AUGMENTATION COMPONENT ANALYSIS: MODELING SIMILARITY VIA THE AUGMENTATION OVERLAPS

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ABSTRACT

Self-supervised learning aims to learn embeddings with which semantically similar samples are close. Contrastive learning methods pull views of samples together and push different samples away, which utilizes semantic invariance of augmentation but ignores the relationship between samples. To better exploit the power of augmentation, we observe that semantically similar samples are more likely to have similar augmented views. So the augmentation feature, composed of the distribution of augmentations, can act as the ideal embedding, and similarity over them reveals how much the augmentations of two samples overlap. Without computational burdens to explicitly estimate its value, we propose Augmentation Component Analysis (ACA) with a contrastive-like loss to learn principal components and an on-the-fly projection loss to embed data. ACA equals an efficient dimension reduction by PCA and extracts low-dimensional embeddings, theoretically preserving the similarity of augmentation distribution between samples. Empirical results show our method can achieve competitive results against various traditional contrastive learning methods on different benchmarks.

1 INTRODUCTION

The rapid development of contrastive learning has pushed self-supervised representation learning to unprecedented success. Many contrastive learning methods surpass traditional pretext-based methods by a large margin and even outperform representation learned by supervised learning (Wu et al., 2018; van den Oord et al., 2018; Tian et al., 2020a; He et al., 2020a; Chen et al., 2020a;c). The key idea of self-supervised contrastive learning is to construct views of samples via modern data augmentations (Chen et al., 2020a). Then discriminative embeddings are learned by pulling together views of the same sample in the embedding space while pushing apart views of others.

Contrastive learning methods utilize the semantic invariance between views of the same sample, but the semantic relationship between samples is ignored. Instead of measuring the similarity between certain augmented views of samples, we claim that it is the similarity between the *augmentation distributions* of samples, *i.e.*, the set of all possible images to be augmented to, that reveals the sample-wise similarity better. As is shown in Figure 1 left, two images of deer create many similar crops, and sets of their augmentation results, *i.e.*, their distributions, overlap much. In contrast, a car image will rarely be augmented to the same crop as a deer, and their augmentation distributions overlap little. In Figure 1 right, we verify the motivation numerically. We approximate the overlaps between image augmentations with a classical image matching algorithm (Zitova & Flusser, 2003), which counts the portion of matched key points of raw images. We find samples of the same class overlap more than different classes on average, supporting our motivation. Therefore, we establish the semantic relationship between samples in an unsupervised manner based on the similarity of augmentation distributions, *i.e.*, how much they overlap.

The augmentation overlap naturally acts as the self-supervision to learn embeddings. Following this idea, we can manually construct an "ideal" (but possibly infinite-dimensional) *augmentation feature* for a sample — whose elements reveal the probability of augmenting that sample to another and encode the augmentation information comprehensively. It seems that we can obtain the solution directly without further learning, however, not only the elements in the augmentation feature are hard to calculate, but also such high-dimensional target embeddings are impractical to use.

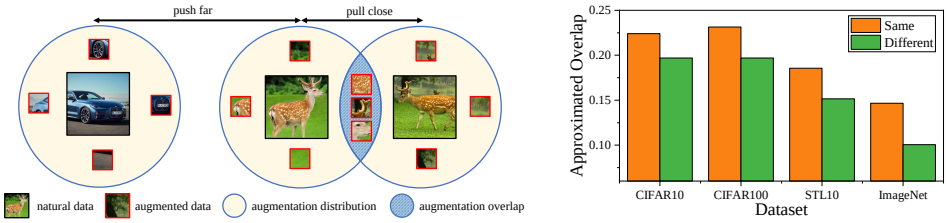

Figure 1: Left: semantically similar samples (*e.g.*, those in the same class) usually create similar augmentations. The right figure indicates the same class images have higher averaged augmentation overlaps than those from different classes on four common datasets. For this reason, we learn embeddings by preserving the similarity between augmentation distributions of samples.

Inspired by the classical strategy to deal with high-dimensional data, we propose Augmentation Component Analysis (ACA), which employs the idea of PCA (Hotelling, 1933) to perform dimension reduction on augmentation features previously mentioned. ACA reformulates the steps of extracting principal components of the augmentation features with a contrastive-like loss. With the learned principal components, another on-the-fly loss embeds samples effectively. ACA learns operable low-dimensional embeddings theoretically preserving the augmentation distribution distances.

In addition, the resemblance between the objectives of ACA and traditional contrastive loss may explain why the latter can learn semantic-related embeddings – they embed samples into spaces that partially preserve augmentation distributions. Experiments on synthetic and real-world datasets demonstrate that our ACA achieves competitive results against various traditional contrastive learning methods. Our contributions are:

- We propose a new self-supervised strategy, which measures the sample-wise similarity via the similarity of augmentation distributions. This new aspect facilitates learning embeddings.
- We propose ACA method that implicitly employs the dimension reduction over the augmentation feature, and the learned embeddings preserve augmentation similarity between samples.
- Benefiting from the resemblance to the contrastive loss, our ACA helps explain the functionality of contrastive learning and why they can learn semantically meaningful embeddings.

2 Related Work

Self-Supervised learning. Learning effective visual representations without human supervision is a long-standing problem. Self-supervised learning methods solve this problem by creating supervision from the data itself instead of human labelers. The model needs to solve a pretext task before it is used for the downstream tasks. For example, in computer vision, the pretext tasks include colorizing grayscale images (Zhang et al., 2016), inpainting images (Pathak et al., 2016), predicting relative patch (Doersch et al., 2015), solving jigsaw puzzles (Noroozi & Favaro, 2016), predicting rotations (Gidaris et al., 2018) and exploiting generative models (Goodfellow et al., 2014; Kingma & Welling, 2014; Donahue & Simonyan, 2019). Self-supervised learning also achieves great success in natural language processing (Mikolov et al., 2013; Devlin et al., 2019).

Contrastive Learning. Contrastive approaches have been one of the most prominent representation learning strategies in self-supervised learning. The idea behind these approaches is to maximize the agreement between positive pairs and minimize the agreement between negative pairs. Positive pairs are commonly constructed by co-occurrence (van den Oord et al., 2018; Tian et al., 2020a; Bachman et al., 2019) or augmentation of the same sample (He et al., 2020a; Chen et al., 2020a;c; Li et al., 2021), while all the other samples are taken as negatives. Most of these methods employ the InfoNCE loss (van den Oord et al., 2018), which acts as a lower bound of mutual information between views. Some methods do not involve negative samples (Grill et al., 2020; Chen & He, 2021).

Although most contrastive methods only utilize the semantic invariance of augmentation and ignore the relationship between samples, their learned embeddings somehow form semantically meaningful clusters (He et al., 2020a; Chen et al., 2020a;c). Some of the previous works explain this phenomenon with the strong assumption that two views are conditionally independent given the label or a hidden variable (Saunshi et al., 2019; Tosh et al., 2021). HaoChen et al. (2021) turn the assumption towards how views of different samples are connected by the augmentation process. Wang et al.

(2022) notice that the augmentation overlap provides a ladder for gradually learning class-separated representations. These analyses motivate us to rethink the role of augmentation and emphasize augmentation distribution as a better self-supervised way of exploiting semantic similarity.

Explanation of Contrastive Learning. Several works provide empirical or theoretical results for explaining the behavior of contrastive learning. Tian et al. (2020b); Xiao et al. (2021) explore the role of augmentation and show contrastive model can extract useful information from views but also can be affected by nuisance information. Zhao et al. (2021) empirically shows that contrastive learning preserves low-level or middle-level instance information. In theoretical studies, Saunshi et al. (2019) provide guarantees of downstream linear classification tasks under conditionally independence assumption. Other works weaken the assumption but are still unrealistic (Lee et al., 2021; Tosh et al., 2021). HaoChen et al. (2021) focus on how views of different samples are connected by the augmentation process and provide guarantees with certain connectivity assumptions. Wang et al. (2022) notice that the augmentation overlap provides a ladder for gradually learning class-separated representations. However, Wang et al. (2022) provides their analysis based on the assumption that common contrastive loss can somehow learn to align overlapped samples. In contrast, we provide a novel contrastive method that ensures preserving augmentation overlap.

3 Notations and Preliminary

Denote the set of all the natural data (the data without augmentation) as $\bar{\mathcal{X}}$, with its size $|\bar{\mathcal{X}}| = N$. We assume the natural data follows a uniform distribution $p(\bar{\boldsymbol{x}})$ on $\bar{\mathcal{X}}$, *i.e.*, $p(\bar{\boldsymbol{x}}) = \frac{1}{N}, \forall \bar{\boldsymbol{x}} \in \bar{\mathcal{X}}$. By applying an augmentation method $\mathcal{A}$, a natural sample $\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}$ could be augmented to another sample $\boldsymbol{x}$ with probability $p_{\mathcal{A}}(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$, so we use $p(\cdot \mid \bar{\boldsymbol{x}})$ to encode the augmentation distribution. [1] For example, if $\bar{\boldsymbol{x}}$ is an image, then $\mathcal{A}$ can be common augmentations like Gaussian blur, color distortion and random cropping (Chen et al., 2020a). Denote the set of all possible augmented data as $\mathcal{X}$. We assume $\mathcal{X}$ has finite size $|\mathcal{X}| = L$ and $L > N$ for ease of exposition. Note that N and L are finite but they can be arbitrarily large.

Self-Supervised Representation Learning (SSL) extracts features with an encoder f_θ, parameterized by θ, which projects a sample $\boldsymbol{x}$ to an embedding vector in $\mathbb{R}^k$. By training on a pretext task, SSL aims to obtain an encoder that provides semantically meaningful representations of any sample and performs well on various downstream tasks. The most popular way of constructing pretext tasks is to utilize the information provided by data augmentation and contrastive learning belongs to this stream. Traditional contrastive learning tasks use augmentation to create views, taking advantage of its semantic invariance property. They pull views of the same samples together and push away views of different samples. A common practice is to use a contrastive loss, *e.g.*, the InfoNCE loss (van den Oord et al., 2018). However, utilizing the semantic invariance property of augmentation only exploits the relationship between views of the same sample, and ignores the semantic relationships between samples. Different from them, we propose a new way to perform contrastive learning with the observation that a strong correlation exists between the similarity of *augmentation distributions* and the similarity of semantics.

4 Learning via Augmentation Overlaps

As we mentioned in Section 1, measuring the similarity between augmentation distributions, *i.e.*, the overlap of all two sample's augmented results, reveals their *semantic* relationship well. For example, in natural language processing, we usually generate augmented sentences by dropping out some words. Then different sentences with similar meanings are likely to contain the same set of words and thus have a high probability of creating similar augmented data. With the help of this self-supervision, <u>we formulate the embedding learning task to meet the following similarity preserving condition</u>:

$$\mathrm{d}_{\mathbb{R}^k}\left(f_{\theta^\star}\left(\bar{\boldsymbol{x}}_1\right),\ f_{\theta^\star}\left(\bar{\boldsymbol{x}}_2\right)\right) \propto \mathrm{d}_{\mathcal{A}}(p(\cdot \mid \bar{\boldsymbol{x}}_1),\ p(\cdot \mid \bar{\boldsymbol{x}}_2))\ . \tag{1}$$

$\mathrm{d}_{\mathbb{R}^k}$ is a distance measure in the embedding space $\mathbb{R}^k$, and $\mathrm{d}_{\mathcal{A}}$ measures the distance between two augmentation distributions. Equation (1) requires the learned embedding with the optimal parameter $\theta^\star$ has the *same similarity comparison* with that measured by the augmentation distributions. In this section, we first introduce the *augmentation feature* for each sample, *which is a manually designed embedding satisfying the condition in Equation* (1). To handle the high dimensionality and complexity

[1] Note that $p(\cdot \mid \bar{\boldsymbol{x}})$ is usually difficult to compute and we can only sample from it. We omit the subscript $\mathcal{A}$ and directly use $p(\cdot \mid \bar{\boldsymbol{x}})$ in the following content for convenient

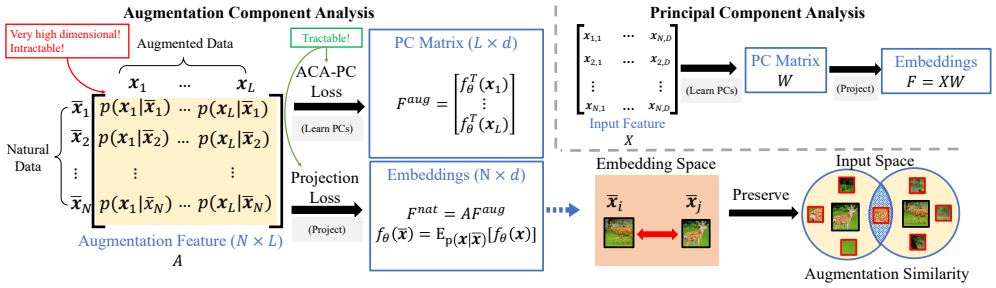

Figure 2: The idea of learning embeddings via Augmentation Component Analysis (ACA). The upper right figure demonstrates the process of PCA. It learns PCs and projects the input feature to get embeddings of data. Similarly, ACA performs PCA on the augmentation feature, which encodes all the information about the augmentation distribution. To overcome the dimensional and computational complexity, ACA employs ACA-PC loss and projection loss to tractably learn PCs and embeddings. Via ACA, our model can learn embeddings that preserve augmentation similarity for natural data.

of the augmentation feature, we further propose our Augmentation Component Analysis (ACA) that learns to reduce the dimensionality and preserve the similarity.

4.1 Augmentation Feature

To reach the goal of similarity preserving in Equation (1), a direct way is to manually construct the feature by the augmentation distributions of each natural sample, *i.e.*, $f(\bar{\boldsymbol{x}}) = [p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}}), \dots, p(\boldsymbol{x}_L \mid \bar{\boldsymbol{x}})]^\top$, where each element $p(\boldsymbol{x}_i \mid \bar{\boldsymbol{x}})$ represents the probability of getting a certain element $\boldsymbol{x}_i$ in space $\mathcal{X}$ by augmenting $\bar{\boldsymbol{x}}$. We omit θ in $f(\bar{\boldsymbol{x}})$ since such *augmentation feature*[2] does not rely on any learnable parameters. In this case, any distance $\mathrm{d}_{\mathbb{R}^L}$ defined in space of f is exactly a valid distribution distance, which reveals the augmentation overlaps as well as the semantic similarity.

Though the constructive augmentation feature naturally satisfies the similarity preserving condition (Equation (1)) (because it directly use the augmentation distribution without loss of information), it is impractical for the following reasons. First, <u>its dimensionality is exponentially high</u>, which is up to L, the number of possible augmented results. For example, even on CIFAR10, the small-scale dataset with image size $32 \times 32 \times 3$, L is up to 256^{3072} (3072 pixels and 256 possible pixel values). Second, <u>the computation of each element is intractable</u>. We may need an exponentially large number of samples to accurately estimate each $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$. The dimensionality and computation problems make the augmentation feature impractical both at inference and training time. Such inconvenience motivates us to (1) conduct certain dimension reduction to preserve the information in low dimensional space (Section 4.2) and (2) develop an efficient algorithm for dimension reduction (Section 4.3).

4.2 Dimension Reduction on Augmentation Features

To deal with the high-dimensional property, we employ the idea of PCA (Hotelling, 1933), which reconstructs the data with principal components.[3] For convenience, we denote the design matrix of augmentation feature by A, where $A \in \mathbb{R}^{N \times L}$, $A_{\bar{\boldsymbol{x}},\boldsymbol{x}} = p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$ (see Figure 2). We perform PCA on a transformed augmentation feature called *normalized augmentation feature*:

$$\hat{A} = AD^{-\frac{1}{2}}, \tag{2}$$

where $D = \mathrm{diag}([d_{\boldsymbol{x}_1}, d_{\boldsymbol{x}_2}, \dots, d_{\boldsymbol{x}_L}])$, $d_{\boldsymbol{x}} = \sum_{\bar{\boldsymbol{x}}} p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$. Based on normalized augmentation feature, we can develop an efficient algorithm for similarity preserving embeddings.

Assume the SVD of $\hat{A} = U\Sigma V^\top$ with $U \in \mathbb{R}^{N \times N}$, $\Sigma \in \mathbb{R}^{N \times L}$, $V \in \mathbb{R}^{L \times L}$, PCA first learns the projection matrix consisting of the top-k right singular vectors, which can be denoted as $\tilde{V} \in \mathbb{R}^{L \times k}$. The vectors in $\tilde{V}$ are called Principal Components (PCs). Then, it projects the feature by $\hat{A}\tilde{V}$ to get the embeddings for each sample. The overall procedure is illustrated in Figure 2. Performing PCA on the augmentation feature will encounter many obstacles. First, the element of augmentation

[2]Following the common knowledge in dimension reduction, we call the raw high dimensional representation as "feature", and learned low-dimensional representation as "embedding".

[3]In this paper, we use the non-centered version (Reyment & Jvreskog, 1996), which is more appropriate for observations than for variables, where the origin matters more.

feature is not possible to accurately estimate. Second, even though we can somehow get the precise value of each $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$, the augmentation feature is exponentially large, making the PCA process too time-consuming. Third, assume we can somehow get the projection matrix $\tilde{V}$, it is also impractical to project high dimensional matrix $\hat{A}$. For this reason, *we propose ACA to make PC learning and projection process efficient without explicitly calculating elements of augmentation feature*.

4.3 Augmentation Component Analysis

Although there are several obstacles when performing PCA on the augmentation features directly, fortunately, it is efficient to sample from the augmentation distribution $p(\bar{\boldsymbol{x}} \mid \boldsymbol{x})$, *i.e.*, by performing augmentation on the natural data $\boldsymbol{x}$ and get an augmented sample $\bar{\boldsymbol{x}}$. Being aware of this, our ACA uses two practical losses to simulate the PCA process efficiently by sampling. The first contrastive-like loss leads the encoder to learn principal components of $\hat{A}$, which can be efficiently optimized by sampling like traditional contrastive methods. The second loss performs on-the-fly projection of $\hat{A}$ through the training trajectory, which solves the difficulty of high dimensional projection.

Learning principal components. ACA learns the principal components by an efficient contrastive-like loss. Besides its projection functionality, these learned principal components can also serve as embeddings that preserve a kind of posterior distribution similarity, as we will show later.

In the SVD view, $U\Sigma$ serves as the PCA projection results for samples and V contains the principal components (Jolliffe, 2002). However, if changing our view, $V\Sigma$ can be seen as the representation of each column. Since each column of $\hat{A}$ encodes probability on augmented data given natural data, $V\Sigma$ preserves certain augmentation relationships, as we will show in Theorem 4.2 later. To leverage the extrapolation power of encoders like deep neural networks, we choose to design a loss that can guide the parameterized encoder f_θ to learn similar embeddings as PCA. Inspired by the rank minimization view of PCA (Vidal et al., 2016), we employ the low-rank approximation objective with matrix factorization, similar to HaoChen et al. (2021):

$$\min_{F \in \mathbb{R}^{L \times k}} \mathcal{L}_{mf} = \|\hat{A}^\top \hat{A} - FF^\top\|_F^2 \ , \tag{3}$$

where columns of F store the scaled version of top-k right singular vectors, and each row can be seen as the embedding of augmented data as will show in Lemma 4.1. According to Eckart–Young–Mirsky theorem (Eckart & Young, 1936), by optimizing $\mathcal{L}_{mf}$, we can get the optimal $\hat{F}$, which has the form $\tilde{V}\tilde{\Sigma}Q$, $Q \in \mathbb{R}^{k \times k}$ is an orthonormal matrix. $\tilde{\Sigma}$ and $\tilde{V}$ contains the top-k singular values and right singular vectors. By expanding Equation (3), we get Augmentation Component Analysis Loss for learning Principal Components (ACA-PC) in the following lemma:

Lemma 4.1 (ACA-PC loss)**.** *Let $F_{\boldsymbol{x},:} = \sqrt{d_{\boldsymbol{x}}} f_\theta^\top(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}$. Minimizing $\mathcal{L}_{mf}$ is equivalent to minimizing the following objective:*

$$\begin{aligned} \mathcal{L}_{\text{ACA-PC}} = &- 2\mathbb{E}_{\substack{\bar{\boldsymbol{x}} \sim p(\bar{\boldsymbol{x}}), \boldsymbol{x}_i \sim p(\boldsymbol{x}_i \mid \bar{\boldsymbol{x}}) \\ \boldsymbol{x}_j \sim p(\boldsymbol{x}_j \mid \bar{\boldsymbol{x}})}} f_\theta(\boldsymbol{x}_i)^\top f_\theta(\boldsymbol{x}_j) \\ &+ N\mathbb{E}_{\boldsymbol{x}_1 \sim p_{\mathcal{A}}(\boldsymbol{x}_1), \boldsymbol{x}_2 \sim p_{\mathcal{A}}(\boldsymbol{x}_2)} \left[\left(f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2) \right)^2 \right] . \end{aligned} \tag{4}$$

The proof can be found in Appendix C. In ACA-PC, the first term is the common alignment loss for augmented data and the second term is a form of uniformity loss (Wang & Isola, 2020). Both the two terms can be estimated by Monte-Carlo sampling. ACA-PC is a kind of contrastive loss. But unlike most of the others, it has theoretical meanings. We note that the form of ACA-PC differs from spectral loss (HaoChen et al., 2021) by adding a constant N before the uniformity term. This term is similar to the noise strength in NCE (Gutmann & Hyvärinen, 2010) or the number of negative samples in InfoNCE (van den Oord et al., 2018). It can be proved that the learned embeddings by ACA-PC preserve the posterior distribution distances between augmented data:

Theorem 4.2 (Almost isometry for posterior distances)**.** *Assume f_θ is a universal encoder, σ_{k+1} is the $(k+1)$-th largest singular value of $\hat{A}$, $d_{\min} = \min_{\boldsymbol{x}} d_{\boldsymbol{x}}$, and $\delta_{\boldsymbol{x}_1 \boldsymbol{x}_2} = \mathbb{I}(\boldsymbol{x}_1 = \boldsymbol{x}_2)$, the minimizer θ^* of $\mathcal{L}_{ACA-PC}$ satisfies:*

$$\mathrm{d}_{\text{post}}^2(\boldsymbol{x}_1, \boldsymbol{x}_2) - \frac{2\sigma_{k+1}^2}{d_{\min}} (1 - \delta_{\boldsymbol{x}_1 \boldsymbol{x}_2}) \leq \|f_{\theta^*}(\boldsymbol{x}_1) - f_{\theta^*}(\boldsymbol{x}_2)\|_2^2 \ \leq \ \mathrm{d}_{\text{post}}^2(\boldsymbol{x}_1, \boldsymbol{x}_2) \ , \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$$

where the posterior distance

$$\mathrm{d}_{\text{post}}^2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(p_{\mathcal{A}}(\bar{\boldsymbol{x}} \mid \boldsymbol{x}_1) - p_{\mathcal{A}}(\bar{\boldsymbol{x}} \mid \boldsymbol{x}_2) \right)^2$$

measures the squared Euclidean distance between the posterior distribution $p_{\mathcal{A}}(\bar{\boldsymbol{x}} \mid \boldsymbol{x}) = \frac{p(\boldsymbol{x}|\bar{\boldsymbol{x}})p(\bar{\boldsymbol{x}})}{p_{\mathcal{A}}(\boldsymbol{x})}$.

We give the proof in Appendix D. Theorem 4.2 states that the optimal encoder for ACA-PC preserves the distance of posterior distributions between augmented data within an error related to embedding size k. As k increase to N, the error decrease to 0. It corresponds to the phenomenon that a larger embedding size leads to better contrastive performance (Chen et al., 2020a). The posterior distribution $p_{\mathcal{A}}(\bar{\boldsymbol{x}} \mid \boldsymbol{x})$ represents the probability that a given augmented sample $\boldsymbol{x}$ is created by a natural sample $\bar{\boldsymbol{x}}$. Augmented data that are only produced by the same natural sample will have the smallest distance, and embeddings of those in overlapped areas will be pulled together by ACA-PC. Since the overlapped area are usually created by two same-class samples, ACA-PC can form semantically meaningful embedding space.

It is also noticeable that the optimal encoder meets the similarity preserving condition (Equation (1)) but concerning the posterior distribution for augmented data not the augmentation distribution for natural data. Since what we care about is the distribution of natural data, we further propose a projection loss that helps learn good embeddings for all the natural data.

On-the-fly Projection. As stated in the previous part, the learned embeddings by ACA-PC not only serve as certain embeddings for augmented data but also contain principal components of normalized augmentation feature. Based on this, we propose to use these embeddings to act as a projection operator to ensure meaningful embeddings for all the natural data. To be specific, denote the embedding matrix for all augmented data as $F^{aug}(\in \mathbb{R}^{L\times k})$, where each row $F^{aug}_{\boldsymbol{x},:} = f_{\theta^*}(\boldsymbol{x})$. From Equation (3) and $\hat{F}_{\boldsymbol{x},:} = \sqrt{d_{\boldsymbol{x}}} f^{\top}_{\theta^*}(\boldsymbol{x})$, it can be easily seen that:

$$F^{aug} = D^{-\frac{1}{2}}\hat{F} = D^{-\frac{1}{2}}\tilde{V}\tilde{\Sigma}Q$$

Similar to PCA (Hotelling, 1933) that project the original feature by V, the principal components, we propose to use F^{aug} to project the augmentation feature to get the embeddings for each natural sample. Denote the embedding matrix for natural data as $F^{nat}(\in \mathbb{R}^{N\times k})$, where each row $F^{nat}_{\bar{\boldsymbol{x}},:}$ represents the embeddings of $\bar{\boldsymbol{x}}$. We compute F^{nat} as follows:

$$F^{nat} = AF^{aug} = \hat{A}D^{\frac{1}{2}}D^{-\frac{1}{2}}\tilde{V}\tilde{\Sigma}Q = (\tilde{U}\tilde{\Sigma})\tilde{\Sigma}Q, \tag{5}$$

where $\tilde{\Sigma}$,$\tilde{U}$ contain the top-k singular values and corresponding left singular vectors. It is noticeable that F^{nat} is exactly the PCA projection result multiplied by an additional matrix $\tilde{\Sigma}Q$. Fortunately, such additional linear transformation does not affect the linear probe performance (HaoChen et al., 2021). With Equation (5), the embedding of each natural sample can be computed as follows:

$$F^{nat}_{\bar{\boldsymbol{x}},:} = A_{\bar{\boldsymbol{x}},:}F^{aug} = \sum_{\boldsymbol{x}} p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}) f^{\top}_{\theta^*}(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}\sim p(\boldsymbol{x}|\bar{\boldsymbol{x}})} f^{\top}_{\theta^*}(\boldsymbol{x}) \tag{6}$$

which is exactly the expected feature over the augmentation distribution. Similar to Theorem 4.2, the embeddings calculated by Equation (6) also present a certain isometry property :

Theorem 4.3 (Almost isometry for weighted augmentation distances)**.** *Assume f_θ is a universal encoder, σ_{k+1} is the $(k+1)$-th largest sigular value of $\hat{A}$,$\delta_{\bar{\boldsymbol{x}}_1\bar{\boldsymbol{x}}_2} = \mathbb{I}(\bar{\boldsymbol{x}}_1 = \bar{\boldsymbol{x}}_2)$, let the minimizer of $\mathcal{L}_{ACA-PC}$ be θ^* and $g(\bar{\boldsymbol{x}}) = \mathbb{E}_{\boldsymbol{x}\sim p(\boldsymbol{x}|\bar{\boldsymbol{x}})} f_{\theta^*}(\boldsymbol{x})$ as in Equation* (6)*, then:*

$$\mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) - 2\sigma^2_{k+1}\left(1 - \delta_{\bar{\boldsymbol{x}}_1\bar{\boldsymbol{x}}_2}\right) \leq \|g(\bar{\boldsymbol{x}}_1) - g(\bar{\boldsymbol{x}}_2)\|^2_{\Sigma_k^{-2}} \;\leq\; \mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2)\;, \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$$

where $\|\cdot\|_{\Sigma_k^{-2}}$ represent the Mahalanobis distance with matrix Σ_k^{-2},$\Sigma_k = \mathrm{diag}([\sigma_1, \sigma_2, \ldots, \sigma_k])$ is the diagonal matrix containing top-k sigular values and the weighted augmentation distance

$$\mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) = \frac{1}{N}\sum_{\boldsymbol{x}\in\mathcal{X}} \frac{\left(p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1) - p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_2)\right)^2}{p_{\mathcal{A}}(\boldsymbol{x})}$$

measures the weighted squared Euclidean distance between the augmentation distribution $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$.

Different from Theorem 4.2, which presents isometry between Euclidean distances in embeddings and augmentation distribution, Theorem 4.3 presents isometry between Mahalanobis distances. The weighted augmentation distances weight the Euclidean distances by $p_{\mathcal{A}}(\boldsymbol{x})$. $\mathrm{d}_{\text{w-aug}}$ can be regarded as a valid augmentation distance measure $\mathrm{d}_{\mathcal{A}}$ as in Equation (1) and F^{nat} preserve such distance.

So our goal to make embeddings of $\bar{\boldsymbol{x}}$ approaches $\mathbb{E}_{p(\boldsymbol{x}|\bar{\boldsymbol{x}})} f_{\theta^\star}(\boldsymbol{x})$. However, as stated before, the additional projection process is not efficient, *i.e.*, we need exponentially many samples from $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$. We notice that samples during the training process of ACA-PC can be reused. For this reason, we propose an on-the-fly projection loss that directly uses the current encoder for projection:

$$\mathcal{L}_{\text{proj}} = \mathbb{E}_{\bar{\boldsymbol{x}} \sim p(\bar{\boldsymbol{x}})} \left[\| f_\theta(\bar{\boldsymbol{x}}) - \mathbb{E}_{p(\boldsymbol{x}|\bar{\boldsymbol{x}})} f_\theta(\boldsymbol{x}) \|_2^2 \right] \tag{7}$$

Full objective of ACA. Based on the discussion of the above parts, ACA simultaneously learns the principal components by ACA-PC and projects natural data by an on-the-fly projection loss. The full objective of ACA has the following form:

$$\mathcal{L}_{\text{ACA-Full}} = \mathcal{L}_{\text{ACA-PC}} + \alpha \mathcal{L}_{\text{proj}} \tag{8}$$

where α is a trade-off hyperparameter. We also find N in Equation (4) too large for stable training, so we replace it with a tunable hyperparameter K. Here, we only display the loss in expectation forms. Details of implementation are described in Appendix A.

5 A Pilot Study

In this section, we experiment with our Augmentation Component Analysis method on a synthetic mixture component data with a Gaussian augmentation method. In this example, we aim to show the relationship between semantic similarity and posterior/weighted augmentation distances. We also show the effectiveness of our method compared to traditional contrastive learning. In this example, the natural data $\bar{\boldsymbol{x}}$ are sampled from a mixture gaussian with c component:

$$p(\bar{\boldsymbol{x}}) = \sum_{i=1}^{c} \pi_i \mathcal{N}(\boldsymbol{\mu}_i, s_i I)$$

We use Gaussian noise as the data augmentation of a natural data sample, *i.e.*, $\mathcal{A}(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}} + \xi$ where $\xi \sim \mathcal{N}(0, s_a I)$. Concretely, we conduct our experiment on 2-D data with $c = 4, \pi_i = \frac{1}{c}, s_i = 1$ and the μ_i uniformly distributed on a circle with radius 2 . For each component, we sample 200 natural data with the index of the component as their label. For each natural datum, we augment it 2 times with $s_a = 4$, which results in totally 1600 augmented data. We compute the augmentation probability for between $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ by $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$ and we normalize the probability for each $\bar{\boldsymbol{x}}$.

First, we plot the distribution of posterior distances for pairs of augmented data and weighted augmentation distances for pairs of natural data in Figure 3 left. The two distances appear similar distributions because the synthetic data are Gaussian. It can be seen that data from the same component tend to have small distances, while from different components, their distances are large. In low-distance areas, there are same-class pairs, meaning that the two distances are reliable metrics for judging semantic similarity. In all, this picture reveals the correlation between semantic similarity and posterior/weighted augmentation distances.

Second, we compare our methods with SimCLR (Chen et al., 2020a), the traditional contrastive method and Spectral (HaoChen et al., 2021), which similarly learns embeddings with spectral theory. We test the learned embeddings using a Logistic Regression classifier and report the error rate of the prediction in Figure 3 right. We also report performance when directly using *augmentation feature* (AF). First, AF has the discriminability for simple linear classifiers. As the embedding size increase, SimCLR and Spectral tend to underperform AF, while our methods consistently outperform. It may be confusing since our method performs dimension reduction on this feature. But we note that as the embedding size increase, the complexity of the linear model also increases, which affect the generalizability. All of the methods in Figure 3 right show degradation of this kind. However, our methods consistently outperform others, which shows the superiority of ACA. Also, by adding projection loss, ACA-Full improves ACA-PC by a margin. Additionally, traditional contrastive learning like SimCLR achieves similar performance as our methods. We think it reveals that traditional contrastive learning has the same functionality as our methods.

6 Experiments

6.1 Setup

Dataset. In this paper, we conduct experiments mainly on the following datasets with RTX-3090 $\times 4$. **CIFAR-10** and **CIFAR-100** (Krizhevsky et al., 2009): two datasets containing totally 500K

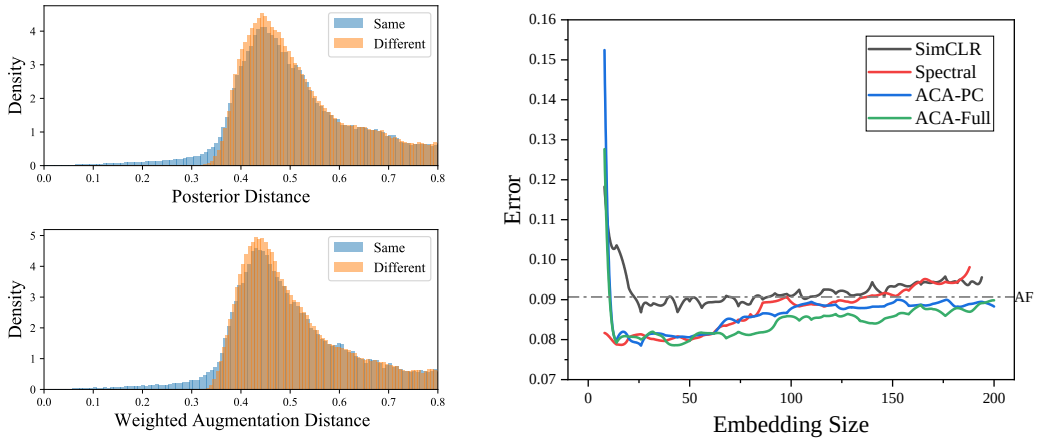

Figure 3: Synthetic experiments on mixture Gaussian data with Gaussian noise as augmentation. (a) The posterior distance and weighted augmentation distances among data that is sampled from the same component and different components. It reveals the correlation between semantic similarity and the two distances, especially when the distance is small. (b) Comparison of linear classification performance among SimCLR, Spectral and our methods with various embedding dimensions ranging from 4 to 200. The dashed line represents the result when directly using *Augmentation Feature* (AF). ACA-PC outperforms SimCLR and Spectral. ACA-Full further improves.

images of size 32×32 from 10 and 100 classes respectively. **STL-10** (Coates et al., 2011): derived from ImageNet (Deng et al., 2009), with 96×96 resolution images with 5K labeled training data from 10 classes. Additionally, 100K unlabeled images are used for unsupervised learning. **Tiny ImageNet**: a reduced version of ImageNet (Deng et al., 2009), composed of 100K images scaled down to 64×64 from 200 classes. **ImageNet-100** (Tian et al., 2020a): a subset of ImageNet, with 100-classes. **ImageNet** (Deng et al., 2009), the large-scale dataset with 1K classes.

Network Structure. Following common practice (Chen et al., 2020a;b;c), we use the encoder-projector structure during training, where the projector projects the embeddings into a low-dimensional space. For CIFAR-10 and CIFAR-100, we use the CIFAR variant of ResNet-18 (He et al., 2016; Chen & He, 2021) as the encoder. We use a two-layer MLP as the projector whose hidden dimension is half of the input dimension and output dimension is 64. For STL-10 and Tiny ImageNet, only the max-pooling layer is disabled following (Chen & He, 2021; Ermolov et al., 2021). For these two datasets, we use the same projector structure except for the output dimension is 128. For ImageNet, we use ResNet-50 with the same projector as Chen et al. (2020a).

Image Transformation. Following the common practice of contrastive learning (Chen et al., 2020a), we apply the following augmentations sequentially during training: (a) crops with a random size; (b) random horizontal flipping; (c) color jittering; (d) grayscaling. For ImageNet-100 and ImageNet, we use the same implementation as (Chen et al., 2020a).

Optimizer and other Hyper-parameters. For datasets except for ImageNet, adam optimizer (Kingma & Ba, 2015) is used for all the datasets. For CIFAR-10 and CIFAR-100, we use 800 epochs with a learning rate of 3×10^{-3}. For Tiny ImageNet and STL-10, we train 1,000 epochs with learning rate 2×10^{-3}. We use a 0.1 learning rate decay at 100, 50, 20 epochs before the end. Due to hardware resource restrictions, we use a mini-batch of size 512. The weight decay is 1×10^{-6} if not specified. Following common practice in contrastive learning, we normalize the projected feature into a sphere. For CIFAR-10, we use $\alpha = 1$. For the rest datasets, we use $\alpha = 0.2$. By default, K is set to 2. For ImageNet, we use the same hyperparameters as (Chen et al., 2020a) except batch size being 256, $\alpha = 0.2$ and $K = 2$.

Evaluation Protocol. We evaluate the learned representation on two most commonly used protocols – linear classification (Zhang et al., 2016; Kolesnikov et al., 2019) and k-nearest neighbors classifier (Chen & He, 2021). In all the experiments, we train the linear classifier for 100 epochs. The learning rate is exponentially decayed from 10^{-2} to 10^{-6}. The weight decay is 1×10^{-6}. We report the classification accuracy on test embeddings as well as the accuracy of a 5-Nearest Neighbors classifier for datasets except for ImageNet.

Table 1: Top-1 linear classification accuracy and 5-NN accuracy on four datasets with a ResNet-18 encoder. We use **bold** to mark the best results and underline to mark the second best results. $^\sharp$ means the results are reproduced by our code.

method	CIFAR-10		CIFAR-100		STL-10		Tiny ImageNet	
	Linear	5-NN	Linear	5-NN	Linear	5-NN	Linear	5-NN
SimCLR$^\sharp$	90.88	88.25	65.53	55.32	89.27	85.44	46.16	**31.43**
BYOL$^\sharp$	91.20	89.52	64.85	54.92	88.64	85.07	44.35	29.66
SimSiam$^\sharp$	91.06	89.43	65.41	54.84	90.04	85.48	45.17	30.41
Spectral$^\sharp$	90.28	87.25	65.42	55.05	89.16	84.23	45.69	29.32
ACA-PC (ours)	90.35	87.38	65.69	54.57	90.08	85.86	46.08	30.97
ACA-Full (ours)	**91.58**	**89.79**	**66.09**	**55.59**	**90.88**	**86.44**	**46.41**	31.39

Table 2: Left: Top-1 classification accuracy and 5-NN accuracy on ImageNet-100 with ResNet-18. †: results are taken from (Wang & Isola, 2020). *: results are taken from (Tian et al., 2020b).$^\sharp$ means the results are reproduced by our code. Right: Top-1 classification accuracy on ImageNet with ResNet-50, results are taken from (Chen & He, 2021; HaoChen et al., 2021). We use **bold** to mark the best results and underline to mark the second best results.

ImageNet-100	Linear	5-NN
MoCo†	72.80	-
$\mathcal{L}_{align} + \mathcal{L}_{uniform}$ †	74.60	-
InfoMin*	74.90	-
SimCLR$^\sharp$	75.62	62.70
Spectral$^\sharp$	75.52	61.80
ACA-PC (ours)	75.80	62.54
ACA-Full (ours)	**76.02**	**63.20**

ImageNet	Linear (100 epochs)
SimCLR	66.5
MoCo v2	67.4
BYOL	66.5
SimSiam	68.1
Spectral	66.97
ACA-PC (ours)	67.21
ACA-Full (ours)	**68.32**

6.2 Performance Comparison with state-of-the-art

In Table 1, we compare our method on various small-scale or mid-scale benchmarks with several state-of-the-art methods including SimCLR (Chen et al., 2020a), BYOL (Grill et al., 2020), SimSiam (Chen & He, 2021) and Spectral (HaoChen et al., 2021). SimCLR uses is a method that uses contrastive loss. BYOL and SimSiam do not use negative samples. Spectral is a similar loss derived from the idea of spectral clustering. From Table 1, we can see that our ACA-Full method achieves competitive results on small-scale or mid-scale benchmarks, achieving either the best or the second-best results on all the benchmarks except the 5-NN evaluation on STL-10. Also, ACA-PC differs from ACA-Full in the projection loss. In all the benchmarks, we can see the projection loss improves performance.

For large-scale benchmarks, we compare several methods on ImageNet-100 and ImageNet. On ImageNet-100, we compare our method additionally to MoCo (He et al., 2020b), $\mathcal{L}_{align} + \mathcal{L}_{uniform}$ (Wang & Isola, 2020) and InfoMin (Tian et al., 2020b). Note that the results of the other three methods are reported when using the ResNet-50 encoder, which has more capacity than ResNet18. Our method can also achieve state-of-the-art results among them. This means our method is also effective with relatively small encoders even on large-scale datasets. On ImageNet, we see that ACA-PC achieves competitive performance against state-of-the-art contrastive methods (Chen et al., 2020a;c; Grill et al., 2020; Chen & He, 2021; HaoChen et al., 2021) and ACA-Full achieves the best.

7 Conclusion

In this paper, we provide a new way of constructing self-supervised contrastive learning tasks by modeling similarity through augmentation overlap, which is motivated by the observation that semantically similar data usually creates similar augmentations. We propose Augmentation Component Analysis to perform PCA on augmentation feature efficiently. Interestingly, our methods have a similar form as the traditional contrastive loss and may explain the ability of contrastive loss. We hope our paper can inspire more thoughts about how to measure similarity in self-supervised learning and how to construct contrastive learning tasks. Currently, we limit our work to discrete augmentation distribution and we put investigating of continuous distribution for future work.

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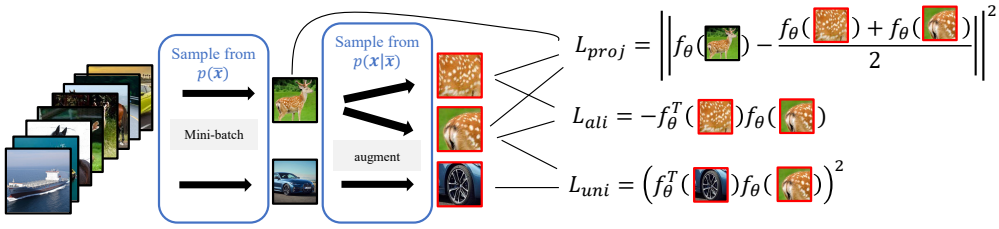


Figure 4: The implementation of ACA. Like traditional contrastive learning methods, ACA samples a mini-batch from the whole natural dataset, then performs two random augmentations on the mini-batch. The mini-batch sampling and augmentation create samples from $p(\bar{\boldsymbol{x}})$ and $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$ respectively. Then we use the samples to estimate the values of $\mathcal{L}_{uni}$,$\mathcal{L}_{uni}$ and $\mathcal{L}_{proj}$ as in the figure.

A Implementation and Further Discussion

In the previous section, we have presented the expected form of ACA. Thanks to its form, we can efficiently optimize ACA by Monte-Carlo sampling, making the problem **tractable**.

For convenience of illustration, we decompose the ACA-PC loss into two parts, *i.e.*, $\mathcal{L}_{\text{ACA-PC}} = \mathcal{L}_{ali} + \mathcal{L}_{uni}$. For the first part, $\mathcal{L}_{ali}$ serves as the alignment loss in traditional contrastive learning, which maximizes the inner product similarity between augmented samples from the same natural data:

$$\mathcal{L}_{\text{ali}} = 2\mathbb{E}_{\bar{\boldsymbol{x}}\sim p(\bar{\boldsymbol{x}}), \substack{\boldsymbol{x}_i\sim p(\boldsymbol{x}_i|\bar{\boldsymbol{x}}) \\ \boldsymbol{x}_j\sim p(\boldsymbol{x}_j|\bar{\boldsymbol{x}})}} - f_\theta(\boldsymbol{x}_i)^\top f_\theta(\boldsymbol{x}_j), \tag{9}$$

we use the mini-batch of natural sample to estimate $\mathbb{E}_{\bar{\boldsymbol{x}}\sim p(\bar{\boldsymbol{x}})}$. And we just use one sample to estimate $\mathbb{E}_{\boldsymbol{x}_i\sim p(\boldsymbol{x}_i|\bar{\boldsymbol{x}})}$ and $\mathbb{E}_{\boldsymbol{x}_j\sim p(\boldsymbol{x}_j|\bar{\boldsymbol{x}})}$ respectively. This leads to the traditional contrastive learning procedure : sample a mini-batch of natural data, augment it twice, compute and maximize the similarity of two augmented data.

For the second part, $\mathcal{L}_{uni}$ minimize the inner product similarity of augmented data from the marginal distribution:

$$\mathcal{L}_{uni} = N\mathbb{E}_{\boldsymbol{x}_1\sim p_{\mathcal{A}}(\boldsymbol{x}_1), \boldsymbol{x}_2\sim p_{\mathcal{A}}(\boldsymbol{x}_2)} \left[\left(f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2) \right)^2 \right]. \tag{10}$$

We use the in-batch augmented data to estimate $\mathbb{E}_{\boldsymbol{x}_1\sim p_{\mathcal{A}}(\boldsymbol{x}_1)}$. Notably, two augmented samples randomly sampled are hardly augmented by the same natural sample. Therefore, following common practice (Chen et al., 2020a), we use two augmented data that are created by augmenting two different natural data to compute this term. Additionally, we find that N in $\mathcal{L}_{uni}$ is too large to perform stable numerical computation. Thus in our implementation, we replace the N with a tunable noise strength K.

For $\mathcal{L}_{proj}$, it is not efficient to fully sample from $p(\boldsymbol{x} \mid \bar{\boldsymbol{x}})$. However, it is notable that:

$$\mathcal{L}_{proj} = \mathbb{E}_{\bar{\boldsymbol{x}}\sim p(\bar{\boldsymbol{x}})} \left[\| f_\theta(\bar{\boldsymbol{x}}) - \mathbb{E}_{p(\boldsymbol{x}|\bar{\boldsymbol{x}})} f_\theta(\boldsymbol{x}) \|_2^2 \right] = \mathbb{E}_{\bar{\boldsymbol{x}}\sim p(\bar{\boldsymbol{x}}), \substack{\boldsymbol{x}_i\sim p(\boldsymbol{x}_i|\bar{\boldsymbol{x}}) \\ \boldsymbol{x}_j\sim p(\boldsymbol{x}_j|\bar{\boldsymbol{x}})}} \left[\| f_\theta(\bar{\boldsymbol{x}}) - \frac{f_\theta(\boldsymbol{x}_i) + f_\theta(\boldsymbol{x}_j)}{2} \|_2^2 \right].$$

It has the same expectation subscript as $\mathcal{L}_{\text{ali}}$. So we can use the same strategy as $\mathcal{L}_{\text{ACA-PC}}$ and reuse the samples. $\mathcal{L}_{proj}$ is computed along with $\mathcal{L}_{\text{ali}}$ during training, *i.e.*, the principal component learning and projection are done simultaneously. That is why we call $\mathcal{L}_{proj}$ "on-the-fly projection".

The overall implementation of ACA is illustrated in Figure 4. And the algorithm is illustrated in Algorithm 1.

Discussion on the relation with traditional contrastive learning. As is described in this section. ACA-PC takes a similar form as traditional contrastive learning methods. Similar to them, ACA-PC maximizes the inner product similarity of two views from the sample by Equation (9), and minimizes the squared inner product similarity of views from different samples. Note that we have proved that the learned embeddings by ACA-PC function as the principal components of the augmentation feature and preserve the posterior augmentation distances (Theorem 4.2). We believe traditional contrastive loss also has the similar functionality as ours. Due to the strong correlation

between augmentation overlap and semantic similarity, this may explain contrastive learning can learn semantically meaningful embeddings even though they ignore the semantic relationship between samples.

Discussion on approximation in implementations. There are several approximations to stabilize the training. First, we replace the factor N in Equation (10) with a tunable noise strength K . Usually, the number of samples is very large in common datasets. When we use a complex model like DNN, it is unstable to involve such a large number in the loss. Therefore, we tune it small and find it works well. But we also note that we use N in our synthetic experiment in Section 5 for the dimensionality and model is not too complex. The superior results proves effectiveness of our theory. Second, we normalize embeddings to project them into a sphere, which equals replacing the inner product with cosine similarity. We find this modification improves the performance from 81.84% to 91.58%.

Algorithm 1 Augmentation Component Analysis Algorithm

Require: Unlabeled natural dataset $\{\boldsymbol{x}_i\}_{i=1}^N$; Augmentation function $\mathcal{A}$; Encoding model f_θ parameterized with θ; projection parameter α; Noise Strength K; Batch size B.

for sampled minibatch $\{\boldsymbol{x}_i\}_{i=1}^B$ **do**
  for $i = 1, 2, \ldots, B$ **do**
    $\boldsymbol{x}_i^{(1)} = \mathcal{A}(\bar{\boldsymbol{x}}_i), \boldsymbol{x}_i^{(2)} = \mathcal{A}(\bar{\boldsymbol{x}}_i)$
  end for
  $\mathcal{L}_{ACA-Full} = -\frac{2}{B}\sum_{i=1}^B f_\theta^\top(\boldsymbol{x}_i^{(1)})f_\theta(\boldsymbol{x}_i^{(2)}) + \frac{K}{B(B-1)}\sum_{i\neq j}(f_\theta^\top(\boldsymbol{x}_i^{(1)})f_\theta(\boldsymbol{x}_j^{(2)}))^2 + \alpha\|f_\theta(\bar{\boldsymbol{x}}) - \frac{f_\theta(\boldsymbol{x}_i^{(1)})+f_\theta(\boldsymbol{x}_i^{(2)})}{2}\|^2$
  update θ with w.r.t $\mathcal{L}_{ACA-Full}$.
end for
return f_θ

B Effect of Augmentation Overlaps

Like contrastive learning, our method relies on the quality of augmentation. Therefore, we investigate the influence of different augmentations and reveal the relationship between distribution difference and the linear probe performance on CIFAR10. The augmentation distribution is estimated by augmenting 10^6 times for a subset of random 2000 pairs of samples with the number of intra-class and inter-class pairs being 1000 respectively. Note that as is stated in Section 4.1, even on CIFAR10, the actual value of L is exponentially large (up to 256^{3072}). It is impossible to accurately estimate a distribution over so many possible values. But we notice that for neural networks, many operators can reduce the possible number of values, like convolutions and poolings. Following this observation and to make the computation efficient, we descrete the color into 8-bit for each channel and use a max pooling operation to get a 4×4 picture. by this kind of approximation, the number of L reduces to 8^{48}. Seems still too large, but it can be noted that the augmentation distribution of each sample covers only a small region. It is enough to estimate the distribution by sampling. For restriction of memory, we can not fully estimate the weighted augmentation distance in Theorem 4.3. Because we can not store all possible values for $p_{\mathcal{A}}(\boldsymbol{x})$. Instead, we use the Hellinger distance as the distribution distance measure:

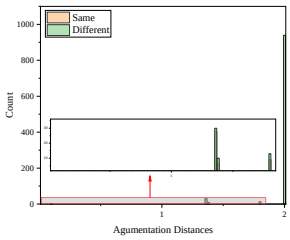
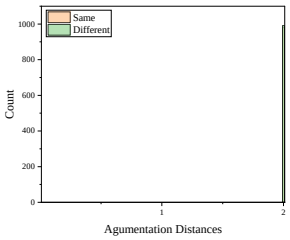
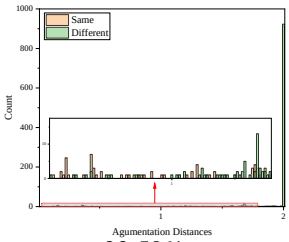
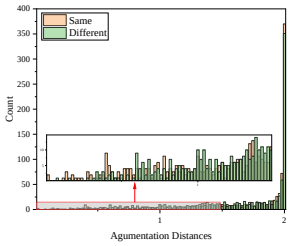
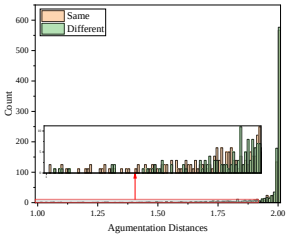
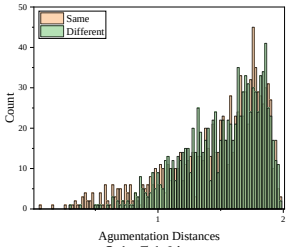
$$\mathrm{d}_{\mathrm{H}}^2(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) = \frac{1}{N}\sum_{\boldsymbol{x}\in\mathcal{X}}\left(\sqrt{p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1)} - \sqrt{p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_2)}\right)^2$$

Hellinger distance ranges $[0, 2]$, making the comparison clear.

We list the experimented augmentation here:

1. **Grayscale**: Randomly change the color into gray with probability of 0.1.
2. **HorizontalFlip**: Randomly flip horizontally with probability 0.5.
3. **Rotation**: Randomly rotate image with uniformly distributed angle in $[0, \pi]$
4. **ColorJitter**: Jitter (brightness, contrast, saturation, hue) with strength (0.4, 0.4, 0.4, 0.1) and probability 0.8.

Table 3: Histogram (HIST) of distribution distances and linear probe accuracy (ACC) when using different augmentations on CIFAR10. Note that HIST is estimated in the input space. It is property of augmentation, regardless of learning algorithm. We aims to investigate the different augmentation overlaps caused by different augmentation, and reveal its connection between learned model. "Same" denotes the distance between samples with the same semantic class, and "Different" means different classes. The existence of overlap and relationship between intra/inter-class distances affects the performance.

	Grayscale	HorizontalFlip	Rotation
HIST			
ACC	27.05%	29.67%	33.50%
	ColorJitter	**ResizedCrop**	**Augs in SimCLR**
HIST			
ACC	59.74%	75.98%	91.51%

5. **ResizedCrop**: Extract crops with a random size from 0.2 to 1.0 of the original area and a random aspect ratio from 3/4 to 4/3 of the original aspect ratio.

6. **Augs in SimCLR**: Sequential combination of 5,4,1,2.

In Table 3, we display the histogram (HIST) of intra- and inter-class augmentation *distribution distances*. ACC displays the linear probe performance on the test set. From the table, the following requirements for a good augmentation can be concluded: (1) **Existence of overlap**. For the upper three augmentations. The "scope" of augmentation is small. As a result, the majority of samples do not overlap. This makes the embeddings lack the discriminative ability for downstream tasks. On the contrary, the lower three create overlaps for most of the samples, leading to much better performance. (2) **Intra-class distance is lower than inter-class**. Compared to ColorJitter, ResizedCrop makes more intra-class samples have lower distance. So ResizedCrop outperforms ColorJitter. SimCLR augmentation surpasses these two for the same reason. Interestingly, we find the same phenomena appear when using other contrastive methods like SimCLR. It shows that these methods somehow utilize the augmentation overlap like our method.

C Proof of Lemma 4.1

For convenient, we define $M := \hat{A}^\top \hat{A}$. The elements of M are:

$$M_{\boldsymbol{x}_1\boldsymbol{x}_2} = \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \frac{p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}}) p(\boldsymbol{x}_2 \mid \bar{\boldsymbol{x}})}{\sqrt{d_{\boldsymbol{x}_1}} \sqrt{d_{\boldsymbol{x}_2}}}, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X} \tag{11}$$

Expanding Equation (3), we get:

$$
\begin{aligned}
\mathcal{L}_{mf} &= \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} (M_{\boldsymbol{x}_1\boldsymbol{x}_2} - F_{\boldsymbol{x}_1}^\top F_{\boldsymbol{x}_2})^2 \\
&= \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} (M_{\boldsymbol{x}_1\boldsymbol{x}_2} - \sqrt{d_{\boldsymbol{x}_1}}\sqrt{d_{\boldsymbol{x}_2}} f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2))^2 \\
&= \text{const} - 2\sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} \sqrt{d_{\boldsymbol{x}_1}}\sqrt{d_{\boldsymbol{x}_2}} M_{\boldsymbol{x}_1\boldsymbol{x}_2} f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2) + \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} d_{\boldsymbol{x}_1} d_{\boldsymbol{x}_2} (f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2))^2 \\
&= \text{const} - 2\sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} \sum_{\bar{\boldsymbol{x}}\in\bar{\mathcal{X}}} p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}}) p(\boldsymbol{x}_2 \mid \bar{\boldsymbol{x}}) f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2) + \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} d_{\boldsymbol{x}_1} d_{\boldsymbol{x}_2} (f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2))^2
\end{aligned}
$$

multiply by $p(\bar{\boldsymbol{x}}) = \frac{1}{N}$ and replace $d_{\boldsymbol{x}}$ with $\sum_{\bar{\boldsymbol{x}}} p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}) = N p_{\mathcal{A}}(\boldsymbol{x})$. The objective becomes:

$$
\begin{aligned}
\min_\theta \quad & -2\sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} \sum_{\bar{\boldsymbol{x}}\in\bar{\mathcal{X}}} p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}}) p(\boldsymbol{x}_2 \mid \bar{\boldsymbol{x}}) p(\bar{\boldsymbol{x}}) f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2) \\
& + N \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}} p_{\mathcal{A}}(\boldsymbol{x}_1) p_{\mathcal{A}}(\boldsymbol{x}_2) (f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2))^2 \\
= & -2\mathbb{E}_{\bar{\boldsymbol{x}}\sim p(\bar{\boldsymbol{x}}), \substack{\boldsymbol{x}_i\sim\mathcal{A}(\boldsymbol{x}_i\mid\bar{\boldsymbol{x}}) \\ \boldsymbol{x}_j\sim\mathcal{A}(\boldsymbol{x}_j\mid\bar{\boldsymbol{x}})}} \left[f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2) \right] \\
& + N\mathbb{E}_{\boldsymbol{x}_1\sim p_{\mathcal{A}}(\boldsymbol{x}_1), \boldsymbol{x}_2\sim p_{\mathcal{A}}(\boldsymbol{x}_2)} \left[(f_\theta(\boldsymbol{x}_1)^\top f_\theta(\boldsymbol{x}_2))^2 \right] \\
= & \mathcal{L}_{\text{ACA-PC}}
\end{aligned}
$$

D Proof of Theorem 4.2

As in Appendix C, we define $M := \hat{A}^\top \hat{A}$. By Eckart–Young–Mirsky theorem (Eckart & Young, 1936), the minimizer $\hat{F}$ of $\|M - FF^\top\|_F^2$, must have the form $\hat{V}\hat{\Sigma}Q$, where $\hat{V}, \hat{\Sigma}$ contain the top-k singular values and corresponding right singular vectors of $\hat{A}$, $Q \in \mathbb{R}^{k\times k}$ is some orthonormal matrix with $Q^\top Q = I$. Since we let $F_{\boldsymbol{x}} = \sqrt{d_x} f_\theta(\boldsymbol{x})$, then the minimizer $\theta^\star$ must satisfy

$$
f_{\theta^\star}(\boldsymbol{x}) = Q\frac{\hat{\boldsymbol{\sigma}} \odot \hat{\boldsymbol{v}}(\boldsymbol{x})}{\sqrt{d_{\boldsymbol{x}}}} = Q\frac{[\sigma_1 v_1(\boldsymbol{x}), \sigma_2 v_2(\boldsymbol{x}), \ldots, \sigma_k v_k(\boldsymbol{x})]^\top}{\sqrt{d_{\boldsymbol{x}}}}.
$$

where $\odot$ is the element-wise multiplication. For convenience, we use σ_i to denote i-th largest singular value, $u_i(\bar{\boldsymbol{x}})$,$v_i(\boldsymbol{x})$ to denote the element of i-th left/right singular value corresponding to $\bar{\boldsymbol{x}}$/$\boldsymbol{x}$. When

$p(\bar{\boldsymbol{x}}) = \frac{1}{N}$, $d_{\boldsymbol{x}} = Np_{\mathcal{A}}(\boldsymbol{x}) = \frac{p_{\mathcal{A}}(\boldsymbol{x})}{p(\bar{\boldsymbol{x}})}$. Then the posterior distance:

$$
\begin{aligned}
\mathrm{d}^2_{\mathrm{post}}(\boldsymbol{x}_1, \boldsymbol{x}_2) &= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(p_{\mathcal{A}}(\bar{\boldsymbol{x}} \mid \boldsymbol{x}_1) - p_{\mathcal{A}}(\bar{\boldsymbol{x}} \mid \boldsymbol{x}_2)\right)^2 \\
&= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(\frac{p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}})p(\bar{\boldsymbol{x}})}{p_{\mathcal{A}}(\boldsymbol{x}_1)} - \frac{p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}})p(\bar{\boldsymbol{x}})}{p_{\mathcal{A}}(\boldsymbol{x}_1)}\right)^2 \\
&= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(\frac{p(\boldsymbol{x}_1 \mid \bar{\boldsymbol{x}})}{d_{\boldsymbol{x}_1}} - \frac{p(\boldsymbol{x}_2 \mid \bar{\boldsymbol{x}})}{d_{\boldsymbol{x}_2}}\right)^2 \\
&= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(\frac{\hat{A}_{\bar{\boldsymbol{x}}\boldsymbol{x}_1}}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{\hat{A}_{\bar{\boldsymbol{x}}\boldsymbol{x}_2}}{\sqrt{d_{\boldsymbol{x}_2}}}\right)^2 \\
&= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(\sum_{i=1}^{N} \frac{\sigma_i u_i(\bar{\boldsymbol{x}}) v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{\sigma_i u_i(\bar{\boldsymbol{x}}) v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}}\right)^2 \\
&= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \left(\sum_{i=1}^{N} \sigma_i u_i(\bar{\boldsymbol{x}}) (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})\right)^2 \\
&= \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} \sum_{i,i'} \sigma_i u_i(\bar{\boldsymbol{x}}) \sigma_{i'} u_{i'}(\bar{\boldsymbol{x}}) (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})(\frac{v_{i'}(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_{i'}(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}}) \\
&= \sum_{i,i'} \sigma_i \sigma_{i'} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})(\frac{v_{i'}(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_{i'}(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}}) \sum_{\bar{\boldsymbol{x}} \in \bar{\mathcal{X}}} u_i(\bar{\boldsymbol{x}}) u_{i'}(\bar{\boldsymbol{x}}) \\
&\overset{(1)}{=} \sum_{i,i'} \sigma_i \sigma_{i'} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})(\frac{v_{i'}(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_{i'}(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}}) \delta_{i,i'} \\
&= \sum_{i=1}^{N} \sigma_i^2 (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2
\end{aligned}
\tag{12}
$$

(1) is due to the orthogonality of singular vectors. Note that:

$$
\begin{aligned}
&\sum_{i=1}^{N} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \\
=&\sum_{i=1}^{L} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 - \sum_{i=N+1}^{L} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \\
\leq&\sum_{i=1}^{L} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \\
=&\sum_{i=1}^{L} \frac{v_i^2(\boldsymbol{x}_1)}{d_{\boldsymbol{x}_1}} + \sum_{i=1}^{L} \frac{v_i^2(\boldsymbol{x}_2)}{d_{\boldsymbol{x}_2}} - 2\sum_{i=1}^{L} \frac{v_i(\boldsymbol{x}_1) v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_1}}\sqrt{d_{\boldsymbol{x}_2}}} \\
=&\frac{1}{d_{\boldsymbol{x}_1}} + \frac{1}{d_{\boldsymbol{x}_2}} - \frac{2\delta_{\boldsymbol{x}_1\boldsymbol{x}_2}}{\sqrt{d_{\boldsymbol{x}_1}}\sqrt{d_{\boldsymbol{x}_2}}} \\
\overset{(2)}{\leq}&(\frac{1}{d_{\boldsymbol{x}_1}} + \frac{1}{d_{\boldsymbol{x}_2}})(1 - \delta_{\boldsymbol{x}_1\boldsymbol{x}_2}) \\
\leq&\frac{2}{d_{\min}} (1 - \delta_{\boldsymbol{x}_1\boldsymbol{x}_2})
\end{aligned}
$$

(2) can be deduced by considering conditions whether $\boldsymbol{x}_1 = \boldsymbol{x}_2$ or not. Then:

$$
\begin{aligned}
&\|f_{\theta^\star}(\boldsymbol{x}_1) - f_{\theta^\star}(\boldsymbol{x}_2)\|^2 \\
=&\sum_{i=1}^{k} \sigma_i^2 (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \\
=&\mathrm{d}^2_{\mathrm{post}}(\boldsymbol{x}_1, \boldsymbol{x}_2) - \sum_{i=k}^{N} \sigma_i^2 (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \quad (\leq \mathrm{d}^2_{\mathrm{post}}(\boldsymbol{x}_1, \boldsymbol{x}_2)) \\
\geq&\mathrm{d}^2_{\mathrm{post}}(\boldsymbol{x}_1, \boldsymbol{x}_2) - \sigma_{k+1}^2 \sum_{i=k+1}^{N} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \\
\geq&\mathrm{d}^2_{\mathrm{post}}(\boldsymbol{x}_1, \boldsymbol{x}_2) - \sigma_{k+1}^2 \sum_{i=1}^{N} (\frac{v_i(\boldsymbol{x}_1)}{\sqrt{d_{\boldsymbol{x}_1}}} - \frac{v_i(\boldsymbol{x}_2)}{\sqrt{d_{\boldsymbol{x}_2}}})^2 \\
\geq&\mathrm{d}^2_{\mathrm{post}}(\boldsymbol{x}_1, \boldsymbol{x}_2) - \frac{2\sigma_{k+1}^2}{d_{\min}} (1 - \delta_{\boldsymbol{x}_1 \boldsymbol{x}_2})
\end{aligned}
$$

Therefore, we have proved Theorem 4.2.

E PROOF OF THEOREM 4.3

similar to Appendix D,

$$
\begin{aligned}
\mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) &= \sum_{\boldsymbol{x} \in \mathcal{X}} \frac{1}{N p_{\mathcal{A}}(\boldsymbol{x})} \left(p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1) - p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_2)\right)^2 \\
&= \sum_{\boldsymbol{x} \in \mathcal{X}} \left(\frac{p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1)}{\sqrt{N p_{\mathcal{A}}(\boldsymbol{x})}} - \frac{p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1)}{\sqrt{N p_{\mathcal{A}}(\boldsymbol{x})}} \right)^2 \\
&= \sum_{\boldsymbol{x} \in \mathcal{X}} \left(\frac{p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1)}{\sqrt{d_{\boldsymbol{x}}}} - \frac{p(\boldsymbol{x} \mid \bar{\boldsymbol{x}}_1)}{\sqrt{d_{\boldsymbol{x}}}} \right)^2 \\
&= \sum_{\boldsymbol{x} \in \mathcal{X}} \left(\hat{A}_{\bar{\boldsymbol{x}}_1 \boldsymbol{x}} - \hat{A}_{\bar{\boldsymbol{x}}_2 \boldsymbol{x}} \right)^2 \\
&= \sum_{\boldsymbol{x} \in \mathcal{X}} \left(\sum_{i=1}^{N} \sigma_i u_i(\bar{\boldsymbol{x}}_1) v_i(\boldsymbol{x}) - \sigma_i u_i(\bar{\boldsymbol{x}}_2) v_i(\boldsymbol{x}) \right)^2 \\
&= \sum_{\boldsymbol{x} \in \mathcal{X}} \left(\sum_{i=1}^{N} \sigma_i (u_i(\bar{\boldsymbol{x}}_1) - u_i(\bar{\boldsymbol{x}}_2)) v_i(\boldsymbol{x}) \right)^2 \\
&= \sum_{\boldsymbol{x} \in \mathcal{X}} \sum_{i,i'} \sigma_i v_i(\boldsymbol{x}) \sigma_{i'} v_{i'}(\boldsymbol{x}) (u_i(\bar{\boldsymbol{x}}_1) - u_i(\bar{\boldsymbol{x}}_2))(u_{i'}(\bar{\boldsymbol{x}}_1) - u_{i'}(\bar{\boldsymbol{x}}_2)) \\
&= \sum_{i,i'} \sigma_i \sigma_{i'} (u_i(\bar{\boldsymbol{x}}_1) - u_i(\boldsymbol{x}_2))(u_{i'}(\bar{\boldsymbol{x}}_1) - u_{i'}(\bar{\boldsymbol{x}}_2)) \sum_{\boldsymbol{x} \in \mathcal{X}} v_i(\boldsymbol{x}) v_{i'}(\boldsymbol{x}) \\
&\overset{(1)}{=} \sum_{i,i'} \sigma_i \sigma_{i'} (u_i(\bar{\boldsymbol{x}}_1) - u_i(\bar{\boldsymbol{x}}_2))(u_{i'}(\bar{\boldsymbol{x}}_1) - u_{i'}(\bar{\boldsymbol{x}}_2)) \delta_{i,i'} \\
&= \sum_{i=1}^{N} \sigma_i^2 (u_i(\boldsymbol{x}_1) - u_i(\boldsymbol{x}_2))^2
\end{aligned}
$$

(1) is due to the orthogonality of singular vectors. And $g(\bar{\boldsymbol{x}})$ takes the following form:

$$
g(\bar{\boldsymbol{x}}) = Q \left[\sigma_1^2 u_1(\boldsymbol{x}), \sigma_2^2 u_2(\boldsymbol{x}), \ldots, \sigma_k^2 u_k(\boldsymbol{x}) \right]^\top .
$$

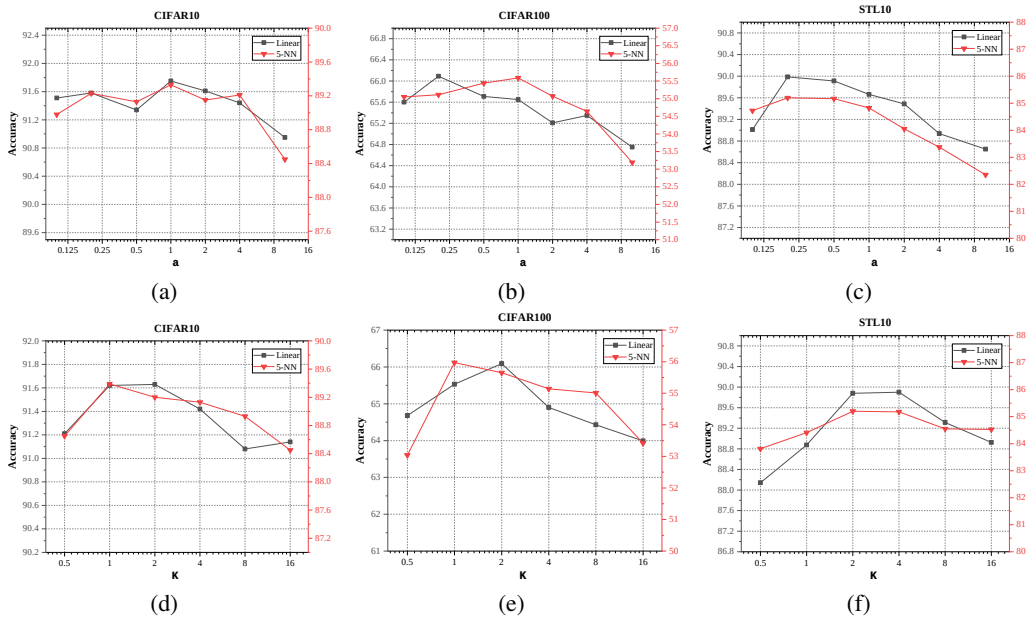


Figure 5: Ablation studies on the effect of α and K. We report linear classification and 5-nearest neighbor accuracy on different datasets with the ResNet-18 encoder. The upper 3 figures illustrate the effect of α on 3 different datasets. The lower 3 figures illustrate the performance v.s. K.

Thus,

$$
\begin{aligned}
\|g(\bar{\boldsymbol{x}}_1) - g(\bar{\boldsymbol{x}}_2)\|^2_{\Sigma_k^{-2}} &= \sum_{i=1}^{k} \sigma_i^2 (u_i(\boldsymbol{x}_1) - u_i(\boldsymbol{x}_2))^2 \\
&= \mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) - \sum_{i=k+1}^{N} \sigma_i^2 (u_i(\boldsymbol{x}_1) - u_i(\boldsymbol{x}_2))^2 \quad (\leq \mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2)) \\
&\geq \mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) - \sigma_{k+1}^2 \sum_{i=1}^{N} (u_i(\boldsymbol{x}_1) - u_i(\boldsymbol{x}_2))^2 \\
&= \mathrm{d}^2_{\text{w-aug}}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2) - 2\sigma_{k+1}^2 (1 - \delta_{\bar{\boldsymbol{x}}_1 \bar{\boldsymbol{x}}_2})
\end{aligned}
$$

F Ablation Study on Parameter α and K

We conduct ablation experiments on the parameter α and K. α is the trade-off parameter between ACA-PC loss and projection loss Equation (8). K act as the noise strength for ACA-PC, which replace N in Equation (4).

Figure 5 shows the effect of α and K on different benchmarks. It can be seen that α is necessary to improve the performance of ACA-PC. A certain value of α helps the model achieve better results. However, a too large value of α degrades the performance. The same phenomenon is the same on K.

G Comparison of Nearest Neighbors

We randomly select 8 samples from the validation set of ImageNet-100 (Tian et al., 2020a). Then we use the encoder learned by our ACA method and SimCLR (Chen et al., 2020a) to extract features and investigate their nearest neighbors of them. The left-most column displays the selected samples and the following columns show the 5 nearest neighbors. The samples that are labeled as different classes have been marked by the red box. We also annotate the distance between the samples

Figure 6: The 5 nearest neighbors of selected samples in the embedding space of ACA and SimCLR. Images are taken from the validation set of ImageNet-100. Distances between selected samples and their nearest neighbors are annotated above each picture. We can see that the embeddings learned by ACA tend to pull together images that are similar in the input space, *i.e.*, creating similar augmentation. While SimCLR sometimes has neighbors that seem different.

and their nearest neighbors. First, we can see that even though utilizing the augmentation in a different way, ACA achieves similar results as traditional contrastive learning. Both of them can learn semantically meaningful embeddings. However, we can see that ACA tends to learn embeddings that pull together images that are similar in the input space, *i.e.*, creating similar augmentation, while SimCLR sometimes has neighbors that seem different.