Generalization Bounds for (Wasserstein) Robust Optimization

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Abstract

(Distributionally) robust optimization has gained momentum in machine learning community recently, due to its promising applications in developing generalizable learning paradigms. In this paper, we derive generalization bounds for robust optimization and Wasserstein robust optimization for Lipschitz and piecewise Hölder smooth loss functions under both stochastic and adversarial setting, assuming that the underlying data distribution satisfies transportation-information inequalities. The proofs are built on new generalization bounds for variation regularization (such as Lipschitz or gradient regularization) and its connection with robustness.

1 Introduction

Consider the following stochastic optimization problem of seeking a function $f : \mathscr{X} \to \mathbb{R}$ from the hypothesis space $\mathscr{F}$ so as to minimize the expected loss:

$$\min_{f\in\mathscr{F}} \left\{\mathcal{L}^{\mathsf{true}}(f) := \mathbb{E}_{X\sim\mathbb{P}_{\mathsf{true}}}[f(X)]\right\}.$$

Here the data X follows some underlying distribution $\mathbb{P}_{\mathsf{true}}$ with support $\mathscr{X} \subset (\mathbb{R}^d, \|\cdot\|)$. In practice, $\mathbb{P}_{\mathsf{true}}$ is often replaced with its empirical estimate $\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\hat{x}^i}$, where $\boldsymbol{\delta}$ is the Dirac measure:

$$\min_{f\in\mathscr{F}} \left\{\mathcal{L}_n^{\mathsf{erm}}(f) := \mathbb{E}_{X\sim\mathbb{P}_n}[f(X)]\right\}.$$

The discrepancy between $\mathbb{P}_{\mathsf{true}}$ and $\mathbb{P}_n$ may be resulted from

(i) The sampling error of $\mathbb{P}_n$ when data are independently drew from $\mathbb{P}_{\mathsf{true}}$;
(ii) Non-i.i.d. data, e.g., x^i's are samples from a Markov chain with a stationary distribution $\mathbb{P}_{\mathsf{true}}$;
(iii) The deploying environment is (adversarially) different from the data-collecting environment.

Such discrepancy largely contributes to the poor generalization behavior of empirical risk minimization. To tackle this issue, *robust optimization* [5, 38, 39] studies

$$\min_{f\in\mathscr{F}} \left\{\mathcal{L}_{n,p}^{\mathsf{ro}}(f;\varrho) := \max_{\{x^i\}_{i=1}^n\in\mathscr{X}} \left\{\frac{1}{n}\sum_{i=1}^n f(x^i) : \left(\frac{1}{n}\sum_{i=1}^n \|x^i - \hat{x}^i\|^p\right)^{\frac{1}{p}} \le \varrho\right\}\right\}. \tag{RO}$$

Here $p \in [1,\infty]$; the uncertainty set, denoted by $\mathscr{X}_p(\varrho)$, describes the data perturbations to hedge against; and the radius $\varrho > 0$ specifies the level of robustness. In a similar spirit, *Wasserstein robust optimization* [16, 8, 20] studies

$$\min_{f\in\mathscr{F}} \left\{\mathcal{L}_{n,p}^{\mathsf{wo}}(f;\varrho) := \max_{\mathbb{P}} \left\{\mathbb{E}_{x\sim\mathbb{P}}[f(x)] : \mathcal{W}_p(\mathbb{P},\mathbb{P}_n) \le \varrho\right\}\right\}. \tag{WO}$$

Here the distributional uncertainty set, denoted by $\mathscr{P}_p(\varrho)$, consists of all probability distributions that are close to $\mathbb{P}_n$ in p-Wasserstein distance. When $p = \infty$, (WO) and (RO) coincide. Both problems have received increasing attentions in machine learning; see [26] for a survey.

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When data is generated stochastically, such as i.i.d. or Markovian (see (i)(ii) above), we would like to choose the radius $\varrho = \varrho_n$ adaptive to the sample size, so that the robust loss can balance between performance and robustness. More precisely, ϱ_n should be large enough so that the robust loss dominates the true loss with high probability, but not be too large to produce an inferior solution. When the testing environment is adversarially different from the training environment (see (iii) above), consider the *adversarial loss* minimization

$$\min_{f \in \mathscr{F}} \left\{ \mathcal{L}_p^{\mathsf{adv}}(f; \varrho) := \max_{\mathbb{P} \in \mathscr{P}_p(\varrho)} \left\{ \mathbb{E}_{x \sim \mathbb{P}}[f(x)] : \mathcal{W}_p(\mathbb{P}, \mathbb{P}_{\mathsf{true}}) \leq \varrho \right\} \right\}, \tag{Adv}$$

where the radius ϱ captures the difference between the training and testing environments. When $\mathbb{P}_{\mathsf{true}}$ is not available, a surrogate problem (WO) based on $\mathbb{P}_n$ is often considered. In this case, ϱ is often fixed and we hope that the robust solution yielding from (WO) generalizes well.

The goal of this paper is to provide a comprehensive analysis of the generalization capability of the above robust learning paradigms. Specifically, in the stochastic setting, under proper choice of the radius ϱ_n, we derive finite-sample guarantees for (RO) and (WO). In the adversarial setting, we derive generalization bounds for (Adv) with a small fixed radius ϱ.

Related Literature

Stochastic setting. The most related work to ours is [18], but we differ in several important ways. First, [18] studies only (WO) while we also consider (RO). Second, [18] studies i.i.d. data while we also consider non-i.i.d. by exploiting a different notion of transport inequality. Third, even in the i.i.d. setting, [18] studies only smooth losses and Wasserstein order $p = 1, 2$. We consider non-smooth losses and $p \in [1, 2]$, which includes important cases like $|\theta^\top x - y|^p$ and CVaR minimization. Moreover, for $p = 1$, we tighten the order of the remainder; for $p = 2$, we obtain a cleaner expression using a different set of assumptions. All these extensions require in-depth analysis on worst-case scenarios.

For (WO) with i.i.d. data, [16] originally proposed to choose the radius ϱ_n such that $\mathbb{P}_{\mathsf{true}}$ is contained in the uncertainty set with high probability, but the resulting performance bound is overly conservative and suffers from the curse of dimensionality. This issue has been mitigated asymptotically for smooth loss functions [7, 9, 34], and non-asymptotically for linear models [33, 13] and for smooth loss functions [18]. As mentioned above, our results apply to non-i.i.d. data and non-smooth losses.

For (RO) in the stochastic setting, its generalization properties were first studied in the pioneer work [40], which established algorithmic generalization generalization bounds for various learning tasks with i.i.d. and Markovian data. Nonetheless, many bounds developed therein suffer from the curse of dimensionality. This framework is then extended to metric learning [4] and ensemble models [42]. Our results circumvent the curse of dimensionality of the bounds developed in [40].

Adversarial setting. For $p \in [1, \infty)$, generalization properties of (WO) with fixed radius was considered first in [28]. The main difference is that we consider different magnitudes of data perturbation. We mainly focus on adversarial robustness where the radius is often a tiny number, and our results in Section 5 suggest that the generalization gap is small when the radius is small. While their motivation was from domain adaptation, and their bound is mostly useful when the Wasserstein radius is not extremely small, which suggests that the generalization gap becomes smaller as the radius grows to infinity (see their Remark 4). The bounds developed in [35, 31] does not depend on the Wasserstein radius, while our result indicates that when the radius is small, the remainder actually demonstrates its linear scaling and therefore tighter. For $p = \infty$, [25, 41, 1] focus on deriving generalization bounds for specific loss function classes such as linear hypotheses and neural networks. Our results apply to generic function classes, and most importantly, the bound suggests the the generalization gap is controlled by not only the complexity of the loss function class, but also the complexity of the gradient norm functions of the loss functions. The intuition is that the gradient norm controls the robustness of the model subject to adversarial perturbations. As far as we know, the bound in such form is new in the literature.

There are also some earlier works focusing on investigating the fundamental limit of adversarial learning and its difference from empirical risk minimization, such as [32, 36, 11, 12]. They consider somewhat different settings than ours, which usually involves a stylized data generating model that facilitates theoretical findings.

Regularization. Part of our analysis is based on the connection between robustness and regularization. The connection between (RO)(WO) and norm regularization was established for various machine learning tasks [39, 38, 6, 7, 13], and the connection between (WO) and Lipschitz/gradient regularization has been established for smooth loss functions [33, 35, 17, 14, 9, 2] and non-smooth loss functions [19]. Our results improve the sandwich bounds in [19] for $p = 1$, and generalizes the results therein to piecewise Hölder smooth loss functions for $p \in (1, \infty]$. Finally, there are also works connecting regularization and distributional robustness under other choices of distance [27, 22, 15, 24].

2 Preliminaries

In this section, we prepare several technical tools that are useful in deriving our main results.

Notations For $p \in [1, \infty]$, we denote by q its Hölder conjugate number, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathscr{P}(\mathscr{X})$ be the set of Borel probability distributions on $\mathscr{X}$, and let $\mathscr{P}_p(\mathscr{X}) = \{\mathbb{P} \in \mathscr{P}(\mathscr{X}) : \mathbb{E}_{\mathbb{P}}[\|x\|^p] < \infty\}$. For a given probability measure $\mathbb{Q}$, denote by $(L^p_{\mathbb{Q}}(\mathscr{X}), \|\cdot\|_{p,\mathbb{Q}})$ the set of L^p functions on $\mathscr{X}$ with respect to $\mathbb{Q}$. Unless stated otherwise, we will not assume the samples are i.i.d. Instead, we use $S_n = \{\hat{x}^i\}_{i=1}^n$ to represent a sample of size n, and denote by μ_{S_n} the sampling distribution and use $\mathbb{E}_{S_n}$ to indicate the expectation with respect to the μ_{S_n}, and $\mathcal{L}^{\mathsf{true}}(f) = \mathbb{E}_{S_n}[\frac{1}{n}\sum_{i=1}^n f(\hat{x}^i)]$. For a fixed set of n points, we denote $\mathbb{P}_n$ as the empirical uniform distribution on these n points. For two real numbers, a and b, we denote $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$.

Transport inequalities In the stochastic setting, we will work with distributions that satisfy the following condition. Let $p \in [1, \infty)$ and $\tau > 0$. A distribution $\mathbb{Q} \in \mathscr{P}_p(\mathscr{X})$ satisfies a *transport inequality*, denoted as $T_p(\tau)$, if for all $\mathbb{P} \in \mathscr{P}_p(\mathscr{X})$ it holds that

$$\mathcal{W}_p^2(\mathbb{P}, \mathbb{Q}) \leq \tau \int_{\mathscr{X}} \log(d\mathbb{P}/d\mathbb{Q}) d\mathbb{P},$$

where $d\mathbb{P}/d\mathbb{Q}$ denotes the Radon-Nikodym derivative. The integral on the right-hand side of the inequality above represents the relative entropy between $\mathbb{P}$ and $\mathbb{Q}$. The transportation-information inequality $T_p(\tau)$ covers a variety of useful classes of distributions. For example, if $\mathscr{X}$ is bounded by B, then any distribution on $\mathscr{X}$ satisfies $T_1(8B^2)$. Gaussian distributions with covariance Σ satisfies $T_2(2\lambda_{\max}(\Sigma))$. Interested readers are referred to [23] for a recent survey.

Variation Regularization Following [19], we define the *local* and *global slope* of a continuous function $f : \mathscr{X} \to \mathbb{R}$ at $x \in \mathscr{X}$, denoted by $|\partial f|(x)$ and $G(f)(x)$ respectively, as

$$|\partial f|(x) = \limsup_{\|\tilde{x} - x\| \to 0} \frac{(f(\tilde{x}) - f(x))_+}{\|\tilde{x} - x\|}, \quad G(f)(x) = \sup_{\tilde{x} \in \mathscr{X}} \frac{(f(\tilde{x}) - f(x))_+}{\|\tilde{x} - x\|}.$$

When f is differentiable at x, we have $|\partial f|(x) = \|\nabla f(x)\|_*$. When f is L-Lipschitz, we have $G(f)(x) \leq L$ for all $x \in \mathscr{X}$ and $\|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}} = \|f\|_{\mathrm{Lip}}$. For a function f, denote by $\|f\|_{\mathrm{Lip}}$ its Lipschitz norm whenever it exists, and by $\||\partial f|\|_q$ the $L^q(\mathbb{P}_{\mathsf{true}})$-norm of its slope functin. We define *variation regularization* as

$$\min_{f \in \mathscr{F}} \mathcal{L}^{\mathsf{vr}}_{n,q}(f; \varrho) := \begin{cases} \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) + \varrho \||\partial f|\|_{q,\mathbb{P}_n}, & q < \infty, \\ \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) + \varrho \|f\|_{\mathrm{Lip}}, & q = \infty. \end{cases} \tag{VR}$$

We introduce the following class of functions consisting of slope functions of loss functions.

$$\partial\mathscr{F}_q := \left\{|\partial f|^q : f \in \mathscr{F}\right\}, \quad \mathscr{N}_q := \left\{\frac{|\partial f|^q}{\||\partial f|\|^q_{q,\mathbb{P}_{\mathsf{true}}}} : f \in \mathscr{F}\right\}. \tag{1}$$

Throughout we utilize the following assumptions, which covers most loss functions that are amenable to optimize efficiently using first-order methods.

Assumption 1. *Every $f \in \mathscr{F}$ is L-Lipschitz.*

Assumption 2. *Every $f \in \mathscr{F}$ is piecewise differentiable. There exists $h, \alpha > 0$ such that for every $f \in \mathscr{F}$ and every $\tilde{x}, x$ in the same piece of f, it holds that $\|\nabla f(\tilde{x}) - \nabla f(x)\|_* \leq h\|\tilde{x} - x\|^\alpha$.*

3 Robustness and Variation Regularization

In this section, we establishes connection between robust optimization (RO) (WO) and variation regularization (VR), which serve as building blocks for results in Section 5.

We make the following assumption, which states that the loss function family has sufficient variation.

Assumption 3. $\eta := \inf_{f\in\mathscr{F}} \||\partial f|\|_{q,\mathbb{P}_{\mathsf{true}}} > 0$.

We also define $\tilde{\eta} := \inf_{f\in\mathscr{F}} \||\partial f|\|_{q,\mathbb{P}_n}$. To simplify notations, we introduce a function $e_n : \mathbb{R}_+ \to \mathbb{R}_+$, defined as

$$e_n(\rho) = \sup_{f\in\mathscr{F}} \left\{ \mathbb{E}_{X\sim\mathbb{P}_n} \left[(\rho - d(X, \mathscr{D}_f))_+ \right] \right\}, \tag{2}$$

which describes the data concentration within a margin of non-differentiable points.

Remark 1. *$\tilde{\eta}$ can be ensured to be positive with high probability using Assumptions 1, 3 and a uniform version of Bennett's inequality (Lemma 6 in the supplementary). $e_n(\varrho)$ vanishes for differentiable families, and is often $\tilde{O}(1/n)$ when $\varrho = \varrho_n = \tilde{O}(1/\sqrt{n})$ (e.g., [19, Lemma EC.11]).*

Our first result in this section bounds the difference between (WO) and (VR) for $p > 1$.

Proposition 1. *Let $p \in (1, \infty]$ and $\varrho > 0$. Assume Assumptions 1, 2 and 3 hold. Then it holds that*

$$|\mathcal{L}^{\mathsf{wo}}_{n,p}(f;\varrho) - \mathcal{L}^{\mathsf{vr}}_{n,q}(f;\varrho)| \le C_0 \varrho^{(\alpha+1)\wedge p} + 2Le_n\big((pL/\tilde{\eta})^{q-1}\varrho\big), \quad \forall f \in \mathscr{F},$$

where C_0 equals h if $p = \infty$, $h\max(1, (Lp/\tilde{\eta})^{\frac{\alpha+1}{p-1}})$ if $p \in (\alpha+1, \infty)$, and $L(h/L)^{\frac{p-1}{\alpha}}$ otherwise.

Proposition 1 shows that (WO) and (VR) are close with a gap controlled by two high-order terms: one term $\varrho^{(\alpha+1)\wedge p}$ depends on the Hölder smoothness parameter α; while the other term e_n depends on the data distribution around the non-differentiable region. The smoother the loss function, the smaller the gap is. Intuitively, the worst-case distribution in (WO) perturbs the data approximately aligned with gradient ascent direction, thus for smoother losses, the gradient norm in (VR) provides a tighter first-order approximation on the difference between the worst-case and the empirical losses. By focusing on Lipschitz loss functions, we are able to obtain a cleaner result comparing to [19, Theorem 1]. Since we consider different assumptions, in the proof we have to make different constructions of approximately worst-case distributions.

Our second result in this section establishes a relationship between (WO) and (RO) for $p > 1$. Recall that they coincide for $p = \infty$.

Proposition 2. *Let $p \in (1, \infty)$. Assume Assumptions 1, 2 and 3 hold. Then it holds that*

$$0 \le \mathcal{L}^{\mathsf{wo}}_{n,p}(f;\varrho) - \mathcal{L}^{\mathsf{ro}}_{n,p}(f;\varrho) \le \tfrac{2L}{n}(\tfrac{L}{\lambda_n})^{\frac{1}{p-1}}, \quad \forall f \in \mathscr{F},$$

where λ_n is the dual minimizer of (WO) satisfying

$$2^p\varrho^p\lambda_n \ge \tilde{\eta}\cdot\varrho - 2^{\alpha+1}\varrho^{\alpha+1}h - 2Le_n(2\varrho) - 2Le_n\big((Lp/\tilde{\eta})^{\frac{1}{p-1}}\varrho\big) - L(\tfrac{h}{L})^{\frac{p-1}{\alpha}}\varrho^p - h(Lp/\tilde{\eta})\varrho^{\alpha+1}.$$

Proposition 2 shows that (WO) and (RO) has a uniform upper bounded gap. Intuitively, both these two robust paradigms perturbs data points aligned with gradient ascent direction. The difference is that (WO) allows to perturb a tiny fraction of probability arbitrarily far away, but (RO) prohibits probability splitting, which results in a gap. The proof is based on careful analysis on the probabilistic nature of the worst-case distribution. The first result of this kind was established in [20] for a single loss function, but the bound therein can be undesirable for a family of losses. Under the conditions in Remark 1, the gap vanishes in the order $O(\varrho/n)$.

Thus far, we have shown that when $p \in (1, \infty]$, $\mathcal{L}^{\mathsf{vr}}_{n,q}(f;\varrho_n)$, $\mathcal{L}^{\mathsf{ro}}_{n,p}(\varrho_n)$ and $\mathcal{L}^{\mathsf{wo}}_{n,p}(\varrho_n)$ only differ by a higher order term that depends on the smoothness and variation of the loss function class.

Next, we consider the case $p = 1$. In light of Example 3.3.1 in [19], it is not always possible to have that $\mathcal{L}^{\mathsf{vr}}_{n,\infty}(\varrho) - \mathcal{L}^{\mathsf{wo}}_{n,1}(\varrho)$ is of a higher order than ϱ. Below we identify conditions in order to bound the gap. For $f \in \mathscr{F}$, we define cumulative distribution functions $H_f, \tilde{H}_f$ as

$$H_f(a) := \mathbb{P}_{\mathsf{true}}\big\{\|f\|_{\mathrm{Lip}} - G(f)(X) \le a\big\}, \;\; a \ge 0,$$
$$\tilde{H}_f(a) := \mathbb{P}_{\mathsf{true}}\big\{\|f\|_{\mathrm{Lip}} - |\partial f|(X) \le a\big\}, \;\; a \ge 0,$$

where the random variable X has distribution $\mathbb{P}_{\mathsf{true}}$, and recall $G(f), |\partial f|$ are the global and local slope functions of f defined in Section 2.

Assumption 4. *Assume there exists $\beta, \bar{a}, c > 0$ such that for all $a \in [0, \bar{a})$ and all $f \in \mathscr{F}$, it holds that $H_f(a^\beta) \geq ca$ for (WO), and $\tilde{H}_f(a^\beta) \geq ca$ for (RO).*

This assumption imposes the concentration of maximal slope, and large β indicates that $G(f)(X)$ or $|\partial f|(X)$ concentrates more on the maximal slope $\|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}}$. For example, if H_f is continuous at 0, then Assumption 4 holds with $\beta = 1$ and imposes a positive lower bound on the density. As another example, if f is a linear function, then the set of global slopes is a singleton and $H_f(a) = \tilde{H}_f(a) = 1$ for all $a \geq 0$, hence Assumption 4 is trivially satisfied with $\bar{a} = \beta = \infty$. More generally, under the assumption that $\limsup_{\|x-\hat{x}\|\to\infty} \frac{(f(x)-f(\hat{x}))_+}{\|x-\hat{x}\|} = \|f\|_{\mathrm{Lip}}$, which is imposed by [18], the global slope $G(f)(x) = \|f\|_{\mathrm{Lip}}$ for all $x \in \mathscr{X}$, thereby $H_f(a) = 1$ for all $a \geq 0$.

Proposition 3. *Let $p = 1$ and $\varrho > 0$. Assume Assumptions 1, 2, 4 hold. Let $\delta \in [0, \frac{1}{2})$. Then it holds that*

$$\mathcal{L}^{\mathsf{ro}}_{n,1}(f;\varrho) \leq \mathcal{L}^{\mathsf{wo}}_{n,1}(f;\varrho) \leq \mathcal{L}^{\mathsf{vr}}_{n,\infty}(f;\varrho),$$

and that

$$\mathcal{L}^{\mathsf{wo}}_{n,1}(f;\varrho) \geq \mathcal{L}^{\mathsf{vr}}_{n,\infty}(f;\varrho) - \epsilon_n, \quad \mathcal{L}^{\mathsf{ro}}_{n,1}(f;\varrho) \geq \mathcal{L}^{\mathsf{vr}}_{n,\infty}(f;\varrho) - \tilde{\epsilon}_n,$$

where

$$\epsilon_n = \varrho(\|f\|_{\mathrm{Lip}} - \Delta_n) + 4hn^{\alpha\delta}\varrho^{\alpha+1} + 2Le_n(\varrho n^\delta),$$
$$\tilde{\epsilon}_n = \varrho(\|f\|_{\mathrm{Lip}} - \tilde{\Delta}_n) + 4hn^{\alpha\delta}\varrho^{\alpha+1} + 2Le_n(\varrho n^\delta),$$

where Δ_n (resp. $\tilde{\Delta}_n$) equals the $\lfloor n^{1-\delta} \rfloor + 1$-th order statistics of the sample $\{G(f)(\hat{x}^i)\}_{i=1}^n$ (resp. $\{|\partial f|(\hat{x}^i)\}_{i=1}^n$).

Remark 2. *Consider data are generated i.i.d. When $G_f(X)$ has a positive mass on the maximal global slope and f has piecewise Lipschitz gradient (i.e., $\alpha = 1$), by setting $\delta = 0$, with high probability, we have $\Delta_n = \|f\|_{\mathrm{Lip}}$, and $\epsilon_n = O(\varrho^2 + e_n(\varrho))$, which is $O(1/n)$. When H_f is continuous, under the condition in Remark 1, as shown in the supplementary, the remainder Δ_n will be $O(1/n)$*

Proposition 3 shows that (RO), (WO) and (VR) are equivalent up to a higher order error term ϵ_n, which depends on the smoothness of the loss function as indicated by α and e_n, and the concentration of slopes as indicated by Δ_n. Ideally, we would like to have a large $\beta \geq 1$, in which case H_f has a Hölder constant less than or equal to 1, and in particular, it has a density at 0 when $\beta = 1$. The smoother the function and the more concentrated on the maximal global slope, the smaller the gap between (WO), (RO) and (VR) is. This improves the sandwich result of [19, Section 3.3.2] in which $\epsilon_n = O(1/\sqrt{n})$. Our finer analysis is based on construction of feasible solutions for (WO) and (RO) obtained from perturbation of empirical data points as well as careful control on the distances of perturbation.

4 Generalization Bounds for Variation Regularization

In this section, we derive generalization bounds for variation regularization in the stochastic setting, which are building blocks to develop the generalization bounds for (WO) and (RO).

We make the following assumptions on the sampling distribution.

Assumption 5. *The sampling distribution μ_{S_n} satisfies $T_p(\tau_n)$.*

When S_n contains i.i.d. samples from some underlying distribution that satisfies $T_p(\tau)$, then by tensorization of transport inequalities [23], μ_{S_n} satisfies $T_p(\tau_n)$ with $\tau_n = \tau n^{2/p-1}$. In Example 2, we will consider Markovian data that satisfies $T_1(\tau_n)$ with $\tau_n = O(n)$.

Suppose we work with a parametric family $\mathscr{F} = \{f_\theta\}_{\theta\in\Theta}$, where the parameter space Θ is endowed with a norm $\|\cdot\|_\Theta$. We impose the assumption on Lipschitz parametric family.

Assumption 6. *There exists $\kappa > 0$ such that $|f_{\theta'}(x) - f_\theta(x)| \leq \kappa\|\theta' - \theta\|$ for all $x \in \mathscr{X}$.*

Proposition 4. *Let $p = 1$. Assume Assumptions 1, 5 and 6 are in force. Further assume Assumption 2 when $p > 1$. Let $t > 0$. Set*

$$\varrho_n = \sqrt{\frac{\tau_n(t + \log \mathcal{N}(\frac{1}{n}; \Theta, \|\cdot\|_\Theta))}{n^{2/p}}}.$$

Then with probability at least $1 - e^{-t}$, for all $f \in \mathscr{F}$, it holds that

$$\mathcal{L}^{\mathsf{true}}(f) \leq \mathcal{L}^{\mathsf{vr}}_{n,\infty}(f; \varrho_n) + \frac{3\kappa}{n}.$$

Proposition 4 shows that by choosing the radius properly, Lipschitz regularization upper bounds the true risk $\mathcal{L}^{\mathsf{true}}(f)$ uniformly with high probability up to some error terms. The radius ϱ_n depends on the sampling distribution parameter and the complexity of the loss function class. When $\tau_n = O(n^{2/p-1})$ and $\log \mathcal{N}(\frac{1}{n}; \Theta, \|\cdot\|_\Theta) = \tilde{O}(1)$, we have $\varrho_n = \tilde{O}(1/\sqrt{n})$, corresponding to the canonical choice. Thereby the error terms are $O(1/n)$. The proof of Theorem 4 follows a similar flow as that of [18, Corollary 2], but our result extends it to non-i.i.d. and non-smooth settings.

For $p > 1$, let define the population counterpart of $e_n(\varrho)$ in (3) as

$$e(\rho) = \sup_{f \in \mathscr{F}} \left\{ \mathbb{E}_{X \sim \mathbb{P}_{\mathsf{true}}} \left[(\rho - d(X, \mathscr{D}_f))_+ \right] \right\}. \tag{3}$$

Proposition 5. *Let $p \in (1, 2]$. Assume Assumptions 1, 2, 5 and 6 are in force. Further assume Assumption 2 when $p > 1$. Let $t > 0$. Set*

$$\varrho_n = \left(1 - 2\mathfrak{R}_n(\mathscr{N}_q) - (L/\eta)^2 \sqrt{\frac{t}{2n}}\right)^{-\frac{1}{q}} \sqrt{\frac{\tau_n(t + \log \mathcal{N}(\frac{1}{n}; \Theta, \|\cdot\|_\Theta))}{n^{2/p}}}.$$

Then with probability at least $1 - e^{-t}$, for all $f \in \mathscr{F}$, it holds that

$$\mathcal{L}^{\mathsf{true}}(f) \leq \mathcal{L}^{\mathsf{vr}}_{n,q}(f; \varrho_n) + \frac{3\kappa}{n} + C_0 \varrho_n^{(\alpha+1)\wedge p} + 2Le\big((pL/\tilde{\eta})^{q-1} \varrho_n\big).$$

Recalling $\mathscr{N}_q$ is the set of normalized slope defined in (1), and the inflation of the radius compared with the case of $p = 1$ is to compensate the discrepancy between $\||\partial f|\|_q$ and $\||\partial f|\|_{q,\mathbb{P}_n}$. The inflation factor is well-defined so long as the Rademacher complexity of $\mathscr{N}_q$ diminishes with respect to sample size. The additional remainder term is similar to the remainder in Proposition 1, which measures the difference between (WO) and (VR) when the nominal distribution is $\mathbb{P}_{\mathsf{true}}$.

Example 1 (Linear Prediction with Polynomial Loss)**.** *Consider a supervised learning problem with feature vector $x \in \mathscr{X} \subset \mathbb{R}^d$ and response variable $y \in \mathscr{Y} \subset \mathbb{R}$. Suppose $\|x\|_2 \leq B_1$ for all $x \in \mathscr{X}$ and $|y| \leq B_2$ for all $y \in \mathscr{Y}$. Let Θ be a bounded set in $\mathbb{R}^d$ such that $\|\theta\|_2 \leq B_3 - 1$ for all $\theta \in \Theta$. Consider a linear predictor with polynomial loss $\mathscr{F} = \{(x, y) \mapsto f_\theta(x, y) = |\theta^\top x - y|^p : \theta \in \Theta\}$. We now discuss how to ensure the assumptions in Propositions 4, with details in the supplementary. It is easy to verify that every $f_\theta(x, y)$ is Lipschitz in both variables and parameters, with $\kappa = p(B_1 + B_2)^p B_3^{p-1}$ in Assumption 6. Suppose the samples are i.i.d. from $\mathbb{P}_{\mathsf{true}}$ satisfying $T_p(\tau)$, which implies Assumption 5 with $\tau_n = \tau n^{\frac{2}{p}-1}$. Hence we can apply Proposition 4, and by [37, Example 5.8], we have $\log \mathcal{N}(\frac{1}{n}; \Theta, \|\cdot\|_\Theta) \leq d \log(B_3(1 + 2n))$.*

Example 2 (Markovian Data)**.** *In this example, we show that our Assumption 5 about μ_{S_n} can be satisfied for Markovian data. Consider a homogeneous Markov chain on $\mathscr{X}$ with a transition kernel $P(dy|x)$. Assume there exist $\alpha_0 > 0, \bar{\beta} < \alpha_0$ and $C < \infty$, such that*

$$\int_{\mathscr{X}} e^{\alpha_0 \|y\|^2} P(dy|x) \leq C e^{\bar{\beta}\|x\|^2}, \quad \forall x \in \mathscr{X}.$$

Let $\{x^i\}_{i=1}^n$ be n consecutive outputs of this Markov chain. Then by Theorem 4.1 in [10], μ_{S_n} satisfies $T_1(\tau_n)$ with $\tau_n = \frac{2n(1+\log C)}{\alpha_0 - \bar{\beta}}$. Thus Assumption 5 is satisfied.

5 Generalization Bounds for (Wasserstein) Robust Optimization

In this section, we develop the generalization bounds for (WO) and (RO), based on the results in previous sections.

5.1 Stochastic Setting

Combining the results in Sections 3 and 4, we immediately obtain the following theorems.

Theorem 1. *Let $p \in (1, 2]$ and $t > 0$. Set the radius*

Under the setup of Propositions 1 and 4, with probability at least $1 - e^{-t}$, for all $f \in \mathscr{F}$, it holds that

$$\mathcal{L}^{\mathsf{true}}(f) \leq \mathcal{L}^{\mathsf{wo}}_{n,p}(f) + 2C_0 \varrho_n^{(\alpha+1)\wedge p} + 2Le_n\big((pL/\tilde{\eta})^{q-1}\varrho_n\big) + 2Le\big((pL/\tilde{\eta})^{q-1}\varrho_n\big) + \frac{3\kappa}{n}.$$

With the additional setup as in Proposition 2, it holds that

$$\mathcal{L}^{\mathsf{true}}(f) \leq \mathcal{L}^{\mathsf{ro}}_{n,p}(f) + 2C_0 \varrho_n^{(\alpha+1)\wedge p} + 2Le_n\big((pL/\tilde{\eta})^{q-1}\varrho_n\big) + 2Le\big((pL/\tilde{\eta})^{q-1}\varrho_n\big) + \frac{3\kappa}{n} + \frac{2L}{n}\big(\frac{L}{\lambda_n}\big)^{\frac{1}{p-1}}.$$

Theorem 2. *Let $p = 1$ and $t > 0$. Under the setup of Propositions 3 and 4, with probability at least $1 - e^{-t}$, for all $f \in \mathscr{F}$, it holds that*

$$\mathcal{L}^{\mathsf{true}}(f) \leq \mathcal{L}^{\mathsf{wo}}_{n,1}(f) + \epsilon_n + \frac{3\kappa}{n},$$
$$\mathcal{L}^{\mathsf{true}}(f) \leq \mathcal{L}^{\mathsf{ro}}_{n,1}(f) + \tilde{\epsilon}_n + \frac{3\kappa}{n},$$

In summary, both Theorems 1 and 2 show that by choosing a reasonably small radius without suffering from the curse of dimensionality (as opposed to most Wasserstein DRO literature), the robust solution resulting from solving (VR), (RO), or (WO) has a good generalization error bound, expressed as the sum of the empirical loss and the empirical variation of the loss multiplied by the radius, plus a high-order term. The high-order term depends mainly on the smoothness of the loss function as indicated by α, the (non-)concentration of probability distributions on non-differentiable points as indicated by e_n, and the concentration of the maximal slope as indicated by Δ_n (resp. $\tilde{\Delta}_n$) in the expression of ϵ_n (resp. $\tilde{\epsilon}_n$) (when the Wasserstein order $p > 1$).

Example 3 (Unsupervised Learning)**.** *Consider the principal component analysis*

$$\max_{W \in \mathbb{R}^{d\times k}} \left\{ \|W^\top X\|_F^2 : \; W^\top W = I_k \right\},$$

that seeks for d principal directions, along which the data has maximized variance. Here X is a $d \times n$ sample matrix consisting of n samples $\{X_i\}_{i=1}^n$ from $\mathbb{P}_{\mathsf{true}}$ on a set $\mathscr{X} \subset \{x \in \mathbb{R}^d : \|x\|_2 \leq B\}$; $\|\cdot\|_F$ denotes the Frobenius norm; I_k denotes the k-dimensional identity matrix; and $W = [w_1, \ldots w_k]$ is the set of k orthonormal principal directions in $\mathbb{R}^d$. Assume that $\mathbb{E}_{x\sim\mathbb{P}_{\mathsf{true}}}[x] = 0$, and the smallest eigenvalue λ of the covariance matrix $\mathbb{E}_{x\sim\mathbb{P}_{\mathsf{true}}}[xx^\top]$ is positive.

Let $p = 2$ and $\mathscr{F} = \{x \mapsto f_W(x) = -\|W^\top x\|_2^2 : W^\top W = I_k\}$. It is easy to verify Assumption 1 holds since f_W is $2Bk$-Lipschitz, and Assumption 2 holds with $\alpha = 1$, and Assumption 6 holds with $\kappa = 2\sqrt{k}B^2$. and satisfies $T_2(\tau)$, which implies Assumption 5 with $\tau_n = \tau n$. The zero-mean and non-singular covariance of $\mathbb{P}_{\mathsf{true}}$ imply Assumption 3. Finally, to compute the covering number of the set $\mathcal{W} = \{W : W^\top W = I_k\}$, observe that each W consists of k orthogonal vectors, each of which belongs to the unit ball in $\mathbb{R}^d$. Hence $\log \mathcal{N}(\frac{1}{n}; \mathcal{W}, \|\cdot\|_2) \leq kd \log((1 + 2n))$. Thus, we improve the bound in [40, Example 8] that depends exponentially on d.

Example 4 (Risk-averse Minimization)**.** *Conditional Value-at-Risk ($\mathbb{C}$V@R) is a popular risk measure that takes risk attitude into account in decision making. The empirical $\mathbb{C}$V@R at level ν is defined as $\mathbb{C}\mathrm{V@R}_{\nu,n}[Z] = \min_{u\in\mathbb{R}} \left\{u + \frac{1}{\nu n}\sum_{i=1}^n (Z_i - u)_+\right\}$. Consider the following robust $\mathbb{C}$V@R minimization*

$$\min_{\|w\|_2 \leq B, u\in\mathbb{R}} \max_{\{x^i\}_{i=1}^n \in \mathscr{X}_1(\varrho)} \left\{ u + \frac{1}{\nu n}\sum_{i=1}^n (w^\top x^i - u)_+ \right\}. \tag{CVaR}$$

Define

$$\mathscr{F} = \left\{ f_\theta(x, y) = u + \tfrac{1}{\nu}(w^\top x - uy)_+ : \; \theta = (w, u) \in \mathbb{R}^{d+1}, \|w\|_2 \leq B \right\}.$$

The objective is piecewise linear, thus Assumptions 1, 2 holds. Assume $\mathbb{P}_{\mathsf{true}}$ satisfies $T_1(\tau)$, then Assumption 5 holds with $\tau_n = \tau n$, and set $B_1 = \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[\|x\|_2]$. In addition, it can be shown that the minimizer of (CVaR), $|u_n(w)| \leq B(B_1 + \sqrt{\tau(t + \log(n))})$ with probability $1 - e^{-t}$. With the same probability, Assumption 6 holds with $\kappa = 1 + \frac{1}{\nu}B(1 + B_1 + \sqrt{\tau(t + \log(n))})$. Finally, the parameter space is a product space, we can compute covering numbers and then take the product, i.e. $\log \mathcal{N}(\frac{1}{n}; \Theta, \|\cdot\|_\Theta) \leq d \log(B(1 + 2n)) + \log(nB(B_1 + \sqrt{\tau(t + \log(n))})$.

Before closing this section, we would like to emphasize that this stochastic setting is drastically different from the adversarial setting in the next subsection, for which many recent studies on the generalization properties of robust optimization. In fact, choosing a good radius scaling scheme with nice finite-sample guarantees is at the core of Wasserstein DRO and remains open for quite some time until the recent work [18]. We generalize this result to non-i.i.d., non-smooth settings as well as for (RO).

5.2 Adversarial Setting

In the adversarial setting (Adv), we are trying to minimize the expected loss under the worst-case scenario when the learning algorithm is being purposely attacked by an adversary. By replacing $\mathbb{P}_{\mathsf{true}}$ with $\mathbb{P}_n$ in (Adv), we obtain the empirical adversarial robustness problem, which is the same as the Wasserstein robust optimization problem (WO). One of the most fundamental questions in adversarial robustness problem is to estimate the generalization gap between (Adv) and its empirical counterpart (WO). We aim to derive generalization bounds for $p \in [1, \infty]$ by virtue of Theorems 1 and 3. Following the literature, in this subsection, we assume $\mathbb{P}_n$ consists of i.i.d. samples from a continuous distribution.

We start with $p = 1$.

Theorem 3. *Let $p = 1$. Under the setup of Proposition 3, assume that $\|f\|_\infty \leq M$ for all $f \in \mathscr{F}$. Let $\delta \in [0, \frac{1}{2})$ and $t > 0$. Then with probability at least $1 - 2e^{-2t^2}$, it holds that for every $f \in \mathscr{F}$,*

$$|\mathcal{L}^{\mathsf{wo}}_{n,1}(f;\varrho) - \mathcal{L}^{\mathsf{adv}}_{1}(f;\varrho)| \leq 2\mathfrak{R}_n(\mathscr{F}) + M\sqrt{\tfrac{t}{2n}} + 2\epsilon_n + C\varrho^{1+\frac{\alpha\beta}{\alpha+\beta}}.$$

On the right-hand side of the inequality in Theorem 3, the first two terms are identical to the generalization bound of empirical risk minimization. The third term ϵ_n from Proposition 3 is linear in the adversarial perturbation level ϱ, which represents the inflation of generalization bound in the adversarial setting as opposed to the stochastic setting, whose dominating factor is the concentration of the maximal slope. The rest terms are of higher-order for small adversarial perturbation level ϱ, that mainly depends on the smoothness of the function.

Next, we consider $p \in (1, \infty]$.

Theorem 4. *Let $p \in (1, \infty]$. Under the setup of Proposition 1, assume that $\|f\|_\infty \leq M$ for all $f \in \mathscr{F}$. Let $\varrho \leq c^{\frac{1}{\delta}}(\frac{\bar{a}}{2})^{\frac{1}{\beta\delta}}$ and $t > 0$. Then there exists $C > 0$, such that with probability at least $1 - 3e^{-t}$, it holds that for every $f \in \mathscr{F}$,*

$$\begin{aligned}|\mathcal{L}^{\mathsf{wo}}_{n,p}(f;\varrho) - \mathcal{L}^{\mathsf{adv}}_{p}(f;\varrho)| \leq 2\mathfrak{R}_n(\mathscr{F}) + M\sqrt{\frac{t}{2n}} + \frac{2\mathfrak{R}_n(\partial\mathscr{F}_q) + L^q\sqrt{\frac{t}{2n}}}{q\cdot(\eta\wedge\tilde{\eta})^{-\frac{q}{p}}}\varrho \\ + C_0\varrho^{(\alpha+1)\wedge p} + Le_n\Big((\eta p L)^{q-1}\varrho\Big) + C\varrho^2,\end{aligned}$$

where C_0 is defined in Proposition 1.

Similar to the case of $p = 1$, the first two terms in the above bound is identical to the bound of empirical risk minimization, and the high-order term in the second row depends on the smoothness of the loss functions. The major difference from $p = 1$ is that, the inflation term in the adversarial setting depends on $\mathfrak{R}_n(\partial\mathscr{F}_q)$, the Rademacher complexity of the class of gradient norm functions. This term appears to be new in the literature. Intuitively, if the complexity of the variation is small, the model is more robust to the adversarial perturbations and thus generalizes better. We will illustrate this result in Section 5.

Theorems 3 and 4 are proved based on a different strategy than that in the stochastic setting. Set $\mathcal{L}^{\mathsf{vr}}_q(\varrho) := \min_{f\in\mathscr{F}}\{\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] + \varrho\||\partial f|\|_{q,\mathbb{P}_{\mathsf{true}}}\}$. Using the triangle inequality we decompose the difference $|\mathcal{L}^{\mathsf{wo}}_{n,p}(f;\varrho) - \mathcal{L}^{\mathsf{adv}}_{p}(f;\varrho)|$ into three terms $|\mathcal{L}^{\mathsf{wo}}_{n,p}(f;\varrho) - \mathcal{L}^{\mathsf{vr}}_{n,q}(f;\varrho)|$, $|\mathcal{L}^{\mathsf{vr}}_{n,q}(f;\varrho) - \mathcal{L}^{\mathsf{vr}}_{q}(f;\varrho)|$, $|\mathcal{L}^{\mathsf{vr}}_{q}(f;\varrho) - \mathcal{L}^{\mathsf{adv}}_{p}(f;\varrho)|$ and then bound each term individually.

Example 5 (Adversarial Learning)**.** *Consider a classification problem with loss function class*

$$\mathscr{F}^\phi := \{(x,y) \mapsto \phi(yf(x)) : \ f \in \mathscr{F}\},$$

where $\phi : \mathbb{R} \to [0,1]$ is monotonically non-increasing, L_ϕ-Lipschitz and has h_ϕ-Lipschitz gradient, $y \in \{\pm 1\}$, and $f : \mathscr{X} \to [0,1]$ is a family of L-Lipschitz hypotheses with h-Lipschitz gradient. It is easy to verify Assumption 1, because ϕ is Lipschitz. Particularly, when $\mathscr{F} = \{f_\theta(x) = \theta^\top x : \theta \in \Theta, \|x\| \le B_2\}$, where $\Theta \subset \{\theta \in \mathbb{R}^d : \|\theta\|_2 \le B_1\}$. We can verify Assumption 2 holds with $\alpha = 1$, and the constant $2h_\phi B_1(B_1 + B_1 B_2)$, and Assumption 6 with $\kappa = (L_\phi + h_\phi B_1 B_2)$, and the covering number $\log \mathcal{N}(\frac{1}{n}; \Theta, \|\cdot\|_\Theta) \le d \log(B_1(1+2n))$ can be computed in the same way as in Example 1.

6 Concluding Remarks

In this paper, we thoroughly analyzed the finite-sample performance bounds for robust optimization, drawing connection with transport inequality and local Rademacher complexity. Our results generalize existing results from various perspectives, and particularly, to non-smooth loss functions and to non-i.i.d. data, both of which requires non-trivial analysis. Our results help to better understand the regularization effect of robust learning and explain their superior empirical performances. We hope our results can inspire regularization schemes for other applications as well. Our bounds involve the Rademacher complexities of (normalized) gradient norm function $\partial\mathscr{F}_q, \mathscr{N}_q$ and margin to discontinuity $\mathscr{I}_\varrho$, which are non-standard; we have provided various examples and will investigate in more depth for future work.

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Supplementary Material

A Auxiliary Results

Lemma 1 ([20, 21])**.** *Suppose every $f \in \mathscr{F}$ is L-Lipschitz continuous. When $p \in [1, \infty)$, it holds that*

$$\mathcal{L}^{\mathsf{wo}}_{n,p}(f;\varrho) = \min_{\lambda \geq 0} \left\{ \lambda \varrho^p + \frac{1}{n} \sum_{i=1}^{n} \sup_{x \in \mathscr{X}} \left\{ f(x) - \lambda \|x - \hat{x}^i\|^p \right\} \right\}.$$

When $p = \infty$, it holds that

$$\mathcal{L}^{\mathsf{wo}}_{n,\infty}(f;\varrho) = \mathcal{L}^{\mathsf{ro}}_{n,\infty}(f;\varrho) = \frac{1}{n} \sum_{i=1}^{n} \sup_{x \in \mathscr{X}, \|x - \hat{x}^i\| \leq \varrho} f(x).$$

Lemma 2 (Lemma 2 in [30])**.**

$$\mathbb{E}_S \left[\sup_{f \in \mathscr{F}} (\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] - \mathbb{E}_n[f]) \right] \leq 2\mathfrak{R}_n(\mathscr{F}).$$

Lemma 3. *Suppose $f(x) \in [0, M]$ for all $f \in \mathscr{F}$, then for any $t > 0$, with probability $1 - e^{-2t}$, we have:*

$$\sup_{f \in \mathscr{F}} |\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] - \mathbb{E}_{\mathbb{P}_n}[f]| \leq 2\mathfrak{R}_n(\mathscr{F}) + M\sqrt{\frac{t}{2n}}.$$

If we have $\|f\|_\infty < M$, then by replacing f by $f + M$, we can get the same result by replacing M by $2M$.

Lemma 4 (Contraction Lemma)**.** *Let f be a L-Lipschitz function, and $\mathcal{F}$ a family of functions on $\mathscr{X}$. Then:*

$$\mathfrak{R}_n(f \circ \mathscr{F}) \leq L \cdot \mathfrak{R}_n(\mathscr{F}),$$

where $f \circ \mathscr{F} = \{f \circ g : g \in \mathscr{F}\}$.

B Proofs for Section 4

Lemma 5 (Theorem 2.1 in [3])**.** *Assume Assumption 1 holds. Let $t > 0$. Then with probability at least $1 - e^{-t}$, for all $f \in \mathscr{F}$, it holds that*

$$|||\partial f|||^q_{q,\mathbb{P}_n} \geq |||\partial f|||^q_{q,\mathbb{P}_{\mathsf{true}}} - 3\mathfrak{R}_n(\partial \mathscr{F}_q) - L\sqrt{\tfrac{2t}{n}} - \tfrac{7Lt}{3n}.$$

Therefore, as long as $\mathfrak{R}_n(\partial \mathscr{F}_q) \to 0$ as $n \to \infty$, the right-hand side is strictly positive for sufficiently large n.

We bound $|||\partial f|||_{q,\mathbb{P}_n}$ and $|||\partial f|||_{q,\mathbb{P}_{\mathsf{true}}}$ in both absolute difference and relative difference.

Lemma 6. *Assume Assumptions 1 and 3 hold. Let $t > 0$. Then with probability at least $1 - e^{-t}$,*

$$|||\partial f|||_{q,\mathbb{P}_{\mathsf{true}}} \leq |||\partial f|||_{q,\mathbb{P}_n} \left(1 - 2\mathfrak{R}_n(\mathscr{N}_q) - (L/\eta)^q \sqrt{\frac{t}{2n}} \right)^{-\frac{1}{q}}.$$

Proof. Using McDiarmid's inequality, with probability at least $1-e^{-t}$, for every $f \in \mathscr{F}$,

$$\sup_{f\in\mathscr{F}} \left| \mathbb{E}_{\mathbb{P}_n}\left[\frac{|\partial f|(x)^q}{\||\partial f|\|^q_{\mathbb{P}_{\mathsf{true}},q}}\right] - 1 \right| \le \mathbb{E}_{S_n}\left[\sup_{f\in\mathscr{F}} \left| \mathbb{E}_{\mathbb{P}_n}\left[\frac{|\partial f|(x)^q}{\||\partial f|\|^q_{\mathbb{P}_{\mathsf{true}},q}}\right] - 1 \right| \right] + (L/\eta)^q \sqrt{\frac{t}{2n}},$$

which implies that

$$\frac{\||\partial f|(x)\|^q_{\mathbb{P}_n,q}}{\||\partial f|(x)\|^q_{\mathbb{P}_{\mathsf{true}},q}} - 1 \ge -2\mathfrak{R}_n(\partial\mathscr{F}_q) - (L/\eta)^q\sqrt{\frac{t}{2n}}.$$

Thus, it holds that

$$\||\partial f|\|_{\mathbb{P}_{\mathsf{true}},q} \le \||\partial f|\|_{\mathbb{P}_n,q}\Big(1 - 2\mathfrak{R}_n(\partial\mathscr{F}_q) - (L/\eta)^q\sqrt{\frac{t}{2n}}\Big)^{-\frac{1}{q}}.$$

B.1 Proof of Proposition 4

In the sequel, we set $\mathbf{x}^n := (x^1,\dots,x^n)$, and define a metric $\mathbf{d}$ on $\mathscr{X}^n$ as $\mathbf{d}(\tilde{\mathbf{x}}^n,\mathbf{x}^n) = (\sum_{i=1}^n \|\tilde{x}^i - x^i\|^p)^{1/p}$. The following result follows from the proof of Lemma 5 in [18] by replacing $\mathbb{E}_\otimes$ and τ therein with $\mathbb{E}_{S_n}$ and $\tau_n n^{1-\frac{2}{p}}$, respectively.

Lemma 7. *Let $p \in [1,2]$. Assume Assumption 5 holds. Let $F : \mathscr{X}^n \to \mathbb{R}$. Assume $\mathbb{E}_{S_n}[F] = 0$ and there exist $M, L > 0$ and $\mathbf{x}_0^n \in \mathscr{X}^n$ such that*

$$F(\mathbf{x}^n) \le M + \frac{L}{n}\mathbf{d}(\mathbf{x}^n,\mathbf{x}_0^n)^p, \quad \forall \mathbf{x}^n \in \mathscr{X}^n.$$

Define $\mathcal{R}(\cdot;F) : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\mathcal{R}_{S_n,p}(\varrho;F) = \min_{\lambda\ge 0}\left\{\lambda\varrho^p + \mathbb{E}_{S_n}\left[\sup_{\tilde{\mathbf{x}}^n\in\mathscr{X}^n}\left\{F(\tilde{\mathbf{x}}^n) - F(\mathbf{x}^n) - \frac{\lambda}{n}\mathbf{d}(\tilde{\mathbf{x}}^n,\mathbf{x}^n)^p\right\}\right]\right\}.$$

Let $t > 0$. Then with probability at least $1-e^{-t}$,

$$F(\mathbf{x}^n) \le \mathcal{R}_{S_n,p}\left(\sqrt{\frac{\tau_n t}{n^{\frac{2}{p}}}};\, F\right).$$

Proof of Proposition 4. We first fix $f \in \mathscr{F}$. Set $F(\mathbf{x}^n) = \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] - \mathbb{E}_{\mathbb{P}_n}[f]$. By definition $\mathbb{E}_{S_n}[F(\mathbf{x}^n)] = 0$. By Assumption 1 we have

$$\begin{aligned}
F(\tilde{\mathbf{x}}^n) - F(\mathbf{x}^n) &\le \sup_{f\in\mathscr{F}}\left\{\frac{1}{n}\sum_{i=1}^n f(\tilde{x}^i) - f(x^i)\right\} \\
&\le \frac{1}{n}\sum_{i=1}^n L\|\tilde{x}^i - x^i\| \\
&\le \frac{1}{n}\sum_{i=1}^n L(1 + \|\tilde{x}^i - x^i\|^p) \\
&= L + L\mathbf{d}(\tilde{\mathbf{x}}^n,\mathbf{x}^n)^p.
\end{aligned}$$

Hence, by Lemma 7, with probability at least $1 - e^{-t}$, we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] - \mathbb{E}_{\mathbb{P}_n}[f] \\
&\leq F(\mathbf{x}^n) \\
&\leq \mathcal{R}_{S_n,p}\left(\sqrt{\frac{\tau_n t}{n^{\frac{2}{p}}}}; F\right) \\
&= \min_{\lambda \geq 0}\left\{\lambda\left(\sqrt{\frac{\tau_n t}{n^{\frac{2}{p}}}}\right)^p + \mathbb{E}_{S_n}\left[\sup_{\tilde{\mathbf{x}}^n \in \mathscr{X}^n}\left\{F(\tilde{\mathbf{x}}^n) - F(\mathbf{x}^n) - \frac{\lambda}{n}\mathbf{d}(\tilde{\mathbf{x}}^n, \mathbf{x}^n)^p\right\}\right]\right\} \\
&\leq \min_{\lambda \geq 0}\left\{\lambda\left(\sqrt{\frac{\tau_n t}{n^{\frac{2}{p}}}}\right)^p + \mathbb{E}_{S_n}\left[\frac{1}{n}\sup_{\tilde{\mathbf{x}}^n \in \mathscr{X}^n}\sum_{i=1}^n\big(-f(\tilde{x}^i) + f(x^i) - \lambda\|\tilde{x}^i - x^i\|^p\big)\right]\right\} \\
&= \min_{\lambda \geq 0}\left\{\lambda\left(\sqrt{\frac{\tau_n t}{n^{\frac{2}{p}}}}\right)^p + \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}\left[\sup_{\tilde{x} \in \mathscr{X}}\big\{-f(\tilde{x}) + f(x) - \lambda\|\tilde{x} - x\|^p\big\}\right]\right\} \\
&= \mathcal{L}_p^{\mathsf{wo}}(-f; \varrho_n) - \mathcal{L}^{\mathsf{true}}(-f).
\end{aligned}
$$

With slight abuse of notation, we denote the last line as $\mathcal{R}_p\left(\varrho_n; -f\right)$.

To obtain a uniform bound, by Assumption 6, for any distribution $\mathbb{P}$ it holds that

$$\mathbb{E}_{\mathbb{P}}[f_{\theta'}] - \mathbb{E}_{\mathbb{P}}[f_\theta] \leq \kappa\|\theta' - \theta\|_\Theta.$$

Thus

$$\mathcal{R}_p(\varrho_n; -f_{\theta'}) - \mathcal{R}_p(\varrho_n; -f_\theta) \leq \kappa\|\theta' - \theta\|_\theta.$$

Let $\epsilon > 0$ and Θ_ϵ be an ϵ-cover of Θ. We have that

$$
\begin{aligned}
&\mathbb{P}_{\mu_{S_n}}\left\{\exists\theta \in \Theta, s.t.\ \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f_\theta] > \mathbb{E}_{\mathbb{P}_n}[f_\theta] + \mathcal{R}_p(\varrho_n; -f_\theta) + 3\kappa\epsilon\right\} \\
&\leq \mathbb{P}_{\mu_{S_n}}\left\{\exists\theta' \in \Theta_\epsilon, s.t.\ \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f_{\theta'}] > \mathbb{E}_{\mathbb{P}_n}[f_{\theta'}] + \mathcal{R}_p(\varrho_n; -f_{\theta'})\right\} \\
&\leq \sum_{\theta' \in \Theta_\epsilon}\mathbb{P}_{\mu_{S_n}}\left\{\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f_{\theta'}] > \mathbb{E}_{\mathbb{P}_n}[f_{\theta'}] + \mathcal{R}_p(\varrho_n; -f_{\theta'})\right\} \\
&\leq \mathscr{N}(\epsilon; \Theta, \|\cdot\|_\Theta)e^{-t}.
\end{aligned}
$$

Letting $\epsilon = 1/n$ yields that with probability at least $1 - \mathscr{N}(\epsilon; \Theta, \|\cdot\|_\Theta)e^{-t}$, for every $\theta \in \Theta$,

$$\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f_\theta] - \mathbb{E}_{\mathbb{P}_n}[f_\theta] \leq \mathcal{R}_p(\varrho_n; -f_\theta) + \frac{3\kappa}{n}.$$

It remains to bound $\mathcal{R}_p(\varrho; -f)$ using variation. When $p = 1$, Assumption 1 implies that for any $\lambda > \|f\|_{\mathrm{Lip}}$,

$$\sup_{\tilde{x} \in \mathscr{X}}\Big\{-f(\tilde{x}) - f(\hat{x}^i) - \lambda\|\tilde{x} - \hat{x}^i\|\Big\} = 0.$$

Consequently by definition $\mathcal{R}_1(\varrho; -f) \leq \varrho\|f\|_{\mathrm{Lip}}$. When $p \in (1, 2]$, using the argument in [2, Remark 9], there exists $C > 0$, such that $\mathcal{R}_p(\varrho; -f) \leq \||\partial f|\|_q + C\varrho^2$. Therefore the proof is completed.

B.2 Proofs for Example 1

Lemma 8. *We gather a few simple facts about Assumption 6:*

(i) If $\{f_\theta : \theta \in \Theta\}$ satisfies Assumption 6 with parameter κ, and ϕ is a L-Lipschitz over the range of all functions in $\{f_\theta : \theta \in \Theta\}$, then $\{\phi \circ f_\theta : \theta \in \Theta\}$ satisfies Assumption 6 with parameter $L\kappa$.

(ii) If both $\{f_\theta : \theta \in \Theta\}$ and $\{g_\theta : \theta \in \Theta\}$ satisfy Assumption 6 with parameters κ_1 and κ_2, then $\{af_\theta + bg_\theta : \theta \in \Theta\}$ satisfies Assumption 6 with parameters $a\kappa_1 + b\kappa_2$ for any constants $a, b \geq 0$.

(iii) $\{x \mapsto \theta^\top x : \theta \in \Theta, \|x\|_2 \le B\}$ *satisfies Assumption 6 with parameter* B *when* $\|\theta\| = \|\cdot\|_2$.

(iv) $\{x \mapsto \|W^\top x\|_2 : W \in \mathcal{W} \subset \mathbb{R}^{d\times k}, \|x\|_2 \le B\}$ *satisfies Assumption 6 with parameter* B *when* $\|W\|_{\mathcal{W}} = \|W\|_F$.

Proof. (i) $|\phi \circ f_{\theta_1} - \phi \circ f_{\theta_2}| \le L|f_{\theta_1} - f_{\theta_2}| \le L\kappa\|\theta_1 - \theta_2\|$.

(ii) $|af_{\theta_1} + bg_{\theta_1} - af_{\theta_2} - bg_{\theta_2}| \le a|f_{\theta_1} - f_{\theta_2}| + b|g_{\theta_1} - g_{\theta_2}| \le (a\kappa_1 + b\kappa_2)\|\theta_1 - \theta_2\|_2$.

(iii) $|\theta_1^\top x - \theta_2^\top x| \le \|\theta_1 - \theta_2\|_2\|x\|_2 \le B\|\theta_1 - \theta_2\|_2$.

(iv) $|\|W_1^\top x\|_2 - \|W_2^\top x\|_2| \le \|(W_1 - W_2)^\top x\|_2 \le \|W_1 - W_2\|_2\|x\|_2 \le B\|W_1 - W_2\|_2$.

□

Set $\tilde\theta = [\theta, -1]$, then $f_{\tilde\theta}(x, y) = |\tilde\theta^\top(x, y)|^p \in \mathcal{F}$. It is clear from the definition that every $f \in \mathcal{F}$ is piece-wise differentiable. We assumed both feature space $\mathscr{X}$ and the weight space Θ are bounded: $\|x\|_2 \le B_1$ for all $x \in \mathscr{X}$, $|y| \le B_2$ for all $y \in \mathscr{Y}$, and $\|\theta\|_2 \le B_3 - 1$ for all $\theta \in \Theta$. Note that $|\cdot|^p$ is Lipschitz, with constant bounded by the upper bound of the gradient norm:

$$p|\tilde\theta^\top(x, y)|^{p-1} \le p\|\tilde\theta\|_2^{p-1}\|(x, y)\|_2^{p-1} \le pB_3^{p-1}(B_1 + B_2)^{p-1}, \tag{4}$$

hence Assumption 1 is verified.

To verify Assumption 6, observe that by (4), $|\cdot|^p$ is Lipschitz over the range of $\tilde\theta^\top(x, y)$, for all $\theta \in \Theta$. The verification follows from Lemma 8 (i) and (iii).

C Proofs for Section 3

C.1 Proof of Proposition 1

Fix $f \in \mathscr{F}$. Consider any $x^i \in \mathscr{X}$. Suppose the first non-differentiable point on the line segment connecting from $\hat x^i$ to x^i is $\hat x^i + t_i(x^i - \hat x^i)$. If there is no non-differentiable point, we simply let $t_i = 1$. Then on differentiable points $\hat x^i$ we have

$$\begin{aligned}
&|f(x^i) - f(\hat x^i) - \langle \nabla f(\hat x^i), x^i - \hat x^i\rangle| \\
&= |\int_0^1 \langle \nabla f(\hat x^i + t(x^i - \hat x^i)) - \nabla f(\hat x^i), x^i - \hat x^i\rangle dt| \\
&\le \int_0^{t_i} |\langle \nabla f(\hat x^i + t(x^i - \hat x^i)) - \nabla f(\hat x^i), x^i - \hat x^i\rangle| dt \\
&\quad + \int_{t_i}^1 |\langle \nabla f(\hat x^i + t(x^i - \hat x^i)) - \nabla f(\hat x^i), x^i - \hat x^i\rangle| dt \\
&\le h \cdot \|x^i - \hat x^i\|^{\alpha+1} + 2L \cdot (1 - t_i)\|x^i - \hat x^i\| \\
&\le h \cdot \|x^i - \hat x^i\|^{\alpha+1} + 2L\big(\|x^i - \hat x^i\| - d(\hat x^i, \mathscr{D}_f)\big)_+,
\end{aligned}$$

where the first term follows from Assumption 2, and the second term is due to that $t_i\|x^i - \hat x^i\| \ge d(\hat x^i, \mathscr{D}_f)$, recalling $d(\hat x^i, \mathscr{D}_f)$ denotes the distance between $\hat x^i$ to the set of non-differentiable points of f. It follows that for all $\delta \ge 0$,

$$|\sup_{x\in\mathscr{X}, \|x - \hat x^i\| \le \delta} f(x) - f(\hat x^i) - |\partial f|(\hat x^i)\delta| \le h\delta^{\alpha+1} + 2L\big(\delta - d(\hat x^i, \mathscr{D}_f)\big)_+, \tag{5}$$

where we have used the definition of the local slope $|\partial f|(\hat x^i)$ and Cauchy-Schwartz inequality.

Below, we start with the simple case of $p = \infty$, for which (WO) and (RO) coincide. Using Lemma 1, we have

$$\mathcal{L}^{\mathsf{wo}}_{n,\infty}(f; \varrho) = \mathcal{L}^{\mathsf{ro}}_{n,\infty}(f; \varrho) = \frac{1}{n}\sum_{i=1}^n \sup_{x\in\mathscr{X}, \|x - \hat x^i\| \le \varrho} f(x).$$

By (3) and (5),

$$\big|\mathcal{L}^{\mathsf{wo}}_{n,\infty}(f; \varrho) - \mathcal{L}^{\mathsf{vr}}_{n,q}(f; \varrho)\big| \le h\varrho^{\alpha+1} + 2Le_n(\varrho).$$

Next we consider $p \in (1, \infty)$. Let us first prove for the upper bound. By Assumption 1,

$$0 \le \sup_{x^i \in \mathscr{X}} \{f(x^i) - f(\hat{x}^i) - \lambda \|x^i - \hat{x}^i\|^p\} \le \sup_{\delta \ge 0} \{L\delta - \lambda \delta^p\},$$

hence the optimal δ satisfies

$$\delta \le (\tfrac{L}{\lambda})^{\frac{1}{p-1}}. \tag{6}$$

If $p < 1 + \alpha$, by (5) we can bound the inner maximization as

$$\begin{aligned}
&\sup_{x^i} \{f(x^i) - f(\hat{x}^i) - \lambda \|x^i - \hat{x}^i\|^p\} \\
&\le \sup_{\delta \ge 0} \{|\partial f|(\hat{x}^i)\delta + h\delta^{\alpha+1} + 2L\big(\delta - d(\hat{x}^i, \mathscr{D}_f)\big)_+ - \lambda \delta^p\} \\
&\le \sup_{\delta \ge 0} \{|\partial f|(\hat{x}^i)\delta - (\lambda - h\delta^{\alpha+1-p})\delta^p + 2L\big(\delta - d(\hat{x}^i, \mathscr{D}_f)\big)_+\} \\
&\le \sup_{\delta \ge 0} \{|\partial f|(\hat{x}^i)\delta - (\lambda - h(\tfrac{L}{\lambda})^{\frac{\alpha+1-p}{p-1}})\delta^p\} + 2L\big((\tfrac{L}{\lambda})^{\frac{1}{p-1}} - d(\hat{x}^i, \mathscr{D}_f)\big)_+.
\end{aligned}$$

It follows from Lemma 1 that

$$\begin{aligned}
&\mathcal{L}^{\mathsf{wo}}_{n,p}(f; \varrho) - \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) \\
&\le \min_{\lambda \ge L(\frac{h}{L})^{\frac{p-1}{\alpha}}} \Big\{\lambda \varrho^p + \tfrac{1}{n}\sum_{i=1}^n \sup_{\delta_i > 0} \{|\partial f|(\hat{x}^i)\delta_i - (\lambda - h(\tfrac{L}{\lambda})^{\frac{\alpha+1-p}{p-1}})\delta_i^p\} + 2L\big((\tfrac{L}{\lambda - L(\frac{h}{L})^{\frac{p-1}{\alpha}}})^{\frac{1}{p-1}} - d(\hat{x}^i, \mathscr{D}_f)\big)_+\Big\} \\
&\le \min_{\lambda \ge L(\frac{h}{L})^{\frac{p-1}{\alpha}}} \Big\{\lambda \varrho^p + \tfrac{1}{n}\sum_{i=1}^n \sup_{\delta_i > 0} \{|\partial f|(\hat{x}^i)\delta_i - (\lambda - L(\tfrac{h}{L})^{\frac{p-1}{\alpha}})\delta_i^p\} + 2L\big((\tfrac{L}{\lambda - L(\frac{h}{L})^{\frac{p-1}{\alpha}}})^{\frac{1}{p-1}} - d(\hat{x}^i, \mathscr{D}_f)\big)_+\Big\} \\
&\overset{\lambda \leftarrow \lambda - L(\frac{h}{L})^{\frac{p-1}{\alpha}}}{=} \min_{\lambda \ge 0} \Big\{\lambda \varrho^p + \tfrac{1}{n}\sum_{i=1}^n \sup_{\delta_i > 0} \{|\partial f|(\hat{x}^i)\delta_i - \lambda \delta_i^p\} + 2L\big((\tfrac{L}{\lambda})^{\frac{1}{p-1}} - d(\hat{x}^i, \mathscr{D}_f)\big)_+\Big\} + L(\tfrac{h}{L})^{\frac{p-1}{\alpha}} \varrho^p \\
&\overset{\lambda = \frac{\||\partial f|\|_{\mathbb{P}_n,q}}{p}\varrho^{1-p}}{\le} \||\partial f|\|_{\mathbb{P}_n,q} \cdot \varrho + L(\tfrac{h}{L})^{\frac{p-1}{\alpha}} \varrho^p + 2L\mathbb{E}_{\mathbb{P}_n}\big[\big((\tfrac{Lp}{\||\partial f|\|_{\mathbb{P}_n,q}})^{\frac{1}{p-1}}\varrho - d(\hat{x}^i, \mathscr{D}_f)\big)_+\big].
\end{aligned} \tag{7}$$

If $p \ge 1 + \alpha$, by (5) and (6), we have

$$\sup_{x^i} \{f(x^i) - f(\hat{x}^i) - \lambda \|x^i - \hat{x}^i\|^p\} \le \sup_{\delta \ge 0} \{|\partial f|(\hat{x}^i)\delta - \lambda \delta^p\} + h(\tfrac{L}{\lambda})^{\frac{\alpha+1}{p-1}} + L\big((\tfrac{L}{\lambda})^{\frac{1}{p-1}} - d(\hat{x}^i, \mathscr{D}_f)\big)_+.$$

It follows that

$$\begin{aligned}
&\mathcal{L}^{\mathsf{wo}}_{n,p}(f; \varrho) - \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) \\
&\le \min_{\lambda \ge 0} \Big\{\lambda \varrho^p + \tfrac{1}{n}\sum_{i=1}^n \sup_{\delta_i > 0} \{|\partial f|(\hat{x}^i)\delta_i - \lambda \delta_i^p\} + h(\tfrac{L}{\lambda})^{\frac{\alpha+1}{p-1}} + L\big((\tfrac{L}{\lambda})^{\frac{1}{p-1}} - d(\hat{x}^i, \mathscr{D}_f)\big)_+\Big\} \\
&\le \||\partial f|\|_{\mathbb{P}_n,q} \cdot \varrho + h(\tfrac{Lp}{\||\partial f|\|_{\mathbb{P}_n,q}})^{\frac{\alpha+1}{p-1}} \varrho^{\alpha+1} + 2L\mathbb{E}_{\mathbb{P}_n}\big[\big((\tfrac{Lp}{\||\partial f|\|_{\mathbb{P}_n,q}})^{\frac{1}{p-1}}\varrho - d(\hat{x}^i, \mathscr{D}_f)\big)_+\big],
\end{aligned} \tag{8}$$

where the last inequality holds by taking a feasible solution $\lambda = \frac{\||\partial f|\|_{\mathbb{P}_n,q}}{p}\varrho^{1-p}$. This completes the proof for the upper bound.

Next, we prove the lower bound. If $p \ge \alpha + 1$, by (5) we have

$$\begin{aligned}
&\frac{1}{n}\sup_{\{x^i\}_i \in \mathscr{X}} \sum_{i=1}^n (f(x^i) - f(\hat{x}^i)) \\
&\ge \tfrac{1}{n}\sum_{i=1}^n \sup_{\delta_i \ge 0} \{|\partial f|(\hat{x}^i)\delta_i - h\delta_i^{\alpha+1} - 2L\big(\delta_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+ : \tfrac{1}{n}\sum_{i=1}^n \delta_i^p \le \varrho^p\} \\
&\ge \tfrac{1}{n}\sup_{\delta_i \ge 0} \{\sum_{i=1}^n |\partial f|(\hat{x}^i)\delta_i - 2L\big(\delta - d(\hat{x}^i, \mathscr{D}_f)\big)_+ : \tfrac{1}{n}\sum_{i=1}^n \delta_i^p \le \varrho^p\} - \tfrac{1}{n}\sup_{\delta_i \ge 0} \sum_{i=1}^n \{h\delta_i^{\alpha+1} : \tfrac{1}{n}\sum_{i=1}^n \delta_i^p \le \varrho^p\} \\
&\ge \||\partial f|\|_{q,\mathbb{P}_n}\varrho - 2L \cdot \mathbb{E}_{\mathbb{P}_n}\Big[\big((\tfrac{|\partial f|}{\||\partial f|\|_{q,\mathbb{P}_n}})^{q-1}\varrho - d(\hat{x}, \mathscr{D}_f)\big)_+\Big] - h\varrho^{\alpha+1},
\end{aligned}$$

where the first term in the last inequality is obtained by taking $\delta_i = \frac{|\partial f|(\hat{x}^i)^{q-1}}{|||\partial f|||_{q,\mathbb{P}_n}^{q-1}}\varrho$, and the second term is due to Hölder's inequality. If $p < \alpha + 1$, using Lemma 1 we have

$$\mathcal{L}_{n,p}^{\mathsf{wo}}(f;\varrho) - \frac{1}{n}\sum_{i=1}^{n} f(\hat{x}^i) \geq \min_{\lambda\geq 0}\Big\{\lambda\varrho^p + \tfrac{1}{n}\sum_{i=1}^{n}\sup_{\delta\geq 0}\{|\partial f|(\hat{x}^i)\delta - h\delta^{\alpha+1} - L\big(\delta - d(\hat{x}^i,\mathscr{D}_f)\big)_+ - \lambda\delta^p\}\Big\},$$

Using the fact that the optimal $\delta < (\frac{L}{h})^{\frac{1}{\alpha}}$ (c.f. (6)), we can bound the inner supremum as

$$\begin{aligned}
&\sup_{\delta\geq 0}\{|\partial f|(\hat{x}^i)\delta - h\delta^{\alpha+1} - 2L\big(\delta - d(\hat{x}^i,\mathscr{D}_f)\big)_+ - \lambda\delta^p\}\\
=&\sup_{\delta}\{|\partial f|(\hat{x}^i)\delta - (\lambda + h\delta^{\alpha+1-p})\delta^p - 2L\big(\delta - d(\hat{x}^i,\mathscr{D}_f)\big)_+\}\\
\geq&\sup_{\delta}\{|\partial f|(\hat{x}^i)\delta - (\lambda + h(\tfrac{L}{h})^{\frac{\alpha+1-p}{\alpha}})\delta^p - 2L\big(\delta - d(\hat{x}^i,\mathscr{D}_f)\big)_+\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lambda\varrho^p + \frac{1}{n}\sum_{i=1}^{n}\sup_{\delta\geq 0}\{|\partial f|(\hat{x}^i)\delta - h\delta^{\alpha+1} - L\big(\delta - d(\hat{x}^i,\mathscr{D}_f)\big)_+ - \lambda\delta^p\}\\
\geq&\ \lambda\varrho^p + \tfrac{1}{n}\sum_{i=1}^{n}\sup_{\delta\geq 0}\{|\partial f|(\hat{x}^i)\delta - (\lambda + h(\tfrac{L}{h})^{\frac{\alpha+1-p}{\alpha}})\delta^p - 2L\big(\delta - d(\hat{x}^i,\mathscr{D}_f)\big)_+\}\\
\geq&\ |||\partial f|||_{q,\mathbb{P}_n}\cdot\varrho - h(\tfrac{L}{h})^{\frac{\alpha+1-p}{\alpha}}\varrho^p - 2L\cdot\mathbb{E}_{\mathbb{P}_n}\left[\big((\tfrac{|\partial f|(\hat{x})}{|||\partial f|||_{q,\mathbb{P}_n}})^{q-1}\varrho - d(\hat{x},\mathscr{D}_f)\big)_+\right],
\end{aligned}$$

where the last inequality holds by taking $\delta = \frac{|\partial f|(\hat{x})^{q-1}}{|||\partial f|||_{q,\mathbb{P}_n}^{q-1}}\varrho$. Thus

$$\mathcal{L}_{n,p}^{\mathsf{wo}}(f;\varrho) - \frac{1}{n}\sum_{i=1}^{n} f(\hat{x}^i) \geq |||\partial f|||_{q,\mathbb{P}_n}\varrho - h(\tfrac{L}{h})^{\frac{\alpha+1-p}{\alpha}}\varrho^p - 2L\cdot\mathbb{E}_{\mathbb{P}_n}\left[\big((\tfrac{|\partial f|(\hat{x})}{|||\partial f|||_{q,\mathbb{P}_n}})^{q-1}\varrho - d(\hat{x},\mathscr{D}_f)\big)_+\right].$$

Finally, using the assumption that $|||\partial f|||_{q,\mathbb{P}_n} \geq \tilde{\eta}$ for all $f \in \mathscr{F}$, we get the desired result.

We remark that the result can be extended to arbitrary nominal distributions using exactly the same idea, with summation replaced by integration.

C.2 Proof of Proposition 2

Since $\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho) - \mathcal{L}_{n,p}^{\mathsf{ro}}(\varrho) \geq 0$ trivially holds, it suffices to prove the other direction. Fixing $f \in \mathscr{F}$, consider the dual problem from Lemma 1,

$$\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho) = \min_{\lambda\geq 0}\Big\{\lambda\varrho^p + \tfrac{1}{n}\sum_{i=1}^{n}\sup_{x^i\in\mathscr{X}}\{f(x^i) - \lambda\|x^i - \hat{x}^i\|^p\}\Big\}.$$

Define

$$v(\lambda) = \lambda\varrho^p + \tfrac{1}{n}\sum_{i=1}^{n}\sup_{x\in\mathscr{X}}\{f(x) - \lambda\|x - \hat{x}^i\|^p\},$$

and let λ_n be a minimizer. If $\lambda_n = 0$, then $\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho) = \mathcal{L}_{n,p}^{\mathsf{ro}}(\varrho)$, since according to the structure of the worst-case distribution [20], there exists a worst-case distribution that supports on n points. In the sequel we consider $\lambda_n > 0$. By Assumption 1 and the structure of the worst-case distribution [20], $\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho)$ is attained at a distribution of the form

$$\frac{1}{n}\sum_{i=1}^{n-1}\boldsymbol{\delta}_{x^i} + \frac{1-\epsilon}{n}\boldsymbol{\delta}_{x_-^n} + \frac{\epsilon}{n}\boldsymbol{\delta}_{x_+^n},$$

where

$$\begin{aligned}
x^i &\in \arg\max_{x\in\mathscr{X}}\{f(x) - \lambda\|x - \hat{x}^i\|^p\}, \quad i = 1,\ldots,n,\\
x_\pm^n &\in \arg\max_{x\in\mathscr{X}}\{f(x) - \lambda\|x - \hat{x}^n\|^p\},
\end{aligned}$$

and

$$\frac{1}{n}\sum_{i=1}^{n-1}\|x^i - \hat{x}^i\|^p + \frac{1-\epsilon}{n}\|x_-^n - \hat{x}^n\|^p + \frac{\epsilon}{n}\|x_+^n - \hat{x}^n\|^p = \varrho^p.$$

By (6), we have that $\|x^i - \hat{x}^i\| \le (\frac{L}{\lambda_n})^{\frac{1}{p-1}}$. Without loss of generality, we assume that $\|x_+^n - \hat{x}^n\| \ge \|x_-^n - \hat{x}^n\|$, which implies $f(x_+^n) \ge f(x_-^n)$. It follows that $\{x^i, i = 1, \dots n-1, x_-^n\} \in \mathscr{X}_p(\varrho)$, and we have

$$\begin{aligned}\max_{\{x^i\}_{i=1}^n \in \mathscr{X}_p(\varrho)} \frac{1}{n}\sum_{i=1}^n f(x^i) &\ge \frac{1}{n}\sum_{i=1}^{n-1} f(x^i) + \frac{1}{n} f(x_-^n) \\ &\ge \mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho) - \frac{\epsilon}{n}(f(x_-^n) - f(x_+^n)) \\ &\ge \mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho) - \tfrac{2L}{n}(\tfrac{L}{\lambda_n})^{\frac{1}{p-1}}.\end{aligned}$$

It remains to prove that with probability $1 - e^{-t}$, we can lower bound λ_n for all $f \in \mathscr{F}$. To this end, observe that by choosing $\lambda_0 = \frac{|||\partial f|||_{\mathbb{P}_n,q}}{p}\varrho^{-p+1}$, by (7) and (8), we have for all $f \in \mathscr{F}$,

$$\begin{aligned}v(\lambda_0) - \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) \le& |||\partial f|||_{\mathbb{P}_n,q} \cdot \varrho + 2Le_n\Big((\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})^{\frac{1}{p-1}}\varrho\Big) \\ &+ \mathbf{1}\{p < \alpha+1\}L(\tfrac{h}{L})^{\frac{p-1}{\alpha}}\varrho^p + \mathbf{1}\{p \ge \alpha+1\}h(\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})\varrho^{\alpha+1},\end{aligned}$$

recalling that e_n is defined in (3). Also note that

$$\begin{aligned}v(\lambda_n) - \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) &\overset{(5)}{\ge} \frac{1}{n}\sum_{i=1}^n \sup_{\delta_i \ge 0}\{|\partial f|(\hat{x}^i)\delta - h\delta_i^{\alpha+1} - 2Le_n(\delta_i) - \lambda_n\delta_i^p\} \\ &\overset{\delta_i = 2\varrho}{\ge} 2|||\partial f|||_{q,\mathbb{P}_n} \cdot \varrho - 2^{\alpha+1}\varrho^{\alpha+1}h - 2^p\varrho^p\lambda_n - 2Le_n(2\varrho).\end{aligned}$$

If

$$\begin{aligned}2^p\varrho^p\lambda_n <& |||\partial f|||_{q,\mathbb{P}_n} \cdot \varrho - 2^{\alpha+1}\varrho^{\alpha+1}h - 2Le_n(2\varrho) \\ &- 2Le_n\Big((\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})^{\frac{1}{p-1}}\varrho\Big) - L(\tfrac{h}{L})^{\frac{p-1}{\alpha}}\varrho^p - h(\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})\varrho^{\alpha+1},\end{aligned}$$

then

$$\begin{aligned}&v(\lambda_n) - \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i) \\ &> |||\partial f|||_{q,\mathbb{P}_n} \cdot \varrho + 2Le_n\Big((\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})^{\frac{1}{p-1}}\varrho\Big) + L(\tfrac{h}{L})^{\frac{p-1}{\alpha}}\varrho^p + h(\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})\varrho^{\alpha+1} \\ &\ge v(\lambda_0) - \frac{1}{n}\sum_{i=1}^n f(\hat{x}^i),\end{aligned}$$

which is contradicted to the optimality of λ_n. Hence, we must have

$$\begin{aligned}2^p\varrho^p\lambda_n \ge& |||\partial f|||_{q,\mathbb{P}_n} \cdot \varrho - 2^{\alpha+1}\varrho^{\alpha+1}h - 2Le_n\Big(2\varrho\Big) \\ &- 2Le_n\Big((\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})^{\frac{1}{p-1}}\varrho\Big) - L(\tfrac{h}{L})^{\frac{p-1}{\alpha}}\varrho^p - h(\tfrac{Lp}{|||\partial f|||_{\mathbb{P}_n,q}})\varrho^{\alpha+1},\end{aligned}$$

hence we complete the proof.

C.3 Proof of Proposition 3

In the sequel, when the loss function $f \in \mathscr{F}$ is clear from the context, we denote $H(x), G(x)$ for $H_f(x), G_f(x)$, and G_i for $G(\hat{x}_i)$ to simplify the notation. The upper bound is straightforward.

To prove the lower bound, we focus on (WO), without loss of generality, we assume that $G(\hat{x}^1) \ge G(\hat{x}^2) \ge \dots G(\hat{x}^n)$, otherwise, we just relabel them. Suppose the maximum in defining G_i is attained at $\tilde{x}^i$ with distance d_i from $\hat{x}^i$; otherwise we can use an approximation argument. Starting from $i = 1$, if $d_i > n^\delta\varrho$, we set $x^i = \tilde{x}^i$, otherwise we set $x^i = \hat{x}^i + n^\delta\varrho u_i$, where u_i is a direction to be determined later, i.e., we perturb $\hat{x}^i$ with distance $\delta_i = \|x^i - \hat{x}^i\| = \max(d_i, n^\delta\varrho)$, in the direction of $\tilde{x}^i - \hat{x}^i$ if $d_i > n^\delta\varrho$, or u_i otherwise. Then we proceed to $i+1$, until we achieve the largest N such that $\frac{1}{n}\sum_{i=1}^N \delta_i \le \varrho$. Note that since each $\delta_i \ge n^\delta\varrho$, we have $N \le \lfloor n^{1-\delta}\rfloor$. If for some i

we cannot perturb it with fully mass $1/n$, we perturb $\frac{\epsilon}{n}$ of $\hat{x}_{N+1}$, so that $\frac{1}{n}\sum_{i=1}^{N}\delta_i + \frac{\epsilon}{n}\delta_{N+1} = \varrho$, where $0 \le \epsilon \le 1$. By construction, $\{x^i\}_{i=1\dots n} \in \mathscr{X}_1(\varrho)$. Hence we have

$$
\begin{aligned}
&\tfrac{1}{n}\sum_{i=1}^{n}(f(x^i) - f(\hat{x}^i)) \\
\ge\ & \tfrac{1}{n}\sum_{i=1}^{N}(f(x^i) - f(\hat{x}^i)) + \frac{\epsilon}{n}(f(x^{N+1}) - f(\hat{x}^{N+1})) \\
=\ & \tfrac{1}{n}\big[\sum_{i:d_i > n^\delta\varrho}(f(x^i) - f(\hat{x}^i)) + \sum_{i:d_i \le n^\delta\varrho}(f(x^i) - f(\hat{x}^i))\big] + \frac{\epsilon}{n}(f(x^{N+1}) - f(\hat{x}^{N+1})) \\
=\ & \tfrac{1}{n}\big[\sum_{i:d_i > n^\delta\varrho}\delta_i G_i + \sum_{i:d_i \le n^\delta\varrho}(f(x^i) - f(\hat{x}^i))\big] + \frac{\epsilon}{n}(f(x^{N+1}) - f(\hat{x}^{N+1})).
\end{aligned}
$$

To bound second sum above involving $d_i \le n^\delta\varrho$, by (5), we choose u_i (and thus x^i) so that

$$
f(x^i) - f(\hat{x}^i) \ge |\partial f|(\hat{x}^i)\delta_i - (h\cdot\delta_i^{\alpha+1} + L\big(\delta_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+). \tag{9}
$$

On the other hand, to bound the first sum above involving the difference between G_i and $|\partial f|(\hat{x}^i)$, by (5) and $\delta_i \ge d_i > 0$ when $d_i \le n^\delta\varrho$, we have

$$
G_i d_i = f(\tilde{x}^i) - f(\hat{x}^i) \le |\partial f|(\hat{x}^i)d_i + h\cdot d_i^{\alpha+1} + L\big(d_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+.
$$

Hence,

$$
\begin{aligned}
(G_i - |\partial f|(\hat{x}^i))\delta_i &= (G_i - |\partial f|(\hat{x}^i))d_i \cdot \frac{\delta_i}{d_i} \\
&\le h\cdot d_i^\alpha\delta_i + L\big(d_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+ \cdot \frac{\delta_i}{d_i} \\
&\le h\cdot\delta_i^{\alpha+1} + L\big(\delta_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+.
\end{aligned} \tag{10}
$$

Therefore we have

$$
\begin{aligned}
&\tfrac{1}{n}\sum_{i=1}^{n}(f(x^i) - f(\hat{x}^i)) \\
\ge\ & \tfrac{1}{n}\Big[\sum_{i:d_i > n^\delta\varrho}\delta_i G_i + \sum_{i:d_i \le n^\delta\varrho} G_i\delta_i - 2\big(h\cdot\delta_i^{\alpha+1} + L\big(\delta_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+\big)\Big] + \frac{\epsilon}{n}\delta_{N+1}G_{N+1} \\
\ge\ & \frac{1}{n}\sum_{i=1}^{N}\delta_i G_i + \frac{\epsilon}{n}\delta_{N+1}G_{N+1} - \frac{2}{n}\sum_{i=1}^{N+1}\big(h\cdot(n^\delta\varrho)^{\alpha+1} + L\big((n^\delta\varrho) - d(\hat{x}^i, \mathscr{D}_f)\big)_+\big) \\
\ge\ & \Delta_n\varrho - \frac{2h(N+1)}{n}(n^\delta\varrho)^{\alpha+1} - 2L\mathbb{E}_{\mathbb{P}_n}\big[\big(n^\delta\varrho - d(x, \mathscr{D}_f)\big)_+\big] \\
\ge\ & \Delta_n\varrho - (4h\varrho^{\alpha+1})n^{\delta\alpha} - 2Le_n(n^\delta\varrho),
\end{aligned}
$$

For (RO), fix $N = \lfloor n^{1-\delta}\rfloor + 1$, and we just perturb the first $\lfloor n^{1-\delta}\rfloor$ points x^i to the distance $\delta_i = \|x^i - \hat{x}^i\| = \frac{n\varrho}{N} \le n^\delta\varrho$, with the direction so that:

$$
f(x^i) - f(\hat{x}^i) \ge |\partial f|(\hat{x}^i)\delta_i - (h\cdot\delta_i^{\alpha+1} + L\big(\delta_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+),
$$

and remain the rest points unchanged. Then,

$$
\begin{aligned}
&\tfrac{1}{n}\sum_{i=1}^{n}(f(x^i) - f(\hat{x}^i)) \\
\ge\ & \tfrac{1}{n}\sum_{i=1}^{n}\Big[|\partial f|(\hat{x}^i)\delta_i - (h\cdot\delta_i^{\alpha+1} + L\big(\delta_i - d(\hat{x}^i, \mathscr{D}_f)\big)_+)\Big] \\
\ge\ & \frac{1}{n}\sum_{i=1}^{N}\delta_i|\partial f|(\hat{x}^i) - \frac{1}{n}\sum_{i=1}^{N}\big(h\cdot(n^\delta\varrho)^{\alpha+1} + L\big((n^\delta\varrho) - d(\hat{x}^i, \mathscr{D}_f)\big)_+\big) \\
\ge\ & |\partial f|(\hat{x}^{N+1})\varrho - \frac{hN}{n}(n^\delta\varrho)^{\alpha+1} - L\mathbb{E}_{\mathbb{P}_n}\big[\big(n^\delta\varrho - d(x, \mathscr{D}_f)\big)_+\big] \\
\ge\ & |\partial f|(\hat{x}^{N+1})\varrho - (2h\varrho^{\alpha+1})n^{\delta\alpha} - L\mathbb{E}_{\mathbb{P}_n}\big[\big(n^\delta\varrho - d(x, \mathscr{D}_f)\big)_+\big].
\end{aligned}
$$

The rest follows the similar arguments as for (WO). □

Now we bound Δ_n. Observe that $H_f(\Delta_n)$ is the $\lfloor n^{1-\delta} \rfloor$-th order statistic of n i.i.d samplings from unit uniform distribution, which follows Beta distribution $B(\lfloor n^{1-\delta} \rfloor, n - \lfloor n^{1-\delta} \rfloor)$, which is sub-Gaussian with proxy variance $\frac{1}{4n}$ [29]. Hence, for any $t > 0$, with probability at least $1 - e^{-2nt^2}$, we have

$$H(\Delta_n) - \frac{\lfloor n^{1-\delta} \rfloor + 1}{n+1} < t.$$

Replacing t with $\frac{t}{\sqrt{n}}$, we have with probability at least $1 - e^{-2t}$,

$$\Delta_n \leq H^{-1}(\frac{t}{\sqrt{n}} + \frac{\lfloor n^{1-\delta} \rfloor + 1}{n+1}) \leq c^{-\beta}(\frac{t}{\sqrt{n}} + \frac{n^{1-\delta} + 1}{n+1})^{\beta},$$

whenever $\frac{t}{\sqrt{n}} + \frac{\lfloor n^{1-\delta} \rfloor + 1}{n+1} \leq \bar{a}$.

D Proofs for Section 5.1

D.1 Proofs for Example 3

Let us verify $\mathscr{F}$ is a family of Lipschitz functions, hence Assumption 1.

$$\begin{aligned} |f_W(x) - f_W(\tilde{x})| &\leq \|W^\top(x - \tilde{x})\|_2(\|W^\top \tilde{x}\|_2 + \|W^\top x\|_2) \\ &\leq 2B\|W\|_F^2\|x - \tilde{x}\|_2, \end{aligned}$$

hence we get that f_W is $2Bk$-Lipschitz.

To verify Assumption 6, observe that $\|W^\top x\|_2 \leq \|W\|_F\|x\|_2 \leq \sqrt{k}B$, hence $(\cdot)^2$ is Lipschitz over the range of $\|W^\top x\|_2$, where $W^\top W = I_k$. Hence, using Lemma 8(i)(iv) we have $\kappa = 2\sqrt{k}B^2$.

Moreover, we have that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_{\text{true}}}[\|\nabla f_W\|_2^2] &= \mathbb{E}_{\mathbb{P}_{\text{true}}}[\|2WW^\top x\|_2^2] \\ &= 4\mathbb{E}_{\mathbb{P}_{\text{true}}}[x^\top WW^\top x] \\ &= 4\text{Tr}(W^\top \mathbb{E}_{\mathbb{P}_{\text{true}}}[xx^\top]W) \\ &= \sum_{j=1}^{k} w_j^\top \mathbb{E}_{\mathbb{P}_{\text{true}}}[xx^\top]w_j \\ &\geq 4k\lambda, \end{aligned}$$

hence Assumption 3 is verified with $\eta = 2\sqrt{k\lambda}$.

D.2 Proofs for Example 4

Since the domain of u is unbounded, we first derive a probabilistic upper bound on the minimizer of (CVaR) for each fixed w, denoted as $u_n(w)$. We have that

$$|u_n(w)| \leq \max_{1 \leq i \leq n} w^\top x^i \leq B \max_{1 \leq i \leq n} \|x^i\|_2.$$

By [18, Lemma 24], we know that with probability at least $1 - e^{-t}$,

$$\max_{1 \leq i \leq n} \|x^i\|_2 \leq \mathbb{E}_{\mathbb{P}_{\text{true}}}[\|x\|_2] + \sqrt{\tau(t + \log(n))},$$

and thus $|u_n(w)| \leq B(\mathbb{E}_{\mathbb{P}_{\text{true}}}[\|x\|_2] + \sqrt{\tau(t + \log(n))})$.

To verify Assumption 6, note that $(\cdot)_+$ is 1-Lipschitz everywhere, also $g_\theta(x, y) = u$ is a trivial function that satisfies Assumption 6, hence we get $\kappa = 1 + \frac{1}{\nu}B(1 + B_1 + \sqrt{\tau(t + \log(n))})$ by Lemma 8 (i)(ii).

E Proofs for Section 5.2

E.1 Proofs for Theorems 3 and 4

Lemma 9. *Assume Assumptions 1, 2 and 4 hold, and $\mathbb{P}_{\mathsf{true}}$ is continuous, and $\varrho \le c^{\frac{1}{\delta}}(\frac{\bar{a}}{2})^{\frac{1}{\beta\delta}}$, where the constants are from Assumption 4. Then*

$$|\mathcal{L}_1^{\mathsf{adv}}(\varrho; f) - \mathcal{L}_\infty^{\mathsf{vr}}(\varrho; f)| \le C\varrho^{(1+\frac{\alpha\beta}{\alpha+\beta})}, \quad \forall f \in \mathscr{F}.$$

for some $C > 0$.

Proof. For every $x \in \mathscr{X}$, let $S(x)$ be such that

$$f(S(x)) - f(x) = d(x, S(x))G(f)(x),$$

where we have assumed the existence of the maximizer defining $G(f)(x)$, otherwise we can argue by approximation. Let $\epsilon, \delta > 0$. Set $\mathscr{X}_\epsilon = \{x \in \mathscr{X} : G(f)(x) > \|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}} - \epsilon\}$.

Define a mapping $T_\epsilon : \mathscr{X} \to \mathscr{X}$ as

$$T_\epsilon(x) = \begin{cases} S(x), & \text{if } x \in \mathscr{X}_\epsilon,\ d(x, S(x)) \ge \varrho^{1-\delta}, \\ x + \varrho^{1-\delta}u, & \text{if } x \in \mathscr{X}_\epsilon,\ d(x, S(x)) < \varrho^{1-\delta}, \\ x, & \text{if } x \in (\mathscr{X} \setminus \mathscr{X}_\epsilon) \cup \mathscr{D}_f, \end{cases}$$

where u is the direction such that

$$f(T_\epsilon(x)) - f(x) \ge |\partial f|(x) \cdot d(x, T_\epsilon(x)) - h \cdot d(x, T_\epsilon(x))^{\alpha+1},$$

which holds due to (5).

Denote the monotonic increasing function

$$M(\epsilon) = \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[d(x, T_\epsilon(x))\mathbf{1}\{x \in \mathscr{X}_\epsilon,\ d(x, S(x)) \ge \varrho^{1-\delta}\}] + \varrho^{1-\delta}\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[\mathbf{1}\{x \in \mathscr{X}_\epsilon,\ d(x, S(x)) < \varrho^{1-\delta}\}],$$

and define

$$\epsilon = \inf_{\varepsilon \ge 0} \big\{M(\varepsilon) \ge \varrho\big\}.$$

By Assumption 4, we have

$$M(\frac{\bar{a}}{2}) \ge \varrho^{1-\delta}\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[\mathbf{1}\{x \in \mathscr{X}_{\frac{\bar{a}}{2}}\}] \ge \varrho^{1-\delta}c(\frac{\bar{a}}{2})^{\frac{1}{\beta}} \ge \varrho.$$

It follows that $\epsilon \le \frac{\bar{a}}{2} < \bar{a}$. For any $\epsilon_0 < \epsilon$, we similarly have

$$\varrho > M(\epsilon_0) \ge \varrho^{1-\delta}\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[\mathbf{1}\{x \in \mathscr{X}_{\epsilon_0}\}] \ge \varrho^{1-\delta}c(\epsilon_0)^{\frac{1}{\beta}}.$$

Taking the limit $\epsilon_0 \to \epsilon$, we get that $\varrho^{1-\delta}c\epsilon^{\frac{1}{\beta}} \le \varrho$, which implies $\epsilon \le (\frac{\varrho^\delta}{c})^\beta$. Define $\epsilon_1 = \epsilon + (\frac{\varrho^\delta}{c})^\beta$. Since $M(\epsilon_1) \ge \varrho$, we choose $r \le 1$ such that $rM(\epsilon_1) = \varrho$. Now we define a distribution $\mathbb{P} = (1-r)\mathbb{P}_{\mathsf{true}} + rT_{\epsilon_1\#}\mathbb{P}_{\mathsf{true}}$. Then

$$\mathbb{E}_{\mathbb{P}}[f(x)] - \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(x)] = r\big(\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(T_{\epsilon_1}(x))] - \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(x)]\big).$$

For $x \in \mathscr{X}_{\epsilon_1}$, if $d(x, S(x)) \ge \varrho^{1-\delta}$, we have $f(T_{\epsilon_1}(x)) - f(x) = G(f)(x) \cdot d(x, T_{\epsilon_1}(x))$, and if $d(x, S(x)) < \varrho^{1-\delta}$, similar to (9) (10) in proof of Proposition 3, it holds that

$$f(T_{\epsilon_1}(x)) - f(x) \ge G(f)(x)d(T_{\epsilon_1}(x), x) - 2h \cdot d(T_{\epsilon_1}(x), x)^{\alpha+1}.$$

It follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[f(x)] - \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(x)] &= r\big(\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(T_{\epsilon_1}(x))] - \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(x)]\big) \\ &\ge rM(\varepsilon_1)(\|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}} - \epsilon_1) - 2rh\varrho^{(1-\delta)(\alpha+1)}\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[\mathbf{1}\{x \in \mathscr{X}_{\epsilon_1}\}] \\ &\ge \varrho\|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}} - 2c^{-\beta}\varrho^{1+\delta\beta} - 2h\varrho^{(1-\delta)(\alpha+1)+\delta}, \end{aligned}$$

where the last row is due to $r\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[\mathbf{1}\{x \in \mathscr{X}_{\epsilon_1}\}]\varrho^{1-\delta} \le rM(\epsilon_1) = \varrho$.

Setting $\delta = \frac{\alpha}{\alpha+\beta}$ yields that

$$\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(T_{\epsilon_1}(x))] - \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f(x)] \ge \varrho\|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}} - C\varrho^{(1+\frac{\alpha\beta}{\alpha+\beta})}.$$

□

Proof of Theorems 3 and 4. Set $\mathcal{L}_q^{\mathsf{vr}}(\varrho; f) := \mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] + \varrho \|\|\partial f\|\|_q$. Observe the following decomposition

$$\begin{aligned} &|\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho; f) - \mathcal{L}_p^{\mathsf{adv}}(\varrho; f)| \\ &\leq |\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho; f) - \mathcal{L}_{n,q}^{\mathsf{vr}}(\varrho; f)| + |\mathcal{L}_{n,q}^{\mathsf{vr}}(\varrho; f) - \mathcal{L}_q^{\mathsf{vr}}(\varrho; f)| + |\mathcal{L}_q^{\mathsf{vr}}(\varrho; f) - \mathcal{L}_p^{\mathsf{adv}}(\varrho; f)|. \end{aligned} \tag{11}$$

Below we bound the three terms on the right-hand side separately. For the last term, by [2, Remark 9], we have $|\mathcal{L}_p^{\mathsf{wo}}(\varrho; f) - \mathcal{L}_q^{\mathsf{vr}}(\varrho; f)| \leq C\varrho^2$ for some $C \geq 0$.

We consider $p > 1$ first. For the first term, by Proposition 1, we have

$$|\mathcal{L}_{n,p}^{\mathsf{wo}}(\varrho; f) - \mathcal{L}_{n,q}^{\mathsf{vr}}(\varrho; f)| \leq C_0 \varrho^{(\alpha+1)\wedge p} + 2Le_n\big((pL/\tilde{\eta})^{q-1}\varrho\big).$$

For the second term, it follows from Lemma 3 and Lemma 6 that

$$\begin{aligned} |\mathcal{L}_{n,q}^{\mathsf{vr}}(\varrho; f) - \mathcal{L}_q^{\mathsf{vr}}(\varrho; f)| &\leq |\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] - \mathbb{E}_{\mathbb{P}_n}[f]| + \varrho |\|\|\partial f\|\|_{q,\mathbb{P}_n} - \|\|\partial f\|\|_{q,\mathbb{P}_{\mathsf{true}}}| \\ &\leq 2\mathfrak{R}_n(\mathscr{F}) + M\sqrt{\frac{t}{2n}} + \frac{\varrho}{q}(\eta \wedge \tilde{\eta})^{-\frac{q}{p}}\Big(2\mathfrak{R}_n(\partial\mathscr{F}_q) + L^q\sqrt{\frac{t}{2n}}\Big). \end{aligned}$$

Therefore, we obtain the result.

Next, we consider $p = 1$. By Proposition 3,

$$|\mathcal{L}_{n,1}^{\mathsf{wo}}(\varrho; f) - \mathcal{L}_{n,\infty}^{\mathsf{vr}}(\varrho; f)| \leq \epsilon_n.$$

Moreover,

$$\begin{aligned} |\mathcal{L}_{n,\infty}^{\mathsf{vr}}(\varrho; f) - \mathcal{L}_\infty^{\mathsf{vr}}(\varrho; f)| &\leq \sup_{f \in \mathscr{F}} \{|\mathbb{E}_{\mathbb{P}_{\mathsf{true}}}[f] - \mathbb{E}_{\mathbb{P}_n}[f]| + \varrho |\|G(f)\|_{\infty,\mathbb{P}_{\mathsf{true}}} - \|G(f)\|_{\infty,\mathbb{P}_n}|\} \\ &\leq 2\mathfrak{R}_n(\mathscr{F}) + M\sqrt{\frac{t}{2n}} + (\|f\|_{\mathrm{Lip}} - \Delta_n)\varrho \\ &\leq 2\mathfrak{R}_n(\mathscr{F}) + M\sqrt{\frac{t}{2n}} + \epsilon_n. \end{aligned}$$

Finally, the third term is computed in Lemma 9. Thereby we complete the proof. □

E.2 Proofs for Example 5

In the linear setting, we have:

$$\partial\mathscr{F}_1^\phi = \left\{(x, y) \mapsto |\phi'(y\theta^\top x)| \|\theta\|_2 : \|\theta\|_2 \leq B_1\right\}.$$

To verify Assumption 2, first note that

$$\begin{aligned} |y_1\theta^\top x_1 - y_2\theta^\top x_2| &\leq |y_1\theta^\top x_1 - y_1\theta^\top x_2| + |y_1\theta^\top x_2 - y_2\theta^\top x_2| \\ &\leq |y_1(\theta^\top x_1 - \theta^\top x_2)| + |(y_1 - y_2)\theta^\top x_2| \\ &\leq B_1\|x_1 - x_2\|_2 + |(y_1 - y_2)|B_1B_2 \\ &\leq 2(B_1 + B_1B_2)\|(x_1, y_1) - (x_2, y_2)\|_2, \end{aligned}$$

hence $\phi'(y_1\theta^\top x_1)|\|\theta\|_2 - \phi'(y_2\theta^\top x_2)|\|\theta\|_2 \leq 2h_\phi B_1(B_1 + B_1B_2)\|(x_1, y_1) - (x_2, y_2)\|_2$.

To verify Assumption 6, we note

$$\begin{aligned} |\phi'(y\theta_1^\top x)|\|\theta_1\|_2 - |\phi'(y\theta_2^\top x)|\|\theta_2\|_2 &\leq |\phi'(y\theta_1^\top x)|\|\theta_1\|_2 - |\phi'(y\theta_1^\top x)|\|\theta_2\|_2 + |\phi'(y\theta_1^\top x)|\|\theta_2\|_2 - |\phi'(y\theta_2^\top x)|\|\theta_2\|_2 \\ &\leq |\phi'(y\theta_1^\top x)|(\|\theta_1\|_2 - \|\theta_2\|_2) + (|\phi'(y\theta_1^\top x)| - |\phi'(y\theta_2^\top x)|)\|\theta_2\|_2 \\ &\leq L_\phi(\|\theta_1\|_2 - \|\theta_2\|_2) + h_\phi(\|\theta_1\|_2 - \|\theta_2\|_2)B_1B_2 \\ &\leq (L_\phi + h_\phi B_1B_2)(\|\theta_1\|_2 - \|\theta_2\|_2). \end{aligned}$$