Recurrent Natural Policy Gradient for POMDPs

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Abstract

Solving partially observable Markov decision processes (POMDPs) remains a fundamental challenge in reinforcement learning (RL), primarily due to the curse of dimensionality induced by the non-stationarity of optimal policies. In this work, we study a natural actor-critic (NAC) algorithm that integrates recurrent neural network (RNN) architectures into a natural policy gradient (NPG) method and a multi-step temporal difference (TD) learning method. This framework leverages the representational capacity of RNNs to address non-stationarity in RL to solve POMDPs while retaining the statistical and computational efficiency of natural gradient methods in RL. We provide non-asymptotic theoretical guarantees for this method, including bounds on sample and iteration complexity to achieve global optimality up to function approximation. Additionally, we characterize pathological cases that stem from long-term dependencies, thereby elucidating the fundamental limitations of RNN-based policy optimization in POMDPs.

1 Introduction

Reinforcement learning (RL) for partially-observable Markov decision processes (POMDPs) has been a particularly challenging problem due to the absence of an optimal stationary policy, which leads to a curse of dimensionality as the space of non-stationary policies grows exponentially over time (Krishnamurthy, 2016; Murphy, 2000). To address this curse of dimensionality in solving POMDPs, finite-memory (Yu & Bertsekas, 2008; Yu, 2012; Kara & Yüksel, 2023; Cayci et al., 2024a) and RNN-based (Lin & Mitchell, 1993; Whitehead & Lin, 1995; Wierstra et al., 2010; Mnih et al., 2014; Ni et al., 2021; Lu et al., 2024) model-free RL approaches are widely used to solve POMDPs. Despite the empirical success of RNN-based model-free RL methods, a rigorous theoretical understanding of their performance in the POMDP setting remains limited.

We begin by outlining two key observations that motivate our approach:

Observation 1. Recurrent neural networks (RNNs) have been extensively employed in model-free reinforcement learning (RL) to solve partially-observable Markov decision processes (POMDPs) (Whitehead & Lin, 1995; Wierstra et al., 2010; Mnih et al., 2014). Recent work Ni et al. (2021) demonstrates that RNN-based model-free RL can perform competitively with more sophisticated and structured approaches under appropriate hyperparameter and architecture choices. In Lu et al. (2024), shortcomings of emerging transformers in solving POMDPs were demonstrated, and it was shown, somewhat surprisingly, that particular recurrent architectures can achieve superior practical performance in certain scenarios. However, despite this plethora of works that demonstrate the effectiveness of RNN-based model-free algorithms for solving POMDPs, a concrete theoretical understanding of these methods is still in a nascent stage. This is particularly important since, as noted by Ni et al. (2021), RNN-based model-free RL algorithms are sensitive to optimization parameters, and identification of provably good choices is important for practice.

Observation 2. Natural policy gradient (NPG) framework has been shown to be effective in solving MDPs due to its versatility in encompassing powerful function approximators, such as deep neural networks (Wang et al., 2019; Cayci et al., 2024b). However, a naïve application of such non-recurrent model-free RL algorithms to solve POMDPs has been observed to be ineffective (Ni et al., 2021), which necessitate careful incorporation of recurrent architectures into the policy optimization framework. This calls for the need to incorporate and analyze policy optimization, particularly NPG framework, augmented with recurrent architectures, to obtain a provably effective solution for POMDPs.

Our study is motivated by these observations and guided by the following key questions, each addressed in this work:

Q_1. How can we achieve (i) provably effective and (ii) computation/memory-efficient policy evaluation for non-stationary policies in partially observable environments?
▷ A multi-step temporal difference (TD) learning algorithm with an IndRNN (Rec-TD) overcomes the so-called *perceptual aliasing* problem imperative in memoryless TD learning for POMDPs (Singh et al., 1994), and achieves *near-optimal* policy evaluation, provided a sufficiently large network (Theorem 5.4 and Remark 5.5). Our analysis identifies the *exploding semi-gradients* pathology in policy evaluation, which can significantly increase network and iteration complexities to mitigate perceptual aliasing under long-term dependencies (Remark 5.6), and demonstrates the role of regularization to mitigate this. We also provide empirical results in random-POMDP instances in Appendix C.

Q_2. How can we parameterize non-stationary policies by a rich and practically feasible class of RNNs and perform efficient policy optimization?

▷ We represent non-stationary policies using IndRNNs with SOFTMAX parameterization as a form of finite-state controller, and perform computationally efficient NPG updates (based on path-based compatible function approximation for POMDPs) for policy optimization. The policy optimization update (called Rec-NPG) is aided by Rec-TD as the critic (Section 4).

Q_3. What are the memory, computation and sample complexities of the resulting Rec-NAC method, which employs Rec-NPG for policy updates and Rec-TD for policy evaluation?

▷ Our non-asymptotic analyses of Rec-TD (Theorem 5.4) and Rec-NPG (Theorem 6.3) demonstrate their near-optimality in the large-network limit while highlighting dependencies on memory, long-term POMDP dynamics, and RNN smoothness. Pathological cases with long-term dependencies may require exponentially growing resources (Remarks 5.6-6.4).

These results establish principled and scalable RL solutions for POMDPs, offering insights into the interplay between memory, smoothness, and optimization complexity.

1.1 Previous work

Natural policy gradient method, proposed by Kakade (2001), has been extensively investigated for MDPs (Agarwal et al., 2020; Cen et al., 2020; Khodadadian et al., 2021; Liu et al., 2020; Cayci et al., 2024c), and analyses of NPG with feedforward neural networks (FNNs) have been established by Wang et al. (2019); Liu et al. (2019); Cayci et al. (2024b). As these works consider MDPs, the policies are stationary. In our case, the analysis of RNNs and POMDPs constitute a very significant challenge.

Standard TD learning, which does not have a memory structure, was shown to be suboptimal for POMDPs (Singh et al., 1994). We incorporate RNNs into TD learning as a form of memory to address this problem in this work.

In Yu (2012); Singh et al. (1994); Uehara et al. (2022); Kara & Yüksel (2023); Cayci et al. (2024a), finite-memory policies based on sliding-window approximations of the history were investigated. Bilinear frameworks with memory-based policies (Uehara et al., 2022) and Hilbert space embeddings with deterministic latent dynamics (Uehara et al., 2023) enable sample-efficient learning under specific model structures. In Guo et al. (2022), an offline RL algorithm for the specific class of linear POMDPs was proposed. Unlike these existing works, our approach integrates RNNs with NAC methods, providing a scalable and theoretically grounded framework for general POMDPs without requiring structural assumptions such as deterministic transitions, fixed memory windows, or linear POMDP dynamics. Value- and policy-based model-free RL algorithms based on RNNs have been widely considered in practice to solve POMDPs (Lin & Mitchell, 1993; Whitehead & Lin, 1995; Wierstra et al., 2010; Mnih et al., 2014; Ni et al., 2021; Lu et al., 2024). However, these works are predominantly experimental, thus there is no theoretical analysis of RNN-based RL methods

for POMDPs to the best of our knowledge. In this work, we also present theoretical guarantees for RNN-based NPG for POMDPs. For structural results on the hardness of RL for POMDPs, we refer to (Liu et al., 2022; Singh et al., 1994).

2 Preliminaries on Partially-Observable Markov Decision Processes

In this paper, we consider a discrete-time infinite-horizon partially-observable Markov decision process (POMDP) with the (nonlinear) dynamics

$$\mathbb{P}\left(S_{t+1}=s|S_k,A_k,k\leq t\right)=:\mathcal{P}((S_t,A_t),s),$$
$$\mathbb{P}(Y_t=y|S_t)=:\phi(S_t,y),$$

for any $s\in\mathbb{S}$ and $y\in\mathbb{Y}$, where S_t is an $\mathbb{S}$-valued *state*, Y_t is a $\mathbb{Y}$-valued *observation*, and A_t is an $\mathbb{A}$-valued *control* process with the stochastic kernels $\mathcal{P}:\mathbb{S}\times\mathbb{A}\times\mathbb{S}\to[0,1]$ and $\phi:\mathbb{S}\times\mathbb{Y}\to[0,1]$. We consider finite but arbitrarily large $\mathbb{A}\subset\mathbb{R}^{d_1},\mathbb{Y}\subset\mathbb{R}^{d_2}$ with $\mathbb{Y}\times\mathbb{A}\subset\mathcal{B}_2^{(d_1+d_2)}(0,1)$ and $\mathbb{S}$. In this setting, the state process $(S_t)_{t\in\mathbb{N}}$ is not observable by the controller. Let

$$Z_t=\begin{cases}Y_0, & \text{if } t=0,\\ (Z_{t-1},A_{t-1},Y_t), & \text{if } t>0,\end{cases}\tag{1}$$

be the history process, which is available to the controller at time $t\in\mathbb{N}$, and

$$\bar{Z}_t:=(Z_t,A_t)=(Y_0,A_0,\ldots,Y_t,A_t),$$

be the history-action process.

Definition 2.1 (Admissible policy)**.** An admissible control policy $\pi=(\pi_t)_{t\in\mathbb{N}}$ is a sequence of measurable mappings $\pi_t:(\mathbb{Y}\times\mathbb{A})^t\times\mathbb{Y}\to\Delta(\mathbb{A})$ where $\Delta(\mathbb{A})=\{v\in\mathbb{R}_{\geq 0}^{|\mathbb{A}|}:\sum_{a\in\mathbb{A}}v_a=1\}$ is the set of probability vectors over the set $\mathbb{A}$, and the control at time t is chosen under π_t randomly as

$$\mathbb{P}(A_t=a|Z_t=z_t)=\pi_t(a|z_t),$$

for any $z_t\in(\mathbb{Y}\times\mathbb{A})^t\times\mathbb{Y}$. We denote the class of all admissible policies by Π_{NM}.

If an action a is taken at state a, then a reward $r(s,a)$ is obtained. For simplicity, we assume that the reward is deterministic, and $\max_{s,a}|r(s,a)|\leq r_\infty<\infty$.

Definition 2.2 (Value function, $\mathcal{Q}$-function, advantage function)**.** Let π be an admissible policy, and $\mu\in\Delta(\mathbb{Y})$ be an initial observation distribution. Then, the value function under π with discount factor $\gamma\in(0,1]$ is defined as

$$\mathcal{V}_t^\pi(z_t):=\mathbb{E}^\pi\Big[\sum_{k=t}^{\infty}\gamma^{k-t}r(S_k,A_k)\Big|Z_t=z_t\Big],\tag{2}$$

for any $z_t\in(\mathbb{Y}\times\mathbb{A})^t\times\mathbb{Y}$. Similarly, the state-action value function (also known as $\mathcal{Q}$-function) and the advantage function under π are defined as

$$\begin{aligned}\mathcal{Q}_t^\pi(\bar{z}_t)&:=\mathbb{E}^\pi\Big[\sum_{k=t}^{\infty}\gamma^{k-t}r(S_k,A_k)\Big|\bar{Z}_t=\bar{z}_t\Big],\\ \mathcal{A}_t^\pi(z_t,a)&:=\mathcal{Q}_t^\pi(z_t,a)-\mathcal{V}_t^\pi(z_t),\end{aligned}\tag{3}$$

for any $\bar{z}_t\in(\mathbb{Y}\times\mathbb{A})^{t+1}$, respectively.

Given an initial observation distribution $\mu\in\Delta(\mathbb{Y})$, the optimization problem is

$$\max_{\pi\in\Pi_{\mathsf{NM}}}\mathcal{V}^\pi(\mu),\tag{4}$$

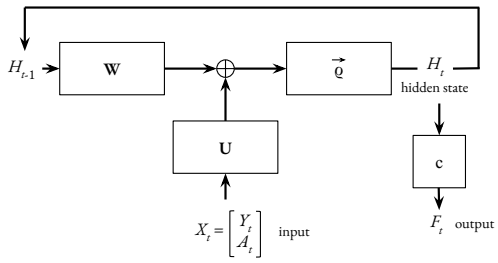

Figure 1: An Elman-type RNN in the RL framework.

where

$$\mathcal{V}^{\pi}(\mu) := \sum_{y \in \mathbb{Y}} \mathcal{V}_0^{\pi}(y_0)\mu(y_0).$$

We denote $\pi^{\star} \in \underset{\pi \in \Pi_{\mathsf{NM}}}{\arg\max}\ \mathcal{V}^{\pi}(\mu)$ as an optimal policy.

Remark 2.3 (Curse of history in RL for POMDPs). Note that the problem in equation 4 is significantly more challenging than its subcase of (fully-observable) MDPs since there may not exist an optimal policy which is (i) stationary, and (ii) deterministic (Krishnamurthy, 2016; Singh et al., 1994). As such, the policy search is over *non-stationary* randomized policies of type $\pi = (\pi_0, \pi_1, \ldots)$ where $\pi_t : (\mathbb{Y} \times \mathbb{A})^t \times \mathbb{Y} \to \Delta(\mathbb{A})$ depends on the history of observations $Z_t = (Y_0, A_0, Y_1, \ldots, A_{t-1}, Y_t)$ for $t \in \mathbb{N}$. In this case, direct extensions of the existing reinforcement learning methods for MDPs become intractable, even for finite $\mathbb{Y}, \mathbb{A}$: the memory complexity of a non-stationary policy $\pi \in \Pi_{\mathsf{NM}}$ at epoch $t \in \mathbb{N}$ is $\mathscr{O}(|\mathbb{Y} \times \mathbb{A}|^{t+1})$, growing exponentially.

In the following section, we formally introduce the RNN architecture that we study in this paper.

3 Independent Recurrent Neural Networks

We consider an independently recurrent neural network (IndRNN) architecture of width $m \in \mathbb{N}$ with a diagonal $\mathbf{W} \in \mathbb{R}^{m \times m}$ and $\mathbf{U} \in \mathbb{R}^{m \times d}$, where $d = d_1 + d_2$, and the rows of $\mathbf{U}$ are denoted as $U_i^{\top}$ for $i = 1, 2, \ldots, m$ (Li et al., 2018; 2019). This architecture has been featured in POPGym (Morad et al., 2023) as it enables RNNs with large sequence lengths by handling longer dependencies. In other works, it has been shown to be effective for POMDPs in practice as well (Lu et al., 2024; Elelimy et al., 2024).

Given a smooth activation function $\varrho : \mathscr{C}^2(\mathbb{R}, \mathbb{R})$ with $\|\varrho\|_{\infty} \leq \varrho_0, \|\varrho'\|_{\infty} \leq \varrho_1, \|\varrho''\|_{\infty} \leq \varrho_2$, we denote $\vec{\varrho} : \mathbb{R}^m \to \mathbb{R}^m : \boldsymbol{z} \mapsto \begin{pmatrix} \varrho(z_1)) \\ \vdots \\ \varrho(z_m) \end{pmatrix}$. Let $X_t = \begin{pmatrix} Y_t \\ A_t \end{pmatrix}$, which is an $\mathbb{R}^d$-valued random variable with $d = d_1 + d_2$. The central structure in an RNN is the sequence of hidden states $H_t \in \mathbb{R}^m$, which evolves according to

$$H_t(\bar{Z}_t; \mathbf{W}, \mathbf{U}) = \vec{\varrho}\Big(\mathbf{W} H_{t-1}(\bar{Z}_{t-1}; \mathbf{W}, \mathbf{U}) + \mathbf{U} X_t\Big), \tag{5}$$

with $H_0(\bar{Z}_0; \mathbf{W}, \mathbf{U}) = \vec{\varrho}(\mathbf{U} X_0)$ and $\bar{Z}_t = (X_0, \ldots, X_t)$ denoting the history. We denote the ith element of H_t as $H_t^{(i)}$ for $i \in [m]$. We consider a linear readout layer with weights $c \in \mathbb{R}^m$, which leads to the output

$$F_t(\bar{Z}_t; \mathbf{W}, \mathbf{U}, c) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} c_i H_t^{(i)}(\bar{Z}_t; \mathbf{W}, \mathbf{U}). \tag{6}$$

The operation of an independent recurrent neural network is illustrated in Figure 1. Following the neural tangent kernel literature, we omit the straightforward task of training the linear output layer $c \in \mathbb{R}^m$ for simplicity, and study the training dynamics of $(\mathbf{W}, \mathbf{U})$, which is the main challenge (Du et al., 2018; Oymak & Soltanolkotabi, 2020; Cai et al., 2019; Wang et al., 2019). Consequently, we denote the learnable parameters

of a hidden unit $i \in [m]$ compactly as $\Theta_i = \begin{pmatrix} W_{ii} \\ U_i \end{pmatrix}$, and denote the learnable parameters of an RNN by $\Theta = \left[W_{11}, U_1^\top, W_{22}, U_2^\top, \ldots, W_{mm}, U_m^\top\right]^\top \in \mathbb{R}^{m(d+1)}$. Given learnable parameters $(\mathbf{W}, \mathbf{U})$, we denote the sequence of recurrent neural network outputs as $\boldsymbol{F}(\cdot; \mathbf{W}, \mathbf{U}) = (F_t(\cdot; \mathbf{W}, \mathbf{U}))_{t \in \mathbb{N}}$, and use Θ and $(\mathbf{W}, \mathbf{U})$ interchangeably throughout the paper.

One key concept is random initialization, which is widely used in practice (Goodfellow et al., 2016) and yields the basis of the kernel analysis (Jacot et al., 2018; Chizat et al., 2019). In this work, we assume that m is even, and use the following symmetric initialization (Chizat et al., 2019; Bai & Lee, 2019; Cayci et al., 2023).

Definition 3.1 (Symmetric random initialization)**.** Let $c_i \sim \mathsf{Rad}(1), V_i \sim \mathsf{Rad}(\alpha), U_i(0) \sim \mathcal{N}(0, I_d)$ independently for all $i \in \{1, 2, \ldots, m/2\}$ and independently from each other, and $c_i = -c_{i-m/2}, V_i = V_{i-m/2}$ and $U_i(0) = U_{i-m/2}(0)$ for $i \in \{m/2+1, \ldots, m\}$. Then, $(\mathbf{W}(0), \mathbf{U}(0), c)$ is called a symmetric random initialization where $\mathbf{W}(0) = \mathsf{diag}_m(V)$ and $U_i^\top(0)$ is the ith-row of $\mathbf{U}(0)$.

Let $\Theta(0)$ denote a random initialization (see Definition 3.1), and let $\rho \in \mathbb{R}^2_{>0}, m \in \mathbb{N}$. We define the parameter set

$$\Omega_{\rho,m} = \Big\{\Theta \in \mathbb{R}^{m(d+1)} : \max_i |W_{ii} - W_{ii}(0)| \leq \frac{\rho_{\mathsf{w}}}{\sqrt{m}}, \max_i \|U_i - U_i(0)\| \leq \frac{\rho_{\mathsf{u}}}{\sqrt{m}}\Big\}.$$

The symmetrization ensures that $F_t(\bar{z}_t; \mathbf{W}(0), \mathbf{U}(0), c) = 0$ for any $t \geq 0$ and input $\bar{z}_t$.

3.1 Neural Tangent Kernel for IndRNNs

We provide the following infinite-width characterization of IndRNNs based on Cayci & Eryilmaz (2024), which will constitute a class of learnable mappings by RNNs in the so-called kernel regime, extending the function classes in previous literature on neural RL methods (Cai et al., 2019; Wang et al., 2019; Cayci et al., 2024b). Let $w_0 \sim \mathsf{Unif}\{-\alpha, \alpha\}$ and $u_0 \sim \mathcal{N}(0, I_d)$ be independent random variables, and $\theta := \begin{pmatrix} w_0 \\ u_0 \end{pmatrix}$. Given $\bar{\boldsymbol{z}} = (x_0, x_1, \ldots) \in (\mathbb{Y} \times \mathbb{A})^{\mathbb{Z}_+}$, let

$$h_t(\bar{z}_t; \theta_0) := \varrho(w_0 h_{t-1}(\bar{z}_{t-1}; \theta_0) + \langle u_0, x_t \rangle), \ t > 0,$$

with $h_{-1} := 0$ (thus $h_0(\bar{z}_0; \theta_0) = \varrho(\langle u_0, x_0 \rangle)$), and $\mathcal{I}_t(\bar{z}_t; \theta_0) := \varrho'(w_0 h_{t-1}(\bar{z}_{t-1}; \theta_0) + \langle u_0, x_t \rangle)$. Then, the neural tangent random feature (NTRF) mapping[1] at time t is defined as (with $\bar{\mathcal{I}}_{t,k}(\bar{z}_t; \theta_0) := \prod_{j=0}^k \mathcal{I}_{t-j}(\bar{z}_{t-j}; \theta_0)$):

$$\psi_t(\bar{z}_t; \theta_0) := \sum_{k=0}^t w_0^k \begin{pmatrix} h_{t-k-1}(\bar{z}_{t-k-1}; \theta_0) \\ x_{t-k} \end{pmatrix} \bar{\mathcal{I}}_{t,k}(\bar{z}_t; \theta_0),$$

For $\Psi(\bar{z}; \theta_0) := \Psi_\infty(\bar{z}; \theta_0)$, we define the NTRF matrix as

$$\Psi_T(\bar{\boldsymbol{z}}; \theta_0) := \begin{pmatrix} \psi_0^\top(\bar{z}_0; \theta_0) \\ \psi_1^\top(\bar{z}_1; \theta_0) \\ \vdots \\ \psi_{T-1}^\top(\bar{z}_{T-1}; \theta_0) \end{pmatrix}, \ T \in \mathbb{N}. \tag{7}$$

Definition 3.2 (Transportation mapping)**.** Let $\mathscr{H}$ be the set of mappings $\boldsymbol{v} : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d} : \theta_0 \mapsto \begin{pmatrix} v_w(w_0) \\ v_u(u_0) \end{pmatrix}$ with

$$\mathbb{E}[|v_w(w_0)|^2] = \frac{1}{2}\Big(|v_w(\alpha)|^2 + |v_w(-\alpha)|^2\Big) < \infty,$$
$$\mathbb{E}[\|v_u(u_0)\|_2^2] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|v_u(u)\|_2^2 e^{-\frac{1}{2}\|u\|_2^2} du < \infty.$$

[1]The feature uses a complicated weighted-sum of all past inputs $x_k, k \leq t$, leading to a discounted memory to tackle non-stationarity. x_{t-k} is scaled with $w_0^k \sim \mathsf{Rad}(\alpha)$, thus it yields a fading memory approximation of the history if $\alpha < 1$.

We call $\boldsymbol{v} \in \mathscr{H}$ a transportation mapping, following Ji & Telgarsky (2019); Ji et al. (2019).

Definition 3.3 (Infinite-width limit)**.** We define the infinite-width limit of Elman-type RNNs as follows:

$$\mathscr{F} := \left\{ (\mathbb{Y} \times \mathbb{A})^{\mathbb{Z}_+} \ni \bar{\boldsymbol{z}} \mapsto \mathbb{E}\left[\Psi(\bar{\boldsymbol{z}}; \theta_0)\boldsymbol{v}(\theta_0)\right] : \boldsymbol{v} \in \mathscr{H} \right\}.$$

$\mathscr{F}$ consists of $f_t^\star(\bar{z}_t; \boldsymbol{v}) = \mathbb{E}[\langle \boldsymbol{v}(\theta_0), \psi_t(\bar{z}_t; \theta_0)\rangle]$ for any $\bar{\boldsymbol{z}} \in (\mathbb{Y} \times \mathbb{A})^{\mathbb{Z}_+}$. The same transportation mapping $\boldsymbol{v}$ is used to define the mapping $f_t^\star$ at each time t, which is a characteristic feature of weight-sharing in RNNs. Also, the input $\bar{\boldsymbol{z}}$ grows over time in a concatenated nature, which implies that $\boldsymbol{f}^\star \in \mathscr{F}$ is a representational assumption on the dynamical structure of the problem.

Remark 3.4 (Reduction to FNNs). When $T = 1$, i.e.,

$$\mathscr{F}_1 := \left\{ \bar{z}_0 \mapsto \mathbb{E}\left[\psi_0^\top(\bar{z}_0; \theta_0)\boldsymbol{v}(\theta_0)\right] : \boldsymbol{v} \in \mathscr{H} \right\},$$

we exactly recover the NTK (and its RKHS) for single-layer feedforward neural networks (FNNs) (Jacot et al., 2018), which constitutes the reference function class in neural RL literature for MDPs (Wang et al., 2019; Liu et al., 2019). $\mathscr{F}_1$ is dense in the space of continuous functions on a compact set (Ji et al., 2019).

4 Rec-NAC Algorithm for POMDPs

In this section, we present a high-level description of our Recurrent Natural Actor-Critic (Rec-NAC) Algorithm with two inner loops, critic (called Rec-TD) and actor (called Rec-NPG), for policy optimization with RNNs. The details of the inner loops of the algorithm will be given in the succeeding sections. We use an admissible policy $\pi = (\pi_t)_{t\in\mathbb{N}}$ that is parameterized by a recurrent neural network $(F_t^{\mathsf{a}}(\cdot; \Phi))_{t\in\mathbb{N}}$ of the form given in equation 6 with a network width $m \in \mathbb{Z}_+$. To that end, for any $t \in \mathbb{N}$, let

$$\pi_t^\Phi(a|z_t) := \frac{\exp\left(F_t^{\mathsf{a}}((z_t, a); \Phi)\right)}{\sum_{a'\in\mathbb{A}} \exp\left(F_t^{\mathsf{a}}((z_t, a'); \Phi)\right)}, \tag{8}$$

for any $z_t \in (\mathbb{Y} \times \mathbb{A})^t \times \mathbb{Y}$ and $a \in \mathbb{A}$ with the parameter $\Phi \in \mathbb{R}^{m(d+1)}$. Rec-NAC operates as follows:

Algorithm 1 Recurrent Natural Actor-Critic

Initialize actor RNN F^{a} with parameter $\Phi(0) \sim \zeta_{\mathsf{init}}$.
for $n = 0, 1, 2, \ldots, N-1$ **do**
  Critic. Obtain estimates $(F_t^{\mathsf{c}})_{t<T}$ of $(\mathcal{Q}_t^{\pi^{\Phi(n)}})_{t<T}$ via Rec-TD in Alg. 2.
  Actor. Apply projected-SGD to obtain

$$\omega_n \in \operatorname*{argmin}_{\omega\in\Omega_{\rho,m}} \mathbb{E} \sum_{t=0}^{T-1} \gamma^t (\nabla \ln \pi_t^n(A_t|Z_t)\omega - \hat{\mathcal{A}}_t^{(n)}(\bar{Z}_t))^2,$$

  where $\hat{\mathcal{Q}}_t^{(n)}(\cdot) = F_t^{\mathsf{c}}(\cdot; \Phi(n))$ and

$$\hat{\mathcal{A}}_t^{(n)}(z_t, a) := \hat{\mathcal{Q}}_t^{(n)}(z_t, a) - \mathbb{E}_{a_t'} \hat{\mathcal{Q}}_t^{(n)}(z_t, a_t'),$$

  with $a_t' \sim \pi_t^{\Phi(n)}(\cdot|z_t)$.
  Policy update $\Phi(n+1) = \mathbf{Proj}_{\Omega_{\rho,m}}\left[\Phi(n) + \eta \cdot \omega_n\right]$.
end for

For information regarding the algorithmic tools, i.e., random initialization and max-norm regularization for RNNs, we refer to Section A.

5 Critic: Recurrent Temporal Difference Learning (Rec-TD)

In this section, we study a policy evaluation method for POMDPs, which will serve as the critic.

Policy evaluation problem. Consider the policy evaluation problem for POMDPs under a given admissible policy $\pi \in \Pi_{\mathsf{NM}}$. Given an initial observation distribution $\mu \in \Delta(\mathbb{Y})$, policy evaluation aims to solve

$$\min_{\Theta \in \Omega_{\rho,m}} \mathcal{R}_T^{\pi}(\Theta) := \mathbb{E}_{\mu}^{\pi} \sum_{t=0}^{T-1} \gamma^t \big(F_t(\bar{Z}_t;\Theta) - \mathcal{Q}_t^{\pi}(\bar{Z}_t)\big)^2, \tag{9}$$

where $T \in \mathbb{N}$ is the truncation level, and $\{F_t : t \in \mathbb{N}\}$ is an IndRNN given in equation 6 – we drop the superscript a for simplicity throughout the discussion. The expectation in $\mathcal{R}_T^{\pi}(\Theta)$ is with respect to the joint probability law $P_T^{\pi,\mu}$ of the stochastic process $\{(S_t, A_t, Y_t) : t \in [0,T]\}$ where $Z_0 \sim \mu$.

5.1 Recurrent TD Learning Algorithm

In this section, we present a multi-step temporal difference learning algorithm for computing the sequence of state-action value functions $\{\mathcal{Q}_t^{\pi} : t \in \mathbb{N}\}$ for large POMDPs.

We assume access to a sampling oracle capable of generating independent trajectories from a given initial state distribution (Bhandari et al., 2018; Cai et al., 2019).

Assumption 5.1 (Sampling oracle)**.** Given an initial state distribution μ, we assume that the system can be independently started from $S_0 \sim \mu$, i.e., independent trajectories $\{(S_t, Y_t, A_t) : t \in [T]\} \sim P_T^{\pi,\mu}$ are obtained.

Rec-TD is presented in Algorithm 2. We study the performance Rec-TD numerically in Section C under long-term and short-term dependencies to validate our theoretical results in Section 5.2.

Algorithm 2 Recurrent TD Learning Algorithm

Input: step-size $\eta > 0$, radius $\rho = (\rho_w, \rho_u)$, truncation level T.
Initialize $\Theta(0)$ randomly according to Definition 3.1. // `Definition 3.1`
for $k = 0, 1, 2, \ldots, K-1$ **do**
  Sample an independent initial state $S_0^k \sim \mu$.
  Observe $Y_0^k \sim \Phi(S_0^k, \cdot)$.
  Choose an action $A_0^k \sim \pi_0(\cdot|Z_0^k)$.
  Set $\check{\nabla}\mathcal{R}_T^k := 0$.
  for $t = 0, 1, \ldots, T$ **do**
    State transition $S_{t+1}^k \sim \mathcal{P}((S_t^k, A_t^k), \cdot)$.
    Observe $Y_{t+1}^k \sim \Phi(S_{t+1}^k, \cdot)$.
    Choose an action $A_{t+1}^k \sim \pi_{t+1}(\cdot|Z_{t+1}^k)$.
    Compute temporal difference $\delta_t(\bar{Z}_t^k, \Theta(k))$ where

$$\delta_t(\bar{z}_{t+1};\Theta) := r_t + \gamma F_{t+1}(\bar{z}_{t+1};\Theta) - F_t(\bar{z}_t;\Theta).$$

    Update stochastic semi-gradient:

$$\check{\nabla}\mathcal{R}_T^k \leftarrow \check{\nabla}\mathcal{R}_T^k + \gamma^t \delta_t(\bar{Z}_{t+1}^k;\Theta(k)).$$

  end for
  Update $\Theta(k+1) = \mathbf{Proj}_{\Omega_{\rho,m}}\big[\Theta(k) + \eta \cdot \check{\nabla}\mathcal{R}_T^k\big]$.
end for

Remark 5.2 (Intuition behind Rec-TD). In a stochastic optimization setting, the loss-minimization for $\mathcal{R}_T(\Theta)$ would be solved by using gradient descent, where the gradient is

$\mathbb{E}_\mu^\pi \sum_{t=0}^{T-1} \gamma^t \left(F_t(\bar{Z}_t;\Theta) - \mathcal{Q}_t^\pi(\bar{Z}_t)\right) \nabla F_t(\bar{Z}_t;\Theta)$. On the other hand, the target function $\mathcal{Q}_t^\pi$ is unknown and to be learned. Following the bootstrapping idea for MDPs in Sutton (1988), we exploit an extended *non-stationary Bellman equation* in Proposition B.3, and use $r_t + \gamma F_{t+1}(\bar{Z}_{t+1};\Theta)$ as a bootstrap estimate for the unknown $\mathcal{Q}_t^\pi(\bar{Z}_t)$. Note that, in the realizable case with $F_t(\cdot;\Theta^\star) = \mathcal{Q}_t^\pi(\cdot),\ t \in \mathbb{Z}_+$ for some $\Theta^\star$, we have $\mathbb{E}_\mu^\pi[\check{\nabla}\mathcal{R}_T(\bar{Z}_T;\Theta^\star)] = 0$, which implies that the stochastic approximation approach for MDPs can be used for the non-stationary setting.

5.2 Theoretical Analysis of Rec-TD: Finite-Time Bounds and Global Near-Optimality

In the following, we prove that Rec-TD with max-norm regularization achieves global optimality in expectation. To characterize the impact of long-term dependencies on the performance of Rec-TD, let $p_t(x) = \sum_{k=0}^{t-1} |x|^k$, and $q_t(x) = \sum_{k=0}^{t-1}(k+1)|x|^k,\ x \in \mathbb{R}, t \in \mathbb{N}$.

In the following, we present a regularity condition on the state-action value functions.

Assumption 5.3 (Regularity of $(\mathcal{Q}_t^\pi)_t$)**.** $\{\mathcal{Q}_t^\pi : t \in \mathbb{N}\} \in \mathscr{F}$ with a transportation mapping $\boldsymbol{v} = (v_\mathsf{w}, v_\mathsf{u}) \in \mathscr{H}$ such that $\sup_{u\in\mathbb{R}^d} \|v_\mathsf{u}(u)\|_2 \le \nu_\mathsf{u}$ and $\sup_{w\in\mathbb{R}} |v_\mathsf{w}(w)| \le \nu_\mathsf{w}$.

Assumption 5.3 is a representational assumption, stating that $(\mathcal{Q}_t^\pi)_t$ lies in the RKHS induced by the random features $\Psi_T(\bar{z};\theta_0)$ defined in equation 7. It directly extends Assumption 4.1 in Wang et al. (2019) and Assumption 2 in Cayci et al. (2024b) to POMDPs, and exactly recovers these assumptions when $T = 1$ (see Remark 3.4).

Theorem 5.4 (Finite-time bounds for Rec-TD)**.** *Under Assumptions 5.1-5.3, for any projection radius $\rho \succeq \nu = (\nu_\mathsf{w}, \nu_\mathsf{u})$ and step-size $\eta > 0$, Rec-TD with max-norm regularization achieves the following error bound:*

$$\mathbb{E}\Big[\frac{1}{K}\sum_{k=0}^{K-1} \mathcal{R}_T^\pi(\Theta(k))\Big] \le \frac{1}{\sqrt{K}}\left(\frac{\|\nu\|_2^2}{(1-\gamma)} + \frac{C_T^{(1)}}{(1-\gamma)^3}\right) + \frac{C_T^{(2)}}{(1-\gamma)^2\sqrt{m}} + \underbrace{\frac{\gamma^T}{(1-\gamma)K}\sum_{k=0}^{K-1}\omega_{T,k}^2}_{(\heartsuit)}. \tag{10}$$

for any $K \in \mathbb{N}$, where

$$C_T^{(1)}, C_T^{(2)} = \mathsf{poly}\left(p_T((\alpha + \rho_w m^{-1/2})\varrho_1), \|\rho\|_2, \|\nu\|_2\right),$$

are instance-dependent constants that do not depend on K, and $\omega_{t,k} := \sqrt{\mathbb{E}[(F_t(\bar{Z}_t;\Theta(k)) - \mathcal{Q}_t^\pi(\bar{Z}_t^k))^2]}$ is a uniformly bounded sequence for $t, k \in \mathbb{N}$. Furthermore, the loss at average-iterate, $\mathbb{E}[\mathcal{R}_T^\pi\left(\frac{1}{K}\sum_{k=0}^{K-1}\Theta(k)\right)]$, admits the same upper bound as the regret upper bound in equation 10, up to a multiplicative factor of 10.

The proof of Theorem 5.4 can be found in Section B.

Remark 5.5 (Overcoming perceptual aliasing with Rec-TD). Memoryless TD learning suffers from a non-vanishing optimality gap in POMDPs, known as perceptual aliasing (Singh et al., 1994). To address this, Rec-TD integrates T-step stochastic approximation with an RNN, enabling it to retain memory. Accordingly, Theorem 5.4 establishes that as $T \to \infty$, Rec-TD reduces $\mathcal{R}_\infty^\pi$ to arbitrarily small values, given sufficiently large network width m and iteration count K.

Remark 5.6 (The impact of long-term dependencies). Note that both constants $C_T^{(1)}, C_T^{(2)}$ polynomially depend on $p_T(\varrho_1\alpha_m)$. As noted in Goodfellow et al. (2016), the spectral radius of $\{\mathbf{W}(k) : k \in \mathbb{N}\}$ determines the degree of long-term dependencies in the problem as it scales H_t. Consistent with this observation, our bounds depend on

$$\alpha_m := \alpha + \frac{\rho_\mathsf{w}}{\sqrt{m}} \ge \lambda_{\mathsf{max}}(\mathbf{W}(k)),$$

for any $k \in \mathbb{N}$. Let $\varepsilon > 0$ be any given target error.

▷ **Short-term memory.** If $\alpha_m < \frac{1}{\varrho_1}$, then it is easy to see that $p_T(\varrho_1\alpha_m) \le \frac{1}{1-\varrho_1\alpha_m}$. Thus, the extra term ($\heartsuit$) in equation 10 vanishes at a geometric rate as $T \to \infty$, yet m (network-width) and K (iteration-complexity) are still $\tilde{\mathscr{O}}(1/\varepsilon^2)$. Rec-TD is very efficient in that case.

▷ **Long-term memory.** If $\alpha_m > \frac{1}{\varrho_1}$, as $T \to \infty$, both m and K grow at a rate $\mathscr{O}\left((\varrho_1\alpha_m)^T/\varepsilon^2\right)$ while the

extra term (♡) in equation 10 vanishes at a geometric rate. As such, the required network size and iterations grow at a geometric rate with T in systems with long-term memory, constituting the pathological case.

The performance of Rec-TD is studied numerically in Random-POMDP instances in Section C.

6 Actor: Recurrent Natural Policy Gradient (Rec-NPG) for POMDPs

The goal is to solve the following problem for a given initial distribution $\mu \in \Delta(\mathbb{Y})$ and $\rho \in \mathbb{R}^2_{>0}$:

$$\underset{\Theta \in \mathbb{R}^{m(d+1)}}{\text{maximize}}\ \mathcal{V}^{\pi^{\Phi}}(\mu) \text{ such that } \Phi \in \Omega_{\rho,m}, \tag{PO}$$

6.1 Recurrent Natural Policy Gradient for POMDPs

In this section, we describe the recurrent natural policy gradient (Rec-NPG) algorithm for non-stationary reinforcement learning. First, we formally establish in Prop. D.2 that the policy gradient under partial observability takes the form

$$\nabla_{\Phi} \mathcal{V}^{\pi^{\Phi}}(\mu) := \mathbb{E}_{\mu}^{\pi^{\Phi}} \sum_{t=0}^{\infty} \gamma^t \mathcal{Q}_t^{\pi^{\Phi}}(Z_t, A_t) \nabla_{\Phi} \ln \pi_t^{\Phi}(A_t|Z_t),$$

where the state S_t in the MDP framework is replaced by the process history Z_t in POMDP. Fisher information matrix under a policy π^{Φ} is defined as

$$G_{\mu}(\Phi) := \mathbb{E}_{\mu}^{\pi^{\Phi}} \sum_{t=0}^{\infty} \gamma^t \nabla \ln \pi_t^{\Phi}(A_t|Z_t) \nabla^{\top} \ln \pi_t^{\Phi}(A_t|Z_t),$$

for an initial observation distribution $\mu \in \Delta(\mathbb{Y})$. Rec-NPG updates the policy parameters by

$$\Phi(n+1) = \Phi(n) + \eta \cdot G_{\mu}^{+}(\Phi(n)) \nabla_{\Phi} \mathcal{V}^{\pi^{\Phi(n)}}(\mu), \tag{11}$$

for an initial parameter $\Phi(0)$ and step-size $\eta > 0$, where G^{+} denotes the Moore-Penrose inverse of a matrix G. This update rule is in the same spirit as the NPG introduced in Kakade (2001), however, due to the non-stationary nature of the partially-observable MDP, it has significant complications that we will address.

In order to avoid computationally-expensive policy updates in equation 11, we utilize the following extension of the compatible function approximation in Kakade (2001) to the case of non-stationary policies for POMDPs.

Proposition 6.1 (Compatible function approximation for non-stationary policies)**.** *For any $\Phi \in \mathbb{R}^{m(d+1)}$ and initial observation distribution μ, let*

$$\mathcal{L}_{\mu}(w; \Phi) = \mathbb{E}_{\mu}^{\pi^{\Phi}} \sum_{t=0}^{\infty} \gamma^t \big(\nabla^{\top} \ln \pi_t^{\Phi}(A_t|Z_t) \omega - \mathcal{A}_t^{\pi^{\Phi}}(\bar{Z}_t) \big)^2, \tag{12}$$

for $\omega \in \mathbb{R}^{m(d+1)}$. Then, we have

$$G_{\mu}^{+}(\Phi) \nabla_{\Phi} \mathcal{V}^{\pi^{\Phi}}(\mu) \in \underset{\omega \in \mathbb{R}^{m(d+1)}}{\arg\min}\ \mathcal{L}_{\mu}(\omega; \Phi). \tag{13}$$

We have the following remark regarding the intricacies of compatible function approximation in the POMDP setting.

Remark 6.2 (Path-based compatible function approximation with truncation)*.* For MDPs, the compatible function approximation error $\mathcal{L}_{\mu}(w; \Phi)$ can be expressed by using the discounted state-action occupancy measure, from which one can obtain unbiased samples (Agarwal et al., 2020; Konda & Tsitsiklis, 2003). Thus, the infinite-horizon can be handled without any loss. On the other hand, for POMDPs as in equation 12,

this simplification is impossible due to the non-stationarity. As such, we use a path-based method under truncation for a given $T \in \mathbb{N}$ with

$$\ell_T(\omega; \Phi, \mathcal{Q}) := \sum_{t=0}^{T-1} \gamma^t (\nabla \ln \pi_t^{\Phi}(A_t|Z_t)\omega - \mathcal{A}_t(Z_t, A_t))^2,$$

where $\mathcal{A}_t(z_t, a_t) = \mathcal{Q}_t(z_t, a_t) - \sum_{a \in \mathbb{A}} \pi_t^{\Phi}(a|z_t)\mathcal{Q}_t(z_t, a)$.

Given a policy with parameter $\Phi(n)$ and the corresponding output of the critic (Rec-TD with the average-iterate $\bar{\Theta}^{(n)} := \frac{1}{K_{\mathsf{td}}} \sum_{k<K_{\mathsf{td}}} \Theta^{(n)}(k)$):

$$\hat{\mathcal{Q}}^{(n)}(\cdot) := F_t(\cdot; \bar{\Theta}^{(n)}),$$

the actor aims to solve the following problem:

$$\min_{\omega \in \Omega_{\rho,m}} \mathbb{E}\Big[\ell_T\left(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)}\right) \Big| \bar{\Theta}^{(n)}, \Phi(n), \ldots, \Phi(0)\Big].$$

We utilize stochastic gradient descent (SGD) to solve the above problem. Let $\bar{Z}_T^{n,k} \sim P_T^{\pi^{\Phi(n)},\mu}$ be an independent random sequence for $k \in \mathbb{N}$, $\hat{\omega}_n(0) = 0$, and

$$\begin{aligned}
&\tilde{\omega}_n(k+1) = \hat{\omega}_n(k) - \eta_{\mathsf{sgd}} \nabla_\omega \ell_T\big(\hat{\omega}_n(k); \Phi(n), \hat{\mathcal{Q}}^{(n)}\big),\\
&\hat{\omega}_n(k+1) = \mathbf{Proj}_{\Omega_{\rho,m}}[\tilde{\omega}_n(k+1)],
\end{aligned}$$

A stochastic estimate of $G_\mu^+(\Phi(n))\nabla_\Phi \mathcal{V}^{\pi^{\Phi(n)}}(\mu)$ is computed as $\omega_n := \frac{1}{K_{\mathsf{sgd}}} \sum_{k<K_{\mathsf{sgd}}} \hat{\omega}_n(k)$, followed by

$$\Phi(n+1) = \Phi(n) + \eta_{\mathsf{npg}} \cdot \omega_n.$$

In the following, we present a theoretical analysis of the above.

6.2 Theoretical Analysis of Rec-NAC for POMDPs

We establish an error bound on the best-iterate for the Rec-NPG. The significance of the following result is two-fold: (i) it will explicitly connect the optimality gap to the compatible function approximation error, and (ii) it will explicitly show the impact of truncation on the performance of path-based policy optimization for the non-stationary case.

Theorem 6.3. *Assume that $P_T^{\pi^\star,\mu}$ is absolutely continuous with respect to $P_T^{\pi^{\Phi(n)},\mu}$ for all $n \in \{0, 1, \ldots, N-1\}$ given $N \in \mathbb{Z}_+$, and let $\kappa := \max_{0\le n<N} \left\| \frac{P_T^{\pi^\star,\mu}}{P_T^{\pi^{\Phi(n)},\mu}} \right\|_\infty$ and*

$$V_n := \mathcal{V}^{\pi^\star}(\mu) - \mathcal{V}^{\pi^{\Phi(n)}}(\mu),\ n < N.$$

Rec-NPG after $N \in \mathbb{Z}_+$ steps with step-size $\eta_{\mathsf{npg}} = \frac{1}{\sqrt{N}}$ and projection radius $\rho \in \mathbb{R}^2_{>0}$ yields

$$\min_{0\le n<N} \mathbb{E}_0[V_n] \lesssim \frac{\ln|\mathbb{A}|}{(1-\gamma)\sqrt{N}} + \frac{\|\rho\|_2^2}{1-\gamma}\frac{p_T(\alpha_m \varrho_1)}{m^{\frac{1}{4}}} + \frac{\gamma^T r_\infty}{(1-\gamma)^2} + \frac{\sqrt{\kappa}}{N\sqrt{1-\gamma}} \sum_{n=0}^{N-1} \mathbb{E}_0\big(\varepsilon_{\mathsf{cfa}}^T(\Phi(n), \omega_n)\big)^{\frac{1}{2}},$$

where $\mathbb{E}_0$ is the conditional expectation given $\Phi(0)$ and

$$\varepsilon_{\mathsf{cfa}}^T(\Phi, \omega) := \sum_{t<T} \gamma^t |\nabla^\top \ln \pi_t^{\Phi}(A_t|Z_t)\omega - \mathcal{A}_t^{\pi^\Phi}(Z_t, A_t)|^2.$$

Remark 6.4. We have the following remarks.

▷ The effectiveness of Rec-NPG is proportional to the approximation power of the RNN used for policy parameterization, as reflected in $\varepsilon_{\mathsf{cfa}}^T$ in Theorem 6.3. We further characterize this error term in Prop. 6.6-6.8.

▷ The terms $L_t, \beta_t, \Lambda_t, \chi_t$ grow at a rate $p_t(\varrho_1\alpha_m)$. Thus, if $\alpha_m > \varrho_1^{-1}$, then m and N should grow at a rate $(\alpha_m\varrho_1)^T$, implying the curse of dimensionality (more generally, it is known as the exploding gradient problem Goodfellow et al. (2016)). On the other hand, if $\alpha_m < \varrho_1^{-1}$, then $L_t, \beta_t, \Lambda_t, \chi_t$ are all $\mathscr{O}(1)$ for all t, implying efficient learning of POMDPs. This establishes a very interesting connection between the memory in the system, the continuity and smoothness of the RNN with respect to its parameters, and the optimality gap under Rec-NPG.

▷ The term $\frac{2\gamma^T r_\infty}{(1-\gamma)^2}$ is due to truncating the trajectory at T, and vanishes with large T.

Remark 6.5. The quantity κ in Proposition 6.8 is the so-called concentrability coefficient in policy gradient methods (Agarwal et al., 2020; Bhandari & Russo, 2019; Wang et al., 2019), and determines the complexity of exploration. Note that it is defined in terms of path probabilities $P_T^{\pi,\mu}$ in the non-stationary setting.

In the following, we decompose the compatible function approximation error $\varepsilon_{\mathsf{cfa}}^T$ into the approximation error for the RNN and the statistical errors. To that end, let

$$\varepsilon_{\mathsf{app},n} = \inf_{\omega\in\Omega_{\rho,m}} \mathbb{E}\sum_{t<T}\gamma^t\left|\nabla^\top F_t(\bar{Z}_t;\Phi(0))\omega - \mathcal{Q}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right|^2,$$

be the approximation error where the expectation is with respect to $P_T^{\pi^{\Phi(n)},\mu}$,

$$\varepsilon_{\mathsf{td},n} = \mathbb{E}[\mathcal{R}_T^{\pi^{\Phi(n)}}(\bar{\Theta}^{(n)})|\Phi(k), k\le n],$$

be the error in the critic (see equation 9), and finally let

$$\varepsilon_{\mathsf{sgd},n} = \mathbb{E}[\ell_T(\omega_n;\Phi(n),\hat{\mathcal{Q}}^{(n)})|\bar{\Theta}^{(n)},\Phi(k),k\le n] - \inf_w \mathbb{E}[\ell_T(\omega;\Phi(n),\hat{\mathcal{Q}}^{(n)})|\bar{\Theta}^{(n)},\Phi(k),k\le n],$$

be the error in the policy update via compatible function approximation.

Proposition 6.6 (Error decomposition for $\varepsilon_{\mathsf{cfa}}^T$)**.** *We have*

$$\mathbb{E}\Big[\mathbb{E}_\mu^{\pi^{\Phi(n)}}\left[\ell_T(\omega_n;\Phi(n),\mathcal{Q}^{(n)})\right]\Big|\Phi(k),k\le n\Big] \le \frac{8\|\rho\|_2^2}{m}\sum_{t=0}^{T-1}\gamma^t\beta_t^2 + 8\varepsilon_{\mathsf{app},n} + 6\varepsilon_{\mathsf{td},n} + 2\varepsilon_{\mathsf{sgd},n}.$$

for any $n\in\mathbb{Z}_+$.

From Theorem 5.4, we have, for $\eta_{\mathsf{td}} = \mathscr{O}(1/\sqrt{K_{\mathsf{td}}})$,

$$\varepsilon_{\mathsf{td},n} \le \mathbf{poly}(p_T(\varrho_1\alpha_m))\mathscr{O}\left(\frac{1}{\sqrt{K_{\mathsf{td}}}} + \frac{1}{\sqrt{m_{\mathsf{critic}}}} + \gamma^T\right),$$

and by Theorem 14.8 in Shalev-Shwartz & Ben-David (2014), we have, for $\eta_{\mathsf{td}} = \mathscr{O}(1/\sqrt{K_{\mathsf{td}}})$,

$$\varepsilon_{\mathsf{sgd},n} \le \mathbf{poly}(p_T(\varrho_1\alpha_m), \|\rho\|_2)\mathscr{O}(1/\sqrt{K_{\mathsf{sgd}}}).$$

As such, the statistical errors in the critic and the policy update (i.e., $\varepsilon_{\mathsf{td},n}, \varepsilon_{\mathsf{sgd},n}$) can be made arbitrarily small by using larger $K_{\mathsf{td}}, K_{\mathsf{sgd}}$ and larger m_{critic}. The remaining quantity to characterize is the approximation error, which is of critical importance for a small optimality gap as shown in Theorem 6.3 and Proposition 6.6. In the following, we will provide a finer characterization of $\varepsilon_{\mathsf{app},n}$ and identify a class of POMDPs that can be efficiently solved using Rec-NPG.

Assumption 6.7. For an index set J and $\nu\in\mathbb{R}^2_{>0}$, we consider a class $\mathscr{H}_{J,\nu}$ of transportation mappings

$$\left\{\boldsymbol{v}^{(j)}\in\mathscr{H} : j\in J, \begin{pmatrix}\sup_{w\in\mathbb{R},j\in J}|v_{\mathsf{w}}^{(j)}(w)| \\ \sup_{u\in\mathbb{R}^d,j\in J}\|v_{\mathsf{u}}^{(j)}(u)\|_2\end{pmatrix} \le \begin{pmatrix}\nu_{\mathsf{w}}\\ \nu_{\mathsf{u}}\end{pmatrix}\right\},$$

and also the corresponding infinite-width limit

$$\mathscr{F}_{J,\nu} := \{\bar{z} \mapsto \mathbb{E}[\Psi(\bar{z};\theta_0)\boldsymbol{v}(\theta_0)] : \boldsymbol{v} \in \mathbf{Conv}(\mathscr{H}_{J,\nu})\},$$

where $\Psi(\cdot;\theta_0)$ is the NTRF matrix, defined in equation 7.

We assume that there exists an index set J and $\nu \in \mathbb{R}^2_{>0}$ such that $\mathcal{Q}^{\pi^{\Phi(n)}} \in \mathscr{F}_{J,\nu}$ for all $n \in \mathbb{N}$.

This representational assumption implies that the $\mathcal{Q}$-functions under all iterate policies $\pi^{\Phi(n)}$ throughout the Rec-NPG iterations $n = 0, 1, \ldots$ can be represented by convex combinations of a *fixed* set of mappings in the NTK function class $\mathscr{F}$ indexed by J. As we will see, the richness of J as measured by a relevant Rademacher complexity will play an important role in bounding the approximation error. To that end, for $\bar{z}_t = (z_t, a_t) \in (\mathbb{Y} \times \mathbb{A})^{t+1}$, let

$$G_t^{\bar{z}_t} := \{\phi \mapsto \nabla_\phi^\top H_t^{(1)}(\bar{z}_t;\phi)\boldsymbol{v}(\phi) : \boldsymbol{v} \in \mathscr{H}_{J,\nu}\},$$

and

$$\mathrm{Rad}_m(G_t^{\bar{z}_t}) := \mathop{\mathbb{E}}_{\substack{\epsilon\sim\mathsf{Rad}^m(1)\\ \Phi(0)\sim\zeta_{\mathsf{init}}}} \sup_{g\in G_t^{\bar{z}_t}} \frac{1}{m}\sum_{i=1}^m \epsilon_i g(\Phi_i(0)).$$

Note that $\boldsymbol{v} \in \mathscr{H}_{J,\nu}$ above can be replaced more with $\boldsymbol{v} \in \mathbf{Conv}(\mathscr{H}_{J,\nu})$ without any loss. In that case, since the mapping $\boldsymbol{v}^{(j)} \mapsto f_t^\star(\bar{z}_t;\boldsymbol{v}^{(j)}) \in G_t^{\bar{z}_t}$ is linear, $G_t^{\bar{z}_t}$ is replaced with $\mathbf{Conv}(G_t^{\bar{z}_t})$ without changing the Rademacher complexity (Mohri et al., 2018).

The following provides a finer characterization of the approximation error.

Proposition 6.8. *Under Assumption 6.7, if $\rho \succeq \nu$, then*

$$\epsilon_{\mathsf{app},n} \le \frac{1}{1-\gamma}\left(2\max_{0\le t<T}\max_{\bar{z}_t\in(\mathbb{Y}\times\mathbb{A})^{t+1}} \mathrm{Rad}_m(G_t^{\bar{z}_t}) + L_T\|\rho\|_2\sqrt{\frac{\ln\left(2T|\mathbb{Y}\times\mathbb{A}|^T/\delta\right)}{m}}\right)^2,$$

for all n simultaneously with probability at least $1-\delta$ over the random initialization for any $\delta \in (0,1)$.

Remark 6.9. An interesting case that lead to a vanishing approximation error (as $m \to \infty$) is $|J| < \infty$. Then, Proposition 6.8 reduces to Cayci et al. (2024b) (with $T = 1$ for FNNs) with the complexity term $\mathscr{O}\left(\sqrt{\frac{\ln(|J|/\delta)}{m}}\right)$ by the finite-class lemma (Mohri et al., 2018). In this case, the $\mathcal{Q}$-functions throughout $n = 0, 1, \ldots$ lie in the convex hull of $|J|$ fixed functions in $\mathscr{F}$ generated by $\{\boldsymbol{v}^{(j)} \in \mathscr{H} : j \in J\}$.

Remark 6.10. As noted in Cayci et al. (2024b), in a *static* problem (e.g., the regression problem in supervised learning or policy evaluation in Section 5) with a target function $f \in \mathscr{F}$, the approximation error is easy to characterize:

$$\left|\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega^\star - f_t(\bar{z}_t)\right| = \mathscr{O}\left(\sqrt{\frac{\ln(1/\delta)}{m}}\right), \tag{14}$$

by Hoeffding inequality with $\omega^\star := \left[\frac{1}{\sqrt{m}}c_i\boldsymbol{v}(\Phi_i(0))\right]_{i\in[m]}$.

In the *dynamical* policy optimization problem, the representational assumption $\mathcal{Q}^{\pi^{\Phi(n)}} \in \mathscr{F}$ does not imply an arbitrarily small approximation error as $m \to \infty$ since the target function $\mathcal{Q}^{\pi^{\Phi(n)}}$ also depends on $\Phi(0)$. Thus,

$$\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega_n^\star = \sum_{i=1}^m \frac{\nabla^\top H_t^{(i)}(\bar{z}_t;\Phi(0))\boldsymbol{v}^{\Phi(n)}(\Phi_i(0))}{m}$$

with $\omega_n^\star := [\frac{1}{\sqrt{m}}c_i\boldsymbol{v}^{\Phi(n)}(\Phi_i(0))]_{i\in[m]}$ for $\boldsymbol{v}^{\Phi(n)} \in \mathscr{H}$ may not converge to the target function $\mathcal{Q}^{\pi^{\Phi(n)}}$ as $m \to \infty$ because of the correlated $\nabla^\top H_t^{(i)}(\bar{z}_t;\Phi(0))\boldsymbol{v}^{\Phi(n)}(\Phi_i(0))$ across $i \in [m]$. To address this, we characterize the uniform approximation error as in Proposition 6.8 for the random features of the actor RNN in approximating all $\mathcal{Q}^{\pi^{\Phi(n)}}$ for all n based on Rademacher complexity.

7 Conclusion

We studied RNN-based policy evaluation and policy optimization methods with finite-time analyses, which demonstrate the effectiveness of the NPG method equipped with RNNs for POMDPs. An important limitation of Rec-NPG is that its memory and sample complexity significantly increases in POMDPs with long-term dependencies as pointed out in Remarks 5.6-6.4. In order to address these issues, as an extension of this work, theoretical analyses of more complicated LSTM- (Hochreiter & Schmidhuber, 1997) and GRU-based (Chung et al., 2014) NPG methods can be considered as a future work.

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A Algorithmic Tools for Recurrent Neural Networks

A.1 Max-Norm Regularization for Recurrent Neural Networks

Max-norm regularization, proposed by Srebro et al. (2004), has been shown to be very effective across a broad spectrum of deep learning problems (Srivastava et al., 2014; Goodfellow et al., 2013). In this work, we incorporate max-norm regularization (around the random initialization) into the recurrent natural policy gradient for sharp convergence guarantees. To that end, given a random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$ as in Definition 3.1 and a vector $\rho = (\rho_{\mathsf{w}}, \rho_{\mathsf{u}})^\top \in \mathbb{R}^2_{>0}$ of projection radii, we define the compactly-supported set of weights $\Omega_{\rho,m} \subset \mathbb{R}^{m(d+1)}$ as

$$\Omega_{\rho,m} = \Big\{\Theta \in \mathbb{R}^{m(d+1)} : \max_i |W_{ii} - W_{ii}(0)| \leq \frac{\rho_{\mathsf{w}}}{\sqrt{m}},\ \max_i \|U_i - U_i(0)\| \leq \frac{\rho_{\mathsf{u}}}{\sqrt{m}}\Big\}. \tag{15}$$

Given any symmetric random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$ and $\rho \in \mathbb{R}^2_{>0}$, the set $\Omega_{\rho,m}$ is a compact and convex subset of $\mathbb{R}^{m(d+1)}$, and for any $\Theta \in \Omega_{\rho,m}$, we have

$$\max_{1\leq i\leq m} |W_{ii} - W_{ii}(0)| \leq \frac{\rho_{\mathsf{w}}}{\sqrt{m}},$$
$$\max_{1\leq i\leq m} \|U_i - U_i(0)\| \leq \frac{\rho_{\mathsf{u}}}{\sqrt{m}}.$$

Let

$$\mathbf{Proj}_{\Omega_{\rho,m}}[\Theta] = \left[\begin{array}{cc} \displaystyle\operatorname*{arg\,min}_{w\in\mathcal{B}_2\left(W_{ii}(0), \frac{\rho_{\mathsf{w}}}{\sqrt{m}}\right)} |W_{ii} - w_i|, & \displaystyle\operatorname*{arg\,min}_{u_i\in\mathcal{B}_2\left(U_i(0), \frac{\rho_{\mathsf{u}}}{\sqrt{m}}\right)} \|\mathbf{U}_i - u_i\|_2 \end{array}\right]_{i\in[m]} \tag{16}$$

As such, the projection operator $\mathbf{Proj}_{\Omega_{\rho,m}}[\cdot]$ onto $\Omega_{\rho,m}$ is called the max-norm projection (or regularization).

Note that we have $\|\mathbf{W} - \mathbf{W}(0)\|_2 \leq \rho_{\mathsf{w}}$, $\|\mathbf{U} - \mathbf{U}(0)\|_2 \leq \rho_{\mathsf{u}}$ and $\|\Theta - \Theta(0)\|_2 \leq \|\rho\|_2$ in the ℓ_2 geometry for any $\Theta \in \Omega_{\rho,m}$. Therefore, although the max-norm parameter class $\Omega_{\rho,m} \subset \{\Theta \in \mathbb{R}^{m(d+1)} : \|\Theta - \Theta(0)\|_2 \leq \|\rho\|_2\}$, the ℓ_2-projected Cai et al. (2019); Wang et al. (2019); Liu et al. (2019) and max-norm projected Cayci et al. (2024b) optimization algorithms recover exactly the same function class (i.e., RKHS associated with the neural tangent kernel studied in Ji et al. (2019); Telgarsky (2021), see Section 3.1).

B Proofs for Section 5

An important quantity in the analysis of recurrent neural networks is the following:

$$\Gamma_t^{(i)}(\bar{z}_t; \Theta) := W_{ii} H_t^{(i)}(\bar{z}_t; \Theta),$$

for any hidden unit $i \in [m]$ and $\Theta \in \mathbb{R}^{m(d+1)}$. The following Lipschitzness and smoothness results for $\Theta_i \mapsto H_t^{(i)}(\bar{z}_t; \Theta)$ and $\Theta_i \mapsto \Gamma_t^{(i)}(\bar{z}_t; \Theta)$.

Lemma B.1 (Local continuity of hidden states; Lemma 1-2 in Cayci & Eryilmaz (2024))**.** *Given $\rho \in \mathbb{R}^2_{>0}$ and $\alpha \geq 0$, let $\alpha_m = \alpha + \frac{\rho_w}{\sqrt{m}}$. Then, for any $\bar{\boldsymbol{z}} \in (\mathbb{Y} \times \mathbb{A})^{\bar{Z}_+}$ with $\sup_{t\in\mathbb{N}} \left\|\begin{pmatrix} y_t \\ a_t \end{pmatrix}\right\|_2 \leq 1$, $t \in \mathbb{N}$ and $i \in [m]$,*

- *$\Theta_i \mapsto H_t^{(i)}(\bar{z}_t; \Theta)$ is L_t-Lipschitz continuous with $L_t = (\varrho_0^2 + 1)\varrho_1^2 \cdot p_t^2(\alpha_m \varrho_1)$,*
- *$\Theta_i \mapsto H_t^{(i)}(\bar{z}_t; \Theta)$ is β_t-smooth with $\beta_t = \mathscr{O}\left(d \cdot p_t(\alpha_m \varrho_1) \cdot q_t(\alpha_m \varrho_1)\right)$,*
- *$\Theta_i \mapsto \Gamma_t^{(i)}(\bar{z}_t; \Theta)$ is Λ_t-Lipschitz with $\Lambda_t = \sqrt{2}(\varrho_0 + 1 + \alpha_m L_t)$,*

- $\Theta_i \mapsto \Gamma_t^{(i)}(\bar{z}_t;\Theta)$ *is* χ_t*-smooth with* $\chi_t = \sqrt{2}(L_t + \alpha_m\beta_t)$,

in $\Omega_{\rho,m}$*. Consequently, for any* $\Theta \in \Omega_{\rho,m}$,

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty} \max_{0\le t\le T} |F_t(\bar{z}_t;\Theta)| \le L_T \cdot \|\rho\|_2,\ T\in\mathbb{N}, \tag{17}$$

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty} |F_t^{\mathsf{Lin}}(\bar{z}_t;\Theta) - F_t(\bar{z}_t;\Theta)| \le \frac{2}{\sqrt{m}}(\varrho_2\Lambda_t^2 + \varrho_1\chi_t)\|\Theta-\Theta(0)\|_2^2,\ t\in\mathbb{N}, \tag{18}$$

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty} \left\langle \nabla F_t(\bar{z}_t;\Theta) - \nabla F_t(\bar{z}_t;\Theta(0)), \Theta-\bar{\Theta}\right\rangle \le \frac{2\beta_t^2\|\rho\|_2^2}{\sqrt{m}}, \tag{19}$$

with probability 1 over the symmetric random initialization $(\mathbf{W}(0),\mathbf{U}(0),c)$.

Lemma B.2 (Approximation error between RNN-NTRF and RNN-NTK)**.** *Let* $f^\star \in \mathscr{F}$ *with the transportation mapping* $\boldsymbol{v}\in\mathscr{H}$*, and let*

$$\bar{\Theta}_i = \Theta_i(0) + \frac{1}{\sqrt{m}}c_i\boldsymbol{v}(\Theta_i(0)), i\in[m]. \tag{20}$$

for any symmetric random initialization $(\mathbf{W}(0),\mathbf{U}(0),c)$ *in Def. 3.1. Let*

$$F_t^{\mathsf{Lin}}(\cdot;\Theta) = \nabla_\Theta F_t(\cdot;\Theta(0))\cdot(\Theta-\Theta(0)).$$

If $P_T^{\pi,\mu}$ *induces a compactly-supported marginal distribution for* $X_t, t\in\mathbb{N}$ *such that* $\|X_t\|_2\le 1$ *a.s. and* $\{\bar{Z}_t : t\in\mathbb{N}\}$ *is independent from the random initialization* $(\mathbf{W}(0),\mathbf{U}(0),c)$*, then we have*

$$\mathbb{E}\left[\mathbb{E}_\mu^\pi\left[\left(f_t^\star(\bar{Z}_t) - F_t^{\mathsf{Lin}}(\bar{Z}_t;\bar{\Theta})\right)^2\right]\right] \le \frac{2\|\nu\|_2^2(1+\varrho_0^2)p_t^2(\alpha\varrho_1)}{m}, \tag{21}$$

where the outer expectation is with respect to the random initialization $(\mathbf{W}(0),\mathbf{U}(0),c)$.

Proof. For any hidden unit $i\in[m]$, let

$$\zeta_i = \left\langle \boldsymbol{v}(\Theta_i(0)), \sum_{k=0}^{t} {W_{ii}}^k(0)\begin{pmatrix} H_{t-k-1}^{(i)}(\bar{Z}_{t-k-1},\Theta_i(0)) \\ X_{t-k}\end{pmatrix}\prod_{j=0}^{k}\mathcal{I}_{t-j}(\bar{Z}_{t-j};\Theta_i(0))\right\rangle.$$

Then, it is straightforward to see that

$$F_t^{\mathsf{Lin}}(\bar{Z}_t;\bar{\Theta}) = \frac{1}{m}\sum_{i=1}^{m}\zeta_i, \tag{22}$$

and $\mathbb{E}[\zeta_i|\bar{Z}_t] = \mathbb{E}[f_t^\star(\bar{Z}_t)|\bar{Z}_t]$ almost surely. Note that $\{\zeta_i : i\in[m/2]\}$ is independent and identically distributed and $\zeta_i = \zeta_{i+m/2}$ for any $i\in[m/2]$. Also, with probability 1 we have

$$\begin{aligned}
|\zeta_i| &\overset{(\spadesuit)}{\le} \|\boldsymbol{v}(\Theta_i(0))\|_2 \cdot \left\| \sum_{k=0}^{t} {W_{ii}}^k(0)\begin{pmatrix} H_{t-k-1}^{(i)}(\bar{Z}_{t-k-1},\Theta_i(0)) \\ X_{t-k}\end{pmatrix}\prod_{j=0}^{k}\mathcal{I}_{t-j}(\bar{Z}_{t-j};\Theta_i(0))\right\|_2, \\
&\overset{(\clubsuit)}{\le} \|\boldsymbol{v}(\Theta_i(0))\|_2 \sum_{k=0}^{t-1}\alpha^k\varrho_1^{k+1}\sqrt{1+\varrho_0^2}, \\
&\overset{(\diamondsuit)}{\le} \|\nu\|_2\cdot\varrho_1\cdot\sqrt{1+\varrho_0^2}\cdot p_t(\alpha\varrho_1),
\end{aligned}$$

where ($\spadesuit$) follows from Cauchy-Schwarz inequality, ($\clubsuit$) follows from the uniform bound $\sup_{z\in\mathbb{R}}|\varrho(z)|\le\varrho_1$ and almost-sure bounds $\|X_k\|_2\le 1$ and $|W_{ii}(0)|\le\alpha$, and ($\clubsuit$) follows from $\boldsymbol{v}\in\mathscr{H}_\nu$. From these bounds,

$$\mathrm{Var}(\zeta_i) \le \mathbb{E}[\mathbb{E}_\mu^\pi[|\zeta_i|^2]] \le \|\nu\|_2^2\varrho_1^2(1+\varrho_0)^2p_t^2(\alpha\varrho_1),\ i\in[m]. \tag{23}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}_\mu^\pi\left[\left(f_t^\star(\bar{Z}_t) - F_t^{\mathsf{Lin}}(\bar{Z}_t;\bar{\Theta})\right)^2\right]\right] &= \mathbb{E}_\mu^\pi\left[\mathbb{E}\left[\left|\frac{1}{m}\sum_{i=1}^{m}\left(\zeta_i - \mathbb{E}[\zeta_i|\bar{Z}_t]\right)\right|^2\right]\right], \\
&= \mathbb{E}_\mu^\pi\left[\mathbb{E}\left[\left|\frac{2}{m}\sum_{i=1}^{m/2}\left(\zeta_i - \mathbb{E}[\zeta_i|\bar{Z}_t]\right)\right|^2\right]\right], \\
&= \frac{4}{m^2}\mathbb{E}_\mu^\pi\sum_{i=1}^{m/2}\sum_{j=1}^{m/2}\mathbb{E}\left[\left(\zeta_i - \mathbb{E}[\zeta_i|\bar{Z}_t]\right)\left(\zeta_j - \mathbb{E}[\zeta_j|\bar{Z}_t]\right)\right], \\
&= \frac{4}{m^2}\mathbb{E}_\mu^\pi\sum_{i=1}^{m/2}\mathrm{Var}(\zeta_i) \leq \frac{2}{m}\|\nu\|_2^2\varrho_1^2(1+\varrho_0)^2 p_t^2(\alpha\varrho_1),
\end{aligned}$$

where the first identity is from Fubini's theorem, the second identity is from the symmetricity of the random initialization, the fourth identity is due to the independent initialization for $i \leq m/2$, and the inequality is from the bound in equation 23.

□

Proposition B.3 (Non-stationary Bellman equation)**.** *For $\pi \in \Pi_{\mathsf{NM}}$, we have*

$$\mathcal{Q}_t^\pi(\bar{z}_t) = \mathbb{E}^\pi\left[r(S_t, A_t) + \gamma\mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1})\middle|\bar{Z}_t = \bar{z}_t\right] = \mathbb{E}^\pi\left[r(S_t, A_t) + \gamma\mathcal{V}_{t+1}^\pi(Z_{t+1})\middle|\bar{Z}_t = \bar{z}_t\right],$$

for any $t \in \mathbb{Z}_+$.

Proof of Theorem 5.4. Since $\{\mathcal{Q}_t^\pi : t \in \mathbb{N}\} \in \mathscr{F}$, let the point of attraction $\bar{\Theta}$ be defined as in equation 20, and the potential function be defined as

$$\Psi(\Theta) = \|\Theta - \bar{\Theta}\|_2^2. \tag{24}$$

Then, from the non-expansivity of the projection operator onto the convex set $\Omega_{\rho,m}$, we have the following inequality:

$$\Psi(\Theta(k+1)) \leq \Psi(\Theta(k)) + 2\eta\sum_{t=0}^{T-1}\gamma^t\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\left\langle\nabla F_t(\bar{Z}_t^k;\Theta(k)), \Theta(k) - \bar{\Theta}\right\rangle + \eta^2\|\check{\nabla}\mathcal{R}_T(\bar{Z}_T^k;\Theta(k))\|_2^2. \tag{25}$$

Let $\check{\mathbb{E}}_t^k[\cdot] := \mathbb{E}[\cdot|\Theta(k), \ldots, \Theta(0), \bar{Z}_t^k]$. Then, we obtain

$$\begin{aligned}
\mathbb{E}[\Psi(\Theta(k+1) - \Psi(\Theta(k))] \leq 2\eta\mathbb{E}\Big[\sum_{t=0}^{T-1}\gamma^t\underbrace{\check{\mathbb{E}}_t^k[\delta_t(\bar{Z}_{t+1}^k;\Theta(k))]\left\langle\nabla F_t(\bar{Z}_t^k;\Theta(k)), \Theta(k) - \bar{\Theta}\right\rangle}_{(\spadesuit)_t}\Big] \\
+ \eta^2\mathbb{E}\underbrace{\|\check{\nabla}\mathcal{R}_T(\bar{Z}_T^k;\Theta(k))\|_2^2}_{(\clubsuit)}.
\end{aligned} \tag{26}$$

Bounding $\mathbb{E}(\spadesuit)_t$. By using the Bellman equation in the non-stationary setting (cf. Proposition B.3), notice that

$$\begin{aligned}
\check{\mathbb{E}}_t^k\delta_t(\bar{Z}_{t+1}^k;\Theta(k)) &= \check{\mathbb{E}}_t^k[r_t^k + \gamma F_{t+1}(\bar{Z}_{t+1}^k;\Theta(k)] - F_t(\bar{Z}_t^k;\Theta(k)), \\
&= \gamma\check{\mathbb{E}}_t^k\left[F_{t+1}(\bar{Z}_{t+1}^k;\Theta(k)) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)\right] + \mathcal{Q}_t^\pi(\bar{Z}_t) - F_t(\bar{Z}_t^k;\Theta(k)).
\end{aligned}$$

Secondly, we perform a change-of-feature as follows:

$$\left\langle\nabla F_t(\bar{Z}_t^k;\Theta(k)), \Theta(k) - \bar{\Theta}\right\rangle = \left\langle\nabla F_t(\bar{Z}_t^k;\Theta(0)), \Theta(k) - \bar{\Theta}\right\rangle + \mathsf{err}_{t,k}^{(1)}, \tag{27}$$

where

$$\mathsf{err}_{t,k}^{(1)} := \left\langle \nabla F_t(\bar{Z}_t^k;\Theta(k)) - \nabla F_t(\bar{Z}_t^k;\Theta(0)), \Theta(k)-\bar{\Theta}\right\rangle, \text{ and } |\mathsf{err}_{t,k}^{(1)}| \le \frac{2\beta_t^2\|\rho\|_2^2}{\sqrt{m}} \le \frac{2\beta_T^2\|\rho\|_2^2}{\sqrt{m}},$$

by Lemma B.1. Furthermore,

$$\left\langle \nabla F_t(\bar{Z}_t^k;\Theta(0)), \Theta(k)-\bar{\Theta}\right\rangle = F_t^{\mathsf{Lin}}(\bar{Z}_t^k;\Theta(k)) - F_t^{\mathsf{Lin}}(\bar{Z}_t^k;\bar{\Theta}), \tag{28}$$

$$= F_t(\bar{Z}_t^k;\Theta(k)) - \mathcal{Q}_t^\pi(\bar{Z}_t^k) + \mathsf{err}_{t,k}^{(2)} + \mathsf{err}_{t,k}^{(3)} \tag{29}$$

where

$$\mathsf{err}_{t,k}^{(2)} := F_t^{\mathsf{Lin}}(\bar{Z}_t^k;\Theta(k)) - F_t(\bar{Z}_t^k;\Theta(k)),$$
$$\mathsf{err}_{t,k}^{(3)} := -F_t^{\mathsf{Lin}}(\bar{Z}_t^k;\bar{\Theta}) + \mathcal{Q}_t^\pi(\bar{Z}_t^k).$$

Thus,

$$(\spadesuit)_t = -(\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\Theta(k)))^2 + \gamma\check{\mathbb{E}}_t^k\left[F_{t+1}(\bar{Z}_{t+1}^k;\Theta(k)) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)\right]\cdot(\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\Theta(k)))$$
$$+ \check{\mathbb{E}}_t^k\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\sum_{j=1}^{3}\mathsf{err}_{t,k}^{(j)}.$$

By equation 17, we have

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty}|\delta_t(\bar{z}_{t+1};\Theta(k))| \le r_\infty + 2L_T\|\rho\|_2 =: \delta_{\mathsf{max}}$$

Now, let $\omega_{t,k} := \left(\mathbb{E}[(\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\Theta(k)))^2]\right)^{1/2}$, where the expectation is over the joint distribution of $\Theta(k)$ and $\bar{Z}_T^k$. Then,

$$\mathbb{E}[(\spadesuit)_t] \le -\omega_{t,k}^2 + \gamma\omega_{t+1,k}\omega_{t,k} + \delta_{\mathsf{max}}\sum_{j=1}^{3}\mathbb{E}|\mathsf{err}_{t,k}^{(j)}|.$$

From equation 18, we have

$$\mathbb{E}|\mathsf{err}_{t,k}^{(2)}| \le \frac{2}{\sqrt{m}}(\varrho_2\Lambda_T^2 + \varrho_1\chi_T)\|\rho\|_2^2.$$

From the approximation bound in Lemma B.2, we get

$$\mathbb{E}|\mathsf{err}_{t,k}^{(3)}| \le \sqrt{\mathbb{E}|\mathsf{err}_{t,k}^{(3)}|^2} \le \frac{2\|\nu\|_2\sqrt{1+\varrho_0^2}\cdot p_T(\alpha\varrho_1)}{\sqrt{m}}.$$

Also, note that $\omega_{t+1,k}\omega_{t,k} \le \frac{1}{2}(\omega_{t,k}^2 + \omega_{t+1,k}^2)$. Putting these together, we obtain the following bound for every $t \in \{0,1,\ldots,T-1\}$:

$$\mathbb{E}[(\spadesuit)_t] \le -\omega_{t,k}^2 + \frac{\gamma}{2}(\omega_{t+1,k}^2 + \omega_{t,k}^2) + \delta_{\mathsf{max}}\cdot\frac{C_T}{\sqrt{m}},$$

where

$$C_T := 2\beta_T^2\|\rho\|_2^2 + 2(\varrho_2\Lambda_T^2 + \varrho_1\chi_T)\|\rho\|_2^2 + 2\|\nu\|_2\sqrt{1+\varrho_0^2}\cdot p_T(\alpha\varrho_1).$$

Hence, we obtain the following upper bound:

$$\sum_{t=0}^{T-1}\gamma^t\mathbb{E}[(\spadesuit)_t] \le -(1-\gamma/2)\sum_{t<T}\gamma^t\omega_{t,k}^2 + \frac{\delta_{\mathsf{max}}\cdot C_T}{(1-\gamma)\sqrt{m}} + \underbrace{\frac{1}{2}\sum_{t<T}\gamma^{t+1}\omega_{t+1,k}^2}_{\le\frac{1}{2}(\sum_{t<T}\gamma^t\omega_{t,k}^2 + \gamma^T\omega_{T,k}^2)}$$

$$\le -\frac{1-\gamma}{2}\sum_{t<T}\gamma^t\omega_{t,k}^2 + \frac{1}{2}\gamma^T\omega_{T,k}^2 + \frac{C_T\cdot\delta_{\mathsf{max}}}{(1-\gamma)\sqrt{m}}. \tag{30}$$

Bounding $\mathbb{E}[(\clubsuit)]$. Using the triangle inequality, we obtain:

$$\|\sum_{t<T}\gamma^t\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\nabla F_t(\bar{Z}_t;\Theta(k))\|_2 \leq \sum_{t<T}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k))|\cdot\|\nabla F_t(\bar{Z}_t;\Theta(k))\|_2.$$

Since $\Theta(k)\in\Omega_{\rho,m}$ for every $k\in\mathbb{N}$ as a consequence of the max-norm regularization, we have

$$|\delta_t(\bar{Z}_{t+1}^k;\Theta(k))| \leq \delta_{\mathsf{max}} = r_\infty + 2L_T\|\rho\|_2,$$
$$\|\nabla F_t(\bar{Z}_t^k;\Theta(k))\|_2^2 = \frac{1}{m}\sum_{i=1}^m\|\nabla_{\Theta_i}H_t^{(i)}(\bar{Z}_t^k;\Theta(k))\|_2^2 \leq L_t^2 \leq L_T^2,$$

for every $t<T$ with probability 1 since $\Theta_i\mapsto H_t^{(i)}(\bar{z}_t;\Theta_i)$ is L_t-Lipschitz continuous by Lemma B.1. Hence, we obtain:

$$\|\check{\nabla}\mathcal{R}_T(\bar{Z}_T^k;\Theta(k))\|_2 \leq \frac{\delta_{\mathsf{max}}L_T}{1-\gamma}. \tag{31}$$

Final step. Now, taking expectation over $(\bar{Z}_t^k,\Theta(k))$ in equation 26, and substituting equation 30 and equation 31, we obtain:

$$\mathbb{E}[\Psi(\Theta(k+1))-\Psi(\Theta(k))] \leq -\eta(1-\gamma)\sum_{t=0}^{T-1}\gamma^t\omega_{t,k}^2 + \eta\gamma^T\omega_{T,k}^2 + \eta\frac{\delta_{\mathsf{max}}\cdot C_T}{(1-\gamma)\sqrt{m}} + \eta^2\frac{\delta_{\mathsf{max}}^2L_T^2}{(1-\gamma)^2},$$

for every $k\in\mathbb{N}$. Note that $\Psi(\Theta(0))\leq\|\nu\|_2^2$. Thus, telescoping sum over $k=0,1,\ldots,K-1$ yields

$$\frac{1}{K}\sum_{k=0}^{K-1}\mathcal{R}_T(\Theta(k)) \leq \frac{\|\nu\|_2^2}{\eta(1-\gamma)K} + \frac{\eta\delta_{\mathsf{max}}^2L_T^2}{(1-\gamma)^3} + \frac{\delta_{\mathsf{max}}\cdot C_T}{(1-\gamma)^2\sqrt{m}} + \frac{\gamma^T}{(1-\gamma)K}\sum_{k=0}^{K-1}\omega_{T,k}^2. \tag{32}$$

The final inequality in the proof stems from the linearization result Lemma B.2, and directly follows from

$$\mathcal{R}_T\left(\frac{1}{K}\sum_{k<K}\Theta(k)\right) \leq \frac{4}{K}\sum_{k<K}\mathcal{R}_T(\Theta(k)) + \frac{6}{\sqrt{m}}\left(\varrho_2\Lambda_T^2+\varrho_1\chi_T\right)\|\rho\|_2^2,$$

which directly follows from Cayci & Eryilmaz (2024), Corollary 1. □

In the following, we study the error under mean-path Rec-TD learning algorithm.

Theorem B.4 (Finite-time bounds for mean-path Rec-TD)**.** *For $K\in\mathbb{N}$, with the step-size choice $\eta=\frac{(1-\gamma)^2}{64L_T^2}$, mean-path Rec-TD learning achieves the following error bound:*

$$\mathbb{E}\left[\frac{1}{K}\sum_{k<K}\mathcal{R}_T^{\boldsymbol{\pi}}(\Theta(k))\right] \leq \frac{2\|\nu\|_2^2}{(1-\gamma)\eta K} + \frac{\gamma^T\omega_{T,k}}{1-\gamma} + \frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)^2\sqrt{m}} + \eta\left(\frac{(C_T')^2}{m} + 16\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2)\right),$$

where C_T' and L_T are terms that do not depend on K.

Theorem B.4 indicates that if a noiseless semi-gradient is used in Rec-TD, then the rate can be improved from $\mathscr{O}\left(\frac{1}{\sqrt{K}}\right)$ to $\mathscr{O}\left(\frac{1}{K}\right)$, indicating the potential limits of using variance-reduction schemes.

Proof of Theorem B.4. At any iteration $k\in\mathbb{N}$, let

$$\bar{\nabla}\mathcal{R}_T(\Theta(k)) := \mathbb{E}_\mu^\pi\Big[\check{\nabla}\mathcal{R}(\bar{Z}_t^k;\Theta(k))\Big], \tag{33}$$

be the ***mean-path semi-gradient***. First, note that

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))\|_2^2 \leq 2\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2 + 2\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2. \tag{34}$$

Bounding $\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2$. For any $k \in \mathbb{N}, t \leq T$, we have

$$\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\bar{Z}_t^k,\Theta(0),c\big] = \gamma\mathbb{E}[F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)|\bar{Z}_t^k,\Theta(0),c] + \mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\bar{\Theta}).$$

Since $\|\nabla F_t(\bar{z}_t;\bar{\Theta})\|_2 \leq L_t$, the following inequality holds:

$$\begin{aligned}\big\|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\big]\big\|_2 &\leq \mathbb{E}\left\|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\bar{Z}_t^k,\Theta(0),c\big]\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\right\|_2,\\ &\leq L_T\mathbb{E}\left|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\bar{Z}_t^k,\Theta(0),c\big]\right|,\\ &\leq L_T\left(\gamma\mathbb{E}\left|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)\right| + \mathbb{E}\left|\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\bar{\Theta})\right|\right), \quad (35)\end{aligned}$$

where we used Jensen's inequality, the law of iterated expectations, and triangle inequality. From the above inequality, we obtain

$$\begin{aligned}\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 &\overset{①}{\leq} \sum_{t=0}^{T-1}\gamma^t\big\|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\big]\big\|_2,\\ &\overset{②}{\leq} L_T\gamma\sum_{t<T}\gamma^t\mathbb{E}|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)| + L_T\sum_{t<T}\gamma^t\mathbb{E}|\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\bar{\Theta})|,\\ &\overset{③}{\leq} \frac{L_T}{\sqrt{1-\gamma}}\left(\gamma\mathbb{E}\sqrt{\sum_{t<T}\gamma^t|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)|^2} + \mathbb{E}\sqrt{\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2}\right),\\ &\overset{④}{\leq} \frac{L_T}{\sqrt{1-\gamma}}\left(\gamma\sqrt{\mathbb{E}\sum_{t<T}\gamma^t|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)|^2} + \sqrt{\mathbb{E}\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2}\right),\\ &\overset{⑤}{\leq} \frac{\sqrt{2}(1+\gamma)L_T}{\sqrt{1-\gamma}}\frac{\|\nu\|_2\sqrt{1+\varrho_0^2}\cdot p_T(\varrho_1\alpha)}{\sqrt{m}}.\end{aligned}$$

where ① follows from triangle inequality, ② follows from equation 35, ③ follows from Cauchy-Schwarz inequality and the monotonicity of the geometric series $T \mapsto \sum_{t<T}\gamma^t$, ④ follows from Jensen's inequality, and finally ⑤ follows from Lemma B.2. Hence, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2 \leq \frac{8L_T^2\|\nu\|_2^2(1+\varrho_0^2)p_T^2(\varrho_1\alpha)}{(1-\gamma)m}. \quad (36)$$

Bounding $\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2$. First, note that

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 = \|\mathbb{E}\Big[\sum_{t<T}\gamma^t\left(\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\nabla F_t(\bar{Z}_t^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\right)\Big]\|_2$$

We make the following decomposition for each $t < T$:

$$\begin{aligned}\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\nabla F_t(\bar{Z}_t^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta}) &= \delta_t(\bar{Z}_{t+1}^k;\Theta(k))\left(\nabla F_t(\bar{Z}_t^k;\Theta(k)) - \nabla F_t(\bar{Z}_t^k;\bar{\Theta})\right)\\ &\quad + \nabla F_t(\bar{Z}_t^k;\Theta(k))\left(\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta}) - \delta_t(\bar{Z}_{t+1}^k;\Theta(k))\right) \quad (37)\end{aligned}$$

By Lemma B.1, we have $|\delta_t(\bar{Z}_{t+1}^k;\Theta)| \leq \delta_{\mathsf{max}}$ and $\|\nabla F_t(\bar{Z}_t^k;\Theta)\|_1 \leq L_t \leq L_T$ almost surely for any $\Theta \in \Omega_{\rho,m}$, which holds for $\Theta(k)$ (due to the max-norm projection) and $\bar{\Theta}$. As such, by triangle inequality,

$$\begin{aligned}\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 &\leq \sum_{t<T}\gamma^t\left(\delta_{\mathsf{max}}\frac{\beta_t^2\mathbb{E}\|\Theta(k)-\bar{\Theta}\|_2^2}{m} + L_t\mathbb{E}|\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta}) - \delta_t(\bar{Z}_{t+1}^k;\Theta(k))|\right),\\ &\leq \underbrace{\frac{\delta_{\mathsf{max}}\beta_T^2(\|\rho\|_2^2+\|\nu\|_2^2)}{m(1-\gamma)}}_{=:\frac{C_T^{(4)}}{m}} + L_T\mathbb{E}\left[\sum_{t=0}^{T-1}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta}) - \delta_t(\bar{Z}_{t+1}^k;\Theta(k))|\right] \quad (38)\end{aligned}$$

Note that

$$\sum_{t<T}\gamma^t|\delta_t(\bar{Z}^k_{t+1};\Theta(k))-\delta_t(\bar{Z}^k_{t+1};\bar{\Theta})| = \sum_{t<T}\gamma^t\Big(|F_{t+1}(\bar{Z}^k_{t+1};\bar{\Theta})-F_{t+1}(\bar{Z}^k_{t+1};\Theta(k))|+|F_t(\bar{Z}^k_t;\bar{\Theta})-F_t(\bar{Z}^k_t;\Theta(k))|\Big),$$
$$\leq 2\sum_{t<T}\gamma^t\Big|F_t(\bar{Z}^k_t;\bar{\Theta})-F_t(\bar{Z}^k_t;\Theta(k))\Big|+\gamma^T L_T\|\Theta(k)-\bar{\Theta}\|_2, \tag{39}$$

where the second line follows from the Lipschitz continuity of $\Theta\mapsto F_t(\cdot;\Theta)$. Then, adding and subtracting $\mathcal{Q}^\pi_t$ to each term, we obtain

$$\sum_{t<T}\gamma^t|\delta_t(\bar{Z}^k_{t+1};\Theta(k))-\delta_t(\bar{Z}^k_{t+1};\bar{\Theta})| \leq 2\sum_{t<T}\gamma^t\left(|F_t(\bar{Z}^k_t;\bar{\Theta})-\mathcal{Q}^\pi_t(\bar{Z}^k_t)|+|\mathcal{Q}^\pi_t(\bar{Z}^k_t)-F_t(\bar{Z}^k_t;\Theta(k))|\right)$$
$$+\gamma^T L_T\|\Theta(k)-\bar{\Theta}\|_2. \tag{40}$$

Taking expectation, we obtain

$$\mathbb{E}\sum_{t<T}\gamma^t|\delta_t(\bar{Z}^k_{t+1};\Theta(k))-\delta_t(\bar{Z}^k_{t+1};\bar{\Theta})| \leq \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathbb{E}\left[\sum_{t<T}\gamma^t|F_t(\bar{Z}^k_t;\Theta(k))-\mathcal{Q}^\pi_t(\bar{Z}^k_t)|^2\right]}$$
$$+\frac{2}{\sqrt{1-\gamma}}\sqrt{\mathbb{E}\left[\sum_{t<T}\gamma^t|F_t(\bar{Z}^k_t;\bar{\Theta})-\mathcal{Q}^\pi_t(\bar{Z}^k_t)|^2\right]}+\gamma^T L_T\|\Theta(k)-\bar{\Theta}\|_2.$$

By Lemma B.2 and equation 18, we have

$$\mathbb{E}|F_t(\bar{Z}^k_t;\bar{\Theta})-\mathcal{Q}^\pi_t(\bar{Z}^k_t)|^2 \leq \frac{4}{m}\|\nu\|_2^2\varrho_1^2(1+\varrho_0)^2p_t^2(\alpha\varrho_1)+\frac{4}{m}(\varrho_2\Lambda_T^2+\varrho_1\chi_T)^2\|\rho\|_2^4,$$

for any $t<T$. Thus,

$$\mathbb{E}\sum_{t<T}\gamma^t|\delta_t(\bar{Z}^k_{t+1};\Theta(k))-\delta_t(\bar{Z}^k_{t+1};\bar{\Theta})| \leq \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathbb{E}\left[\sum_{t<T}\gamma^t|F_t(\bar{Z}^k_t;\Theta(k))-\mathcal{Q}^\pi_t(\bar{Z}^k_t)|^2\right]}$$
$$+\frac{1}{\sqrt{m}}\underbrace{\frac{4}{\sqrt{(1-\gamma)}^3}\left(\|\nu\|_2\varrho_1(1+\varrho_0)p_T(\alpha\varrho_1)+(\varrho_2\Lambda_T^2+\varrho_1\chi_T)\|\rho\|_2^2\right)}_{=:C_T^{(3)}}+\gamma^T L_T\underbrace{\|\Theta(k)-\bar{\Theta}\|_2}_{\leq\|\rho\|_2+\|\nu\|_2}.$$

This results in the following bound:

$$\mathbb{E}\sum_{t<T}\left[\gamma^t|\delta_t(\bar{Z}^k_{t+1};\Theta(k))-\delta_t(\bar{Z}^k_{t+1};\bar{\Theta})|\right] \leq \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathcal{R}_T(\Theta(k))}+\frac{C_T^{(3)}}{\sqrt{m}}+\gamma^T L_T(\|\rho\|_2+\|\nu\|_2). \tag{41}$$

Substituting the local smoothness result in equation 41 into equation 38, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 \leq L_T\left(\frac{2}{\sqrt{1-\gamma}}\sqrt{\mathcal{R}_T(\Theta(k))}+\frac{C_T^{(3)}}{\sqrt{m}}+\gamma^T L_T(\|\rho\|_2+\|\nu\|_2)\right)+\frac{C_T^{(4)}}{m}.$$

Thus, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2 \leq \frac{16L_T^2}{1-\gamma}\mathcal{R}_T(\Theta(k))+\frac{4(C_T^{(3)})^2L_T^2+4(C_T^{(4)})^2}{m}+8\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2). \tag{42}$$

Using equation 36 and equation 42 together, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))\|_2^2 \leq 2\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2+2\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2,$$
$$\leq \frac{32L_T^2\mathcal{R}_T(\Theta(k))}{1-\gamma}+\frac{(C_T')^2}{m}+16\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2). \tag{43}$$

In the final step, we use equation 26, equation 30 and equation 43 together:

$$\mathbb{E}\left[\Psi(\Theta(k+1)) - \Psi(\Theta(k))\right] \leq -\eta(1-\gamma)\mathbb{E}\mathcal{R}_T(\Theta(k)) + \eta\gamma^T\omega_{T,k} + \eta\frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)\sqrt{m}} + \eta^2\left(\frac{32L_T^2\mathbb{E}\mathcal{R}_T(\Theta(k))}{1-\gamma} + \frac{(C_T')^2}{m} + 16\gamma^{2T}L_T^4(\|\rho\|_2^2 + \|\nu\|_2^2)\right), \quad (44)$$

where the expectation is over the random initialization. Choosing $\eta = \frac{(1-\gamma)^2}{64L_T^2}$, we obtain

$$\mathbb{E}[\Psi(\Theta(k+1)) - \Psi(\Theta(k))] \leq -\frac{\eta(1-\gamma)}{2}\mathbb{E}\mathcal{R}_T(\Theta(k)) + \eta\gamma^T\omega_{T,k} + \eta\frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)\sqrt{m}} + \eta^2\left(\frac{(C_T')^2}{m} + 16\gamma^{2T}L_T^4(\|\rho\|_2^2 + \|\nu\|_2^2)\right). \quad (45)$$

Telescoping sum over $k = 0, 1, \ldots, K-1$, and re-arranging terms, we obtain:

$$\mathbb{E}\left[\frac{1}{K}\sum_{k<K}\mathcal{R}_T(\Theta(k))\right] \leq \frac{2\|\nu\|_2^2}{(1-\gamma)\eta K} + \frac{\gamma^T\omega_{T,k}}{1-\gamma} + \frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)^2\sqrt{m}} + \eta\left(\frac{(C_T')^2}{m} + 16\gamma^{2T}L_T^4(\|\rho\|_2^2 + \|\nu\|_2^2)\right). \quad (46)$$

□

C Numerical Experiments for Rec-TD

In the following, we will demonstrate the numerical performance of Rec-TD for a given non-stationary policy π^{greedy}.

POMDP setting. We consider a randomly-generated finite POMDP instance with $|\mathbb{S}| = |\mathbb{Y}| = 8$, $|\mathbb{A}| = 4$, $r(s,a) \sim \mathsf{Unif}[0,1]$ for all $(s,a) \in \mathbb{S} \times \mathbb{A}$. For a fixed ambient dimension $d = 8$, we use a random feature mapping $(y,a) \mapsto \varphi(y,a) \sim \mathcal{N}(0, I_d),\ \forall (y,a) \sim \mathbb{Y} \times \mathbb{A}$.

ϵ-greedy policy. Let

$$j^\star(t) \in \arg\max_{0 \leq j < t} r_j,$$

be the instance before t at which the maximum reward was obtained, and let

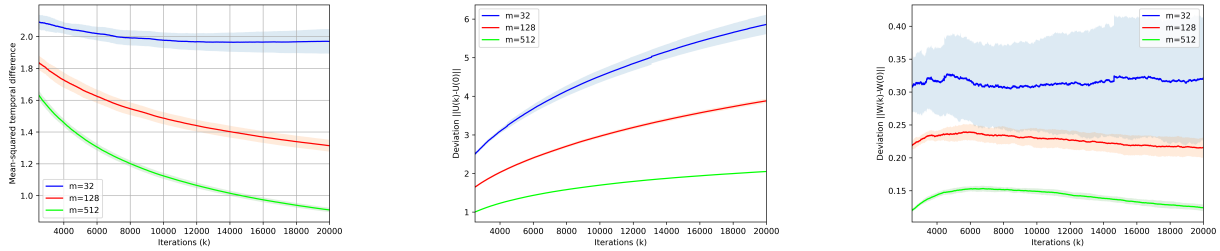
$$\pi_t^{\epsilon-\mathsf{greedy}}(a|Z_t) = \begin{cases} \frac{1}{|\mathbb{A}|}, & \text{w.p. } \min\{\frac{2+t}{10}, p_{\mathsf{exp}}\}, \\ \mathbb{1}_{a=A_{j^\star(t)}}, & \text{w.p. } 1 - \min\{\frac{2+t}{10}, p_{\mathsf{exp}}\}, \end{cases} \quad (47)$$

be the greedy policy with a user-specified exploration probability $p_{\mathsf{exp}} \in (0,1)$. The long-term dependencies in this greedy policy is obviously controlled by p_{exp}: a small exploration probability will make the policy (thus, the corresponding $\mathcal{Q}$-functions) more history-dependent. Since the exact computation of $(\mathcal{Q}_t^\pi)_{t\in\mathbb{N}}$ is highly intractable for POMDPs, we use (empirical) mean-squared temporal difference (MSTD) [2] as a surrogate loss.

Example 1 (Short-term memory). We first consider the performance of Rec-TD with learning rate $\eta = 0.05$, discount factor $\gamma = 0.9$ and RNNs with various choices of network width m. For $p_{\mathsf{exp}} = 0.8$, the performance of Rec-TD is demonstrated in Figure 2. Consistent with the theoretical results in Theorem 5.4, Rec-TD (1) achieves smaller error with larger network width m, (2) requires smaller deviation from the random initialization $\Theta(0)$, which is known as the *lazy training* phenomenon.

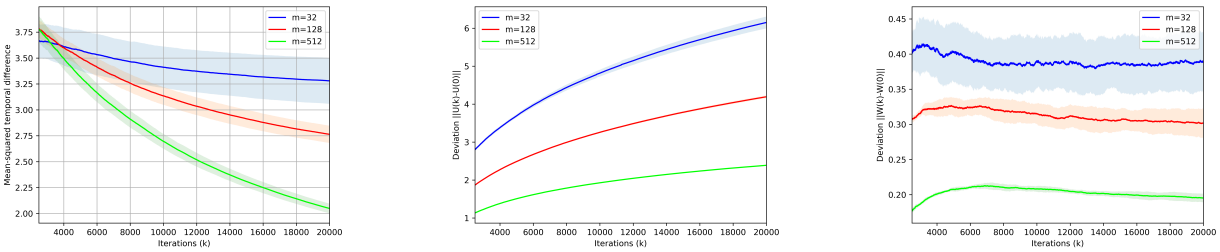
Example 2 (Long-term memory). In the second example, we consider the same POMDP with same random samples, and an RNN with the same neural network initialization. The exploration probability is reduced to $p_{\mathsf{exp}} = 0.25$, which leads to longer dependency on the history. This impact can be observed in Figure 3c, which implies a larger spectral radius compared to Example 1 (in comparison with Figure 2c).

[2] the empirical mean of independently sampled $\left\{\frac{1}{k}\sum_{s<k}\hat{\mathcal{R}}_T^{\mathsf{TD}}(\Theta(s)) : k \in \mathbb{N}\right\}$ where $\hat{\mathcal{R}}_T^{\mathsf{TD}}(\Theta(k)) = \sum_{t=0}^{T-1}\gamma^t\delta_t^2(\bar{Z}_t^k;\Theta(k))$.



(a) Mean-squared TD, $T = 8$ (b) $\frac{1}{m}\sum_i \|U_i(k) - U_i(0)\|$, $T = 8$. (c) $\frac{1}{m}\sum_i |W_{ii}(k) - W_{ii}(0)|$, $T = 8$.

Figure 2: Mean-squared TD and (mean) parameter deviation under Rec-TD for the case $p_{\mathsf{min}} = 0.8$ and $\gamma = 0.9$. The mean curve and confidence intervals (90%) stem from 5 trials.



(a) Mean-squared TD, $T = 8$ (b) $\frac{1}{m}\sum_i \|U_i(k) - U_i(0)\|$, $T = 8$. (c) $\frac{1}{m}\sum_i |W_{ii}(k) - W_{ii}(0)|$, $T = 8$.

Figure 3: Mean-squared TD and (mean) parameter deviation under Rec-TD for the case $p_{\mathsf{min}} = 0.25$ and $\gamma = 0.9$. The mean curve and confidence intervals (90%) stem from 5 trials.

In Figure 4, we investigate the impact of the truncation level T on the MSTD performance with $p_{\mathsf{exp}} = 0.25$, which implies long-term dependency, for an RNN with $m = 256$ units. Increasing T implies a larger MSTD due to long-term dependencies, validating the theoretical results.

D Policy Gradients under Partial Observability

In this section, we will provide basic results for policy gradients under POMDPs, which is critical to develop the natural policy gradient method for POMDPs.

Proposition D.1. *Let $\pi' \in \Pi_{\mathsf{NM}}$ be an admissible policy, and let $\bar{Z}_T \sim P_T^{\pi',\mu}$. Then, for any $t < T$, conditional distribution of S_t given $\bar{Z}_t$ is independent of π'. Furthermore, for any $\pi \in \Pi_{\mathsf{NM}}$, the conditional distribution of $r(S_t, A_t) + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})$ given $\bar{Z}_t$ is independent of π'.*

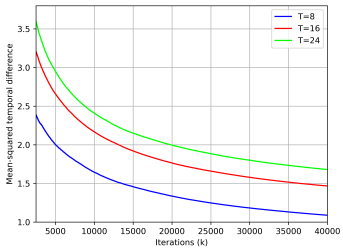


Figure 4: MSTD performance with $m = 256$ with various sequence lengths T with $p_{\mathsf{exp}} = 0.25$. Increasing T implies larger MSTD.

Proof of Prop. D.1. Let the belief at time $t \in \mathbb{N}$ be defined as

$$b_t(s) := \mathbb{P}(S_t = s|\bar{Z}_t). \tag{48}$$

For any non-stationary admissible policy π, the belief function is policy-independent. To see this, note that

$$\begin{aligned}\mathbb{P}(S_t = s_t, \bar{Z}_t = \bar{z}_t) &= \sum_{(s_0,\dots,s_{t-1})\in\mathbb{S}^t} \mathbb{P}(S_0 = s_0|Y_0 = y)\pi_0(a_0|z_0)\prod_{k=0}^{t-1}\mathcal{P}(s_{k+1}|s_k,a_k)\phi(y_{k+1}|s_{k+1})\pi_{k+1}(a_{k+1}|z_{k+1}),\\ &= \left(\prod_{k=0}^{t}\pi_k(a_k|z_k)\right)\sum_{(s_0,\dots,s_{t-1})\in\mathbb{S}^t} \mathbb{P}(S_0 = s_0|Y_0 = y)\prod_{k=0}^{t-1}\mathcal{P}(s_{k+1}|s_k,a_k)\phi(y_{k+1}|s_{k+1}),\end{aligned}$$

since $\prod_{k=0}^{t}\pi_k(a_k|z_k)$ does not depend on the summands $(s_0,\dots,s_{t-1})$ – note that we use the notation $\mathcal{P}(s_{k+1}|s_k,a_k) := \mathcal{P}(s_k,a_k,\{S_{k+1} = s_{k+1}\})$ and $\phi(y_k|s_k) := \phi(s_k,\{Y_k = y_k\})$. Thus,

$$b_t(s_t) = \frac{\sum_{(s_0,\dots,s_{t-1})\in\mathbb{S}^t}\mathbb{P}(S_0 = s_0|Y_0 = y)\prod_{k=0}^{t-1}\mathcal{P}(s_{k+1}|s_k,a_k)\phi(y_{k+1}|s_{k+1})}{\sum_{(s'_0,\dots,s'_{t-1},s'_t)\in\mathbb{S}^{t+1}}\mathbb{P}(S_0 = s'_0|Y_0 = y)\prod_{k=0}^{t-1}\mathcal{P}(s'_{k+1}|s'_k,a_k)\phi(y_{k+1}|s'_{k+1})},$$

independent of π. As such, we have

$$\begin{aligned}\mathbb{E}^{\pi'}[r_t + \gamma\mathcal{V}^{\pi}(Z_{t+1})|\bar{Z}_t] &= \sum_{s\in\mathbb{S}} b_t(s)\mathbb{E}^{\pi'}[r_t + \gamma\mathcal{V}^{\pi}_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t, S_t = s],\\ &= \sum_{s_t,s_{t+1}\in\mathbb{S}}\sum_{y\in\mathbb{Y}} b_t(s_t)\left(r(s_t,A_t) + \gamma\mathcal{P}(s_{t+1}|s_t,A_t)\phi(y|s_{t+1})\mathcal{V}^{\pi}_{t+1}(Z_t,y_{t+1})\right),\\ &= \mathbb{E}[r_t + \gamma\mathcal{V}^{\pi}_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t],\end{aligned}$$

in other words, the conditional distribution of $r(S_t,A_t) + \gamma\mathcal{V}^{\pi}_{t+1}(Z_{t+1})$ given $\{\bar{Z}_t = \bar{z}_t\}$ is independent of π'. We also know from Prop. B.3 that

$$\mathbb{E}^{\pi'}[r_t + \gamma\mathcal{V}^{\pi}_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t] = \mathbb{E}[r_t + \gamma\mathcal{V}^{\pi}_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t] = \mathcal{Q}^{\pi}_t(\bar{z}_t).$$

□

The next result generalizes the policy gradient theorem to POMDPs. We note that there is an extension of REINFORCE-type policy gradient for POMDPs in Wierstra et al. (2010). The following result is a different and improved version as it ① provides a variance-reduced unbiased estimate of the policy gradient for POMDPs, and more importantly ② yields the compatible function approximation (Prop. 6.1) that yields natural policy gradient (NPG) for POMDPs.

Proposition D.2 (Policy gradient – POMDPs)**.** *For any $\Phi \in \mathbb{R}^{m(d+1)}$, we have*

$$\nabla_\Phi\mathcal{V}^{\pi^\Phi}(\mu) = \mathbb{E}^{\pi^\Phi}_{\mu}\left[\sum_{t=0}^{\infty}\gamma^t\cdot\mathcal{Q}^{\pi^\Phi}_t(Z_t,A_t)\cdot\nabla_\Phi\ln\pi^\Phi_t(A_t|Z_t)\right], \tag{49}$$

for any $\mu \in \Delta(\mathbb{Y})$.

Proof of Prop. D.2. For any $t \in \mathbb{N}$, we have

$$\mathcal{V}^{\pi^\Phi}_t(z_t) = \sum_{a_t}\pi^\Phi_t(a_t|z_t)\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t), \tag{50}$$

by Prop. B.3. Thus, we obtain

$$\begin{aligned}\nabla\mathcal{V}^{\pi^\Phi}_t(z_t) &= \sum_{a_t}\pi^\Phi_t(a_t|z_t)\nabla\ln\pi^\Phi_t(a_t|z_t)\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t) + \sum_{a_t}\pi^\Phi_t(a_t|z_t)\nabla\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t),\\ &= \mathbb{E}^{\pi^\Phi}[\nabla\ln\pi^\Phi_t(A_t|Z_t)\mathcal{Q}^{\pi^\Phi}_t(Z_t,A_t) + \nabla\mathcal{Q}^{\pi^\Phi}_t(Z_t,A_t)|Z_t = z_t].\end{aligned} \tag{51}$$

Now, note that

$$\begin{aligned}\mathcal{Q}_t^{\pi^\Phi}(z_t, a_t) &= \mathbb{E}[r(S_t, A_t) + \gamma \mathcal{V}_{t+1}^{\pi^\Phi}(Z_{t+1}) | \bar{Z}_t = (z_t, a_t)], \\ &= \sum_{s_t} b_t(s_t) \left(r(s_t, a_t) + \gamma \sum_{s_{t+1}} \mathcal{P}(s_{t+1}|s_t, a_t) \sum_{y_{t+1}} \phi(y_{t+1}|s_{t+1}) \mathcal{V}_{t+1}^{\pi^\Phi}(z_{t+1}) \right),\end{aligned}$$

where $z_{t+1} = (z_t, a_t, y_{t+1})$. As a consequence of Prop. D.1, we have $\nabla_\Phi \sum_{s_t} b_t(s_t) r(s_t, a_t) = 0$, and also

$$\begin{aligned}\nabla_\Phi \mathcal{Q}_t^{\pi^\Phi}(z_t, a_t) &= \gamma \sum_{s_t} b_t(s_t) \sum_{s_{t+1}} \mathcal{P}(s_{t+1}|s_t, a_t) \sum_{y_{t+1}} \phi(y_{t+1}|s_{t+1}) \nabla_\Phi \mathcal{V}_{t+1}^{\pi^\Phi}(z_{t+1}), \\ &= \gamma \mathbb{E}[\nabla \ln \pi_{t+1}^\Phi(A_{t+1}|Z_{t+1}) \mathcal{Q}_{t+1}^{\pi^\Phi}(Z_{t+1}, A_{t+1}) + \nabla_\Phi \mathcal{Q}_{t+1}^{\pi^\Phi}(Z_{t+1}, A_{t+1}) | \bar{Z}_t = (z_t, a_t)], \\ &= \gamma \mathbb{E}^{\pi^\Phi}\Big[\sum_{k=t+1}^{\infty} \gamma^{k-t-1} \nabla_\Phi \ln \pi_k^\Phi(A_k|Z_k) \mathcal{Q}_k^{\pi^\Phi}(Z_k, A_k) \Big| \bar{Z}_t = (z_t, a_t) \Big].\end{aligned}$$

Using the above recursive formula for $\nabla_\Phi \mathcal{Q}_t^{\pi^\Phi}$ along with the law of iterated expectations in equation 51, we obtain

$$\nabla_\Phi \mathcal{V}_t^{\pi^\Phi}(z_t) = \mathbb{E}^{\pi^\Phi}\Big[\sum_{k=t}^{\infty} \gamma^{k-t} \nabla_\Phi \ln \pi_k^\Phi(A_k|Z_k) \mathcal{Q}_k^{\pi^\Phi}(Z_k, A_k) \Big| Z_t = z_t \Big]. \tag{52}$$

Since we have $\mathcal{V}^\pi := \mathcal{V}_0^\pi$, and also $\nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu) = \nabla_\Phi \sum_{z_0} \mu(z_0) \mathcal{V}^{\pi^\Phi}(z_0) = \sum_{z_0} \mu(z_0) \nabla_\Phi \mathcal{V}^{\pi^\Phi}(z_0)$ by the linearity of gradient, we conclude the proof.

Note on the baseline. Similar to the case of fully-observable MDPs, adding a baseline $q_t^{\pi^\Phi}(z_t)$ to the $\mathcal{Q}$-function does not change the policy gradients since $\sum_a \pi_t(a|z_t) \nabla \ln \pi_t^\Phi(a|z_t) q_t^{\pi^\Phi}(z_t) = q_t^{\pi^\Phi}(z_t) \sum_a \nabla \pi_t^\Phi(a|z_t) = q_t^{\pi^\Phi}(z_t) \nabla \sum_a \pi_t^\Phi(a|z_t) = 0$. Thus, we also have

$$\nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu) = \mathbb{E}_\mu^{\pi^\Phi}\left[\sum_{t=0}^{\infty} \gamma^t \mathcal{A}_t^{\pi^\Phi}(Z_t, A_t) \nabla_\Phi \ln \pi_t^\Phi(A_t|Z_t) \right], \tag{53}$$

which uses $q_t^{\pi^\Phi} = \mathcal{V}_t^{\pi^\Phi}$ as the baseline, akin to the fully-observable case. □

The following result extends the compatible function approximation theorem in Kakade (2001) to POMDPs.

Proof of Prop. 6.1. The proof is identical to Kakade (2001). By first-order condition for optimality, we have

$$2\mathbb{E}_\mu^{\pi^\Phi} \sum_{t=0}^{\infty} \gamma^t \nabla \ln \pi_t^\Phi(A_t|Z_t) \left(\nabla^\top \ln \pi_t^\Phi(A_t|Z_t) \omega^\star - \mathcal{A}_t^{\pi^\Phi}(\bar{Z}_t) \right) = 2 \left(G_\mu(\Phi) \omega^\star - \nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu) \right) = 0,$$

which concludes the proof. □

E Theoretical Analysis of Rec-NPG

First, we prove structural results for RNNs in the kernel regime, which will be key in the analysis later.

E.1 Log-Linearization of SOFTMAX Policies Parameterized by RNNs

The key idea behind the neural tangent kernel (NTK) analysis is linearization around the random initialization. To that end, let

$$F_t^{\mathsf{Lin}}(\bar{z}_t; \Theta) := \langle \nabla F_t(\bar{z}_t; \Theta(0)), \Theta - \Theta(0) \rangle, \tag{54}$$

for any $\Theta \in \mathbb{R}^{m(d+1)}$. We define the log-linearized policy as follows:

$$\tilde{\pi}_t^{\Phi}(a|z_t) := \frac{\exp(F_t^{\mathsf{Lin}}(z_t, a; \Phi))}{\sum_{a' \in \mathbb{A}} \exp(F_t^{\mathsf{Lin}}(z_t, a'; \Phi))}, \ t \in \mathbb{N}. \tag{55}$$

The first result bounds the Kullback-Leibler divergence between π_t^{Φ} and its log-linearized version $\tilde{\pi}_t^{\Phi}$. In the case of FNNs with ReLU activation functions, a similar result was presented in Cayci et al. (2024b). The following result extends this idea to (i) RNNs, and (ii) smooth activation functions.

Proposition E.1 (Log-linearization error)**.** *For any $t \in \mathbb{N}$ and $(z_t, a) \in (\mathbb{Y} \times \mathbb{A})^{t+1}$, we have*

$$\sup_{(z_t,a) \in (\mathbb{Y} \times \mathbb{A})^{t+1}} \left| \ln \frac{\tilde{\pi}_t^{\Phi}(a|z_t)}{\pi_t^{\Phi}(a|z_t)} \right| \leq \frac{6}{\sqrt{m}} \left(\Lambda_t^2 \varrho_2 + \chi_t \varrho_1 \right) \|\Phi - \Phi(0)\|_2^2, \tag{56}$$

for any $t \in \mathbb{N}$. Consequently, we have $\pi_t(\cdot|z_t) \ll \tilde{\pi}_t(\cdot|z_t)$ and $\tilde{\pi}_t(\cdot|z_t) \ll \pi_t(\cdot|z_t)$, and

$$\max \left\{ \mathscr{D}_{\mathsf{KL}}(\pi_t^{\Phi}(\cdot|z_t) \| \tilde{\pi}_t^{\Phi}(\cdot|z_t)), \mathcal{D}_{\mathsf{KL}}(\tilde{\pi}_t^{\Phi}(\cdot|z_t) \| \pi_t^{\Phi}(\cdot|z_t)) \right\} \leq \frac{6}{\sqrt{m}} \left(\Lambda_t^2 \varrho_2 + \chi_t \varrho_1 \right) \|\Phi - \Phi(0)\|_2^2, \tag{57}$$

for all $z_t \in (\mathbb{Y} \times \mathbb{A})^{t+1}$ and $t \in \mathbb{N}$.

Proof. Fix $(z_t, a) \in (\mathbb{Y} \times \mathbb{A})^{t+1}$. By the log-sum inequality Cover & Thomas (2006), we have

$$\ln \frac{\sum_a \exp(F_t^{\mathsf{Lin}}(z_t, a; \Phi))}{\sum_a \exp(F_t(z_t, a; \Phi))} \leq \sum_{a \in \mathbb{A}} \tilde{\pi}_t^{\Phi}(a|z_t) \left(F_t^{\mathsf{Lin}}(z_t, a; \Phi) - F_t(z_t, a; \Phi) \right).$$

Using the same argument, we obtain

$$\left| \ln \frac{\sum_a \exp(F_t^{\mathsf{Lin}}(z_t, a; \Phi))}{\sum_a \exp(F_t(z_t, a; \Phi))} \right| \leq \sum_{a \in \mathbb{A}} \left(\tilde{\pi}_t^{\Phi}(a|z_t) + \pi_t^{\Phi}(a|z_t) \right) \cdot \left| F_t^{\mathsf{Lin}}(z_t, a; \Phi) - F_t(z_t, a; \Phi) \right|. \tag{58}$$

Thus, we have

$$\left| \ln \frac{\tilde{\pi}_t^{\Phi}(a|z_t)}{\pi_t^{\Phi}(a|z_t)} \right| \leq (1 + \tilde{\pi}_t^{\Phi}(a|z_t) + \pi_t^{\Phi}(a|z_t)) \left| F_t^{\mathsf{Lin}}(z_t, a; \Phi) - F_t(z_t, a; \Phi) \right|.$$

By using Lemma B.1, we have $\sup_{\bar{z}_t \in (\mathbb{Y} \times \mathbb{A})^{t+1}} \left| F_t^{\mathsf{Lin}}(\bar{z}_t'; \Phi) - F_t(\bar{z}_t'; \Phi) \right| \leq \frac{2}{\sqrt{m}} (\Lambda_t^2 \varrho_2 + \chi_t \varrho_1) \|\Phi - \Phi(0)\|_2^2$. By using the last two inequalities together, and noting that $1 + \tilde{\pi}_t^{\Phi}(a|z_t) + \pi_t^{\Phi}(a|z_t) \leq 3$, we conclude that

$$\left| \ln \frac{\tilde{\pi}_t^{\Phi}(a|z_t)}{\pi_t^{\Phi}(a|z_t)} \right| \leq \frac{6}{\sqrt{m}} (\Lambda_t^2 \varrho_2 + \chi_t \varrho_1) \|\Phi - \Phi(0)\|_2^2.$$

Since the righthand-side of the above inequality is independent of (z_t, a), we deduce that the result holds for all (z_t, a), thus concluding the proof. □

The following result will be important in establishing the Lyapunov drift analysis of Rec-NPG.

Proposition E.2 (Smoothness of $\ln \tilde{\pi}_t^{\Phi}(a|z_t)$)**.** *For any $t \in \mathbb{N}$, we have*

$$\sup_{(z_t,a) \in (\mathbb{Y} \times \mathbb{A})^{t+1}} \|\nabla \ln \tilde{\pi}_t^{\Phi}(a|z_t) - \nabla \ln \tilde{\pi}_t^{\Phi'}(a|z_t)\|_2 \leq L_t^2 \|\Phi - \Phi'\|_2,$$

for any $\Phi, \Phi' \in \mathbb{R}^{m(d+1)}$.

Proof. Consider a general log-linear parameterization

$$p_\theta(x) \propto \exp(\phi_x^\top \theta), \ x \in \mathbf{X}.$$

Then, if $\sup_{x\in\mathbf{X}} \|\phi_x\|_2 \le B < \infty$, then $\theta \mapsto \ln p_\theta(x)$ has B^2-Lipschitz continuous gradients for each $x \in \mathbf{X}$ Agarwal et al. (2020). The remaining part is to prove a uniform upper bound for $\|\nabla_\Phi F_t(\bar{z}_t;\Phi(0))\|_2$. To that end, notice that

$$\nabla_{\Phi_i} F_t(\bar{z}_t;\Phi(0)) = \frac{1}{\sqrt{m}} c_i \nabla H_t^{(i)}(\bar{z}_t;\Phi(0)),\ \bar{z}_t \in (\mathbb{Y}\times\mathbb{A})^{t+1}, i\in[m].$$

From the local Lipschitz continuity result in Lemma B.1, we have $\sup_{\bar{z}_t:\max_{j\le t}\|(y_j,a_j)\|_2\le 1} \|\nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t;\Phi(0))\|_2 \le L_t$ for any $i\in[m]$. Thus, for any $\bar{z}_t$, we have

$$\|\nabla_\Phi F_t(\bar{z}_t;\Phi(0))\|_2^2 = \frac{1}{m}\sum_{i=1}^m \|\nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t;\Phi(0))\|_2^2 \le L_t^2. \tag{59}$$

□

E.2 Theoretical Analysis of Rec-NPG

For any $\pi \in \Pi_{\mathsf{NM}}$, we define the potential function as

$$\mathscr{L}(\pi) := \mathbb{E}_\mu^{\pi^\star}\left[\sum_{t=0}^{T-1}\gamma^t \mathscr{D}_{\mathsf{KL}}\left(\pi_t^\star(\cdot|Z_t)\|\pi_t(\cdot|Z_t)\right)\right]. \tag{60}$$

Then, we have the following drift inequality.

Proposition E.3 (Drift inequality)**.** *For any $n\in\mathbb{N}$, the drift can be bounded as follows:*

$$\begin{aligned}\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \le -\eta_{\mathsf{npg}}(\mathcal{V}^{\pi^\star}(\mu) - \mathcal{V}^{\pi^{\Phi(n)}}(\mu)) &\underbrace{-\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\left[\sum_{t=0}^{T-1}\gamma^t\left(\nabla^\top \ln \pi_t^{\Phi(n)}(A_t|Z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right)\right]}_{①}\\ &\underbrace{+\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\sum_{t=T}^{\infty}\gamma^t\mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)}_{②}\underbrace{-\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\left(\nabla\ln\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t) - \nabla\ln\pi_t^{\Phi(n)}(A_t|Z_t)\right)^\top\omega_n}_{③}\\ &+\frac{1}{2}\eta_{\mathsf{npg}}^2\|\rho\|_2^2\sum_{t=0}^{T-1}\gamma^t L_t^2 + \frac{12\|\rho\|_2^2}{\sqrt{m}}\sum_{t=0}^{T-1}\gamma^t(\Lambda_t^2\varrho_2 + \chi_t\varrho_1).\end{aligned}$$

Proof. First, note that the drift can be expressed as

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) = \mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\sum_{a\in\mathbb{A}}\pi_t^\star(A_t|Z_t)\ln\frac{\pi_t^{\Phi(n)}(A_t|Z_t)}{\pi_t^{\Phi(n+1)}(A_t|Z_t)}.$$

Then, with a log-linear transformation,

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) = \mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\sum_{a\in\mathbb{A}}\pi_t^\star(A_t|Z_t)\left(\ln\frac{\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t)}{\tilde{\pi}_t^{\Phi(n+1)}(A_t|Z_t)} + \ln\frac{\pi_t^{\Phi(n)}(A_t|Z_t)}{\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t)} + \ln\frac{\tilde{\pi}_t^{\Phi(n+1)}(A_t|Z_t)}{\pi_t^{\Phi(n+1)}(A_t|Z_t)}\right).$$

By using the log-linearization bound in Prop. E.1 twice in the above inequality, we obtain

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \le \mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\sum_{a\in\mathbb{A}}\pi_t^\star(A_t|Z_t)\ln\frac{\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t)}{\tilde{\pi}_t^{\Phi(n+1)}(A_t|Z_t)} + \frac{12}{\sqrt{m}}\sum_{t=0}^{T-1}\gamma^t(\Lambda_t^2\varrho_2 + \chi_t\varrho_1)\|\rho\|_2^2. \tag{61}$$

By the smoothness result in Prop. E.2, we have

$$|\ln \tilde{\pi}_t^{\Phi(n+1)}(a_t|z_t) - \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)(\Phi(n+1) - \Phi(n))| \leq \frac{1}{2} L_t^4 \|\Phi(n+1) - \Phi(n)\|_2^2.$$

Thus, we obtain

$$-\eta_{\mathsf{npg}}^2 L_t^4 \|\rho\|_2^2 \leq -\eta_{\mathsf{npg}}^2 L_t^4 \|\omega_n\|_2^2 \leq -\ln \frac{\tilde{\pi}_t^{\Phi(n)}(a_t|z_t)}{\tilde{\pi}_t^{\Phi(n+1)}(a_t|z_t)} - \eta_{\mathsf{npg}} \nabla^\top \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)\omega_n,$$

because of the max-norm gradient clipping that yields $\|\omega_n\|_2 \leq \|\rho\|_2$ and $\Phi(n+1) = \Phi(n) + \eta_{\mathsf{npg}}\omega_n$ for any $n \in \mathbb{N}$. Using this in equation 61, we get

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \leq -\eta_{\mathsf{npg}} \mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \nabla^\top \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)\omega_n + \frac{12}{\sqrt{m}} \sum_{t=0}^{T-1} \gamma^t (\Lambda_t^2 \varrho_2 + \chi_t \varrho_1) \|\rho\|_2^2 + \frac{1}{2}\eta_{\mathsf{npg}}^2 L_t^4 \|\rho\|_2^2. \tag{62}$$

An important technical result that will be useful in our analysis is the *pathwise* performance difference lemma, which was originally developed in Kakade & Langford (2002) for fully-observable MDPs.

Lemma E.4 (Pathwise Performance Difference Lemma)**.** *Let $\Phi, \Phi' \in \mathbb{R}^{m(d+1)}$ be two parameters. Then, we have*

$$\mathcal{V}^{\pi^{\Phi'}}(\mu) - \mathcal{V}^{\pi^{\Phi}}(\mu) = \mathbb{E}_\mu^{\pi^{\Phi'}} \sum_{t=0}^{\infty} \gamma^t \mathcal{A}_t^{\pi^{\Phi}}(Z_t, A_t).$$

The proof of Lemma E.4 is an extension of Agarwal et al. (2020) to non-stationary policies, and can be found at the end of this subsection.

Using Lemma E.4 in equation 62, we obtain

$$\begin{aligned}\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \leq -\eta_{\mathsf{npg}}(\mathcal{V}^{\pi^\star}(\mu) - \mathcal{V}^{\pi^{\Phi(n)}}(\mu)) - \eta_{\mathsf{npg}} \mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \left(\nabla^\top \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t) \right) \\ + \eta_{\mathsf{npg}} \mathbb{E}_\mu^{\pi^\star} \sum_{t=T}^{\infty} \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t) + \frac{12}{\sqrt{m}} \sum_{t=0}^{T-1} \gamma^t (\Lambda_t^2 \varrho_2 + \chi_t \varrho_1) \|\rho\|_2^2 + \frac{1}{2}\eta_{\mathsf{npg}}^2 L_t^4 \|\rho\|_2^2.\end{aligned} \tag{63}$$

Finally, we replace the term $\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)$ with $\nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)$ by including the corresponding error term, and conclude the proof by considering the telescoping sum, and noting that $\mathscr{L}(\pi^{\Phi(0)}) = \log|\mathbb{A}|$ since $F_t(\cdot; \Phi(0)) = 0$ by symmetric initialization. □

Proof of Theorem 6.3. We prove Theorem 6.3 by bounding the numbered terms in Prop. E.3.

Bounding ① in Prop. E.3. Recall that $p_T(\gamma) = \sum_{t<T} \gamma^t$. Then, by using Jensen's inequality,

$$\begin{aligned}\mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \left(\nabla^\top \ln \pi_t^{\Phi(n)}(A_t|Z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t) \right) &\leq \sqrt{p_T(\gamma) \mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \left| \nabla^\top \ln \pi_t^{\Phi(n)}(A_t|Z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t) \right|^2}, \\ &=: \sqrt{p_T(\gamma)} \sqrt{\kappa \varepsilon_{\mathsf{cfa}}^T(\Phi(n), \omega_n)},\end{aligned}$$

where κ yields a change-of-measure argment from $P_T^{\pi^\star,\mu}$ to $P_T^{\pi^{\Phi(n)},\mu}$.

Bounding ② in Prop. E.3. $\sup_{s,a} |r(s,a)| \leq r_\infty$, therefore $|\mathcal{A}_t^\pi(\bar{z}_t)| \leq \frac{2r_\infty}{1-\gamma}$ for any $t \in \mathbb{N}, \bar{z}_t \in (\mathbb{Y} \times \mathbb{A})^{t+1}$, and $\pi \in \Pi_{\mathsf{NM}}$.

Bounding ③ in Prop. E.3. For any $t \in \mathbb{N}$, Cauchy-Schwarz inequality implies

$$\left(\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t) \right)^\top \omega_n \leq \|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \|\rho\|_2.$$

Recall that

$$\nabla \ln \tilde{\pi}_t^{\Phi}(a_t|z_t) = \nabla F_t(z_t, a_t; \Phi(0)) - \sum_{a'} \tilde{\pi}_t^{\Phi}(a'|z_t) \nabla F_t(z_t, a'; \Phi(0)),$$
$$\nabla \ln \pi_t^{\Phi}(a_t|z_t) = \nabla F_t(z_t, a_t; \Phi) - \sum_{a'} \pi_t^{\Phi}(a'|z_t) \nabla F_t(z_t, a'; \Phi).$$

First, from local β_t-Lipschitzness of $\Phi_i \mapsto \nabla H_t^{(i)}(\bar{z}_t; \Phi_i)$ for $\Phi \in \Omega_{\rho,m}$ by Lemma B.1, we have

$$\begin{aligned}\|\nabla_{\Phi_i} F_t(\bar{z}_t; \Phi(n)) - \nabla_{\Phi_i} F_t(\bar{z}_t; \Phi(0))\|_2 &= \frac{1}{\sqrt{m}} \|\nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t; \Phi_i(n)) - \nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t; \Phi_i(0))\|_2, \\ &\leq \frac{\beta_t \|\rho\|_2}{m},\end{aligned}$$

for any $n \in \mathbb{N}$ since $\max_i \|\Phi_i(n) - \Phi_i(0)\|_2 \leq \frac{\|\rho\|_2}{\sqrt{m}}$ by max-norm projection. Thus,

$$\|\nabla_{\Phi} F_t(\bar{z}_t; \Phi(n)) - \nabla_{\Phi} F_t(\bar{z}_t; \Phi(0))\|_2 \leq \frac{\beta_t \|\rho\|_2}{\sqrt{m}}, \ t \in \mathbb{N}. \tag{64}$$

Thus,

$$\begin{aligned}\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{\beta_t \|\rho\|_2}{\sqrt{m}} &+ \sum_a |\pi_t^{\Phi(n)}(a|z_t) - \tilde{\pi}_t^{\Phi(n)}(a|z_t)| \|\nabla F_t(\bar{z}_t; \Phi(0))\|_2 \\ &+ \sum_a \pi_t^{\Phi(n)}(a|z_t) \|\nabla F_t(z_t, a; \Phi(n)) - \nabla F_t(z_t, a; \Phi(0))\|_2.\end{aligned}$$

From equation 59, we have

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{2\beta_t \|\rho\|_2}{\sqrt{m}} + 2L_t \mathscr{D}_{\mathsf{TV}} \left(\pi_t^{\Phi(n)}(\cdot|z_t) \| \tilde{\pi}_t^{\Phi(n)}(\cdot|z_t) \right),$$

where $\mathscr{D}_{\mathsf{TV}}$ denotes the total-variation distance between two probability measures. By Pinsker's inequality Cover & Thomas (2006), we obtain

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{2\beta_t \|\rho\|_2}{\sqrt{m}} + \sqrt{2} L_t \sqrt{\mathscr{D}_{\mathsf{KL}} \left(\pi_t^{\Phi(n)}(\cdot|z_t) \| \tilde{\pi}_t^{\Phi(n)}(\cdot|z_t) \right)}. \tag{65}$$

By the log-linearization result in Prop. E.1, we have

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{2\beta_t \|\rho\|_2}{\sqrt{m}} + \sqrt{12} L_t \|\rho\|_2 \sqrt{\frac{\Lambda_t^2 \varrho_2 + \chi_t \varrho_1}{\sqrt{m}}}. \tag{66}$$

Thus, we have

$$\left(\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t) \right)^{\top} \omega_n \leq \|\rho\|_2^2 \left(\frac{2\beta_t}{\sqrt{m}} + \sqrt{12} L_t \frac{\sqrt{\Lambda_t \varrho_2 + \chi_t \varrho_1}}{m^{1/4}} \right).$$

□

Proof of Lemma E.4. For any $y_0 \in \mathbb{Y}$, we have:

$$\begin{aligned}\mathcal{V}^{\pi'}(y_0) - \mathcal{V}^{\pi}(y_0) &= \mathbb{E}_{\mu}^{\pi'} \Big[\sum_{t=0}^{\infty} \gamma^t r_t \Big| Z_0 = y_0 \Big] - \mathcal{V}^{\pi}(y_0), \\ &= \mathbb{E}_{\mu}^{\pi'} \Big[\sum_{t=0}^{\infty} \gamma^t \Big(r_t + \mathcal{V}_t^{\pi}(Z_t) - \mathcal{V}_t^{\pi}(Z_t) \Big) \Big| Z_0 = y_0 \Big] - \mathcal{V}^{\pi}(y_0), \\ &= \mathbb{E}_{\mu}^{\pi'} \Big[\sum_{t=0}^{\infty} \gamma^t (r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1}) - \mathcal{V}_t^{\pi}(Z_t) \Big| Z_0 = y_0 \Big],\end{aligned}$$

where $r_t = r(S_t, A_t)$ and the last identity holds since

$$\sum_{t=0}^{\infty} \gamma^t \mathcal{V}_t^{\pi}(z_t) = \mathcal{V}_0^{\pi}(z_0) + \gamma \sum_{t=0}^{\infty} \gamma^t \mathcal{V}_{t+1}^{\pi}(z_{t+1}).$$

Then, letting $r_t = r(s_t, a_t)$ and by using law of iterated expectations,

$$\mathcal{V}^{\pi'}(y_0) - \mathcal{V}^{\pi}(y_0) = \mathbb{E}_{\mu}^{\pi'}\Big[\sum_{t=0}^{\infty} \gamma^t \Big(\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})|\bar{Z}_t, S_t] - \mathcal{V}_t^{\pi}(Z_t)\Big)\Big| Z_0 = y_0\Big], \tag{67}$$

which holds because

$$\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}^{\pi}(Z_{t+1})|\bar{Z}_t] = \mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}^{\pi}(Z_{t+1})|\bar{Z}_t, Z_0].$$

The conditional expectation of $r_t + \gamma \mathcal{V}_{t+1}^{\pi}$ given $\{\bar{Z}_t = \bar{z}_t\}$ is independent of π':

$$\begin{aligned}
\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}^{\pi}(Z_{t+1})|\bar{Z}_t] &= \sum_{s \in \mathbb{S}} b_t(s) \mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})|\bar{Z}_t = \bar{z}_t, S_t = s], \\
&= \sum_{s_t, s_{t+1} \in \mathbb{S}} \sum_{y \in \mathbb{Y}} b_t(s_t) \left(r(s_t, A_t) + \gamma \mathcal{P}(s_{t+1}|s_t, A_t)\phi(y|s_{t+1}) \mathcal{V}_{t+1}^{\pi}(Z_t, y_{t+1})\right), \\
&= \mathbb{E}[r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})|\bar{Z}_t = \bar{z}_t],
\end{aligned}$$

based on Prop. D.1. We also know from Prop. B.3 that

$$\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})|\bar{Z}_t = \bar{z}_t] = \mathbb{E}[r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})|\bar{Z}_t = \bar{z}_t] = \mathcal{Q}_t^{\pi}(\bar{z}_t).$$

Using the above identity in equation 67, we obtain

$$\mathcal{V}^{\pi'}(y_0) - \mathcal{V}^{\pi}(y_0) = \mathbb{E}_{\mu}^{\pi'}\Big[\sum_{t=0}^{\infty} \gamma^t \Big(\mathcal{Q}_t^{\pi}(\bar{Z}_t) - \mathcal{V}^{\pi}(Z_t)\Big)\Big| Z_0 = y_0\Big], \tag{68}$$

which concludes the proof. □

Proof of Prop. 6.6. For any ω, we have

$$\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}}) \leq 2\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)}) + 2\sum_{t=0}^{\infty} \gamma^t (\mathcal{A}_t^{\pi^{\Phi(n)}}(Z_t, A_t) - \hat{\mathcal{A}}_t^{(n)}(Z_t, A_t))^2. \tag{69}$$

Let $\mathcal{G}_n := \sigma(\Phi(k), k \leq n)$ and $\mathcal{H}_n := \sigma(\bar{\Theta}^{(n)}, \Phi(k), k \leq n)$. Then, since

$$\varepsilon_{\mathsf{sgd},n} = \mathbb{E}[\ell_T(\omega_n; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n] - \inf_{\omega \in \mathcal{B}_{2,\infty}^{(m)}(0,\rho)} \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n],$$

we obtain

$$\mathbb{E}[\ell_T(\omega_n; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] \leq 2\mathbb{E}\Big[\inf_{\omega} \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n]\Big|\mathcal{G}_n\Big] + 2(\varepsilon_{\mathsf{td},n} + \varepsilon_{\mathsf{sgd},n}), \tag{70}$$

which uses the fact that $Var(X|\mathcal{G}_n) \leq \mathbb{E}[|X|^2|\mathcal{G}_n]$ for any square-integrable X. We also have

$$\inf_{\omega} \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n] \leq 2\inf_{\omega} \mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] + 2\sum_{t=0}^{\infty} \gamma^t (\mathcal{A}_t^{\pi^{\Phi(n)}}(Z_t, A_t) - \hat{\mathcal{A}}_t^{(n)}(Z_t, A_t))^2, \tag{71}$$

which further implies that

$$\mathbb{E}[\inf_{\omega} \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n]|\mathcal{G}_n] \leq 2\mathbb{E}[\inf_{\omega} \mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n]|\mathcal{G}_n] + 2\varepsilon_{\mathsf{td},n}.$$

Thus,

$$\mathbb{E}[\ell_T(\omega_n;\Phi(n),\mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] \le 4\mathbb{E}\Big[\inf_{\omega}\mathbb{E}[\ell_T(\omega;\Phi(n),\mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n]\Big|\mathcal{G}_n\Big] + 6\varepsilon_{\mathsf{td},n} + 2\varepsilon_{\mathsf{sgd},n}. \tag{72}$$

For any $\omega \in \mathcal{B}^{(m)}_{2,\infty}(0,\rho)$,

$$\begin{aligned}\mathbb{E}[\ell_T(\omega;\Phi(n),\mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] &\le \mathbb{E}[\sum_{t<T}\gamma^t(\nabla_\Phi^\top F_t(\bar{Z}_t;\Phi(n))\omega - \mathcal{Q}_t^{\pi^{\Phi(n)}}(\bar{Z}_t))^2|\mathcal{H}_n],\\ &\le 2\mathbb{E}[\sum_{t<T}\gamma^t(\nabla_\Phi^\top F_t(\bar{Z}_t;\Phi(0))\omega - \mathcal{Q}_t^{\pi^{\Phi(n)}}(\bar{Z}_t))^2 + (\nabla F_t(\bar{Z}_t;\Phi(n)) - \nabla F_t(\bar{Z}_t;\Phi(0))^\top\omega)^2|\mathcal{H}_n],\end{aligned}$$

which implies that

$$\begin{aligned}\inf_{\omega}\mathbb{E}[\ell_T(\omega;\Phi(n),\mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] &\le 2\varepsilon_{\mathsf{app},n} + 2\|\rho\|_2^2\mathbb{E}[\sum_{t<T}\gamma^t\|\nabla F_t(\bar{Z}_t;\Phi(n)) - \nabla F_t(\bar{Z}_t;\Phi(0)\|_2^2|\mathcal{H}_n],\\ &\le 2\varepsilon_{\mathsf{app},n} + \frac{2\|\rho\|_2^4}{m}\sum_{t<T}\gamma^t\beta_t^2,\end{aligned}$$

using equation 64. Hence,

$$\mathbb{E}[\ell_T(\omega_n;\Phi(n),\mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] \le \frac{8\|\rho\|_2^4}{m}\sum_{t<T}\gamma^t\beta_t^2 + 8\varepsilon_{\mathsf{app},n} + 6\varepsilon_{\mathsf{td},n} + 2\varepsilon_{\mathsf{sgd},n},$$

concluding the proof. □

Proof of Prop. 6.8. Under Assumption 6.7, consider $f_t^{(j)}(\bar{z}_t) := \mathbb{E}[\psi_t^\top(\bar{z}_t;\phi_0)\boldsymbol{v}^{(j)}(\phi_0)]$ for $\boldsymbol{v}^{(j)} \in \mathscr{H}_{\mathcal{J},\nu}$. Let

$$\omega_i^{(j)} := \frac{1}{\sqrt{m}}c_i\boldsymbol{v}^{(j)}(\Phi_i(0)),\ i = 1,2,\ldots,m, \tag{73}$$

for any $j \in \mathcal{J}$. Since $\|\omega^{(j)}\|_2 \le \|\nu\|_2$ and $\rho \succeq \nu$, we have

$$\inf_{\omega\in\mathcal{B}^{(m)}_{2,\infty}(0,\rho)}\left|\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega - f_t^{(j)}(\bar{z}_t)\right| \le \left|\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega^{(j)} - f_t^{(j)}(\bar{z}_t)\right|. \tag{74}$$

Thus, we aim to find a uniform upper bound for the second term over $j \in \mathcal{J}$. For each $\bar{z}_t$, we have

$$\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega^{(j)} = \frac{1}{m}\sum_{i=1}^m \nabla_{\Phi_i}^\top H_t^{(i)}(\bar{z}_t;\Phi_i(0))\boldsymbol{v}^{(j)}(\Phi_i(0)),$$

thus $\mathbb{E}[\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega^{(j)}] = f_t^{(j)}(\bar{z}_t)$. Furthermore, from Lemma B.1, since $\Phi(0) \in \Omega_{\rho,m}$ obviously, we have

$$\max_{1\le i\le m}\|\nabla_{\Phi_i}^\top H_t^{(i)}(\bar{z}_t;\Phi_i(0))\boldsymbol{v}^{(j)}(\Phi_i(0))\|_2 \le L_t\|\nu\|_2 \le L_t\|\rho\|_2,\ \text{a.s.}.$$

Thus, by McDiarmid's inequality Mohri et al. (2018), we have with probability at least $1-\delta$,

$$\sup_{j\in\mathcal{J}}\left|\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega^{(j)} - f_t^{(j)}(\bar{z}_t)\right| \le 2\mathrm{Rad}_m(G_t^{\bar{z}_t}) + L_t\|\rho\|_2\sqrt{\frac{\log(2/\delta)}{m}}, \tag{75}$$

for each $t < T$ and $\bar{z}_t$. By union bound,

$$\begin{aligned}\sup_{j\in\mathcal{J}}\max_{\bar{z}_t}\left|\nabla^\top F_t(\bar{z}_t;\Phi(0))\omega^{(j)} - f_t^{(j)}(\bar{z}_t)\right| &\le 2\max_{\bar{z}_t}\mathrm{Rad}_m(G_t^{\bar{z}_t}) + L_t\|\rho\|_2\sqrt{\frac{\log(2T|\mathbb{Y}\times\mathbb{A}|^{t+1}/\delta)}{m}}, &(76)\\ &\le 2\max_{0\le t<T}\max_{\bar{z}_t}\mathrm{Rad}_m(G_t^{\bar{z}_t}) + L_T\|\rho\|_2\sqrt{\frac{\log(2T|\mathbb{Y}\times\mathbb{A}|^{T}/\delta)}{m}}, &(77)\end{aligned}$$

simultaneously for all $t < T$ with probability $\geq 1 - \delta$. Therefore,

$$\inf_{\omega} \mathbb{E}_{\mu}^{\pi^{\Phi(n)}} \sum_{t<T} \gamma^t |\nabla^\top F_t(\bar{Z}_t; \Phi(0))\omega - f_t^{(j)}|^2 \leq \mathbb{E}_{\mu}^{\pi^{\Phi(n)}} \sum_{t<T} \gamma^t \sup_{j \in \mathcal{J}} |\nabla^\top F_t(\bar{Z}_t; \Phi(0))\omega^{(j)} - f_t^{(j)}|^2,$$

$$\leq \frac{1}{1-\gamma} \left(2 \max_{0 \leq t < T} \max_{\tilde{z}_t} \mathrm{Rad}_m(G_t^{\tilde{z}_t}) + L_T \|\rho\|_2 \sqrt{\frac{\log(2T|\mathbb{Y} \times \mathbb{A}|^T/\delta)}{m}} \right)^2 .$$

□