LEARNING STRUCTURED REPRESENTATIONS BY EMBEDDING CLASS HIERARCHY

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ABSTRACT

Existing models for learning representations in supervised classification problems are permutation invariant with respect to class labels. However, structured knowledge about the classes, such as hierarchical label structures, widely exists in many real-world datasets, e.g., the ImageNet and CIFAR benchmarks. How to learn representations that can preserve such structures among the classes remains an open problem. To approach this problem, given a tree of class hierarchy, we first define a tree metric between any pair of nodes in the tree to be the length of the shortest path connecting them. We then provide a method to learn the hierarchical relationship of class labels by approximately embedding the tree metric in the Euclidean space of features. More concretely, during supervised training, we propose to use the Cophenetic Correlation Coefficient (CPCC) as a regularizer for the cross-entropy loss to correlate the tree metric of classes and the Euclidean distance in the class-conditioned representations. Our proposed regularizer is computationally lightweight and easy to implement. Empirically, we demonstrate that this approach can help to learn more interpretable representations due to the preservation of the tree metric, and leads to better in-distribution generalization as well as under sub-population shifts over six real-world datasets.

1 INTRODUCTION

In supervised learning, the cross-entropy loss is often used for classification tasks. As a common practice in deep learning, in order to train a model for classification, practitioners build a linear layer over the representation to obtain the logit score of each class. A softmax transformation is then applied to convert the logits into a vector belonging to the probability simplex. As a result, we can randomly permute the representations of any classes without affecting the performance of the original classification task. However, in many real-world datasets, as we move towards fine-grained classification, labels are not independent from each other anymore: ImageNet (Deng et al., 2009) inherits label relationship from WordNet (Fellbaum, 1998), that contains both semantic and lexical connections; iNaturalist (Van Horn et al., 2017) borrows the biological taxonomy so that each image contains seven labels that reflect the morphological characteristic of the organism. Many existing works (Deng et al., 2014; Yan et al., 2014; Ristin et al., 2015; Guo et al., 2018; Chen et al., 2019) investigated how to leverage this hierarchical information for various purposes, but how to explicitly project this knowledge onto representations remains unexplored.

In this paper, we focus on the most common label relationship: tree hierarchy. As illustrated in Fig. 1b, given a tree hierarchy of classes, our goal is to learn representations in feature space such that the Euclidean distances between different class centers approximate the distances between these classes in the tree. More concretely, we shall first define a *tree metric* to be the length of the shortest path connecting two subset of classes in the tree hierarchy. Based on this tree metric, we then propose a regularizer, the cophenetic correlation coefficient (CPCC) between sequences of tree metric and Euclidean distance of the feature space, to ensure that the class-conditional representations inherit the tree structure of the classes. Different from the original cross-entropy loss with softmax activation, the proposed CPCC regularizer helps to break the symmetry of permutation invariance among the classes, and thus also improves the interpretability of the learned representations.

We show that the proposed CPCC regularizer is computationally lightweight with negligible overhead, and can be applied to a wide range of supervised learning paradigms, including empirical

risk minimization (ERM), multitask learning, curriculum learning, and a hierarchical learning loss. For generalization, over six real-world datasets, we demonstrate that our proposed CPCC regularizer leads to significantly better generalization performance when there is only limited amount of labeled data as well as on unseen tasks with sub-population shifts.

2 PRELIMINARIES

In this section we first introduce the notations used throughout the paper, formulate our learning problem, and then briefly review the CPCC score to quantify the correlation of two sequences.

Notations and Setup We shall use X and Y to denote the input and target random variables, living in spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. In this work, we mainly focus on the supervised classification setting where for each input data point $x \in \mathcal{X} \subseteq \mathbb{R}^d$, there is a ground-truth label $y \in \mathcal{Y} = [k] := \{1, \ldots, k\}$, where k is the number of output classes. We let μ be the joint distribution over (X, Y) from where the data is sampled. During the learning process, the learner has access to a dataset $\mathcal{D} = \{(x_j, y_j)\}_{j=1}^n$ of size n sampled from μ.

In the context of representation learning, a learned representation $z = f_\theta(x)$ is obtained by applying a feature encoder $f_\theta : \mathcal{X} \to \mathcal{Z}$ parametrized by θ to x, where $\mathcal{Z} \subseteq \mathbb{R}^p$ denotes the feature space. Upon feature vector z, we further apply a linear predictor $g : \mathcal{Z} \to \Delta_k$, where we use Δ_k to denote the $(k-1)$-dimensional probability simplex. The cross-entropy loss is our objective function. Specifically, let $q_y \in \Delta_k$ be a one-hot vector with the y-th component being 1. The cross-entropy loss, $\ell_{\text{CE}}(\cdot, \cdot)$ between the prediction $g \circ f(x)$ and the label y is given by $\ell_{\text{CE}}(g \circ f(x), y) := \sum_{i \in [k]} q_i \log(g(f(x))_i)$. For z, z', $\|z - z'\|_2$ denotes the Euclidean distance between them.

2.1 CLASS HIERARCHY

In classification problems, the target label $Y \in [k]$ is treated as a categorical random variable that can take k different nominal values. However, there is no particular ordering among these k categories, i.e., for different categories $i, j \in [k]$, one can only compare whether $i = j$ or not. Formally, letting $d_H(i, j) = 0$ if $i = j$ and $d_H(i, j) = 1$ otherwise defines a metric $d_H(\cdot, \cdot)$ over $\mathcal{Y}$.

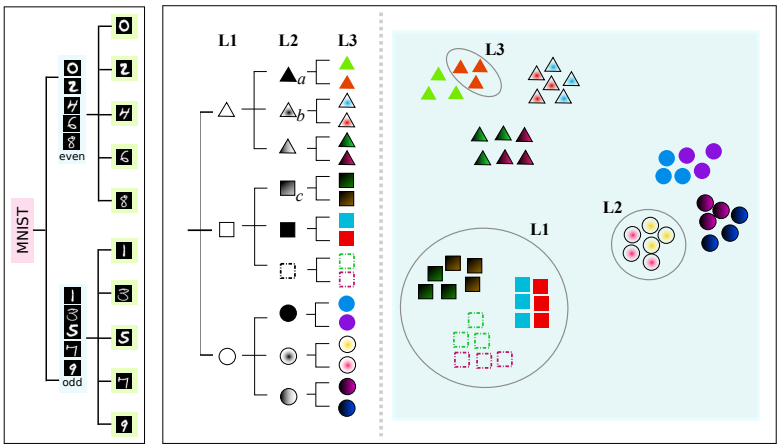
However, in many real-world applications the similarity between different classes is not binary. Consider object classification in ImageNet (Deng et al., 2009) as an example. Intuitively, one would think the distance between the classes corgi and chihuahua to be smaller than that between corgi and panda. One way to characterize this distance between different classes is through a tree of class hierarchy, also known as a dendrogram. An example is shown in Fig. 1a. Formally, let $\mathcal{T} := (V, E, d)$ be a weighted tree, where V is the set of nodes, E the set of weighted edges in $\mathcal{T}$, and $d : V \times V \to \mathbb{R}_+$ specifies the distance between nodes in the tree. Each node v_S in $\mathcal{T}$ is associated with a subset of class labels $S \subseteq [k]$, and can be recursively defined as follows:

1. For each class $i \in [k]$, there is a corresponding leaf node $v_i \in V$. Conversely, each leaf node $v_i \in V$ is identified with a single class label $i \in [k]$.
2. For some $S \subseteq [k]$, if $v_S \in V$ is not a leaf node in $\mathcal{T}$, then its children form a partition of S. In other words, if $v_{S_1}, \ldots, v_{S_c}$ are the children of v_S, then $\forall i \neq j$, $S_i \cap S_j = \varnothing$ and $\cup_{i \in [c]} S_i = S$.
3. The root node of $\mathcal{T}$ is $v_{[k]}$.

At a colloquial level, the tree $\mathcal{T}$ specifies a hierarchy of class labels that represents the structured knowledge among them. For example, as shown in Fig. 1a of the MNIST dataset, the two children of the root node correspond to the odd and even numbers, respectively. Accordingly, the distance between digits 1 and 3 is smaller than that between digits 1 and 2.

2.2 COPHENETIC CORRELATION COEFFICIENT (CPCC)

In the context of clustering, Sokal & Rohlf (1962) introduced the cophenetic correlation coefficient (CPCC) to evaluate the correspondence between two dendrograms. The CPCC is the Pearson's correlation coefficient between two sequences of pairwise distances. For a class hierarchy $\mathcal{T}$ and a node $v \in V$, the depth $dt(v)$ of v is the length of the shortest path from v to the root of $\mathcal{T}$. In the original applications of CPCC, the "dendrogrammatic " ground-truth distance $t(v_i, v_j)$ between a



(a) (b)

Figure 1: Fig. 1a: MNIST class hierarchy. The root node contains all the 10 digit classes. The two children nodes of the root node correspond to the *coarse* classes of odd and even digits, respectively. Each leaf node in this class hierarchy corresponds to a *fine* class label (digit). Fig. 1b: An example of a class hierarchy tree $\mathcal{T}$ along with a visualization of the data in the feature space. The CPCC score computes the correlation coefficient of the tree metric from $\mathcal{T}$ in the left panel and the corresponding Euclidean distance obtained from the feature space in the right panel.

pair of nodes v_i, v_j in $\mathcal{T}$ is defined as follows:

$$t(v_i, v_j) := \max\{dt(v_i), dt(v_j)\} - dt(\text{LCA}\ (v_i, v_j)),$$

where LCA (v_i, v_j) is the least common ancestor (LCA) of v_i and v_j. In Fig. 1a, the LCA of 1 and 3 is "odd" while the LCA of 1 and 0 is the root node. As an example of ground-truth distance, let us consider the class hierarchy tree $\mathcal{T}$ and clustering of classes given in Fig. 1b. In Fig. 1b, $t(\triangle_a, \triangle_b) = 1$ since they share an LCA $\triangle$ at L1, and $t(\triangle_a, \square_c) = 2$ as they go up 2 levels to meet at the the root node, which is their LCA.

Now consider a dataset $\mathcal{D}$. For a node $v_i \in \mathcal{T}$, since v_i corresponds to a subset of classes, we use $\mathcal{D}_i \subseteq \mathcal{D}$ to denote the subset of data points whose class label belongs to v_i. The pairwise distance between $\mathcal{D}_i$ and $\mathcal{D}_j$ is then defined as the Euclidean distance between the center of $\mathcal{D}_i$ and $\mathcal{D}_j$: $\rho(v_i, v_j) := \left\| \frac{1}{n_i} \sum_{x \in \mathcal{D}_i} x - \frac{1}{n_j} \sum_{x' \in \mathcal{D}_j} x' \right\|_2$, where $n_i = |\mathcal{D}_i|$ and $n_j = |\mathcal{D}_j|$ are the number of points in each cluster. Then, the CPCC score $\text{CPCC}(t, \rho)$, between distances t and ρ is defined as:

$$\text{CPCC}(t, \rho) := \frac{\sum_{i<j} (t(v_i, v_j) - \bar{t})(\rho(v_i, v_j) - \bar{\rho})}{\left(\sum_{i<j} (t(v_i, v_j) - \bar{t})^2 \right)^{1/2} \left(\sum_{i<j} (\rho(v_i, v_j) - \bar{\rho})^2 \right)^{1/2}}, \tag{1}$$

where $\bar{t} := 2 \sum_{i<j} t(v_i, v_j) \,/\, k(k-1)$, $\bar{\rho} := 2 \sum_{i<j} \rho(v_i, v_j) / k(k-1)$ are the averages of all the pairwise distances for t and ρ.

3 Our Method

In this section we first define an alternative tree metric used to measure the distance between two nodes in a tree. Then, we proceed to discuss our method that uses the proposed tree metric to learn structured representations when a class hierarchy $\mathcal{T}$ is available during supervised learning.

3.1 Tree Metric

Given a tree $\mathcal{T} = (V, E, d)$ forming a class hierarchy, we first formally define the *tree metric* $d_{\mathcal{T}}$ between any pair of nodes as follows.

Definition 3.1 (Tree Metric)**.** The tree metric $d_{\mathcal{T}}(v, v')$ for any pair of nodes $v, v' \in V$ is the weighted length of the shortest path in $\mathcal{T}$ connecting v and v'.

Proposition 3.1. For any undirected weighted graph $\mathcal{G}$, $d_{\mathcal{T}}(\cdot, \cdot)$ is a well-defined metric over $\mathcal{Y}$, if all edge weights of $\mathcal{G}$ are positive and $\mathcal{G}$ is connected.

Our main motivation to consider the tree metric $d_{\mathcal{T}}(\cdot, \cdot)$ instead of $t(\cdot, \cdot)$ as defined in Section 2.2 is two-folds.

First, while each edge in the class hierarchy $\mathcal{T}$ corresponds to a subset relationship, there are other kinds of structured relationships between classes that go beyond the subset relationship. For example, the benchmark dataset MetaShift (Liang & Zou, 2021) also contains a graph to describe the relationship between different subsets of classes. However, in MetaShift the weighted edges do not correspond to the subset relationship as in the case of a class hierarchy, but rather to a similarity/discrepancy measure between them. In this case, the relationship between classes corresponds to an undirected and weighted graph, where the notion of LCA does not apply any more. In fact, even in the case of a tree, the LCA is subject to change depending on which node in the tree is chosen as the root node. However, the tree metric $d_{\mathcal{T}}$ is invariant to rotations of the tree, and applies to both trees and general graphs as shown in Proposition 3.1.

Second, the definition of $t(\cdot, \cdot)$ does not account for the weights of edges in $\mathcal{T}$. This implies that all the fine-grained classes under a given super-class are the same. However, in many applications, different fine-grained classes may have different proportions or importance, for a given super-class. In these cases, the tree metric is more adapted since it also takes into account the edge weights.

Nevertheless, when $\mathcal{T}$ is an unweighted class hierarchy tree, we have the following relationship between the proposed tree metric $d_{\mathcal{T}}(\cdot, \cdot)$ and $t(\cdot, \cdot)$:

Proposition 3.2. Let $\mathcal{T}$ be an unweighted tree with a fixed root node. Then for any pair of nodes $v_i, v_j \in V$, $d_{\mathcal{T}}(v_i, v_j) = 2t(v_i, v_j) - |dt(v_i) - dt(v_j)|$.

The proof is deferred to App. A. In particular, Proposition 3.2 states that if the depths of v_i and v_j are the same in $\mathcal{T}$, then our tree metric reduces to twice of $t(v_i, v_j)$. Multiplying a variable by a constant does not affect its correlation with others. Henceforth, in what follows, we propose to use the tree metric $d_{\mathcal{T}}(\cdot, \cdot)$ in replacement of $t(\cdot, \cdot)$ to compute CPCC.

3.2 Structured Representations by Embedding the Tree Metric

Now that we have defined our tree metric $d_{\mathcal{T}}(\cdot, \cdot)$, we are interested in learning representations $f_\theta(\cdot)$, by optimizing the model parameter θ, such that the Euclidean distance between the representations of any pair of class centers $i, j \in [k]$ approximates $d_{\mathcal{T}}(v_i, v_j)$ in the tree $\mathcal{T}$. Consider a dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$ of size n for classification problem over Δ_k. We propose to use the CPCC between $\rho_{\mathcal{Z}}(\cdot, \cdot)$ and $d_{\mathcal{T}}(\cdot, \cdot)$ as a regularizer to the cross-entropy loss, resulting in the loss function:

$$\mathcal{L}(\mathcal{D}) = \sum_{(x,y)\in\mathcal{D}} \ell_{\text{CE}}(y, g(f_\theta(x))) - \lambda \cdot \text{CPCC}(d_{\mathcal{T}}, \rho_{\mathcal{Z}}), \tag{2}$$

where $\lambda > 0$ is the coefficient used to control the strength of the regularizer. The Euclidean distance $\rho_{\mathcal{Z}}$ is computed in feature space. Concretely, we first apply the encoder f_θ to $\mathcal{D}$ and obtain a set of points in $\mathcal{Z} \times \mathcal{Y}$: $\mathcal{D}_{\mathcal{Z}} := \{(f_\theta(x_i), y_i)\}_{i=1}^n$. Then, we partition $\mathcal{D}_{\mathcal{Z}}$ into k subsets according to the ground-truth labels, and consider the same tree structure $\mathcal{T}$ on $\mathcal{D}_{\mathcal{Z}}$. Note the negative sign before coefficient λ in the above formulation, as we wish to maximize the CPCC score.

In practice, at each iteration during training, since stochastic optimization methods are used, we process a batch of inputs instead of the whole data set $\mathcal{D}$. For each incoming batch, we track the number of finest classes represented in the batch before any pairwise calculation. When all the inputs in a batch come from the same coarse class, the CPCC score is not well-defined due to the 0 variance of $d_{\mathcal{T}}$. This can happen when the batch size is relatively small. In such cases, we fix the value of the CPCC regularizer to 0 to avoid the numerical division by zero error.

Time Complexity of the CPCC Regularizer The computation of our CPCC regularizer is lightweight. For a feature space with p dimensions, for each training iteration, there will be at most $O(p \min(b^2, k^2))$ additional computations, where b is the batch size. Such an overhead is often negligible when compared with the computations needed to train a neural network.

3.3 THE BENEFITS OF STRUCTURED REPRESENTATIONS

In what follows, we describe two potential benefits of learning structured representations with the proposed CPCC regularizer, before providing thorough empirical validation in Section 4.

Interpretability As we briefly discussed before, one potential drawback of the representations learned through supervised learning is the lack of interpretability. Recent work (Papyan et al., 2020; Han et al., 2021) have both empirically and theoretically (under certain assumptions) shown that under the cross-entropy loss, when enough training has happened, the learned representations will have reduced variance within each class, and the set of features corresponding to different classes will converge to the so-called simplex equiangular tight frame (ETF). Yet, the vertices of the simplex ETF are symmetric (in the sense of being permutation-invariant), hence the class features do not necessarily reflect the similarities/differences between different classes, even in feature space.

By enforcing the Euclidean distances in feature space between different classes to be close to the tree metric through our CPCC regularization, we attempt to break the symmetry in learning the features. This can potentially lead to more interpretable features, as closer classes (in the sense of the tree metric) are closer to each other in feature space.

Generalization Another by-product of structured representations is potentially better generalization both in-distribution when only limited amount of labels is available, or under sub-population shifts (Santurkar et al., 2020). To see this, note that the goal of our CPCC regularizer is consistent with classification accuracy: it essentially pushes data from different classes away proportionally to their distance in the tree. Consequently, for sub-population shifts, future unseen fine-grained classes from the same coarse category will be further away from those under a different coarse category. This could help generalize to unseen fine-grained classes in zero or few-shot learning.

4 EXPERIMENTS

In this section, we apply our proposed method on real-world datasets to: (i) study how using CPCC during training affects the representation learnt under various training objectives (Section 4.3), and (ii) see how the learned structured representations can help generalization (Section 4.4).

4.1 DATA

We conduct our experiments on MNIST (Lecun et al., 1998), CIFAR100 (Krizhevsky, 2009), and BREEDS (Santurkar et al., 2020). By using this variety of datasets and hierarchies, we get a comprehensive overview of the usefulness of CPCC as a regularizer.

MNIST contains handwritten digits from 0 to 9. We define odd and even digits to be two *coarse* classes (Figure 1a). The digits in the leaves are called *fine* classes below. The artificial level based on odd- and even-ness corresponds to concepts that are not visually observable.

CIFAR100 comes with a predefined hierarchy: its coarse level has 20 classes, each containing 5 fine classes (*e.g.*, beaver, dolphin, otter, seal and whale belongs to aquatic mammals). While the hierarchies are semantically meaningful, the coarse level labels are not purely defined by visual similarities. For instance, it is hard to tell the size of an animal from its image (making the coarse classes large omnivores/herbivores and mid-size mammals difficult to distinguish).

BREEDS is a benchmark built on ImageNet (Deng et al., 2009). It contains a manually calibrated label hierarchy, based solely on shared visual characteristics. Santurkar et al. (2020) proposed four tasks: *LIVING17*, *ENTITY13*, *ENTITY30*, and *NONLIVING26*. For each task, we consider their leaf nodes, which are ImageNet classes, as our fine level classes, and define the coarse levels to be their "superclasses" at different granularities in the hierarchy. We end up with trees that only contain the root node and two levels of the initial hierarchy, and hide any other intermediate relationships.

4.2 BASELINES AND METRICS

As mentioned above, we operate in a fully supervised setting. We denote our neural network as a function $h : \mathcal{X} \to \Delta_k$. $h \doteq g \circ f_\theta$ can be decomposed into a feature extractor f_θ, and a linear

classifier g to which the softmax is applied. Training is performed using the following four objectives, on the fine-coarse hierarchy of MNIST, CIFAR, and BREEDS, with and without CPCC as a regularization (see the App. D for additional implementation details).

- **ERM**: Train using ℓ_{CE} on fine classes only, without leveraging any hierarchical information:

$$\mathcal{L}(x, y_{\text{fine}}) = \sum_{(x,y)\in\mathcal{D}} \ell_{\text{CE}}(y_{\text{fine}}, h(x)). \tag{3}$$

- **Multi-task Learning**: We jointly train a two-headed network to treat fine and coarse as two separate tasks. The network then becomes $h : \mathcal{X} \to \Delta_{k_1} \times \Delta_{k_2}$ with a shared feature encoder $f_\theta : \mathcal{X} \to \mathcal{Z}$ and two classifiers $g_{\text{coarse}} : \mathcal{Z} \to \Delta_{k_1}, g_{\text{fine}} : \mathcal{Z} \to \Delta_{k_2}$. k_1 is the number of coarse classes and k_2 the number of fine classes. The loss function is the sum of the cross-entropies on the fine and coarse classification tasks, and we simply set the weight of the two parts to 1:

$$\mathcal{L}(x, y_{\text{coarse}}, y_{\text{fine}}) = \sum_{(x,y)\in\mathcal{D}} \ell_{\text{CE}}(y_{\text{coarse}}, g_{\text{coarse}}(f(x)) + \ell_{\text{CE}}(y_{\text{fine}}, g_{\text{fine}}(f(x))). \tag{4}$$

- **Curriculum Learning**: In the spirit of curriculum learning (Bengio et al., 2009), we first train on the coarse classes using Eq. 3 and use y_{coarse} instead of y_{fine}. In the second step, we remove the linear classifier and fine tune a new one on the fine level labels with Equation 3 as the loss function.
- **Sum Loss**: We define a hierarchical Sum Loss as

$$\mathcal{L}(x, y_{\text{coarse}}, y_{\text{fine}}) = \sum_{(x,y)\in\mathcal{D}} \ell_{\text{CE}}(y_{\text{coarse}}, \mathbf{W}h(x)) + \ell_{\text{CE}}(y_{\text{fine}}, h(x)). \tag{5}$$

  $\mathbf{W}$ is a k_1 by k_2 matrix representing the relationships in the label tree: if a fine class i belongs to a coarse class j, then $\mathbf{W}_{ji}$ is 1, otherwise the entry is set to 0. In other words, the probability of belonging to a given coarse class is simply the sum of the probabilities of its descendants (*i.e.*, fine classes) in the tree. This way, we use the information from both levels with one head only.

Metrics To evaluate the representation structure learnt with the various loss functions, as well as the influence of CPCC, we use (1) silhouette scores (Rousseeuw, 1987) to measure the salience of clustering patterns at the coarse level, (2) CPCC as a metric to measure how is the whole representation structure similar to the fine-coarse hierarchy, (3) t-SNE (Van der Maaten & Hinton, 2008) for visualization of the learnt embeddings in 2D, (4) a symmetric distance matrix to evaluate the hierarchical structure. Specifically, we calculate the Euclidean distance between the mean representation vectors for each pair of fine classes. The matrix is organized in a way where fine classes from the same coarse class are grouped together, so that the coarse within-cluster distance is shown around the diagonal while other entries present coarse level between-cluster distances.

Though our goal is not the state-of-the-art, we also report test accuracy as a non-hierarchical metric. Ideally, enforcing a hierarchical representation does not hurt fine level accuracy. We calculate the coarse accuracy by appropriately marginalizing the fine classes probabilities.

4.3 Structure of the learnt representations

Figures 2 and 3 show the effects of training with CPCC, which matches our expectations shown in Fig. 1b. In the distance matrix, we can see that the within-coarse cluster distance is much smaller than the between-coarse cluster distance (corresponding to off-diagonal 5 by 5 blocks). This fact is verified qualitatively in the t-SNE plots from Fig. 3, where coarse groups tend to be better separated. Similar patterns are observed when CPCC is paired with the other loss functions described above. We want to point out that although other setups have a more structured representation to some extent, these are not as perfect as when paired with CPCC (see App. B for the figures).

In Table 1, we see that the objectives leveraging the hierarchical information tend to increase the CPCC. This is particularly true of the multi-task setting. But, rather obviously, directly optimizing CPCC score gives the largest gains, on both CPCC and silhouette scores.

Of particular note, adding our CPCC regularizer leads to better test accuracy at both fine and coarse levels, across objectives and datasets, with gains sometimes exceeding 1%. According to Goyal et al. (2021), such a performance gain is rarely observed when hierarchical information is leveraged. Overall, our findings suggest that when such information is available, using CPCC as a regularizer can prove beneficial.

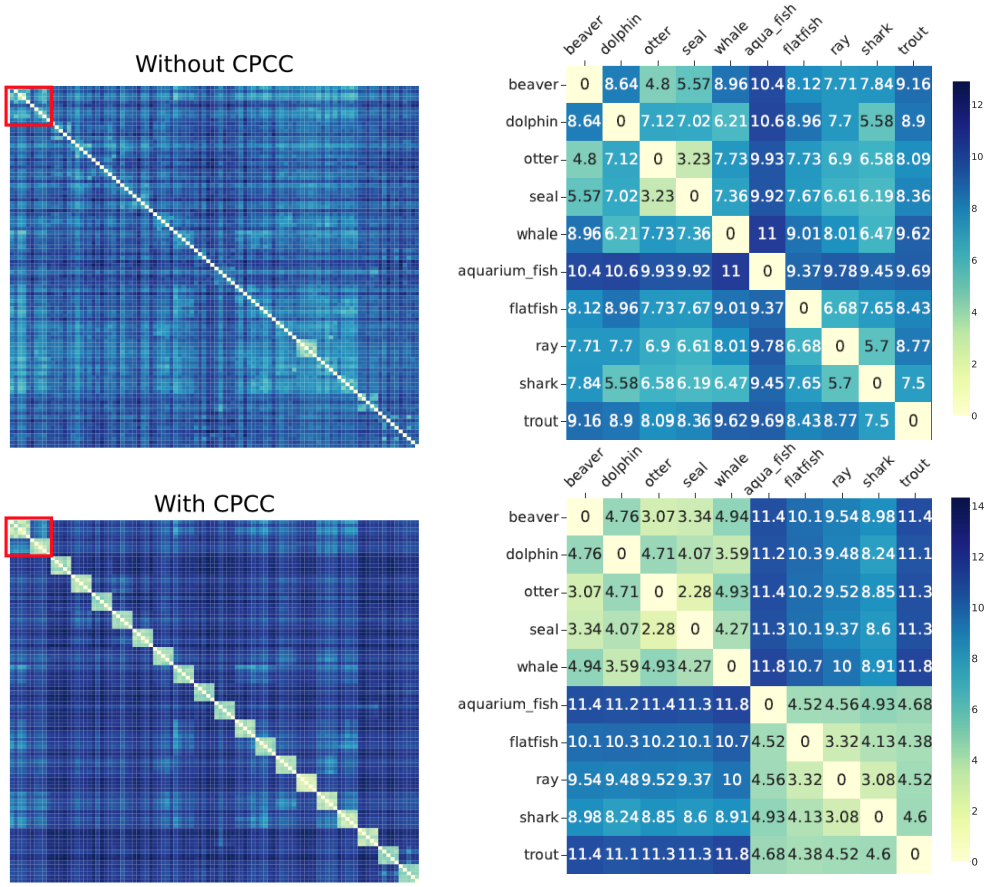


Figure 2: Distance between fine CIFAR100 classes with and without CPCC for the one-head fine **ERM** baseline corresponding to the training objective from Eq.3. The light diagonal blocks with CPCC correspond to the coarse classes. Annotated matrices on the right are zoomed-in matrices corresponding to the red boxes at the left-top of matrices on the left.

4.4 ONE-SHOT GENERALIZATION TO NEW LEVELS IN THE HIERARCHY

Given a representation f_θ trained following 4.2, we want to see if structured representations help transfer knowledge to other levels of the tree hierarchy. To that end, we freeze f_θ and fine tune a linear classifier g_{mid} or g_{coarser} on a single image from each label in the new levels (one-shot generalization).

New levels Based on the coarse and fine levels in the hierarchy, we insert a *mid* level between the coarse and fine ones, as well as a *coarser* level between the coarse one and the root node. This results in classes verifying $k_{\text{coarser}} < k_{\text{coarse}} < k_{\text{mid}} < k_{\text{fine}}$. In MNIST, the mid classes are 1,3,5 (odd numbers ≤ 5), 7,9 (odd numbers > 5), 0,2,4 (even numbers ≤ 5), and 6,8 (even numbers > 5). We do not consider a coarser level (it is trivial to train on the root node). In CIFAR100, each coarse level (containing 5 fine classes) is split into 2/3, creating 40 classes in the mid level. 2 arbitrary coarse classes are merged into 1 coarser label, creating 10 coarser labels. Since BREEDS contains 8 non-root levels in total, and all 4 datasets' coarse levels have a depth ≥ 2, we use the original hierarchy and let the mid level be one level above the fine classes, and the coarser level be one level above the coarse classes. See App. E for the full hierarchies in all datasets.

Results are shown in the last two columns of Table 1. On the coarser levels, we see that without CPCC, objectives (multi-task in particular) that leverage hierarchical information improve performance. Adding our CPCC regularizer provides consistent and significant gains, across datasets and objectives. Intuitively, as all fine classes are grouped together within coarse groups, if one data point is randomly selected, then other data points in the same coarse class will readily be assigned the same coarser label. Without this structure in the representation, generalization will be

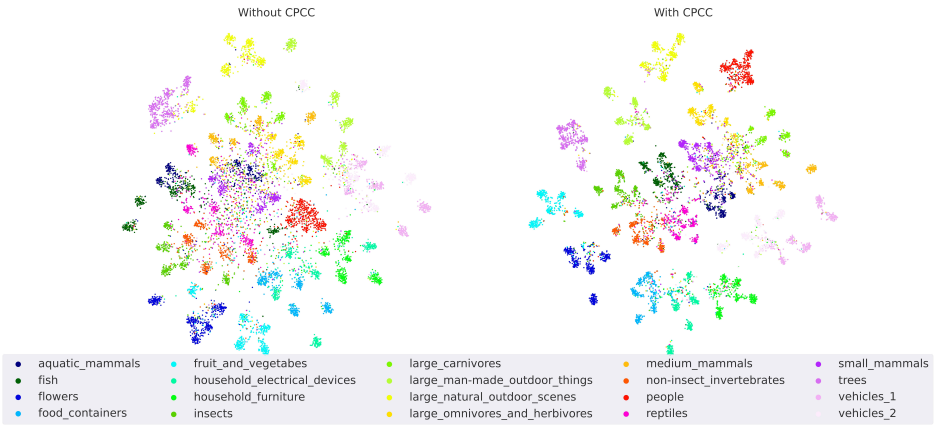


Figure 3: t-SNE visualization of the representations learnt using **ERM** with and without CPCC regularization. The patterns from Figure 2 are confirmed: when CPCC is used, fine classes from the same coarse classes tend to be closer, and coarse classes tend to be further apart.

more difficult as all fine labels are evenly distributed. These findings extend to the mid level task, with the exception of *ENTITY13*, where CPCC hurts the multi-task performance. We posit that it is due to the hierarchical structure of that dataset, where each coarse label has many mid level children and each mid label has very few fine level children (see App. E for the precise hierarchy). In such a case, when forcing the representation to be grouped by coarse level, it becomes difficult to recover the decision boundary of mid classes as it is close to that of fine classes within the same coarse label. On the other hand, when coarse classes have fewer children as in the other datasets, our CPCC regularizer provides a significant boost over the baselines.

5 Related Work

There were many works exploiting label hierarchy and we only refer to the most related ones. However, to the best of our knowledge, none of the previous work set learning structured representations as main objective or embedded the tree metric under this context.

Background of Baseline Methods The most basic label hierarchy tree contains only two level, coarse and fine. One way to set up baseline models is to treat them as two tasks, which can be trained jointly or sequentially. The former originates from **Multi-task Learning** (MTL) where part of single network is shared for multiple heads for each task during training (Caruana, 1997). To deal with hierarchical labels, algorithms were developed to adjust the weight between different levels of supervision in the loss function which is the sum of weighted cross-entropy for each task (Zhao et al., 2020; Inoue et al., 2020). The latter echoes with **Curriculum Learning** (CL) firstly proposed in Bengio et al. (2009), where it is claimed that pretraining with a easy task will help the convergence and performance on a hard task. We define coarse level classes as the easier task due to the intuition that coarse level contains less classes and more images for each class as recommended in Ahn et al. (2021); Peterson et al. (2018); Wang & Cottrell (2015) to emulate human learning behavior. Another approach to use multi-level labels, **Sum Loss**, is to directly modify the loss function by marginalizing fine classes probability for coarse classes. Su & Maji (2021); Hu et al. (2018) defined a similar version to address some partially labeled tasks. Given that training with naive cross-entropy loss tends to separate classes evenly, we hope that including an extra level of supervision via these baseline approaches will not only push fine classes apart, but also present coarse level separation, although as we have shown that all three baselines are not as perfect as using CPCC as a regularizer regarding the representation.

Learning with Label Hierarchy The most common motivation of hierarchical models is to improve the fine-level accuracy (Deng et al., 2014; Guo et al., 2018; Chen et al., 2019; Yan et al., 2014). Interestingly, the results of accuracy improvement was mixed: while most work claimed to gain performance improvement, Wang & Cottrell (2015) stated that this improvement was limited, and Goyal et al. (2021) claimed most hierarchical models lead to worse performance on non-hierarchical accuracy metrics. Additionally, using coarse level labels often appears in a

Table 1: Results of experiments building structured representation, mean % and standard deviation (in parenthesis) over 5 seeds. For objectives with CPCC, we set the regularization factor $\lambda = 1$. The best result based on mean value is highlighted in bold. Overall, CPCC leads to consistent and significant improvements on both the structured representation metrics and generalization performance.

Dataset	Objective	CPCC	Silhouette	Fine Acc	Coarse Acc	Mid Acc	Coarser Acc
MNIST	ERM	10.80 (1.49)	13.97 (0.72)	99.05 (0.23)	99.49 (0.08)	57.87 (5.30)	N/A
	ERM + CPCC	**99.96 (0.01)**	**61.33 (0.42)**	**99.28 (0.08)**	99.61 (0.03)	68.23 (6.52)	
	Multi-task	42.65 (2.96)	20.87 (1.02)	99.21 (0.05)	**99.62 (0.02)**	66.61 (3.94)	
	Multi-task + CPCC	**99.96 (0.01)**	60.60 (0.50)	99.26 (0.05)	99.55 (0.04)	69.53 (3.08)	
	Curriculum	31.22 (4.99)	19.09 (1.17)	99.20 (0.12)	99.59 (0.06)	67.68 (6.63)	
	Curriculum + CPCC	**99.96 (0.01)**	60.90 (0.31)	99.18 (0.10)	**99.62 (0.03)**	**73.69 (2.09)**	
	Sum Loss	23.76 (4.10)	17.05 (0.62)	99.11 (0.08)	99.60 (0.06)	61.01 (3.81)	
	Sum Loss + CPCC	**99.96 (0.01)**	60.07 (0.75)	99.10 (0.12)	99.56 (0.06)	72.64 (5.64)	
CIFAR100	ERM	24.38 (0.57)	5.59 (0.02)	76.82 (0.30)	85.59 (0.35)	41.41 (0.34)	31.06 (0.97)
	ERM + CPCC	84.20 (0.39)	**34.40 (0.11)**	77.47 (0.27)	86.95 (0.17)	49.66 (0.91)	50.48 (1.93)
	Multi-task	39.75 (0.33)	8.09 (0.08)	76.56 (0.20)	85.79 (0.20)	47.82 (0.57)	42.06 (1.69)
	Multi-task + CPCC	84.88 (0.58)	31.58 (0.23)	76.90 (0.32)	87.11 (0.19)	49.86 (0.66)	52.86 (2.07)
	Curriculum	23.81 (0.60)	5.25 (0.11)	76.84 (0.20)	85.72 (0.16)	42.56 (0.24)	31.38 (0.79)
	Curriculum + CPCC	**85.32 (0.51)**	34.08 (0.23)	**77.48 (0.44)**	**87.15 (0.19)**	**51.84 (0.44)**	**53.44 (2.28)**
	Sum Loss	29.85 (0.63)	4.93 (0.07)	76.78 (0.20)	85.88 (0.25)	42.60 (0.48)	34.22 (0.34)
	Sum Loss + CPCC	84.78 (0.64)	31.16 (0.13)	77.26 (0.12)	86.99 (0.07)	49.77 (0.60)	49.79 (1.13)
LIVING17	ERM	36.74 (0.92)	8.89 (0.08)	83.66 (0.51)	89.72 (0.26)	61.38 (0.74)	51.88 (0.79)
	ERM + CPCC	**93.56 (0.46)**	48.26 (0.51)	84.97 (0.68)	91.13 (0.61)	**84.15 (1.24)**	63.66 (1.21)
	Multi-task	45.83 (0.78)	11.66 (0.09)	83.87 (0.28)	90.29 (0.27)	68.51 (1.33)	54.48 (2.12)
	Multi-task + CPCC	90.71 (4.28)	45.05 (0.54)	84.77 (0.59)	91.53 (0.30)	80.86 (0.87)	64.59 (3.70)
	Curriculum	40.19 (0.71)	10.95 (0.12)	83.87 (0.40)	90.19 (0.49)	65.70 (2.02)	53.05 (1.41)
	Curriculum + CPCC	93.41 (0.34)	**51.36 (0.46)**	**85.21 (0.42)**	91.59 (0.65)	82.78 (1.05)	63.76 (1.54)
	Sum Loss	39.02 (1.01)	0.90 (0.27)	83.49 (0.63)	89.92 (0.31)	61.85 (2.52)	49.57 (2.40)
	Sum Loss + CPCC	93.03 (0.46)	44.36 (0.28)	85.05 (0.23)	**91.79 (0.25)**	81.47 (0.66)	**66.58 (1.38)**
ENTITY13	ERM	34.97 (0.43)	3.12 (0.03)	82.36 (0.38)	90.43 (0.34)	17.79 (0.24)	41.89 (1.08)
	ERM + CPCC	90.98 (0.22)	38.68 (0.17)	83.37 (0.37)	91.50 (0.09)	20.39 (2.45)	65.73 (0.88)
	Multi-task	43.55 (0.71)	5.19 (0.11)	82.47 (0.28)	91.13 (0.10)	**22.95 (0.73)**	48.59 (1.23)
	Multi-task + CPCC	90.73 (0.34)	36.62 (0.27)	82.83 (0.23)	91.51 (0.24)	16.26 (1.08)	66.39 (2.47)
	Curriculum	37.03 (0.51)	3.98 (0.05)	83.41 (0.29)	91.15 (0.15)	21.04 (1.21)	46.67 (2.90)
	Curriculum + CPCC	**92.38 (0.25)**	**41.06 (0.15)**	**83.86 (0.25)**	**91.90 (0.20)**	21.24 (1.48)	61.54 (2.25)
	Sum Loss	37.99 (0.26)	3.51 (0.02)	82.50 (0.23)	90.83 (0.25)	19.52 (1.16)	44.39 (2.57)
	Sum Loss + CPCC	89.66 (0.30)	35.08 (0.23)	82.97 (0.43)	91.56 (0.10)	20.65 (1.64)	**68.26 (1.20)**
ENTITY30	ERM	24.29 (0.30)	3.07 (0.06)	80.81 (0.35)	86.00 (0.25)	19.52 (1.31)	35.04 (1.21)
	ERM + CPCC	73.55 (0.77)	29.43 (0.12)	82.00 (0.31)	87.60 (0.13)	32.44 (1.48)	**65.73 (0.88)**
	Multi-task	30.55 (0.26)	5.25 (0.12)	81.31 (0.32)	86.95 (0.15)	24.70 (1.73)	48.59 (1.23)
	Multi-task + CPCC	72.00 (0.52)	25.92 (0.19)	81.90 (0.17)	87.66 (0.15)	30.84 (0.91)	61.51 (2.12)
	Curriculum	27.09 (0.20)	4.55 (0.05)	82.19 (0.45)	87.52 (0.25)	23.18 (0.83)	38.85 (1.02)
	Curriculum + CPCC	**77.80 (0.50)**	**34.08 (0.19)**	**82.86 (0.20)**	**88.55 (0.09)**	**33.66 (2.11)**	61.89 (2.12)
	Sum Loss	26.05 (0.29)	3.18 (0.09)	81.11 (0.17)	86.62 (0.14)	16.42 (1.94)	37.02 (0.64)
	Sum Loss + CPCC	66.10 (0.63)	21.38 (0.20)	81.76 (0.26)	87.63 (0.25)	29.79 (1.19)	63.82 (1.20)
NONLIVING26	ERM	26.95 (0.31)	5.90 (0.13)	80.79 (0.36)	85.16 (0.35)	49.92 (0.61)	44.64 (0.69)
	ERM + CPCC	**82.20 (0.44)**	34.63 (0.31)	82.96 (0.35)	87.42 (0.36)	76.14 (0.90)	60.62 (0.90)
	Multi-task	30.38 (0.73)	7.36 (0.10)	82.06 (0.24)	86.42 (0.22)	55.46 (0.77)	46.02 (0.71)
	Multi-task + CPCC	79.49 (0.66)	30.55 (0.22)	82.84 (0.61)	87.60 (0.39)	75.20 (0.44)	**62.01 (1.13)**
	Curriculum	28.81 (0.41)	7.68 (0.08)	81.82 (0.65)	86.23 (0.73)	53.64 (0.98)	44.97 (1.18)
	Curriculum + CPCC	81.70 (0.71)	**36.92 (0.25)**	**83.54 (0.28)**	**88.06 (0.29)**	**76.75 (1.12)**	60.38 (1.17)
	Sum Loss	28.57 (0.69)	5.65 (0.15)	81.57 (0.97)	86.28 (0.63)	45.96 (2.22)	45.48 (1.76)
	Sum Loss + CPCC	79.04 (0.95)	29.14 (0.36)	82.60 (0.51)	87.42 (0.41)	74.22 (0.59)	59.44 (1.04)

weakly supervised setting, where coarse classes are always available but fine class labels are only accessible for part of data, to reduce annotation cost at a finer level (Taherkhani et al., 2019; Lei et al., 2017; Ristin et al., 2015). Many other works built hierarchy from dataset (Murdock et al., 2016; Li et al., 2010; Verma et al., 2012; Han et al., 2018; Zheng et al., 2017). We only name a few since they are very different from our setting where the hierarchy is defined before training.

6 Conclusion

How to include label relation into representation is an open question. In this paper, in the context of tree label hierarchies, we use the cophenetic correlation coefficient as a regularizer to embed this hierarchical relationship into representations, and outperform other baseline methods. CPCC has multiple advantages, including low time complexity, better interpretability, flexibility on any supervised learning paradigms, and it can be applied on any common label relation graphs. We also demonstrate that it leads to better generalization performance on several downstream tasks. All these benefits show that our method provides an interesting solution to this important problem.

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A PROOF OF PROPOSITIONS

We first restate propositions and then provide the proof.

Proposition 3.1. For any undirected weighted graph $\mathcal{G}$, $d_{\mathcal{T}}(\cdot,\cdot)$ is a well-defined metric over $\mathcal{Y}$, if all edge weights of $\mathcal{G}$ are positive and $\mathcal{G}$ is connected.

Proof. Given $v_i, v_j \in V$, if $d_{\mathcal{T}}(v_i, v_j)$ is a well-defined metric function, it means $d_{\mathcal{T}}$ must satisfy the following three properties:

1. if $v_i \neq v_j$, $d_{\mathcal{T}}(v_i, v_j) > 0$. Otherwise, $d_{\mathcal{T}}(v_i, v_j) = 0$.
2. $d_{\mathcal{T}}(v_i, v_j) = d_{\mathcal{T}}(v_j, v_i)$.
3. $\forall v_k \in V, d_{\mathcal{T}}(v_i, v_j) \leq d_{\mathcal{T}}(v_i, v_k) + d_{\mathcal{T}}(v_k, v_j)$.

By definition $d_{\mathcal{T}}(v_i, v_j)$ is the distance of the shortest path from v_i to v_j. Since $\mathcal{G}$ has positive weights, the first statement holds.

There is always a path p from v_i to v_j because $\mathcal{G}$ is connected. Since $\mathcal{G}$ is undirected, there is also a path p' from v_j to v_i by backtracking vertices on p. The sum of edge weights on p is the same as that on p'. p' must be the shortest path from v_j to v_i: if there exists a shorter path q from v_j to v_i, backtracking q will give a shorter path q' from v_i to v_j. Thus, $d_{\mathcal{T}}(v_i, v_j) = d_{\mathcal{T}}(v_j, v_i)$.

If the shortest path p from v_i to v_j contains v_k, $d_{\mathcal{T}}(v_i, v_j) = d_{\mathcal{T}}(v_i, v_k) + d_{\mathcal{T}}(v_k, v_j)$. If p does not contain v_k, if $d_{\mathcal{T}}(v_i, v_j) > d_{\mathcal{T}}(v_i, v_k) + d_{\mathcal{T}}(v_k, v_j)$, $v_i \dots v_k \dots v_j$ is a valid path that is shorter than p. Contradiction. ■

Proposition 3.2. Let $\mathcal{T}$ be an unweighted tree with a fixed root node. Then for any pair of nodes $v_i, v_j \in V$, $d_{\mathcal{T}}(v_i, v_j) = 2t(v_i, v_j) - |dt(v_i) - dt(v_j)|$.

Proof. Since $\mathcal{T}$ is an unweighted tree, there is a unique path connecting v_i and v_j. Let the path p be $v_i, v_1, \dots, v_{L-1}, v_j$ of length L. Let $v_l :=$ LCA (v_i, v_j). By definition of LCA , there exists a path from v_l to v_i and a path from v_l to v_j as well. Because the path p between v_i and v_j is unique, v_l must be on the path p as well.

Now, since v_l is the least common ancestor, the depth of v_i, v_j can be written as:

$$dt(v_i) = dt(v_l) + l, \quad dt(v_j) = dt(v_l) + L - l.$$

Hence, we have

$$\begin{aligned}
2t(v_i, v_j) - |dt(v_i) - dt(v_j)| &= 2\max\{dt(v_i), dt(v_j)\} - 2dt(\text{LCA }(v_i, v_j)) - |dt(v_i) - dt(v_j)| \\
&= 2\max\{dt(v_l) + l, dt(v_l) + L - l\} - 2dt(v_l) - |L - 2l| \\
&= 2\max\{l, L - l\} - |L - 2l| \\
&= \max\{l, L - l\} + \min\{l, L - l\} \\
&= L = d_{\mathcal{T}}(v_i, v_j),
\end{aligned}$$

where for the second to last equality we use the fact that $\max\{a, b\} - |a - b| = \min\{a, b\}$ for any $a, b \in \mathbb{R}$. ■

B ADDITIONAL FIGURES

We include t-SNE visualization of representations and distance matrices for the remaining objectives (Multi-task, Curriculum, Sum Loss) of Table 1 in this section.

C ADDITIONAL EXPERIMENTS

In this section, we discuss two extra settings where CPCC can be applied.

C.1 CPCC FOR SUB-POPULATION SHIFT

Santurkar et al. (2020) proposed BREEDS benchmark for sub-population shift. Compared to 4.3, the main difference between two tasks is that BREEDS was only trained on coarse level, while 4.3 is trained on fine level for all objectives without CPCC. From the definition of sub-population shift, if we train on the train split of source dataset, there will be a big drop of performance of test performance on source set and target set. Since models are always trained on coarse labels,

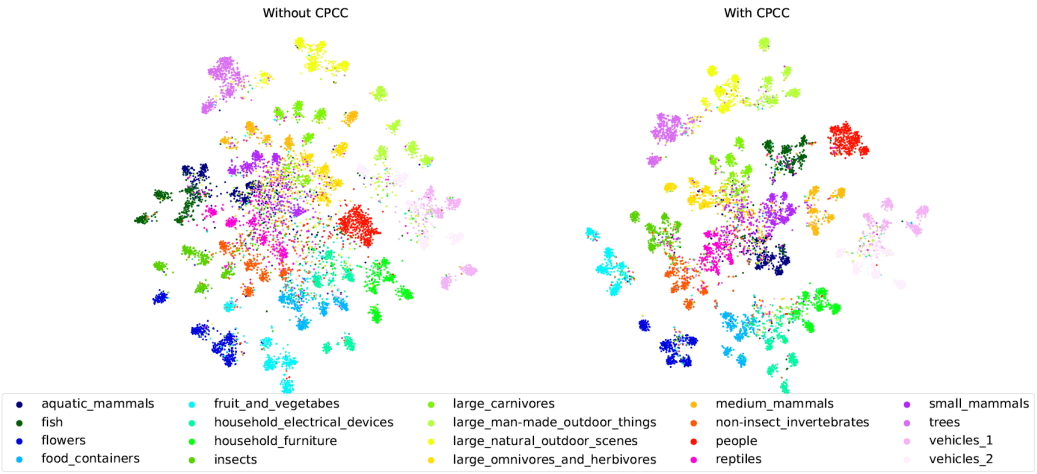


Figure 4: t-SNE visualization of the representations learnt by **Multi-task** with and without CPCC regularization.

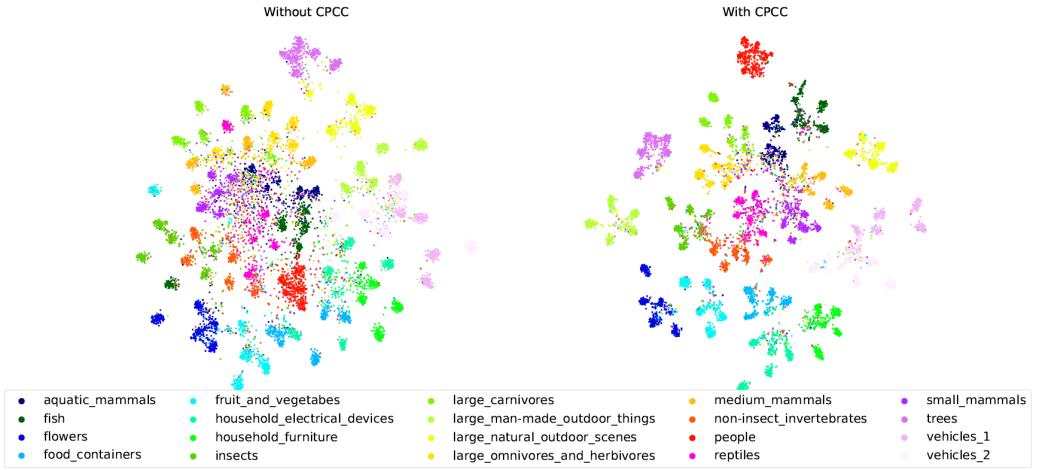


Figure 5: t-SNE visualization of the representations learnt by **Curriculum** with and without CPCC regularization.

fine tuning is unnecessary. In Table 2, we show the performance gain of using CPCC on the ERM method. Since BREEDS contains more than two levels of hierarchy, we present the result of using CPCC on coarse-fine classes, coarse-coarser classes as examples. The result shows CPCC alleviates the sub-population shift problem by using multiple labels in hierarchy, as shown in Mukherjee et al. (2022) who used other methods.

Objective	*LIVING17*	*ENTITY13*	*ENTITY30*	*NONLIVING26*
ERM (coarse)	56.01 (0.71)	60.49 (0.35)	47.72 (0.27)	40.25 (0.66)
ERM + CPCC (coarse, fine)	**57.74 (0.34)**	**61.32 (0.18)**	**48.99 (0.29)**	**42.31 (0.87)**
ERM + CPCC (coarser, coarse)	57.20 (0.90)	60.84 (0.25)	48.45 (0.67)	41.94 (0.90)

Table 2: BREEDS's performance for sub-population shift. Each entry is the target validation accuracy of coarse level classes. Using CPCC consistently outperforms naive ERM.

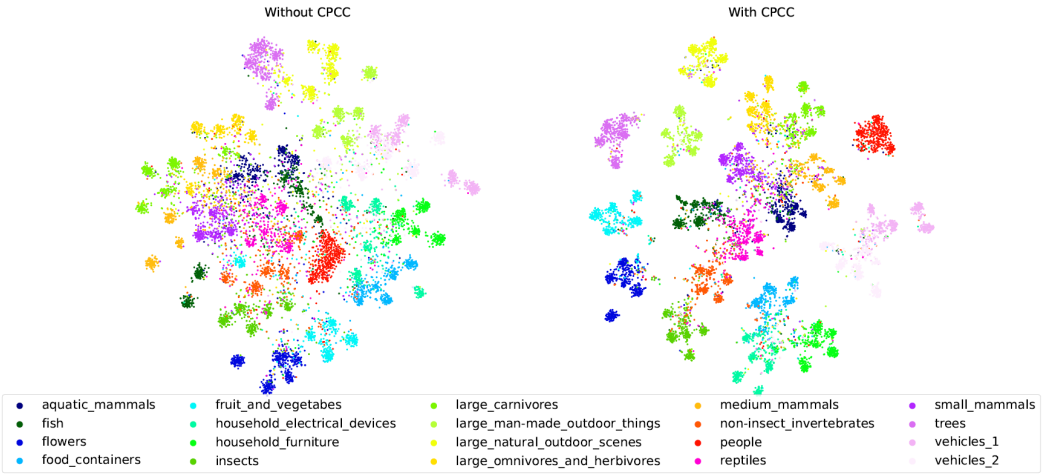


Figure 6: t-SNE visualization of the representations learnt by **Sum Loss** with and without CPCC regularization.

C.2 Generalization to new classes in the hierarchy

Assume kids have learned cat and dog to be animals. Does knowing the animal concept help them understand giraffe, horse, tiger, and other animals better? Since BREEDS divides coarse label into source and target sets, we are able to straightforwardly investigate this question. Instead of evaluating on coarse classes, we take the representations in 4.3 which is learned on the source train test, fine tune a linear classifier on the target train test, and finally evaluate on target validation set. Thus, we can test the generalizability to new classes in the same level of the hierarchy. Similar to the other generalization task, we also use the one-shot setting during fine tuning.

Compared to generalizing to new levels, even if we assume the hierarchy is well-defined, it is a much harder task as we must learn on new classes whose decision boundary might be hard to infer from seen labels, let alone when the number of labels is limited. Thus, we can see a very low test accuracy. Yet, using CPCC is still beneficial: when we have a limited train set, leveraging hierarchical information can be a good hint for classification.

Objective	*LIVING17*	*ENTITY13*	*ENTITY30*	*NONLIVING26*
ERM (fine)	26.38 (1.08)	1.643 (0.39)	2.430 (0.44)	3.069 (0.69)
ERM + CPCC (coarse, fine)	**30.68 (1.73)**	**3.212 (0.77)**	**5.770 (0.50)**	**3.354 (0.49)**

Table 3: One shot transfer from source population to target population by fine tuning the last layer on BREEDS. We use target set fine level validation accuracy as the metric.

However, from some preliminary trials, this minor advantage disappears as we use the full target data during training. Additionally, the result is sensitive to the definition of hierarchy: if we perform this task on CIFAR100 by creating an artificial source target split, using CPCC reduces performance, which might be due to the fact that not all coarse classes in CIFAR100 are defined based on visual characteristics (such as vehicles_1 and vehicles_2).

D Implementation Details

In this section, we describe the training details for our experiments. We want to highlight again that we use these parameters to show CPCC's advantage over other methods in controlled settings, but not to reproduce SOTA results.

All experiments were trained on NVIDIA RTX A6000. Experiments in Sections 4.3 and 4.4 have the same training setting, except that the latter's learning rate starts from 1/10 of the first task's initial learning rate for fine tuning tasks.

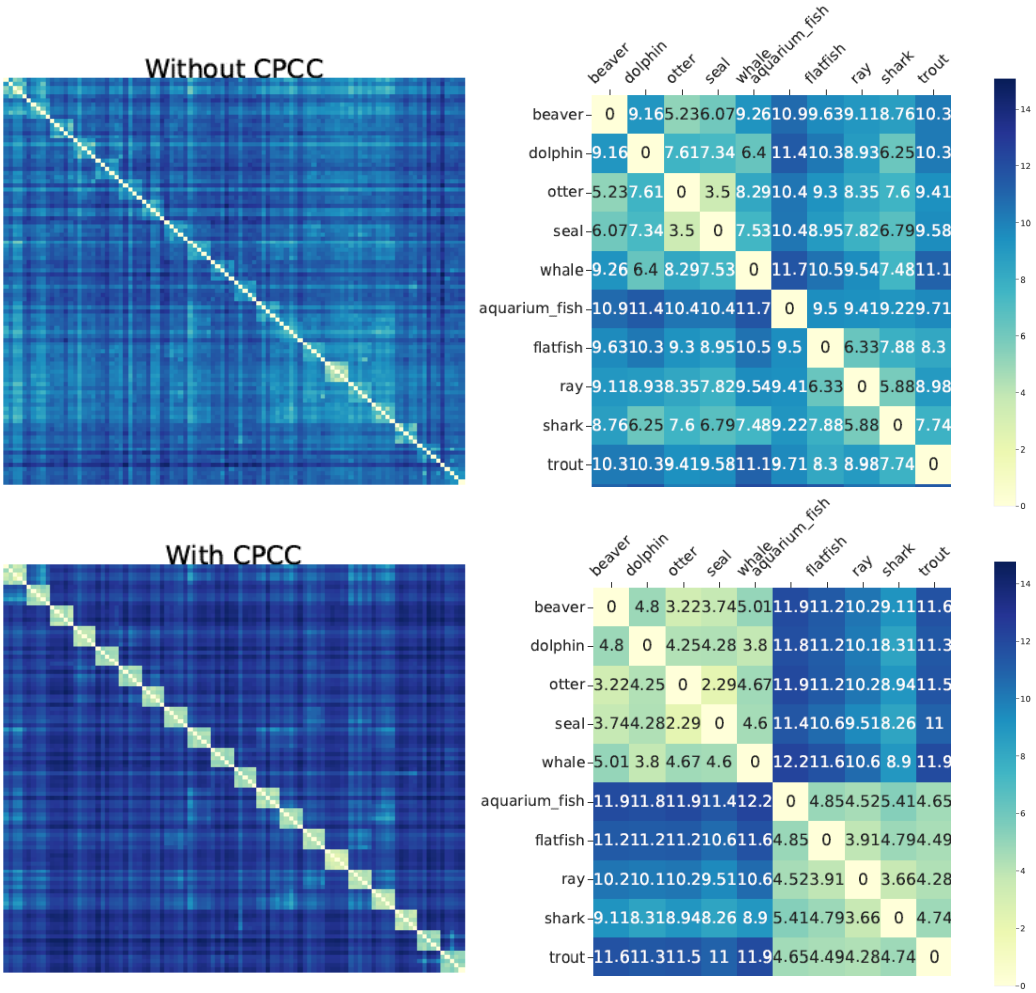


Figure 7: Distance between fine CIFAR100 classes with and without CPCC for **Multi-task**.

Compared with other approaches, curriculum learning in 4.3 contains two stages. Empirically, we find that a few first stage training iterations on coarse labels give a better initialization of the second stage on fine labels. The epochs of the first stage do not have a huge impact on the final performance, but less epochs on the second stage can harm the result. Therefore, we train more epochs on curriculum learning settings so that the fine accuracy for all without CPCC objectives are similar. We spend approximately the same number of epochs on fine labels for all experiments, and add some extra iterations for curriculum learning to train on coarse labels. Training epochs are the same for generalization experiments, given the fact that all representations have similar fine and coarse accuracy.

MNIST The feature encoder is composed of these layers sequentially: a 5×5 convolutional layer, 2×2 max pooling layer, a 5×5 convolutional layer, 2×2 max pooling layer, fully connected layer with output 120 units, and a fully connected layer with output 50 units. We set batch size as 256, epochs as 70, learning rate start from 0.05, divided by 5 every 25 steps, with momentum of 0.9, weight decay 5×10^{-4} of SGD optimizer. No data augmentation is applied. For curriculum learning, we train 18 epochs on coarse labels and 72 epochs on fine labels.

CIFAR100 We train ResNet18 from scratch. Following the original ResNet (He et al., 2015) paper, the first convolution layer has a 3×3 kernel size, and the max pooling layer is not included. We set batch size as 128, epochs as 200, learning rate starts from 0.1, divided by 5 every 60 steps, with momentum of 0.9, weight decay 5×10^{-4} of SGD optimizer. All images are normalized by

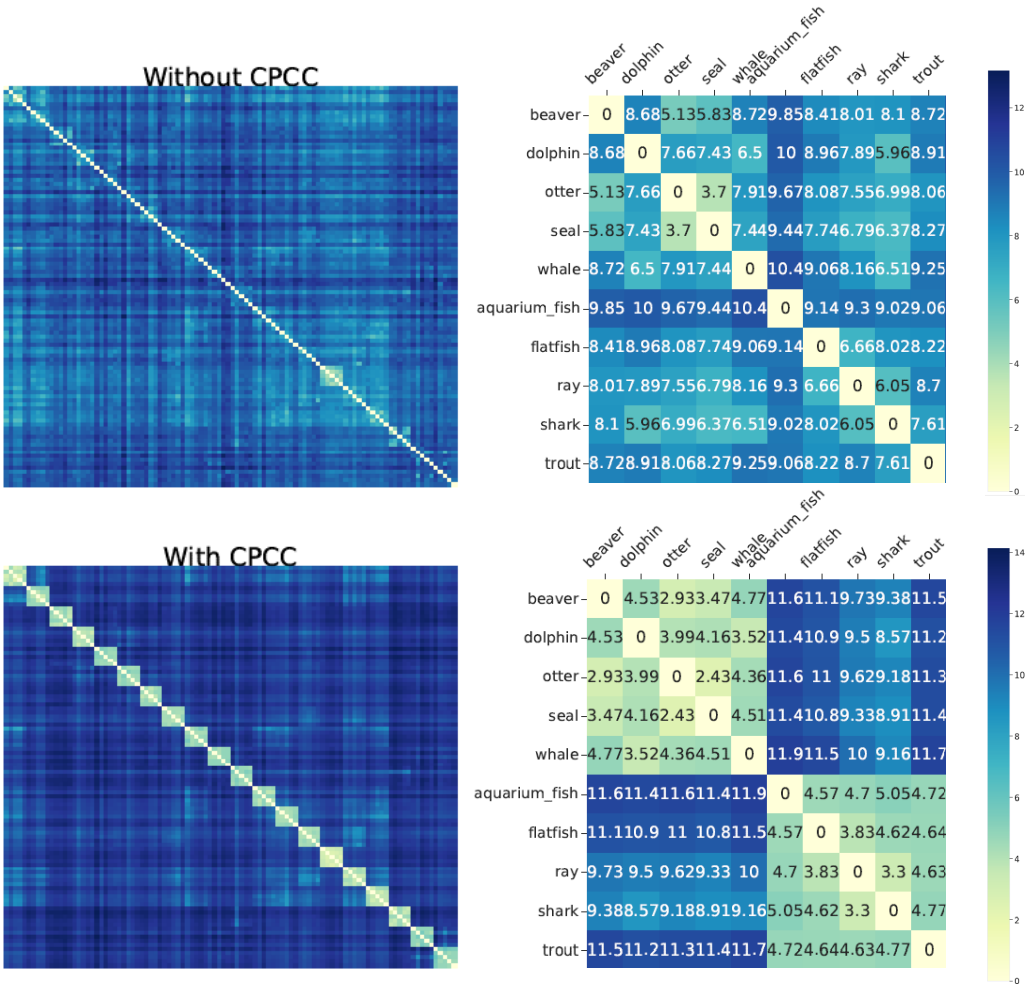


Figure 8: Distance between fine CIFAR100 classes with and without CPCC for **Curriculum**.

CIFAR100's mean and standard deviation. The following data augmentations are applied on train set sequentially: images are zero padded by 4 pixels and cropped into original size; images are flipped horizontally by the chance of 50%; images are randomly rotated by -15 to 15 degrees. For curriculum learning, we train 50 epochs on coarse labels and 200 epochs on fine labels.

BREEDS We train ResNet18 from scratch. The first convolution layer has a 7×7 kernel size. We follow BREEDS training setting: we set batch size as 128, initial learning rate as 0.1, weight decay 10^{-4} of SGD optimizer. On *ENTITY13* and *ENTITY30*, we train 300 epochs, learning rate divided by 10 every 100 steps; on *LIVING17* and *NONLIVING26*, we train 450 epochs, learning rate divided by 10 every 150 steps. For curriculum learning, on *ENTITY13* and *ENTITY30*, we train 75 epochs on coarse labels and 300 epochs on fine labels; on *LIVING17* and *NONLIVING26*, we train 108 epochs on coarse labels and 432 epochs on fine labels. Train set is augmented by a RandomResizedCrop into 224×224, flipped horizontally by the chance of 50 %, and randomly change the brightness, contrast, saturation by 0.9 to 1.1 (ColorJitter). Test set is first resized into 256×256 and CenterCropped into 224×224.

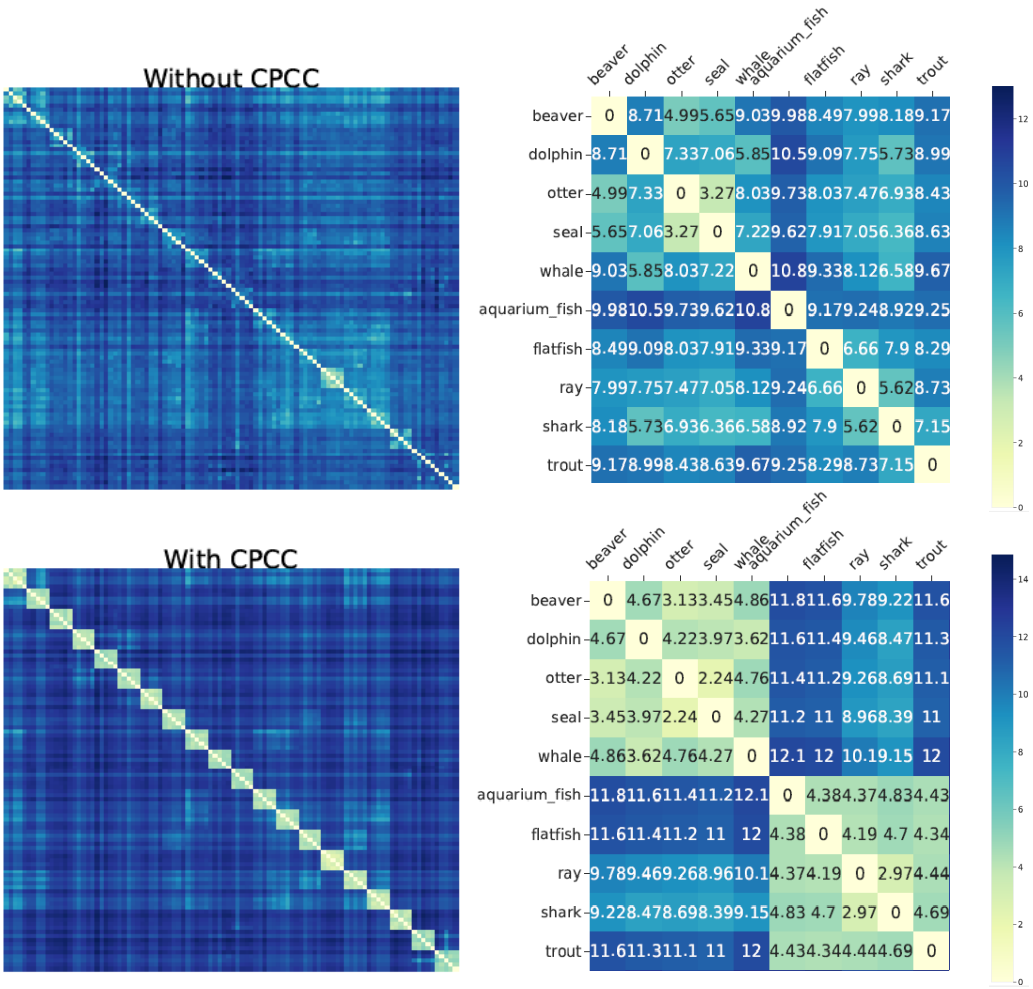


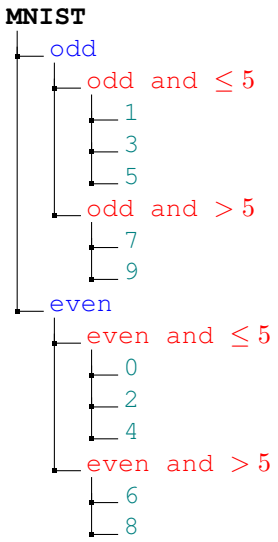
Figure 9: Distance between fine CIFAR100 classes with and without CPCC for **Sum Loss**.

E Dataset Hierarchies

We visualize 4 levels of hierarchy (coarser, coarse, mid, fine) of MNIST, CIFAR, and BREEDS in Fig 10, and provide more details for Section 4.1 and 4.4 about how we constructed the hierarchy based on original labels. We can notice in BREEDS hierarchies, there are some repetitive labels[1]. One case is that the parent node is a dummy node which shares the same name as its child. For example, dummy 52 and n02773037 (WordNet ID) are both called "bags". Santurkar et al. (2020) inserted dummy nodes into the ImageNet hierarchy to adjust the granularity of each level. Since we create mid labels by backtracking the fine classes by 1 level, the other case of repetitive labels happens when a coarse label coincides with the mid label (*i.e.*, if the coarse label is exactly one level higher than the fine label). Unlike on MNIST and CIFAR, we do not want to create some artificial new classes on BREEDS. Therefore, we can also treat repetitive nodes as dummy nodes as in the first case, which only affects mid class results.

[1]All label names of BREEDS can be found in `https://github.com/MadryLab/BREEDS-Benchmarks/blob/master/imagenet_class_hierarchy/modified/node_names.txt`

Figure 10: Hierarchies for all datasets.



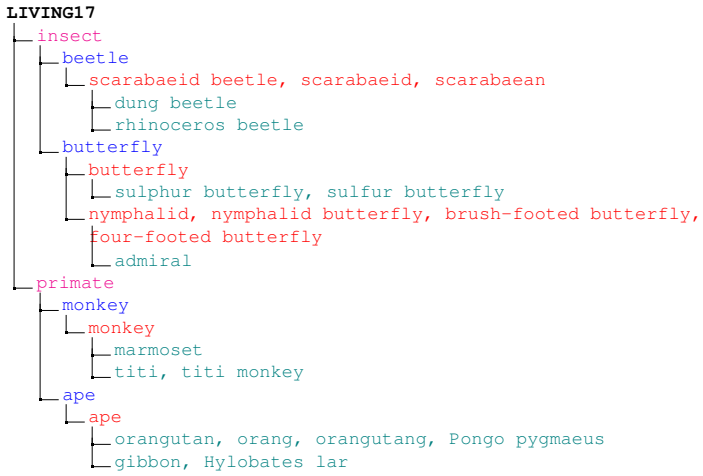
(a) MNIST hierarchy.

```
CIFAR100
├── aquatic mammals + fish
│   ├── aquatic mammals
│   │   ├── beaver + dolphin
│   │   └── otter + seal + whale
│   └── fish
│       ├── aquarium fish + flatfish
│       └── ray + shark + trout
├── flowers + food containers
│   ├── flowers
│   │   ├── orchids + poppies
│   │   └── roses + sunflowers + tulips
│   └── food containers
│       ├── bottles + bowls
│       └── cans + cups + plates
├── fruit and vegetables + household electrical devices
│   ├── fruit and vegetables
│   │   ├── apples + mushrooms
│   │   └── oranges + pears + sweet peppers
│   └── household electrical devices
│       ├── clock + computer keyboard
│       └── lamp + telephone + television
├── household furniture + insects
│   ├── household furniture
│   │   ├── bed + chair
│   │   └── couch + table + wardrobe
│   └── insects
│       ├── bee + beetle
│       └── butterfly + caterpillar + cockroach
├── large carnivores + large man-made outdoor things
│   ├── large carnivores
│   │   ├── bear + leopard
│   │   └── lion + tiger + wolf
│   └── large man-made outdoor things
│       ├── bridge + castle
│       └── house + road + skyscraper
├── large natural outdoor scenes + large omnivores and herbivores
│   ├── large natural outdoor scenes
│   │   ├── cloud + forest
│   │   └── mountain + plain + sea
│   └── large omnivores and herbivores
│       ├── camel + cattle
│       └── chimpanzee + elephant + kangaroo
├── medium-sized mammals + non-insect invertebrates
│   ├── medium-sized mammals
│   │   ├── fox + porcupine
│   │   └── possum + raccoon + skunk
│   └── non-insect invertebrates
│       ├── crab + lobster
│       └── snail + spider + worm
├── people + reptiles
│   ├── people
│   │   ├── baby + boy
│   │   └── girl + man + woman
│   └── reptiles
│       ├── crocodile + dinosaur
│       └── lizard + snake + turtle
├── small mammals + trees
│   ├── small mammals
│   │   ├── hamster + mouse
│   │   └── rabbit + shrew + squirrel
│   └── trees
│       ├── maple + oak
│       └── palm + pine + willow
└── vehicles 1 + vehicles 2
    ├── vehicles 1
    │   ├── bicycle + bus
    │   └── motorcycle + pickup truck + train
    └── vehicles 2
        ├── lawn-mower + rocket
        └── streetcar + tank + tractor
```

(b) CIFAR hierarchy. The leaf nodes (100 fine classes) are evident from the mid level node names (they are simply separated by a +). We create the mid-level split based on alphabetical order. Label names are from `https://www.cs.toronto.edu/~kriz/cifar.html`.

```
LIVING17
├─ dummy6
│  └─ parrot
│     ├─ parrot
│     │  └─ macaw
│     └─ lory
│        └─ lorikeet
├─ dummy8
│  └─ salamander
│     ├─ salamander
│     │  └─ axolotl, mud puppy, Ambystoma mexicanum
│     └─ newt, triton
│        └─ eft
├─ serpentes
│  └─ snake, serpent, ophidian
│     └─ snake, serpent, ophidian
│        ├─ garter snake, grass snake
│        └─ night snake, Hypsiglena torquata
├─ chelonian, chelonian reptile
│  └─ turtle
│     ├─ turtle
│     │  └─ box turtle, box tortoise
│     └─ sea turtle, marine turtle
│        └─ leatherback turtle, leatherback, leathery turtle,
│           Dermochelys coriacea
├─ saurian
│  └─ lizard
│     ├─ teiid lizard, teiid
│     │  └─ whiptail, whiptail lizard
│     └─ anguid lizard
│        └─ alligator lizard
├─ arachnid, arachnoid
│  └─ spider
│     └─ spider
│        ├─ black and gold garden spider, Argiope aurantia
│        └─ tarantula
├─ gallinaceous bird, gallinacean
│  └─ grouse
│     └─ grouse
│        ├─ ptarmigan
│        └─ prairie chicken, prairie grouse, prairie fowl
├─ crustacean
│  └─ crab
│     └─ crab
│        ├─ Dungeness crab, Cancer magister
│        └─ fiddler crab
└─ carnivore
   ├─ dog, domestic dog, Canis familiaris
   │  └─ dog, domestic dog, Canis familiaris
   │     ├─ Pekinese, Pekingese, Peke
   │     └─ bloodhound, sleuthhound
   ├─ wolf
   │  └─ wolf
   │     ├─ red wolf, maned wolf, Canis rufus, Canis niger
   │     └─ coyote, prairie wolf, brush wolf, Canis latrans
   ├─ fox
   │  └─ fox
   │     ├─ Arctic fox, white fox, Alopex lagopus
   │     └─ grey fox, gray fox, Urocyon cinereoargenteus
   ├─ domestic cat, house cat, Felis domesticus, Felis catus
   │  └─ domestic cat, house cat, Felis domesticus, Felis catus
   │     ├─ tiger cat
   │     └─ Egyptian cat
   └─ bear
      └─ bear
         ├─ American black bear, black bear, Ursus americanus,
         │  Euarctos americanus
         └─ sloth bear, Melursus ursinus, Ursus ursinus
```

(c) LIVING17 hierarchy (part 1).



(d) LIVING17 hierarchy (part 2).

```
ENTITY13
├── structure, place
│   └── man-made structure, construction
│       ├── area
│       │   └── bell cote, bell cot
│       ├── dummy47
│       │   └── prison, prison house
│       ├── dummy67
│       │   └── fountain
│       ├── dummy46
│       │   └── planetarium
│       ├── barrier
│       │   └── breakwater, groin, groyne, mole, bulwark, seawall,
│       │       jetty
│       ├── reservoir
│       │   └── water tower
│       ├── signboard, sign
│       │   └── traffic light, traffic signal, stoplight
│       ├── dwelling, home, domicile, abode, habitation, dwelling
│       │   house
│       │   ├── cliff dwelling
│       │   ├── monastery
│       │   └── castle
└── paraphernalia
    ├── instrument
    │   ├── medical instrument
    │   │   └── syringe
    │   ├── percussion instrument, percussive instrument
    │   │   └── steel drum
    │   ├── piano, pianoforte, forte-piano
    │   │   └── upright, upright piano
    │   ├── bottle opener
    │   │   └── corkscrew, bottle screw
    │   ├── scientific instrument
    │   │   └── abacus
    │   ├── timer
    │   │   └── parking meter
    │   ├── tool
    │   │   └── lighter, light, igniter, ignitor
    │   ├── knife
    │   │   └── cleaver, meat cleaver, chopper
    │   ├── lock
    │   │   └── padlock
    │   ├── measuring instrument, measuring system, measuring device
    │   │   └── scale, weighing machine
    ├── equipment
    │   ├── ball
    │   │   ├── volleyball
    │   │   ├── basketball
    │   │   └── croquet ball
    │   ├── photographic equipment
    │   │   └── tripod
    │   ├── digital computer
    │   │   ├── hand-held computer, hand-held microcomputer
    │   │   └── notebook, notebook computer
    │   ├── electronic equipment
    │   │   ├── monitor
    │   │   └── projector
    │   ├── weight, free weight, exercising weight
    │   │   └── barbell
    │   ├── gymnastic apparatus, exerciser
    │   │   └── balance beam, beam
```

(e) ENTITY13 hierarchy (part 1).

```
ENTITY13
└─animal, animate being, beast, brute, creature, fauna
  ├─bird
  │ ├─passerine, passeriform bird
  │ │ └─indigo bunting, indigo finch, indigo bird, Passerina
  │ │   cyanea
  │ ├─hawk
  │ │ └─kite
  │ ├─phasianid
  │ │ └─quail
  │ ├─parrot
  │ │ └─African grey, African gray, Psittacus erithacus
  │ ├─cuckoo
  │ │ └─coucal
  │ ├─coraciiform bird
  │ │ └─bee eater
  │ ├─aquatic bird
  │ │ ├─spoonbill
  │ │ └─limpkin, Aramus pictus
  │ ├─coot
  │ │ └─American coot, marsh hen, mud hen, water hen, Fulica
  │ │   americana
  │ └─penguin
  │   └─king penguin, Aptenodytes patagonica
  └─reptile, reptilian
    ├─turtle
    │ └─mud turtle
    ├─sea turtle, marine turtle
    │ └─loggerhead, loggerhead turtle, Caretta caretta
    ├─agamid, agamid lizard
    │ └─agama
    ├─venomous lizard
    │ └─Gila monster, Heloderma suspectum
    ├─chameleon, chamaeleon
    │ └─African chameleon, Chamaeleo chamaeleon
    ├─ceratopsian, horned dinosaur
    │ └─triceratops
    ├─snake, serpent, ophidian
    │ ├─thunder snake, worm snake, Carphophis amoenus
    │ ├─green snake, grass snake
    │ └─water snake
    └─cobra
      └─Indian cobra, Naja naja
    └─arthropod
      ├─insect
      │ ├─bee
      │ ├─walking stick, walkingstick, stick insect
      │ └─leafhopper
      ├─crab
      │ ├─rock crab, Cancer irroratus
      │ └─fiddler crab
      ├─beetle
      │ ├─tiger beetle
      │ └─ground beetle, carabid beetle
      └─spider
        ├─black and gold garden spider, Argiope aurantia
        ├─barn spider, Araneus cavaticus
        └─black widow, Latrodectus mactans
```

(f) ENTITY13 hierarchy (part 2).

```
ENTITY13
├─animal, animate being, beast, brute, creature, fauna
│  └─mammal, mammalian
│     ├─sea cow, sirenian mammal, sirenian
│     │  └─dugong, Dugong dugon
│     ├─carnivore
│     │  └─tiger, Panthera tigris
│     ├─dog, domestic dog, Canis familiaris
│     │  ├─Irish wolfhound
│     │  ├─Norwegian elkhound, elkhound
│     │  └─Samoyed, Samoyede
│     ├─domestic cat, house cat, Felis domesticus, Felis catus
│     │  ├─Persian cat
│     │  └─Siamese cat, Siamese
│     ├─ungulate, hoofed mammal
│     │  └─hippopotamus, hippo, river horse, Hippopotamus
│     │    amphibius
│     ├─goat, caprine animal
│     │  └─ibex, Capra ibex
│     └─monkey
│        └─colobus, colobus monkey
├─food, nutrient
│  └─produce, green goods, green groceries, garden truck
│     ├─vegetable, veggie, veg
│     │  ├─broccoli
│     │  ├─cauliflower
│     │  └─cucumber, cuke
│     ├─summer squash
│     │  └─spaghetti squash
│     ├─winter squash
│     │  ├─acorn squash
│     │  └─butternut squash
│     ├─sweet pepper
│     │  └─bell pepper
│     └─fruit
│        ├─fig
│        ├─corn
│        └─orange
└─apparel, toiletries
   └─garment
      ├─raincoat, waterproof
      │  └─trench coat
      ├─shirt
      │  └─jersey, T-shirt, tee shirt
      ├─skirt
      │  └─miniskirt, mini
      ├─swimsuit, swimwear, bathing suit, swimming costume,
      │  bathing costume
      │  ├─swimming trunks, bathing trunks
      │  └─bikini, two-piece
      ├─coat
      │  ├─abaya
      │  ├─cloak
      │  └─poncho
      ├─dress, frock
      │  └─gown
      └─garment
         └─military uniform
```

(g) ENTITY13 hierarchy (part 3).

```
ENTITY13
├── apparel, toiletries
│   └── accessory, accoutrement, accouterment
│       ├── cap
│       │   └── bathing cap, swimming cap
│       ├── tights, leotards
│       │   └── maillot
│       ├── face mask
│       │   └── gasmask, respirator, gas helmet
│       ├── helmet
│       │   └── crash helmet
│       └── jewelry, jewellery
│           └── necklace
├── conveyance, transport
│   ├── wheeled vehicle
│   │   ├── cycles
│   │   │   └── unicycle, monocycle
│   │   ├── motorcycle, bike
│   │   │   └── motor scooter, scooter
│   │   ├── bus, autobus, coach, charabanc, double-decker, jitney,
│   │   │   motorbus, motorcoach, omnibus, passenger vehicle
│   │   │   ├── minibus
│   │   │   └── recreational vehicle, RV, R.V.
│   │   ├── car, auto, automobile, machine, motorcar
│   │   │   ├── racer, race car, racing car
│   │   │   └── jeep, landrover
│   │   ├── car, railcar, railway car, railroad car
│   │   │   └── passenger car, coach, carriage
│   │   ├── truck, motortruck
│   │   │   ├── trailer truck, tractor trailer, trucking rig, rig,
│   │   │   │   articulated lorry, semi
│   │   │   └── snowplow, snowplough
│   │   └── handcart, pushcart, cart, go-cart
│   │       └── shopping cart
│   └── craft
│       ├── airplane, aeroplane, plane
│       │   └── airliner
│       ├── passenger ship
│       │   └── liner, ocean liner
│       ├── boat
│       │   ├── fireboat
│       │   ├── yawl
│       │   └── speedboat
│       ├── sailboat, sailing boat
│       │   ├── catamaran
│       │   └── trimaran
│       ├── cargo ship, cargo vessel
│       │   └── container ship, containership, container vessel
│       ├── spacecraft, ballistic capsule, space vehicle
│       │   └── space shuttle
│       └── warship, war vessel, combat ship
│           └── aircraft carrier, carrier, flattop, attack aircraft
│               carrier
└── furnishing
    └── furniture, piece of furniture, article of furniture
        ├── baby bed, baby's bed
        │   └── bassinet
        ├── bed
        │   └── four-poster
        ├── cabinet
        │   └── file, file cabinet, filing cabinet
        ├── screen
        │   ├── shoji
        │   ├── fire screen, fireguard
        │   └── mosquito net
        ├── seat
        │   └── toilet seat
        ├── chair
        │   └── folding chair
        ├── chest of drawers, chest, bureau, dresser
        │   └── chiffonier, commode
        └── wall unit
            └── wardrobe, closet, press
```

(h) ENTITY13 hierarchy (part 4).

```
ENTITY30
├── garment
│   └── garment
│       ├── garment
│       │   └── apron
│       ├── coat
│       │   ├── academic gown, academic robe, judge's robe
│       │   └── fur coat
│       └── skirt
│           └── miniskirt, mini
├── bird
│   └── passerine, passeriform bird
│       ├── passerine, passeriform bird
│       │   ├── brambling, Fringilla montifringilla
│       │   ├── goldfinch, Carduelis carduelis
│       │   └── water ouzel, dipper
│       └── titmouse, tit
│           └── chickadee
│   └── aquatic bird
│       ├── aquatic bird
│       │   ├── albatross, mollymawk
│       │   └── crane
│       ├── stork
│       │   └── white stork, Ciconia ciconia
│       └── sandpiper
│           └── red-backed sandpiper, dunlin, Erolia alpina
├── reptile, reptilian
│   └── serpentes
│       ├── snake, serpent, ophidian
│       │   ├── thunder snake, worm snake, Carphophis amoenus
│       │   ├── king snake, kingsnake
│       │   └── garter snake, grass snake
│       └── black mamba, Dendroaspis augusticeps
│           └── green mamba
│   └── saurian
│       ├── iguanid, iguanid lizard
│       │   └── American chameleon, anole, Anolis carolinensis
│       ├── anguid lizard
│       │   └── alligator lizard
│       ├── venomous lizard
│       │   └── Gila monster, Heloderma suspectum
│       └── lacertid lizard, lacertid
│           └── green lizard, Lacerta viridis
└── arthropod
    └── arachnid, arachnoid
        ├── arachnid, arachnoid
        │   ├── harvestman, daddy longlegs, Phalangium opilio
        │   └── scorpion
        └── spider
            ├── black widow, Latrodectus mactans
            └── barn spider, Araneus cavaticus
    └── crustacean
        ├── crustacean
        │   ├── crayfish, crawfish, crawdad, crawdaddy
        │   └── hermit crab
        ├── crab
        │   └── Dungeness crab, Cancer magister
        └── lobster
            └── spiny lobster, langouste, rock lobster, crawfish,
                crayfish, sea crawfish
    └── insect
        ├── insect
        │   ├── fly
        │   ├── grasshopper, hopper
        │   └── lacewing, lacewing fly
        └── butterfly
            └── sulphur butterfly, sulfur butterfly
```

(i) ENTITY30 hierarchy (part 1).

```
ENTITY30
├── mammal, mammalian
│   ├── ungulate, hoofed mammal
│   │   ├── ungulate, hoofed mammal
│   │   │   ├── zebra
│   │   │   └── llama
│   │   ├── cattle, cows, kine, oxen, Bos taurus
│   │   │   └── ox
│   │   └── antelope
│   │       └── gazelle
│   ├── primate
│   │   ├── monkey
│   │   │   ├── baboon
│   │   │   └── howler monkey, howler
│   │   ├── lemur
│   │   │   └── Madagascar cat, ring-tailed lemur, Lemur catta
│   │   └── ape
│   │       └── chimpanzee, chimp, Pan troglodytes
│   └── carnivore
│       ├── carnivore
│       │   └── black-footed ferret, ferret, Mustela nigripes
│       └── dog, domestic dog, Canis familiaris
│           ├── Italian greyhound
│           ├── Bedlington terrier
│           └── basenji
├── fish
│   └── bony fish
│       ├── bony fish
│       │   └── tench, Tinca tinca
│       ├── salmon
│       │   └── coho, cohoe, coho salmon, blue jack, silver salmon,
│       │       Oncorhynchus kisutch
│       ├── butterfly fish
│       │   └── rock beauty, Holocanthus tricolor
│       └── scorpaenid, scorpaenid fish
│           └── lionfish
└── accessory, accoutrement, accouterment
    ├── headdress, headgear
    │   ├── clip
    │   │   └── hair slide
    │   ├── cap
    │   │   └── shower cap
    │   ├── hat, chapeau, lid
    │   │   └── bonnet, poke bonnet
    │   └── helmet
    │       └── pickelhaube
    ├── neckwear
    │   ├── brace
    │   │   └── neck brace
    │   ├── scarf
    │   │   └── feather boa, boa
    │   ├── necktie, tie
    │   │   └── Windsor tie
    │   └── neckwear
    │       └── bib
    └── footwear, legwear
        ├── footwear, legwear
        │   └── clog, geta, patten, sabot
        ├── tights, leotards
        │   └── maillot
        └── shoe
            ├── Loafer
            └── running shoe
```

(j) ENTITY30 hierarchy (part 2).

```
ENTITY30
├── appliance
│   └── home appliance, household appliance
│       ├── cooker
│       │   └── Crock Pot
│       └── home appliance, household appliance
│           ├── washer, automatic washer, washing machine
│           ├── microwave, microwave oven
│           └── vacuum, vacuum cleaner
├── craft
│   └── vessel, watercraft
│       ├── boat
│       │   └── lifeboat
│       ├── sailboat, sailing boat
│       │   └── trimaran
│       ├── warship, war vessel, combat ship
│       │   └── aircraft carrier, carrier, flattop, attack aircraft
│       │       carrier
│       └── cargo ship, cargo vessel
│           └── container ship, containership, container vessel
├── equipment
│   ├── electronic equipment
│   │   ├── telephone, phone, telephone set
│   │   │   └── pay-phone, pay-station
│   │   └── electronic equipment
│   │       ├── printer
│   │       ├── computer keyboard, keypad
│   │       └── microphone, mike
│   └── sports equipment
│       ├── sports equipment
│       │   ├── ski
│       │   └── racket, racquet
│       ├── ball
│       │   └── croquet ball
│       └── weight, free weight, exercising weight
│           └── dumbbell
└── instrument
    ├── kitchen utensil
    │   ├── kitchen utensil
    │   │   ├── spatula
    │   │   └── measuring cup
    │   ├── knife
    │   │   └── cleaver, meat cleaver, chopper
    │   └── pot
    │       └── coffeepot
    ├── measuring instrument, measuring system, measuring device
    │   ├── compass
    │   │   └── magnetic compass
    │   ├── timer
    │   │   └── parking meter
    │   ├── watch, ticker
    │   │   └── digital watch
    │   └── clock
    │       └── analog clock
    ├── musical instrument, instrument
    │   ├── percussion instrument, percussive instrument
    │   │   └── maraca
    │   ├── piano, pianoforte, forte-piano
    │   │   ├── upright, upright piano
    │   │   └── grand piano, grand
    │   └── wind instrument, wind
    │       └── French horn, horn
    └── tableware
        ├── bottle
        │   └── water bottle
        ├── bowl
        │   └── mixing bowl
        ├── glass, drinking glass
        │   └── beer glass
        └── jug
            └── water jug
```

(k) ENTITY30 hierarchy (part 3).

```
ENTITY30
├── instrument
│   └── tool
│       ├── lock
│       │   ├── combination lock
│       │   └── padlock
│       ├── tool
│       │   └── screw
│       └── pen
│           └── quill, quill pen
├── man-made structure, construction
│   ├── barrier
│   │   ├── gate
│   │   │   └── turnstile
│   │   ├── fence, fencing
│   │   │   └── picket fence, paling
│   │   └── barrier
│   │       ├── bannister, banister, balustrade, balusters, handrail
│   │       └── breakwater, groin, groyne, mole, bulwark, seawall,
│   │           jetty
│   └── building, edifice
│       ├── place of worship, house of prayer, house of God, house
│       │   of worship
│       │   └── mosque
│       ├── mercantile establishment, retail store, sales outlet,
│       │   outlet
│       │   ├── bookshop, bookstore, bookstall
│       │   └── butcher shop, meat market
│       └── dwelling, home, domicile, abode, habitation, dwelling
│           house
│           └── castle
├── wheeled vehicle
│   └── motor vehicle, automotive vehicle
│       ├── bus, autobus, coach, charabanc, double-decker, jitney,
│       │   motorbus, motorcoach, omnibus, passenger vehicle
│       │   └── school bus
│       ├── car, auto, automobile, machine, motorcar
│       │   ├── limousine, limo
│       │   └── convertible
│       └── minibike, motorbike
│           └── moped
├── cooked food, prepared food
│   └── dish
│       ├── dish
│       │   ├── pizza, pizza pie
│       │   └── potpie
│       ├── hamburger, beefburger, burger
│       │   └── cheeseburger
│       └── potato, white potato, Irish potato, murphy, spud, tater
│           └── mashed potato
└── produce, green goods, green groceries, garden truck
    ├── vegetable, veggie, veg
    │   ├── vegetable, veggie, veg
    │   │   ├── cucumber, cuke
    │   │   └── artichoke, globe artichoke
    │   ├── summer squash
    │   │   └── zucchini, courgette
    │   └── winter squash
    │       └── butternut squash
    └── fruit
        ├── fruit
        │   ├── strawberry
        │   ├── pineapple, ananas
        │   └── jackfruit, jak, jack
        └── eating apple, dessert apple
            └── Granny Smith
```

(l) ENTITY30 hierarchy (part 4).

```
NONLIVING26
├── armor
│   └── body armor, body armour, suit of armor, suit of armour,
│       coat of mail, cataphract
│       └── body armor, body armour, suit of armor, suit of armour,
│           coat of mail, cataphract
│           ├── breastplate, aegis, egis
│           └── bulletproof vest
├── bag
│   └── bag
│       └── bag
│           ├── plastic bag
│           └── purse
├── area
│   └── roof
│       └── roof
│           ├── dome
│           └── vault
├── barrier
│   └── fence, fencing
│       ├── fence, fencing
│       │   └── chainlink fence
│       └── rail fence
│           └── worm fence, snake fence, snake-rail fence, Virginia
│               fence
├── building, edifice
│   ├── dwelling, home, domicile, abode, habitation, dwelling house
│   │   └── dwelling, home, domicile, abode, habitation, dwelling
│   │       house
│   │       ├── monastery
│   │       └── palace
│   ├── mercantile establishment, retail store, sales outlet,
│   │   outlet
│   │   └── mercantile establishment, retail store, sales outlet,
│   │       outlet
│   │       ├── butcher shop, meat market
│   │       └── barbershop
│   └── outbuilding
│       ├── shed
│       │   └── apiary, bee house
│       └── outbuilding
│           └── greenhouse, nursery, glasshouse
├── electronic equipment
│   └── digital computer
│       └── digital computer
│           ├── desktop computer
│           └── laptop, laptop computer
├── garment
│   ├── coat
│   │   └── coat
│   │       ├── fur coat
│   │       └── lab coat, laboratory coat
│   └── skirt
│       └── skirt
│           ├── miniskirt, mini
│           └── hoopskirt, crinoline
├── headdress, headgear
│   └── hat, chapeau, lid
│       └── hat, chapeau, lid
│           ├── bearskin, busby, shako
│           └── bonnet, poke bonnet
├── kitchen utensil
│   └── pot
│       └── pot
│           ├── teapot
│           └── Dutch oven
└── measuring instrument, measuring system, measuring device
    └── timepiece, timekeeper, horologe
        ├── timer
        │   └── stopwatch, stop watch
        └── watch, ticker
            └── digital watch
```

(m) NONLIVING26 hierarchy (part 1).

```
NONLIVING26
├─motor vehicle, automotive vehicle
│ ├─bus, autobus, coach, charabanc, double-decker, jitney,
│ │ motorbus, motorcoach, omnibus, passenger vehicle
│ │ └─bus, autobus, coach, charabanc, double-decker, jitney,
│ │   motorbus, motorcoach, omnibus, passenger vehicle
│ │   ├─minibus
│ │   └─trolleybus, trolley coach, trackless trolley
│ ├─car, auto, automobile, machine, motorcar
│ │ └─car, auto, automobile, machine, motorcar
│ │   ├─Model T
│ │   └─racer, race car, racing car
│ └─truck, motortruck
│   └─truck, motortruck
│     ├─pickup, pickup truck
│     └─fire engine, fire truck
├─musical instrument, instrument
│ ├─percussion instrument, percussive instrument
│ │ └─percussion instrument, percussive instrument
│ │   ├─marimba, xylophone
│ │   └─steel drum
│ ├─wind instrument, wind
│ │ └─wind instrument, wind
│ │   ├─oboe, hautboy, hautbois
│ │   └─sax, saxophone
│ ├─keyboard instrument
│ │ ├─piano, pianoforte, forte-piano
│ │ │ └─grand piano, grand
│ │ └─keyboard instrument
│ │   └─organ, pipe organ
│ └─stringed instrument
│   ├─stringed instrument
│   │ └─banjo
│   └─guitar
│     └─electric guitar
├─seat
│ └─chair
│   ├─chair of state
│   │ └─throne
│   └─chair
│     └─folding chair
├─sports equipment
│ └─ball
│   └─ball
│     ├─volleyball
│     └─punching bag, punch bag, punching ball, punchball
├─tableware
│ └─bottle
│   └─bottle
│     ├─pop bottle, soda bottle
│     └─beer bottle
├─vessel, watercraft
│ ├─boat
│ │ ├─boat
│ │ │ └─gondola
│ │ └─sailboat, sailing boat
│ │   └─trimaran
│ └─ship
│   └─ship
│     ├─schooner
│     └─pirate, pirate ship
└─vegetable, veggie, veg
  └─squash
    ├─summer squash
    │ └─spaghetti squash
    └─winter squash
      └─acorn squash
```

(n) NONLIVING26 hierarchy (part 2).