum_{s=0}^{m_r-1}\kappa_b\kappa^3(1-\gamma)^s \leq$$
$$\gamma^{-1}\frac{\kappa_w\kappa^3 + (\kappa_r m_r + \kappa_w m_w)(1-\gamma)^{-1}\kappa^6\kappa_b^2}{1-\kappa^2(1-\gamma)^H} + \frac{(\kappa_w+\kappa_r)\kappa_b\kappa^3}{\gamma} = D.$$

Similarly,

$$\|\tilde{u}_k^\pi(Y_{H,k})\| = \|K\tilde{x}_k^\pi(Y_{H,k-1}) + \sum_{t=1}^{m_w} M^{[t-1]}w_{k-t} + \sum_{s=0}^{m_r-1} P^{[s]}r_{k-s}\| \leq$$
$$\kappa\|\tilde{x}_k^\pi(Y_{H,k-1})\| + \kappa_w\sum_{t=1}^{m_w}\kappa_b\kappa^3(1-\gamma)^{(t-1)} + \kappa_r\sum_{s=0}^{m_r-1}\kappa_b\kappa^3(1-\gamma)^s \leq$$
$$\gamma^{-1}\kappa_w\kappa^3 + \gamma^{-1}(\kappa_w m_w + \kappa_r m_r)\kappa^6\kappa_b^2(1-\gamma)^{-1} + \frac{(\kappa_w+\kappa_r)\kappa_b\kappa^3}{\gamma} \leq D.$$

To bound the difference between the actual and truncated state, from equation 16 and equation 32, one has

$$\|x_k^\pi(Y_{0:k-1}) - \tilde{x}_k^\pi(Y_{H,k-1})\| = \|\tilde{A}_K^H x_{k-H}^\pi(Y_{0:k-H-1})\| \leq \kappa^2(1-\gamma)^H D$$

which gives

$$\|u_k^\pi(Y_{0:k}) - \tilde{u}_k^\pi(Y_{H,k})\| \leq \|K\|\|\tilde{A}_K^H x_{k-H}^\pi(Y_{0:k-H-1})\| \leq \kappa^3(1-\gamma)^H D.$$

This completes the proof.

To make it evident that the truncated cost function explicitly depends on a fixed-size history of the disturbance and reference values, we define

$$f_k(Y_{H,k}) = \tilde{c}_k(\tilde{x}_k^\pi - r_k, \tilde{u}_k^\pi). \tag{39}$$

Lemma 6 *Define $Y_{H,k} = [Y_1, ..., Y_t, ..., Y_{2H}] = [M_{k-H:k}\ P_{k-H:k}]$ and $\tilde{Y}_{H,k} = [Y_1, ..., \tilde{Y}_t, ..., Y_{2H}]$ where $\tilde{Y}$ has all its elements the same as $Y_{H,k}$, except one element. Then, the truncated cost function in equation 39 satisfies the following Lipschitz condition*

$$|f_k(Y_1, , ..., Y_t, ..., Y_{2H}) - f_k(Y_1, , ..., \tilde{Y}_t, ..., Y_{2H})| \leq L_f\|Y_t - \tilde{Y}_t\|$$

where

$$L_f 3\, G_c D\, \kappa_b\kappa^3(\kappa_r + \kappa_w). \tag{40}$$

Proof: Using equation 39 and based on Assumption 4, one has

$$\begin{aligned}&|f_k(Y_1,,...,Y_t,...,Y_{2H}) - f_k(Y_1,,...,\tilde{Y}_t,...,Y_{2H})| = \\ &|\tilde{c}_k^\pi(\tilde{x}_k^\pi(Y_{H,k}), r_k, \tilde{u}_k^\pi(Y_{H,k})) - \tilde{c}_k^\pi(\tilde{x}_k^\pi(\tilde{Y}_{H,k}), r_k, \tilde{u}_k^\pi(\tilde{Y}_{H,k}))| \leq \\ &G_c D\|\tilde{e}_k^\pi(Y_{H,k}) - \tilde{e}_k^\pi(\tilde{Y}_{H,k})\| + G_c D\|\tilde{u}_k^\pi(Y_{H,k}) - \tilde{u}_k^\pi(\tilde{Y}_{H,k})\|.\end{aligned} \tag{41}$$

where $\tilde{e}_k^\pi = \tilde{x}_k^\pi - r_k$. Using

$$\tilde{e}_k^\pi(Y_{H,k}) - \tilde{e}_k^\pi(\tilde{Y}_{H,k}) = \tilde{x}_k^\pi(Y_{H,k}) - r_k + \tilde{x}_k^\pi(\tilde{Y}_{H,k}) - r_k = \tilde{x}_k^\pi(Y_{H,k}) - \tilde{x}_k^\pi(\tilde{Y}_{H,k})$$

in equation 41 yields

$$\begin{aligned}&|f_k(Y_1,,...,Y_t,...,Y_{2H}) - f_k(Y_1,,...,\tilde{Y}_t,...,Y_{2H})| \leq \\ &G_c D\|\tilde{x}_k^\pi(Y_{H,k}) - \tilde{x}_k^\pi(\tilde{Y}_{H,k})\| + G_c D\|\tilde{u}_k^\pi(Y_{H,k}) - \tilde{u}_k^\pi(\tilde{Y}_{H,k})\|.\end{aligned}$$

Based on equation 16, if $Y_{H,k}$ and $\tilde{Y}_{H,k}$ differs in an M_t element, one has

$$\begin{aligned}&\|\tilde{x}_k^\pi(Y_{H,k}) - \tilde{x}_k^\pi(\tilde{Y}_{H,k})\| = \|\tilde{A}_K^t B \sum_{i=0}^{m_w+H-1} \big(M_t^{i-t} - \tilde{M}_t^{i-t}\big) w_{k-i}\mathbb{I}_{1\leq i-t\leq m_w}\| \\ &\leq \kappa_b\kappa^2(1-\gamma)^t\kappa_w \sum_{i=1}^{2H} \big(Y_t^{[i]} - \tilde{Y}_t^{[i]}\big).\end{aligned}$$

On the other hand, if $Y_{H,k}$ and $\tilde{Y}_{H,k}$ differs in an P element, one has

$$\begin{aligned}&\|\tilde{x}_k^\pi(Y_{H,k}) - \tilde{x}_k^\pi(\tilde{Y}_{H,k})\| = \|\tilde{A}_K^t B \sum_{i=0}^{m_r+H-1} \big(P_t^{i-t} - \tilde{P}_t^{i-t}\big) r_{k-i}\mathbb{I}_{1\leq i-t\leq m_w}\| \leq \\ &\kappa_b\kappa^2(1-\gamma)^t\kappa_r \sum_{i=1}^{2H} \big(Y_t^{[i]} - \tilde{Y}_t^{[i]}\big).\end{aligned}$$

Therefore, in general, one has

$$\|\tilde{x}_k^\pi(Y_{H,k}) - \tilde{x}_k^\pi(\tilde{Y}_{H,k})\| \leq \kappa_b\kappa^2(1-\gamma)^t(\kappa_r+\kappa_w) \sum_{i=1}^{2H} \big(Y_t^{[i]} - \tilde{Y}_t^{[i]}\big) \leq \kappa_b\kappa^2(\kappa_r+\kappa_w)\|Y_t - \tilde{Y}_t\|. \tag{42}$$

On the other hand, based on equation 17, one has

$$\begin{aligned}&\|\tilde{u}_k^\pi(Y_{H,k}) - \tilde{u}_k^\pi(\tilde{Y}_{H,k})\| = \|K(\tilde{x}_k^\pi(Y_{H,k}) - \tilde{x}_k^\pi(\tilde{Y}_{H,k})) + \sum_{i=1}^{m_w+m_r} \big(Y_t^{[i]} - \tilde{Y}_t^{[i]}\big)\| \\ &\leq \big(\kappa_b\kappa^3(1-\gamma)^t(\kappa_r+\kappa_w) + 1\big) \sum_{i=1}^{2H} \big(Y_t^{[i]} - \tilde{Y}_t^{[i]}\big) \leq 2\kappa_b\kappa^3(\kappa_r+\kappa_w))\|Y_t - \tilde{Y}_t\|\end{aligned} \tag{43}$$

where the first equality is obtained based on the fact that K is (κ,γ)-stable (see Definition 1). Using equation 42 and equation 43 in equation 41 completes the proof.

Lemma 7 *Let Assumption 5 is satisfied. Then, the following gradient bound is satisfied*

$$\|\nabla_{Y_{H,k}} f_k(Y_{H,k})\|_F \leq 6Hd^2G_c\,(\kappa_r+\kappa_w)\kappa_b\kappa^3\gamma^{-1}G_f \tag{44}$$

where $d = \max(n,m)$.

Proof: To prove the claim, We bound $\nabla_{Y_{p,q}^{[l]}} f_k(Y_{H,k})$ for every $p \in \{1, ..., m\}$, $q \in \{1, ..., n\}$ and $l \in \{1, ..., 2H\}$. We find the bound for the two cases: when $Y_{p,q}^{[l]} = M_{p,q}^{[l_1]}$, for which $l_1 \in \{1, ..., m_w\}$, and when $Y_{p,q}^{[l_2]} = P_{p,q}^{[l]}$, for which $l_2 \in \{1, ..., m_r\}$. For the first case, similar to Zhao et al. (2022), one can show that

$$|\nabla_{M_{p,q}^{[l]}} f_k(Y_{H,k})| \leq 3G_c \kappa_w \kappa_b \kappa^3 \gamma^{-1}. \tag{45}$$

For the second case, using the same procedure as in Zhao et al. (2022), one can show that

$$|\nabla_{P_{p,q}^{[l]}} f_k(Y_{H,k})| \leq 3G_c \kappa_r \kappa_b \kappa^3 \gamma^{-1}. \tag{46}$$

Therefore, since $pq \leq \max(n, m)^2 = d^2$, and $l_1 + l_2 = 2H$, one has $\|\nabla_{Y_{H,k}} f_k(Y_{H,k})\|_F \leq 6Hd^2 G_c (\kappa_r + \kappa_w)\kappa_b \kappa^3 \gamma^{-1}$.

E Proof of Theorem 2

The solution to the system in equation 1 using the linear feedback controller in equation 3 is given in equation 28. Using equation 28, the linear feedback controller in equation 3 reads

$$\begin{aligned} u_k^{\text{lin}} &= K_{fb} x^{\text{lin}} + K_{ff} \sum_{j=0}^{l-1} N^{[l-j]} r_{k-j} \\ &= \sum_{j=1}^{k} \sum_{q=0}^{\min(l-1, j-1)} K_{fb} (A + BK_{fb})^{j-q-1} B K_{ff} N^{[l-q]} r_{k-j} \\ &+ \sum_{i=1}^{k} K_{fb} (A + BK_{fb})^{i-1} w_{k-i} + \sum_{j=0}^{l-1} K_{ff} N^{[l-j]} r_{k-j} \end{aligned} \tag{47}$$

We aim to approximate equation 47 with a linear history-based policy in equation 12. Replacing equation 28 in the linear history-based policy equation 12, we get

$$\begin{aligned} u_k^{\pi} &= Kx_t + \sum_{i=1}^{m_w} M^{[i-1]} w_{k-i} + \sum_{j=0}^{m_r - 1} P^{[j]} r_{k-j} \\ &= \sum_{j=1}^{k} \sum_{q=0}^{\min(l-1, j-1)} K (A + BK_{fb})^{j-q-1} B K_{ff} N^{[l-q]} r_{k-j} \\ &+ \sum_{i=1}^{k} K (A + BK_{fb})^{i-1} w_{k-i} + \sum_{i=1}^{m_w} M^{[i-1]} w_{k-i} + \sum_{j=0}^{m_r - 1} P^{[j]} r_{k-j}. \end{aligned} \tag{48}$$

Now, we derive $u_k^{\text{lin}} - u_k^{\pi}$

$$\begin{aligned} u_k^{\text{lin}} - u_k^{\pi} &= \sum_{i=1}^{m_w} [(K_{fb} - K)(A + BK_{fb})^{i-1} - M^{[i-1]}] w_{k-i} \\ &+ \sum_{i=m_w+1}^{k} (K_{fb} - K)(A + BK_{fb})^{i-1} w_{k-i} \\ &+ \sum_{j=1}^{k} \sum_{q=0}^{\min(l-1, j-1)} (K_{fb} - K)(A + BK_{fb})^{j-q-1} B K_{ff} N^{[l-q]} r_{k-j} \\ &+ \sum_{j=0}^{l-1} K_{ff} N^{[l-j]} r_{k-j} - \sum_{j=0}^{m_r - 1} P^{[j]} r_{k-j}. \end{aligned} \tag{49}$$

Select $M_w^{[i-1]} = (K_{fb} - K)(A + BK_{fb})^{i-1}$ for $i = 1, ..., m_w$. This makes the coefficients of w_{k-i} equal to zero for $i = 1, ..., m_w$.

Similarly, we try to make the coefficients of r_{k-j} equal to zero. We have three cases. a) $j = 0$. r_k appears in the fourth line of equation 49 and its coefficient becomes zero by selecting $P^{[0]} = K_{ff}N^{[l]}$. b) $0 < j < l$. In this case, r_{k-j} appears in the third and fourth lines of equation 49 and $\min(l-1, j-1) = j-1$. Setting the coefficient of r_{k-j} equal to zero

$$\sum_{q=0}^{j-1}(K_{fb} - K)(A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]} + K_{ff}N^{[l-j]} - P^{[j]} = 0,$$

we have

$$P^{[j]} = \sum_{q=0}^{j-1}(K_{fb} - K)(A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]} + K_{ff}N^{[l-j]}.$$

Finally c) $l \leq j < m_r$. In this case, r_{k-j} appears in the third and forth lines of equation 49 and $\min(l - 1, j - 1) = l - 1$. Setting the coefficient of r_{k-j} equal to zero

$$\sum_{q=0}^{l-1}(K_{fb} - K)(A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]} - P^{[j]} = 0,$$

we have

$$P^{[j]} = \sum_{q=0}^{l-1}(K_{fb} - K)(A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]}.$$

The aforementioned results are summarized in equation 19. If we select M, P according to equation 19, then $u_k^{\text{lin}} - u_k^{\pi}$ reads,

$$\begin{aligned} u_k^{\text{lin}} - u_k^{\pi} = &\sum_{i=m_w+1}^{k}(K_{fb} - K)(A + BK_{fb})^{i-1}w_{k-i} \\ + \sum_{j=m_r}^{k}\sum_{q=0}^{l-1}&(K_{fb} - K)(A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]}r_{k-j} = \sum_{i=m_w+1}^{k}M^{[i-1]}w_{k-i} + \sum_{j=m_r}^{k}P^{[j]}r_{k-j}. \end{aligned} \tag{50}$$

As a result

$$\begin{aligned} \|u_k^{\text{lin}} - u_k^{\pi}\| \leq &\sum_{i=m_w+1}^{k}\kappa_b\kappa^3(1-\gamma)^{i-1}\kappa_w + \sum_{j=m_r}^{k}\kappa_b\kappa^3(1-\gamma)^{j}\kappa_r \\ \leq &(k - m_w)\kappa_b\kappa^3(1-\gamma)^{m_w}\kappa_w + (k - m_r + 1)\kappa_b\kappa^3(1-\gamma)^{m_r}\kappa_r. \end{aligned}$$

F Proof of Theorem 3

Using u_k^{lin} in equation 3, x_{k+1}^{lin} can be written as

$$x_{k+1}^{\text{lin}} = Ax_k^{\text{lin}} + Bu_k^{\text{lin}} + w_k = \tilde{A}_K x_k^{\text{lin}} + B(K_{fb} - K)x_k^{\text{lin}} + BK_{ff}\sum_{j=0}^{l-1}N^{[l-j]}r_{k-j} + w_k. \tag{51}$$

Using u_k^{π} in equation 12, x_{k+1}^{π} can be written as

$$x_{k+1}^{\pi} = \tilde{A}_K x_k^{\pi} + B\sum_{t=1}^{m_w}M^{[t-1]}w_{k-t} + w_k + B\sum_{s=0}^{m_r-1}P^{[s]}r_{k-s}. \tag{52}$$

Based on equation 51-equation 52

$$x_{k+1}^{\text{lin}} - x_{k+1}^{\pi} = \tilde{A}_K(x_k^{\text{lin}} - x_k^{\pi}) + B(K_{fb} - K)x_k^{\text{lin}} + BK_{ff}\sum_{j=0}^{l-1} N^{[l-j]}r_{k-j}$$
$$- B\sum_{t=1}^{m_w} M^{[t-1]}w_{k-t} - B\sum_{s=0}^{m_r-1} P^{[s]}r_{k-s}.$$

Replace x_k^{lin} from equation 28 in $B(K_{fb} - K)x_k^{\text{lin}}$

$$x_{k+1}^{\text{lin}} - x_{k+1}^{\pi} = \tilde{A}_K(x_k^{\text{lin}} - x_k^{\pi}) + B(K_{fb} - K)\sum_{j=1}^{k}\sum_{q=0}^{\min(l-1,j-1)} (A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]}r_{k-j}$$
$$+ B(K_{fb} - K)\sum_{i=1}^{k}(A + BK_{fb})^{i-1}w_{k-i} + BK_{ff}\sum_{j=0}^{l-1} N^{[l-j]}r_{k-j} - B\sum_{t=1}^{m_w} M^{[t-1]}w_{k-t} -$$
$$B\sum_{s=0}^{m_r-1} P^{[s]}r_{k-s}. \tag{53}$$

Using M_w in equation 19, the terms containing w in the above equation read

$$t_w = B(K_{fb} - K)\sum_{i=1}^{k}(A + BK_{fb})^{i-1}w_{k-i} - B\sum_{t=1}^{m_w} M^{[t-1]}w_{k-t} =$$
$$B(K_{fb} - K)\sum_{i=1}^{k}(A + BK_{fb})^{i-1}w_{k-i} - B(K_{fb} - K)\sum_{t=1}^{m_w}(A + BK_{fb})^{t-1}w_{k-t}$$
$$= B(K_{fb} - K)\sum_{t=m_w+1}^{k}(A + BK_{fb})^{t-1}w_{k-t}.$$

Using P in equation 19, the terms containing r in the above equation read

$$t_r = B(K_{fb} - K)\sum_{j=1}^{k}\sum_{q=0}^{\min(l-1,j-1)} (A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]}r_{k-j} + BK_{ff}\sum_{j=0}^{l-1} N^{[l-j]}r_{k-j}$$
$$- B\sum_{s=0}^{m_r-1} P^{[s]}r_{k-s} = B(K_{fb} - K)\sum_{j=1}^{k}\sum_{q=0}^{\min(l-1,j-1)} (A + BK_{fb})^{j-q-1}BK_{ff}N^{[l-q]}r_{k-j} +$$
$$BK_{ff}\sum_{j=0}^{l-1} N^{[l-j]}r_{k-j} - B(K_{fb} - K)\sum_{s=1}^{m_r-1}\sum_{q=0}^{\min(l-1,s-1)} (A + BK_{fb})^{s-q-1}BK_{ff}N^{[l-q]}r_{k-s} -$$
$$BK_{ff}\sum_{s=0}^{l-1} N^{[l-s]}r_{k-s} = B(K_{fb} - K)\sum_{s=m_r}^{k}\sum_{q=0}^{l-1}(A + BK_{fb})^{s-q-1}BK_{ff}N^{[l-q]}r_{k-s}.$$

Using t_w, t_r in equation 53, $x_{k+1}^{\text{lin}} - x_{k+1}^{\pi}$ reads

$$x_{k+1}^{\text{lin}} - x_{k+1}^{\pi} = \tilde{A}_K(x_k^{\text{lin}} - x_k^{\pi}) + B(K_{fb} - K)\sum_{t=m_w+1}^{k}(A + BK_{fb})^{t-1}w_{k-t}$$
$$+ B(K_{fb} - K)\sum_{s=m_r}^{k}\sum_{q=0}^{l-1}(A + BK_{fb})^{s-q-1}BK_{ff}N^{[l-q]}r_{k-s}.$$

Since $\tilde{A}_K$ is stable, $x_k^{\text{lin}} - x_k^{\pi}$ reads

$$x_k^{\text{lin}} - x_k^{\pi} = \sum_{i=1}^{k} \tilde{A}_K^{i-1} B(K_{fb} - K) \sum_{t=m_w+1}^{k-i} (A + BK_{fb})^{t-1} w_{k-i-t}$$
$$+ \sum_{i=1}^{k} \tilde{A}_K^{i-1} B(K_{fb} - K) \sum_{s=m_r}^{k-i} \sum_{q=0}^{l-1} (A + BK_{fb})^{s-q-1} BK_{ff} N^{[l-q]} r_{k-i-s}.$$

Since it should hold that $k - i \geq m_w + 1$ in the second line and $k - i \geq m_r$ in the third line,

$$x_k^{\text{lin}} - x_k^{\pi} = \sum_{i=1}^{k-m_w-1} \tilde{A}_K^{i-1} B(K_{fb} - K) \sum_{t=m_w+1}^{k-i} (A + BK_{fb})^{t-1} w_{k-i-t}$$
$$+ \sum_{i=1}^{k-m_r} \tilde{A}_K^{i-1} B(K_{fb} - K) \sum_{s=m_r}^{k-i} \sum_{q=0}^{l-1} (A + BK_{fb})^{s-q-1} BK_{ff} N^{[l-q]} r_{k-i-s}.$$

Using equation 19, we have

$$x_k^{\text{lin}} - x_k^{\pi} = \sum_{i=1}^{k-m_w-1} \tilde{A}_K^{i-1} B \sum_{t=m_w+1}^{k-i} M^{[t-1]} w_{k-i-t} + \sum_{i=1}^{k-m_r} \tilde{A}_K^{i-1} B \sum_{s=m_r}^{k-i} P^{[s]} r_{k-i-s}. \tag{54}$$

Based on (κ, γ)-stability of K and Assumption 5

$$\begin{aligned}
\|x_k^{\text{lin}} - x_k^{\pi}\| \leq & \sum_{i=1}^{k-m_w-1} \kappa^2 \kappa_b (1-\gamma)^{i-1} \sum_{t=m_w+1}^{k-i} \kappa_b \kappa^3 (1-\gamma)^{t-1} \kappa_w \\
& + \sum_{i=1}^{k-m_r} \kappa^2 \kappa_b (1-\gamma)^{i-1} \sum_{s=m_r}^{k-i} \kappa_b \kappa^3 (1-\gamma)^{s} \kappa_r \\
\leq & \sum_{i=1}^{k-m_w-1} \kappa^2 \kappa_b (1-\gamma)^{i-1} (k-1-m_w) \kappa_b \kappa^3 (1-\gamma)^{m_w} \kappa_w \\
& + \sum_{i=1}^{k-m_r} \kappa^2 \kappa_b (1-\gamma)^{i-1} (k-1-m_r) \kappa_b \kappa^3 (1-\gamma)^{m_r} \kappa_r \\
\leq & \gamma^{-1} (k-1-m_w) \kappa^5 \kappa_b^2 (1-\gamma)^{m_w} \kappa_w + \gamma^{-1} (k-1-m_r) \kappa^5 \kappa_b^2 (1-\gamma)^{m_r} \kappa_r.
\end{aligned}$$

G Proof of Theorem 4

Before proving the regret bound, we present Lemma 8 which provides an upper bound for the difference between the costs using optimal linear controller and optimal memory-augmented control policy.

Lemma 8 *Let Assumptions 1-5 hold. Let $K_f^* = [K_{fb}^* \, K_{ff}^*]$ denote the optimal linear gain. Let $x_k^{lin}(K_f^*)$ denote the state using the optimal linear controller $u_k^{lin}(K_f^*)$. Set $H = m_w = m_r$ and let $Y^* = [M^{[0]*}, P^{[0]*}, ..., M^{[H-1]*}, P^{[H-1]*}]$ denote the optimal weights learned by Algorithm 1. Let $\tilde{x}_k^{\pi}(Y^*)$, $\tilde{u}_k^{\pi}(Y^*)$ denote the truncated states and control using the optimal weights according to equation 16-equation 17. Then*

$$\begin{aligned}
& |c_k(\tilde{x}_k^{\pi}(Y^*) - r_k, \tilde{u}_k^{\pi}(Y^*)) - c_k\big(x_k^{lin}(K_f^*) - r_k, u_k^{lin}(K_f^*)\big)| \\
& \qquad \leq 2G_c D \gamma^{-1} H (\kappa_w + \kappa_r) \kappa^6 \kappa_b^2 (1-\gamma)^{(H-1)}.
\end{aligned} \tag{55}$$

Proof of Lemma 8: Based on equation 13

$$x_k^{lin}(K_f^*) = \sum_{y=0}^{k-1} \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1} + \sum_{z=0}^{k-1} \psi_{k,z}^{K_{fb}^*,k} r_{k-z} = \sum_{y=0}^{H-1} \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1} + \sum_{z=0}^{H-1} \psi_{k,z}^{K_{fb}^*,k} r_{k-z}$$
$$+ \sum_{y=H}^{k-1} \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1} + \sum_{z=H}^{k-1} \psi_{k,z}^{K_{fb}^*,k} r_{k-z} = \sum_{y=H}^{k-1} \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1} + \sum_{z=H}^{k-1} \psi_{k,z}^{K_{fb}^*,k} r_{k-z} + x_H^{lin}(K_f^*).$$

As a result, $\|x_k^{lin}(K_f^*) - \tilde{x}_k^{\pi}(Y^*)\|$ reads

$$\|x_k^{lin}(K_f^*) - \tilde{x}_k^{\pi}(Y^*)\| \leq \|x_H^{lin}(K_f^*) - \tilde{x}_H^{\pi}(Y^*)\| + \|\sum_{y=H}^{k-1} \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1}\| + \|\sum_{z=H}^{k-1} \psi_{k,z}^{K_{fb}^*,k} r_{k-z}\|.$$

By equation 54, $\|x_H^{lin}(K_f^*) - \tilde{x}_H^{\pi}(Y^*)\| = 0$. By Lemma 5

$$\begin{aligned} \|\sum_{y=H}^{k-1} \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1}\| &\leq \sum_{y=H}^{k-1} H\kappa^5\kappa_b^2(1-\gamma)^{y-1}\kappa_w \leq \gamma^{-1}H\kappa^5\kappa_b^2(1-\gamma)^{H-1}\kappa_w, \\ \|\sum_{y=H}^{k-1} \psi_{k,z}^{K_{fb}^*,k} r_{k-z}\| &\leq \sum_{y=H}^{k-1} H\kappa^5\kappa_b^2(1-\gamma)^{y-1}\kappa_r \leq \gamma^{-1}H\kappa^5\kappa_b^2(1-\gamma)^{H-1}\kappa_r. \end{aligned} \tag{56}$$

Hence,

$$\|x_k^{lin}(K_f^*) - \tilde{x}_k^{\pi}(Y^*)\| \leq \gamma^{-1}H(\kappa_w + \kappa_r)\kappa^5\kappa_b^2(1-\gamma)^{H-1}.$$

Next, we find $\|u_k^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\|$. In equation 17, set $K \equiv K_{fb}^*$. Then, one has

$$u_k^{lin}(K_f^*) = K_{fb}^* x_k^{lin} + \sum_{s=0}^{H-1} P^* r_{k-s} = \sum_{y=H}^{k-1} K_{fb}^* \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1} + \sum_{z=H}^{k-1} K_{fb}^* \psi_{k,z}^{K_{fb}^*,k} r_{k-z}$$
$$+ K_{fb}^* x_H^{lin} + \sum_{s=0}^{H-1} P^* r_{k-s} = \sum_{y=H}^{k-1} K_{fb}^* \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1} + \sum_{z=H}^{k-1} K_{fb}^* \psi_{k,z}^{K_{fb}^*,k} r_{k-z} + u_H^{lin}(K_f^*).$$

As a result, one can write

$$\|u_k^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\| \leq \|u_H^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\| + \sum_{y=H}^{k-1} \|K_{fb}^* \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1}\| +$$
$$\sum_{z=H}^{k-1} \|K_{fb}^* \psi_{k,z}^{K_{fb}^*,k} r_{k-z}\| = \|u_H^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\|$$
$$+ \sum_{y=H}^{k-1} \|K_{fb}^* \Psi_{k,y}^{K_{fb}^*,k} w_{k-y-1}\| + \sum_{z=H}^{k-1} \|K_{fb}^* \psi_{k,z}^{K_{fb}^*,k} r_{k-z}\|.$$

By equation 50, $\|u_H^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\| = 0$. Using equation 56

$$\|u_k^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\| \leq \gamma^{-1}H(\kappa_w + \kappa_r)\kappa^6\kappa_b^2(1-\gamma)^{H-1}.$$

Therefore,

$$\begin{aligned} &|c_k(\tilde{x}_k^{\pi}(Y^*) - r_k, \tilde{u}_k^{\pi}(Y^*)) - c_k\big(x_k^{lin}(K_f^*) - r_k, u_k^{lin}(K_f^*)\big)| \\ \leq& G_c D\|x_k^{lin}(K_f^*) - \tilde{x}_k^{\pi}(Y^*)\| + G_c D\|u_k^{lin}(K_f^*) - \tilde{u}_k^{\pi}(Y^*)\| \\ \leq& 2G_c D\gamma^{-1}H(\kappa_w + \kappa_r)\kappa^6\kappa_b^2(1-\gamma)^{(H-1)}. \end{aligned}$$

Proof of Theorem 4: The regret reads

$$\begin{aligned}\text{Regret} &= \sum_{k=1}^{T} c_k(e_k, u_k) - T \min_{K_f \in \mathcal{K}} J_T(K_f) \\ &= \underbrace{\sum_{k=1}^{T} c_k(e_k(Y_{0:k-1}), u_k(Y_{0:k-1})) - \sum_{k=1}^{T} f_k(Y_{H,k})}_{\alpha_T} \\ &+ \underbrace{\sum_{k=1}^{T} f_k(Y_{H,k}) - \min_{Y^* \in \mathbb{Y}} \sum_{k=1}^{T} f_k(Y^*)}_{\beta_T} \\ &+ \underbrace{\min_{Y^* \in \mathbb{Y}} \sum_{k=1}^{T} f_k(Y^*) - \sum_{k=1}^{T} c_k\big(x_k^{lin}(K_f^*) - r_k, u_k^{lin}(K_f^*,)\big)}_{\zeta_T}\end{aligned} \tag{57}$$

where $Y^* = [M^{[0]*}, P^{[0]*}, ..., M^{[H-1]*}, P^{[H-1]*}] \in (\mathbb{R}^{m \times n})^{2H}$, and $\mathbb{Y}$ is the set of Y^* for which its elements satisfy Assumption 5.

The regret analysis is split into three parts: α_T denotes the difference between the cost of Algorithm 1 and the truncated cost. β_T denotes the difference between the truncated and optimal truncated costs. ζ_T denotes the difference between the optimal truncated cost and the optimal linear control policy.

We now bound the first term α_T. One has

$$\begin{aligned}&|c_k(e_k, u_k) - f_k(Y_{H,k})| \leq G_c\, D\, \|(x_k^K(Y_{0:k-1}) - r_k) - (\tilde{x}_k^\pi(Y_{H,k}) - r_k)\| \\ &+ G_c D\, \|u_k^K(Y_{0:k-1}) - \tilde{u}_k^\pi(Y_{H,k})\| \leq 2G_c D^2 \kappa^3 (1-\gamma)^H\end{aligned}$$

where we have used Lemma 5 to get the above result. Therefore,

$$\|\alpha_T\| = \|\sum_{k=1}^{T} c_k(e_k, u_k) - \sum_{k=1}^{T} f_k(Y_{H,k})\| \leq 2T\, G_c D^2 \kappa^3 (1-\gamma)^H \leq \mathcal{O}(\sqrt{T}) \tag{58}$$

where the last equality is obtained based on $H = \mathcal{O}(log\, T)$.

We can bound the term β_T by Theorem 4.6 of Agarwal et al. (2019)

$$\sum_{k=1}^{T} f_k(Y_{H,k}) - \min_{Y^* \in \mathbb{Y}} \sum_{k=1}^{T} f_k(Y^*) \leq \frac{1}{\eta} M_b^2 + T G_f^2 \eta + L_f H^2 \eta G_f T \tag{59}$$

where $M_b := 2\sqrt{d}\kappa_b \kappa^3 \gamma^{-1}$, $d = \max(n, m)$. By selecting $\eta = \mathcal{O}(\frac{1}{\sqrt{T}})$, $H = \mathcal{O}(log\, T)$, we have $\beta_T = \mathcal{O}(\sqrt{T})$.

We now bound the third term ζ_T. Based on equation 58, one has,

$$\begin{aligned}\min_{Y^* \in \mathbb{Y}} \sum_{k=1}^{T} f_k(Y^*) - \sum_{k=1}^{T} c_k\big(x_k^{lin}(K_f*) - r_k, u_k^{lin}(K_f*)\big) &\leq \sum_{k=1}^{T} c_k(\tilde{x}_k^\pi(Y^*) - r_k, \tilde{u}_k^\pi(Y^*)) \\ &- \sum_{k=1}^{T} c_k\big(x_k^{lin}(K_f^*) - r_k, u_k^{lin}(K_f^*)\big) + 2T\, G_c D^2 \kappa^3 (1-\gamma)^H.\end{aligned}$$

Using Lemma 8,

$$\begin{aligned}&\min_{Y^* \in \mathbb{Y}} \sum_{k=1}^{T} f_k(Y^*) - \sum_{k=1}^{T} c_k\big(x_k^{lin}(K_f*) - r_k, u_k^{lin}(K_f*)\big) \leq 2TG_c D\gamma^{-1} H(\kappa_w + \kappa_r)\kappa^6 \kappa_b^2 (1-\gamma)^{(H-1)} \\ &+ 2T\, G_c D^2 \kappa^3 (1-\gamma)^H = \mathcal{O}(\sqrt{T})\end{aligned} \tag{60}$$

where the last equality is obtained based on $H = \mathcal{O}(log\, T)$.