A Second-order Regression Model Shows Edge of Stability Behavior

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Abstract

Recent studies of learning algorithms have shown that there is a regime with an initial increase in the largest eigenvalue of the loss Hessian (progressive sharpening), followed by a stabilization of the eigenvalue near the maximum value which allows convergence (edge of stability). We consider a class of predictive models that are quadratic in the parameters, which we call second-order regression models. This is in contrast with the neural tangent kernel regime, where the predictive function is linear in the parameters. For quadratic objectives in two dimensions, we prove that this second order regression model exhibits both progressive sharpening and edge of stability behavior. We then show that in higher dimensions, the model shows this behavior generically without the structure of a neural network, due to a non-linearity induced in the learning dynamics. This suggests that progressive sharpening and edge of stability aren't uniquely features of neural network models, and could be a more general property of learning in high-dimensional, non-linear models. Finally, we show that edge of stability behavior in neural networks is correlated with the behavior in quadratic regression models.

1. Introduction

A recent trend in the theoretical understanding of deep learning has focused on the *linearized* regime, where the Neural Tangent Kernel (NTK) controls the learning dynamics [10, 12]. The NTK describes learning dynamics of all networks over short enough time horizons, and can describe the dynamics of wide networks over large time horizons. In the NTK regime, there is a function-space ODE which allows for explicit characterization of the network outputs [10, 12, 18]. This approach has been used across the board to gain insights into wide neural networks, but it suffers a major limitation: the model is linear in the parameters, so it describes a regime with relatively trivial dynamics that cannot capture feature learning and cannot accurately represent the types of complex training phenomena often observed in practice.

While other large-width scaling regimes can preserve some non-linearity and allow for certain types of feature learning [2, 19], such approaches tend to focus on the small learning-rate or continuous-time dynamics. In contrast, recent empirical work has highlighted a number of important phenomena arising from the non-linear discrete dynamics in training practical networks with large learning rates [5, 6, 8, 15]. In particular, main experiments have shown the tendency for networks to display *progressive sharpening* of the largest curvature towards the *edge of stability*, in which the maximum eigenvalue of the loss Hessian increases over the course of training until it stabilizes at a value equal to roughly two divided by the learning rate, corresponding to the largest eigenvalue for which gradient descent would converge in a quadratic potential [3, 4, 7, 17].

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In order to build a better understanding of this behavior, we introduce a class of models which display all the relevant phenomenology, yet are simple enough to admit numerical and analytic understanding. In particular, we propose a simple *quadratic regression model* with a quartic loss function which fulfills both these goals. We prove that under the right conditions, this simple model shows both progressive sharpening *and* edge of stability behavior. We then empirically analyze a more general model which shows these behaviors *generically* in the large datapoint, large model limit. Finally, we conduct a numerical analysis on the properties of a real neural network and use tools from our theoretical analysis to show that edge of stability behavior "in the wild" shows some of the same patterns as the theoretical models.

2. Basic quartic loss function

2.1. Model definition

We consider the optimization of the quadratic loss function $\mathcal{L}(\boldsymbol{\theta}) = z^2/2$, where z a quadratic function on the $P \times 1$-dimensional parameter vector $\boldsymbol{\theta}$ and $\mathbf{Q}$ is a $P \times P$ symmetric matrix:

$$z = \frac{1}{2}\left[\boldsymbol{\theta}^\top \mathbf{Q}\boldsymbol{\theta} - E\right] . \tag{1}$$

This can be interpreted either as a model in which the predictive function is quadratic in the input parameters, or as a second-order approximation to a more complicated non-linear function such as a deep network.

We are interested in understanding the *edge of stability* (EOS) behavior in this model: gradient descent (GD) trajectories where the maximum eigenvalue of the NTK, $\mathbf{J}\mathbf{J}^\top$, remains close to the critical value $2/\eta$. (Note: we define edge of stability with respect to the maximum NTK eigenvalue; for any twice-differentiable model trained with squared loss, this is equivalent to the maximum eigenvalue of the loss Hessian used in [3] as the model converges to a stationary point [11].)

When $\mathbf{Q}$ has both positive and negative eigenvalues, the loss landscape is the square of a hyperbolic parabaloid (Figure 1, left). As suggested by the gradient flow analysis, this causes some trajectories to increase their curvature before convergence. This causes the final curvature to depend on both the initialization and learning rate. One of the challenges in analyzing the gradient descent (GD) dynamics is that they rapidly and heavily oscillate around minima for large learning rates. One way to mitigate this issue is to consider only every other step (Figure 1, right). We will use this observation to analyze the gradient descent (GD) dynamics directly to find configurations where these trajectories show edge of stability behavior.

In the eigenbasis coordinates, the gradient descent equations are

$$\tilde{z}_{t+1} - \tilde{z}_t = -\tilde{z}_t \sum_{i=1}^{P} \tilde{J}(\omega_i)_t^2 + \frac{1}{2}(\tilde{z}_t^2) \sum_{i=1}^{P} \omega_i \tilde{J}(\omega_i)_t^2 \tag{2}$$

$$\tilde{J}(\omega_i)_{t+1}^2 - \tilde{J}(\omega_i)_t^2 = -\tilde{z}_t \omega_i (2 - \tilde{z}_t \omega_i) \tilde{J}(\omega_i)_t^2 \text{ for all } 1 \leq i \leq P. \tag{3}$$

We'll find it convenient in the following to write the dynamics in terms of weighted averages of $\tilde{J}(\omega_i)^2$ instead of the modes $\tilde{J}(\omega_i)$:

$$T(\alpha) = \sum_{i=1}^{P} \omega_i^\alpha \tilde{J}(\omega_i)^2 . \tag{4}$$

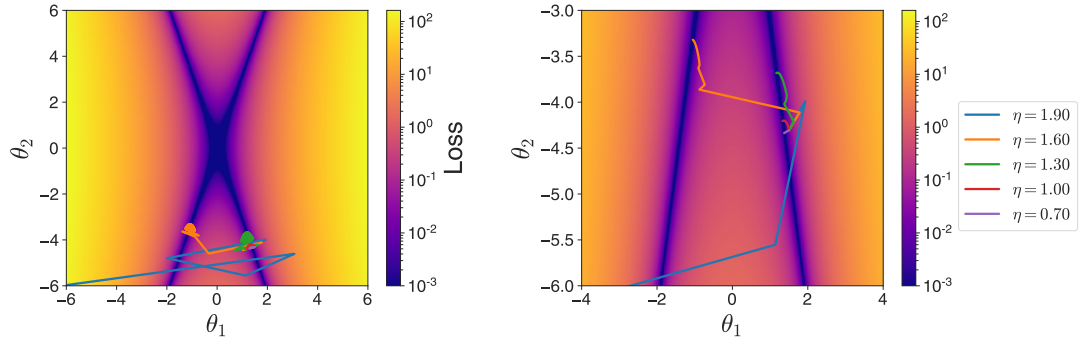

Figure 1: Quartic loss landscape $\mathcal{L}(\cdot)$ as a function of the parameters $\boldsymbol{\theta}$, where $D = 2, E = 0$ and $\mathbf{Q}$ has eigenvalues 1 and -0.1. The GD trajectories converge to minima with larger curvature than at initialization and therefore show progressive sharpening (left). The two-step dynamics, in which we consider only even iteration numbers, exhibit fewer oscillations near the edge of stability (right).

The dynamical equations become:

$$\tilde{z}_{t+1} - \tilde{z}_t = -\tilde{z}_t T_t(0) + \frac{1}{2}(\tilde{z}_t^2) T_t(1) \tag{5}$$

$$T_{t+1}(k) - T_t(k) = -\tilde{z}_t (2T_t(k+1) - \tilde{z}_t T_t(k+2)) \,. \tag{6}$$

If $\mathbf{Q}$ is invertible, then we have $E = T_t(-1) - 2\tilde{z}_t$. Note that by definition $T_t(0) = \eta \mathbf{J}_t \mathbf{J}_t^\top$ is the (rescaled) NTK. Edge of stability behavior corresponds to dynamics which keep $T_t(0)$ near the value 2 as $\tilde{z}_t$ goes to 0.

2.1.1. REDUCTION TO CATAPULT DYNAMICS

If the eigenvalues of $\mathbf{Q}$ are $\{-\omega, \omega\}$, and $E = 0$, the model becomes equivalent to a single hidden layer linear network with one training datapoint (Appendix A.1) - also known as the catapult phase dynamics. This model doesn't exhibit sharpening or edge of stability behavior [13].

2.1.2. EDGE OF STABILITY REGIME

In this section, we consider the case in which $\mathbf{Q}$ has two eigenvalues - one of which is large and positive, and the other one small and negative. Without loss of generality, we assume that the largest eigenvalue of $\mathbf{Q}$ is 1. We denote the second eigenvalue by $-\epsilon$, for $0 < \epsilon \leq 1$. With this notation we can write the dynamical equations (Appendix B.1) as

$$\tilde{z}_{t+1} - \tilde{z}_t = -\tilde{z}_t T_t(0) + \frac{1}{2}(\tilde{z}_t^2)((1-\epsilon)T_t(0) + \epsilon(2\tilde{z}_t + E)) \tag{7}$$

$$T_{t+1}(0) - T_t(0) = -2\tilde{z}_t(\epsilon(2\tilde{z}_t + E) + (1-\epsilon)T_t(0)) + \tilde{z}_t^2 \left[T_t(0) + \epsilon(\epsilon - 1)(T_t(0) - E - 2\tilde{z}_t)\right] \,. \tag{8}$$

For small ϵ, there are trajectories where $\lambda_{\max}$ is initially away from $2/\eta$ but converges towards it (Figure 2, left) - in other words, EOS behavior. We used a variety of step sizes η but initialized at pairs initialized at pairs $(\eta z_0, \eta T_0(0))$ to show the universality of the $\tilde{z}$-$T(0)$ coordinates.

In order to quantitatively understand the progressive sharpening and edge of stability, it is useful to look at the two-step dynamics. One additional motivation for studying the two-step dynamics follows from the analysis of gradient descent on linear least squares (i.e., linear model) with a large step size λ. For every coordinate $\tilde{\theta}$, the one-step and two-step dynamics are

$$\tilde{\theta}_{t+1} - \tilde{\theta}_t = -\lambda\tilde{\theta}_t \text{ and } \tilde{\theta}_{t+2} - \tilde{\theta}_t = (1-\lambda)^2\tilde{\theta}_t \qquad \text{(GD in quadratic potential)}. \tag{9}$$

While the dynamics converge for $\lambda < 2$, if $\lambda > 1$ the one-step dynamics oscillate when approaching minimum, whereas the the two-step dynamics maintain the sign of $\tilde{\theta}$ and the trajectories exhibit no oscillations.

Likewise, plotting every other iterate in the two parameter model more clearly demonstrates the phenomenology. For small ϵ, the dynamics shows the distinct phases described in [14]: an initial increase in $T(0)$, a slow increase in $\tilde{z}$, then a decrease in $T(0)$, and finally a slow decrease of $\tilde{z}$ while $T(0)$ remains near 2 (Figure 2, middle).

Unfortunately, the two-step version of the dynamics defined by Equations 7 and 8 are more complicated – they are 3rd order in $T(0)$ and 9th order in $\tilde{z}$; see Appendix B.2 for a more detailed discussion. However we can still analyze the dynamics as $\tilde{z}$ goes to 0. In order to understand the mechanisms of the EOS behavior, it is useful to understand the *nullclines* of the two step dynamics. The nullcline $f_{\tilde{z}}(\tilde{z})$ of $\tilde{z}$ and $f_T(\tilde{z})$ of $T(0)$ are defined implicitly by

$$(\tilde{z}_{t+2} - \tilde{z}_t)(\tilde{z}, f_{\tilde{z}}(\tilde{z})) = 0,\ (T_{t+2}(0) - T_t(0))(\tilde{z}, f_T(\tilde{z})) = 0 \tag{10}$$

where $\tilde{z}_{t+2} - \tilde{z}_t$ and $T_{t+2}(0) - T_t(0)$ are the aforementioned high order polynomials in $\tilde{z}$ and $T(0)$. Since these polynomials are cubic in $T(0)$, there are three possible solutions as $\tilde{z}$ goes to 0. We are particularly interested in the solution that goes through $\tilde{z} = 0$, $T(0) = 2$ - that is, the critical point corresponding to EOS.

Calculations detailed in Appendix B.2 show that the distance between the two nullclines is linear in ϵ, so they become close as ϵ goes to 0. (Figure 2, middle). In addition, the trajectories stay near $f_{\tilde{z}}$ - which gives rise to EOS behavior. This suggests that the dynamics are slow near the nullclines, and trajectories appear to be approaching an attractor. We can find the structure of the attractor by changing variables to $y_t \equiv T_t(0) - f_{\tilde{z}}(\tilde{z}_t)$ - the distance from the $\tilde{z}$ nullcline. To lowest order in $\tilde{z}$ and y, the two-step dynamical equations become (Appendix B.3):

$$\tilde{z}_{t+2} - \tilde{z}_t = 2y_t\tilde{z}_t + O(y_t^2\tilde{z}_t) + O(y_t\tilde{z}_t^2) \tag{11}$$

$$y_{t+2} - y_t = -2(4 - 3\epsilon + 4\epsilon^2)y_t\tilde{z}_t^2 - 4\epsilon\tilde{z}_t^2 + \epsilon O(\tilde{z}_t^3) + O(y^2\tilde{z}_t^2) \tag{12}$$

We immediately see that $\tilde{z}$ changes slowly for small y - since we chose coordinates where $\tilde{z}_{t+2} - \tilde{z}_t = 0$ when $y = 0$. We can also see that $y_{t+2} - y_t$ is $O(\epsilon)$ for $y_t = 0$ - so for small ϵ, the y dynamics is slow too. Moreover, we see that the coefficient of the $\epsilon\tilde{z}_t^2$ term is negative - the changes in $\tilde{z}$ tend to drive y (and therefore $T(0)$) to decrease. The coefficient of the y_t term is negative as well; the dynamics of y tends to be contractive. The key is that the contractive behavior takes y to an $O(\epsilon)$ fixed point at a rate proportional to $\tilde{z}^2$, while the dynamics of $\tilde{z}$ are proportional to ϵ. This suggests a separation of timescales if $\tilde{z}^2 \gg \epsilon$, where y first equilibrates to a fixed value, and then $\tilde{z}$ converges to 0 (Figure 2, right). This intuition for the lowest order terms can be formalized, and gives us a prediction of $\lim_{t\to\infty} y_t = -\epsilon/2$, confirmed numerically in the full model (Appendix B.5).

We can prove the following theorem about the long-time dynamics of $\tilde{z}$ and y when the higher order terms are included (Appendix B.4):

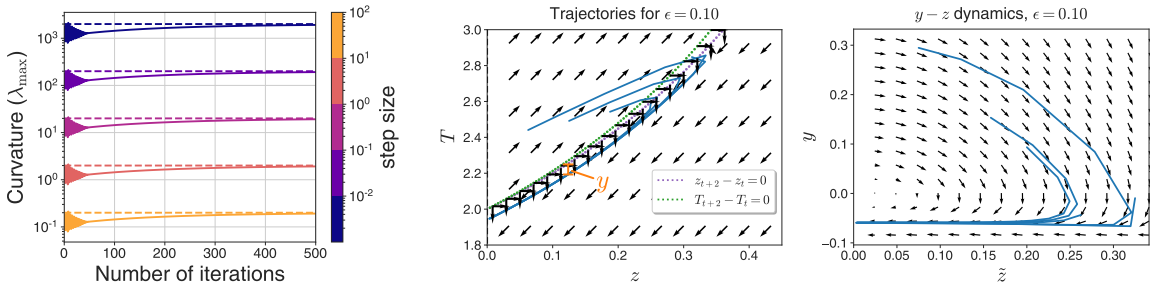

Figure 2: For small ϵ, two-eigenvalue model shows EOS behavior for various step sizes ($\epsilon = 5 \cdot 10^{-3}$, left). Trajectories are the same up to scaling because corresponding rescaled coordinates $\tilde{z}$ and $T(0)$ are the same at initialization. Plotting every other iterate, we see that trajectories in $\tilde{z} - T(0)$ space stay near the nullcline $(\tilde{z}, f_{\tilde{z}}(\tilde{z}))$ - the curve where $\tilde{z}_{t+2} - \tilde{z}_t = 0$ (middle). Changing variables to $y = T(0) - f_{\tilde{z}}(\tilde{z})$ shows quick concentration to a curve of near-constant, small, negative y (right).

Theorem 1 *There exists an $\epsilon_c > 0$ such that for a quadratic regression model with $E = 0$ and eigenvalues $\{-\epsilon, 1\}$, $\epsilon \leq \epsilon_c$. there exists a neighborhood $U \subset \mathbb{R}^2$ and interval $[\eta_1, \eta_2]$ such that for initial $\boldsymbol{\theta} \in U$ and learning rate $\eta \in [\eta_1, \eta_2]$, the model displays edge of stability behavior:*

$$2/\eta - \delta_\lambda \leq \lim_{t \to \infty} \lambda_{\max} \leq 2/\eta \tag{13}$$

for δ_λ of $O(\epsilon)$.

Therefore, unlike the catapult phase model, the small ϵ provably has EOS behavior - whose mechanism is well-understood by the $\tilde{z} - y$ coordinate transformation.

3. Quadratic regression model

3.1. General model

While the model defined in Equation 1 provable displays edge of stability behavior, it required tuning of the eigenvalues of $\mathbf{Q}$ to demonstrate it. We can define a more general model which exhibits edge of stability behavior with less tuning. We define the *quadratic regression model* as follows. Given a P-dimensional parameter vector $\boldsymbol{\theta}$, the D-dimensional output vector $\mathbf{z}$ is given by

$$\mathbf{z} = \mathbf{y} + \mathbf{G}^\top \boldsymbol{\theta} + \frac{1}{2}\mathbf{Q}(\boldsymbol{\theta}, \boldsymbol{\theta}) \tag{14}$$

Here $\mathbf{y}$ is a D-dimensional vector, $\mathbf{G}$ is a $D \times P$-dimensional matrix, and $\mathbf{Q}$ is a $D \times P \times P$-dimensional tensor symmetric in the last two indices - that is, $\mathbf{Q}(\cdot, \cdot)$ takes two P-dimensional vectors as input, and outputs a D-dimensional vector verifying $\mathbf{Q}(\boldsymbol{\theta}, \boldsymbol{\theta})_\alpha = \boldsymbol{\theta}^\top \mathbf{Q}_\alpha \boldsymbol{\theta}$. If $\mathbf{Q} = \mathbf{0}$, the model corresponds to linearized learning (as in the NTK regime). When $\mathbf{Q} \neq \mathbf{0}$, we obtain the first correction to NTK regime. We note that:

$$\mathbf{G}_{\alpha i} = \left.\frac{\partial \mathbf{z}_\alpha}{\partial \boldsymbol{\theta}_i}\right|_{\boldsymbol{\theta}=0}, \ \mathbf{Q}_{\alpha ij} = \frac{\partial^2 \mathbf{z}_\alpha}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j}, \rightarrow \mathbf{J} = \mathbf{G} + \mathbf{Q}(\boldsymbol{\theta}, \cdot)\,, \tag{15}$$

for the $D \times P$ dimensional Jacobian $\mathbf{J}$. For $D = 1$, we recover the model of Equation 1. In the remainder of this section, we will study the limit as D and P increase with fixed ratio D/P.

The quadratic regression model corresponds to a model with a constant second derivative with respect to parameter changes - or a second order expansion of a more complicated ML model. Quadratic expansions of shallow MLPs have been previously studied [1, 20], but we see that even random, unstructured quadratic regression models lead to EOS behavior (Appendix C). This model is related to, but not equivalent to the second order expansion in the neural tangent hierarchy [9] (see Appendix A.3 for details).

4. Connection to real world models

In this section we examine how representative is the proposed model and the developed theory to the behavior of "real world" models. Following [3], we trained a 2-hidden layer $\tanh$ network using the squared loss on 5000 examples from CIFAR10 with learning rate 10^{-2} - a setting which shows edge of stability behavior. Close to the onset of EOS, we approximately computed λ_1, the largest eigenvalue of $\mathbf{J}\mathbf{J}^\top$, and its corresponding eigenvector $\mathbf{v}_1$ using a Lanczos method [6, 16]. We use $\mathbf{v}_1$ to compute $z_1 = \mathbf{v}_1^\top \mathbf{z}$, where $\mathbf{z}$ is the vector of residuals $f(\mathbf{X}, \boldsymbol{\theta}) - \mathbf{Y}$ for neural network function f, training inputs $\mathbf{X}$, labels $\mathbf{Y}$, and parameters $\boldsymbol{\theta}$. The EOS behavior in the NTK is similar to the EOS behavior defined with respect to the full Hessian in [3] (Figure 3, left and right). One again, plotting the trajectories at every other step gets rid of the oscillations (Figure 3, middle). Unlike the $D = 1$, $P = 2$ model, there are multiple crossings of the critical line $\lambda_{\max} = 2/\eta$ line.

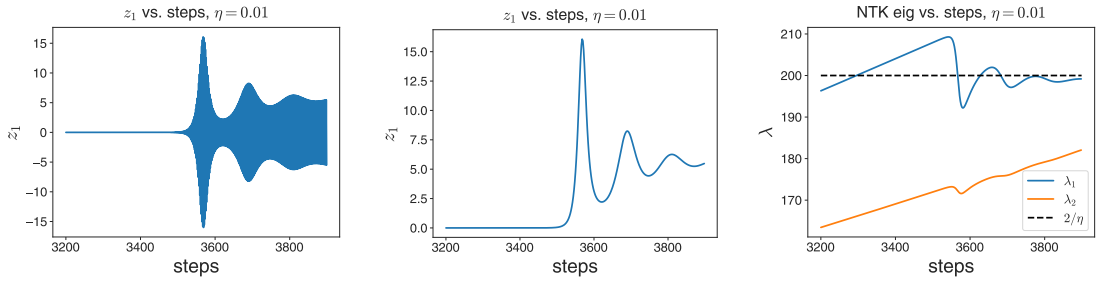

Figure 3: A FCN trained on CIFAR shows multiple cycles of sharpening and edge of stability behavior. z_1, the projection of the training set residuals $f(\mathbf{X}, \boldsymbol{\theta}) - \mathbf{Y}$ onto the top NTK eigenmode $\mathbf{v}_1$, increases in magnitude and oscillates around 0 (left). Plotting dynamics every two steps removes high frequency oscillations (middle). The largest eigenvalue λ_1 crosses the edge of stability multiple times, but the second largest eigenvalue λ_2 remains below the edge of stability.

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Appendix A. Connection to other models

A.1. One-hidden layer linear network

Consider a one hidden layer network with a scalar output:

$$f(\mathbf{x}) = \mathbf{v}^\top \mathbf{U}\mathbf{x} \tag{16}$$

where $\mathbf{x}$ is an input vector of length N, $\mathbf{U}$ is a $K \times N$ dimensional matrix, and $\mathbf{v}$ is a K dimensional vector. We note that

$$\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{v}_i \partial \mathbf{v}_j} = \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{U}_{ij} \partial \mathbf{U}_{kl}} = 0, \ \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{v}_i \partial \mathbf{U}_{jk}} = \delta_{ij} \mathbf{x}_k \tag{17}$$

where δ_{ij} is the Kroenecker delta. For a fixed training set, this second derivative is constant; therefore, the one-hidden layer linear network is a quadratic regression model of the type studied in Section 3.

In the particular case of a single datapoint $\mathbf{x}$, we can compute the eigenvectors of the $\mathbf{Q}$ matrix. Let $(\mathbf{w}, \mathbf{W})$ be an eigenvector of $\mathbf{Q}$, representing the $\mathbf{v}$ and $\mathbf{U}$ components respectively. The eigenvector equations are

$$\omega \mathbf{w}_i = \mathbf{x}_m \delta_{ij} \mathbf{W}_{jm} \tag{18}$$

$$\omega \mathbf{W}_{jm} = \mathbf{x}_m \delta_{ij} \mathbf{w}_i \tag{19}$$

Simplifying, we have:

$$\omega \mathbf{w} = \mathbf{W}\mathbf{x} \tag{20}$$

$$\omega \mathbf{W} = \mathbf{w}\mathbf{x}^\top \tag{21}$$

We have two scenarios. The first is that $\omega = 0$. In this case, we have $\mathbf{w} = 0$, and $\mathbf{W}$ is a matrix with $\mathbf{x}$ in its nullspace. The latter condition gives us M constraints on $M \times N$ equations - for a total of $M(N-1)$ of our $M(N+1)$ total eigenmodes.

If $\omega \neq 0$, then combining the equations we have the conditions:

$$\omega^2 \mathbf{w} = (\mathbf{x} \cdot \mathbf{x})\mathbf{w} \tag{22}$$

$$\omega^2 \mathbf{W} = \mathbf{W}\mathbf{x}\mathbf{x}^\top \tag{23}$$

This gives us $\omega = \pm\sqrt{\mathbf{x} \cdot \mathbf{x}}$. We know from Equation 21 that $\mathbf{W}$ is low rank. Therefore, we can guess a solution of the form

$$\mathbf{W}_{\pm,i} = \pm \mathbf{e}_i \mathbf{x}^\top \tag{24}$$

where the $\mathbf{e}_i$ are the M coordinate vectors. This suggests that we have

$$\mathbf{w}_{\pm,i} = (\sqrt{\mathbf{x} \cdot \mathbf{x}})\mathbf{e}_i \tag{25}$$

This gives us our final $2M$ eigenmodes.

We can analyze the initial values of of the $\tilde{J}(\omega_i)$ as well. The components of the Jacobian can be written as:

$$(\mathbf{J}_v)_i \equiv \frac{\partial f(\mathbf{x})}{\partial \mathbf{v}_i} = \mathbf{U}_{im}\mathbf{x}_m \tag{26}$$

$$(\mathbf{J}_U)_{jm} \equiv \frac{\partial f(\mathbf{x})}{\partial \mathbf{U}_{jm}} = \mathbf{v}_j \mathbf{x}_m \tag{27}$$

From this form, we can deduce that $\mathbf{J}$ is orthogonal to the 0 modes. We can also compute the conserved quantity. Let J_+^2 be the total weight in the positive eigenmodes, and J_-^2 be the total weight in the negative eigenmodes. A direct calculation shows that

$$\omega^{-1}(J_+^2 - J_-^2) = 2f(\mathbf{x}) \tag{28}$$

which implies that $E = 0$.

Therefore, the single-hidden layer linear model on one datapoint is equivalent to the quartic loss model with $E = 0$ and eigenvalues $\pm\sqrt{\mathbf{x} \cdot \mathbf{x}}$.

A.2. Connection to [2]

Since the one-hidden layer linear model has constant $\mathbf{Q}$, the models in Section F.1 of [2] fall into the quadratic regression class. In the case of Section F.1.1, Equation 67, we can make the mapping to a $D = 1$ model explicit. The dynamics are equivalent to said model with a single eigenvalue ω_0 if we make the identifications

$$\Delta = \tilde{z},\ H_y = J_0^2,\ \gamma_0 = \sqrt{2\omega},\ y = -E/2 \tag{29}$$

A.3. Connection to NTH

The Neural Tangent Hierarchy (NTH) equations extend the NTK dynamics to account for changes in the tangent kernel by constructing an infinite sequence of higher order tensors which control the non-linear dynamics of learning [9]. Truncation of the NTH equations at 3rd order is related to, but not the same as the quadratic regression model, as we will show here.

The 3rd order NTH equation describes the change in the tangent kernel $\mathbf{J}\mathbf{J}^\top$. Consider the $D \times D \times D$-dimensional kernel $\mathbf{K}_3$ whose elements are given by

$$(\mathbf{K}_3)_{\alpha\beta\gamma} = \frac{\partial^2 \mathbf{z}_\alpha}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j} \mathbf{J}_{i\gamma} \mathbf{J}_{j\beta} + \frac{\partial^2 \mathbf{z}_\beta}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j} \mathbf{J}_{i\gamma} \mathbf{J}_{j\alpha} \tag{30}$$

where repeated indices are summed over. In the NTH, for squared loss the change in the NTK $\mathbf{J}\mathbf{J}^\top$ is given by

$$\frac{d}{dt}\left(\mathbf{J}\mathbf{J}^\top\right)_{\alpha\beta} = -\eta (\mathbf{K}_3)_{\alpha\beta\gamma} \mathbf{z}_\gamma \tag{31}$$

For fixed $\mathbf{Q} = \frac{\partial^2 \mathbf{z}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}$, this equation is identical to the GF equations for the NTK in the quadratic regression model. We note that $\mathbf{K}_3$ is not constant under the quadratic regression model. Conversely, for fixed $\mathbf{K}_3$, $\frac{\partial^2 \mathbf{z}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}$ is not constant either. Therefore, the two methods can be used to construct different low-order expansions of the dynamics.

Appendix B. 2 parameter model

B.1. Derivation of $\tilde{z}$-$T(0)$ equations

We can use the conserved quantity E to write the dynamics in terms of $\tilde{z}$ and $T(0)$ only. Without loss of generality, let the eigenvalues are 1 and λ, with $-1 \leq \lambda \leq 1$. (We can achieve this by rescaling $\tilde{z}$.) Recall the dynamical equations

$$\tilde{z}_{t+1} - \tilde{z}_t = -\tilde{z}_t T_t(0) + \frac{1}{2}(\tilde{z}_t^2) T_t(1) \tag{32}$$

$$T_{t+1}(0) - T_t(0) = -\tilde{z}_t (2T_t(1) - \tilde{z}_t T_t(2)) \tag{33}$$

We will find substitutions for $T(1)$ and $T(2)$ in terms of $\tilde{z}$ and $T(0)$. Recall that we have

$$T(-1) = E + 2\tilde{z} \tag{34}$$

where E is conserved throughout the dynamics (and indeed is a property of the landscape). We will use this definition to solve for $T(1)$ and $T(2)$.

Since $P = 2$, we can write $T(-1) = bT(0) + aT(1)$, for coefficients a and b which are valid for all combinations of $\tilde{J}$. If $\tilde{J}(\lambda) = 0$, we have $b = 1 - a$. If $\tilde{J}(1) = 0$, we have $1 = \lambda(1-a) + \lambda^2 a$. Solving, we have:

$$T(-1) = (1-a)T(0) + aT(1) \text{ for } a = -\frac{1}{\lambda} \tag{35}$$

The restrictions on λ translate to $a \notin (-1, 1)$. In terms of the conserved quantity $E = T(-1) - 2\tilde{z}$, we have:

$$T(-1) = E + 2\tilde{z} \tag{36}$$

In order to convert the dynamics, we need to solve for $T(1)$ and $T(2)$ in terms of $T(0)$ and $\tilde{z}$. We have:

$$T(1) = \frac{1}{a}\left(T(-1) + (a-1)T(0)\right) = \frac{1}{a}\left(E + 2\tilde{z} + (a-1)T(0)\right) \tag{37}$$

We also have

$$T(2) = T(0) + \left(\frac{1-a}{a^2}\right)\left(T(0) - E - 2\tilde{z}\right) \tag{38}$$

This gives us

$$\tilde{z}_{t+1} - \tilde{z}_t = -\tilde{z}_t T_t(0) + \frac{1}{2a}(\tilde{z}_t^2)((a-1)T_t(0) + 2\tilde{z}_t + E) \tag{39}$$

$$T_{t+1}(0) - T_t(0) = -\frac{2}{a}\tilde{z}_t(2\tilde{z}_t + E + (a-1)T_t(0)) + z_t^2\left[T_t(0) + \left(\frac{1-a}{a^2}\right)\left(T_t(0) - E - 2\tilde{z}_t\right)\right] \tag{40}$$

If $\lambda = -\epsilon$ (that is, $a = \epsilon^{-1}$) we recover the equations from the main text.

The non-negativity of $\tilde{J}^2$ gives us constraints on the values of $\tilde{z}$ and T. For $a > 1$ (small negative second eigenvalue), the constraints are:

$$T > 2\tilde{z} + E,\ T > -(2\tilde{z} + E)/a \tag{41}$$

This is an upward-facing cone with vertex at $\tilde{z} = -E/2$ (Figure 5, left). For $a < -1$, the constraints are

$$-(2\tilde{z} + E)/a < T < 2\tilde{z} + E \tag{42}$$

This is a sideways facing cone with vertex at $\tilde{z} = -E/2$ (Figure 5, right). We see that in this case, there is a limited set of values of T to converge to. Indeed, for $E = 0$, there is no convergence except at $T(0) = 0$. This why we focus on the case of one positive and one negative eigenvalue.

We can also solve for the nullclines - the curves where either $\tilde{z}_{t+1} - \tilde{z}_t = 0$ (blue in Figure 5), or $T_{t+1}(0) - T_t(0) = 0$ (orange in Figure 5). The nullcline $(\tilde{z}, f_{\tilde{z}}(\tilde{z}))$ for $\tilde{z}$ is given by

$$f_{\tilde{z}}(\tilde{z}) = \frac{\tilde{z}(2\tilde{z} + E)}{2a - (a-1)\tilde{z}} \tag{43}$$

The nullcline $(\tilde{z}, f_T(\tilde{z}))$ for $T(0)$ is given by

$$f_T(\tilde{z}) = -\frac{(a-1)\tilde{z} - 2a}{(a^2 - a + 1)\tilde{z} - 2a(a-1)}(2\tilde{z} + E) \tag{44}$$

The line $\tilde{z} = 0$ is also a nullcline.

For the symmetric model $\epsilon = 1$, the structure of the nullclines determines the presence or lack of progressive sharpening. For $E = 0$, there is no sharpening; the phase portrait (Figure 4, left) confirms this as the nullcline in $T_t(0)$ divides the space into two halves, one which converges, and the other which doesn't. However, when $E \neq 0$, the nullclines split, and there is a small region where progressive sharpening can occur (Figure 4, middle). However, there is still no edge of stability behavior in this case - there is no region where the trajectories cluster near $\lambda_{\max} = 2/\eta$ (Figure 4, right).

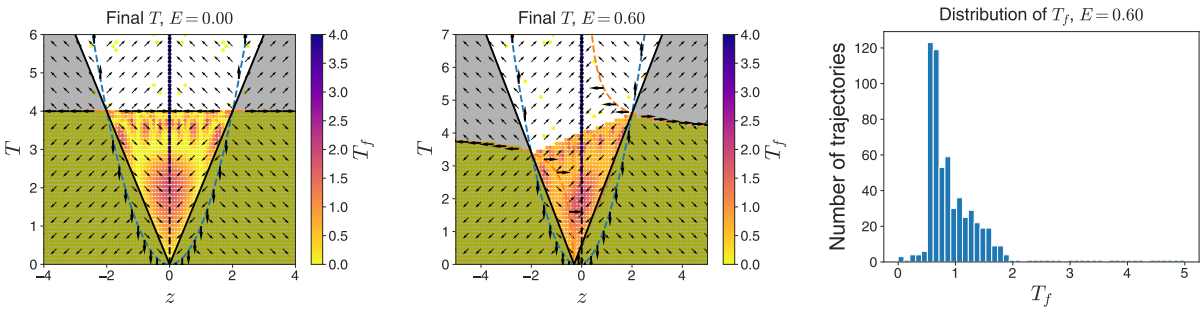

Figure 4: Phase portraits for symmetric model. Arrows indicate signs of changes in $\mathbf{z}$ and T, and grey area represents disallowed coordinates. Dynamics are run from an evenly spaced grid of initializations, and the final value of the curvature $T(0)$ is recorded. Nullclines representing $\tilde{z}_{t+1} - \tilde{z}_t = 0$ (blue) and $T_{t+1}(0) - T_t(0) = 0$ (orange) depend on E. Trajectories show progressive sharpening but no edge of stability effect (right).

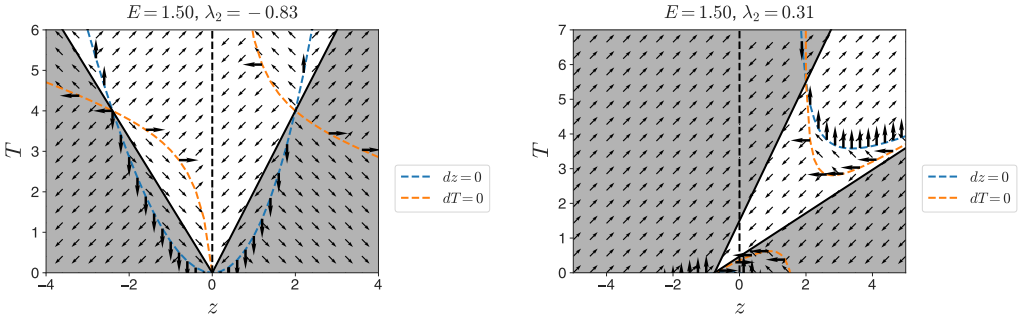

Figure 5: Phase planes of $D = 1$, $P = 2$ model. Grey region corresponds to parameters forbidden by positivity constraints on $\tilde{J}(\omega_i)^2$. For $\lambda > 0$, allowed region is smaller and intersects $\tilde{z} = 0$ at a small range only. Nullclines can be solved for analytically.

B.2. Two-step dynamics

The two-step difference equations can be derived by iterating Equations 7 and 8. We have

$$\tilde{z}_{t+2} - \tilde{z}_t = p_0(\tilde{z}_t, \epsilon) + p_1(\tilde{z}_t, \epsilon)T_t(0) + p_2(\tilde{z}_t, \epsilon)T_t(0)^2 + p_3(\tilde{z}_t, \epsilon)T_t(0)^3 \tag{45}$$

$$T(0)_{t+2} - T_t(0) = q_0(\tilde{z}_t, \epsilon) + q_1(\tilde{z}_t, \epsilon)T_t(0) + q_2(\tilde{z}_t, \epsilon)T_t(0)^2 + q_3(\tilde{z}_t, \epsilon)T_t(0)^3 \tag{46}$$

Here the p_i and q_i are polynomials in $\tilde{z}$, maximum 9th order in $\tilde{z}$ and 6th order in ϵ. They can be computed explicitly but we choose to omit the exact forms for now.

For fixed ϵ, we can solve for the $\tilde{z}$ two-step nullclines ($\tilde{z}_{t+2} - \tilde{z}_t = 0$) and the T nullclines ($T_{t+2}(0) - T_t(0) = 0$) using Cardano's formula to solve for T as a function of $\tilde{z}$. In particular, each nullcline equation has a solution that goes through $\tilde{z} = 0$, $T(0) = 2$, independent of ϵ. This is the family of solutions that we will focus on.

Let $(\tilde{z}, f_{\tilde{z},\epsilon}(\tilde{z}))$ be the nullcline of $\tilde{z}$, and let $(\tilde{z}, f_{T,\epsilon}(\tilde{z}))$ be the nullcline of $T(0)$. We will show that the T values of the nullclines, as a function of $\tilde{z}$ and ϵ, is differentiable around $\tilde{z} = 0$, $\epsilon = 0$.

The nullclines are defined by the implicit equations

$$\begin{aligned} 0 &= 6\tilde{z}^3\epsilon - 2T\tilde{z} - 3T\tilde{z}^2(\epsilon-1) - T\tilde{z}^3(\epsilon+2)(2\epsilon+1) + T^2\tilde{z} + \frac{7}{2}T^2\tilde{z}^2(\epsilon-1) \\ &\quad + \frac{1}{2}T^2\tilde{z}^3\left(9\epsilon^2 - 10\epsilon + 9\right) - \frac{1}{2}T^3\tilde{z}^2(\epsilon-1) - \frac{1}{2}T^3\tilde{z}^3\left(3\epsilon^2 - 4\epsilon + 3\right) + O(\tilde{z}^4) \end{aligned} \tag{47}$$

$$\begin{aligned} 0 &= -8\tilde{z}^2\epsilon - 12\tilde{z}^3(\epsilon-1)\epsilon + 4T\tilde{z}(\epsilon-1) + 2T\tilde{z}^2\left(3\epsilon^2 - \epsilon + 3\right) + 4T\tilde{z}^3(\epsilon-1)\left(\epsilon^2 + 4\epsilon + 1\right) \\ &\quad - 2T^2\tilde{z}(\epsilon-1) - T^2\tilde{z}^2\left(7\epsilon^2 - 8\epsilon + 7\right) - T^2\tilde{z}^3(\epsilon-1)\left(9\epsilon^2 - \epsilon + 9\right) + T^3\tilde{z}^2\left(\epsilon^2 - \epsilon + 1\right) \\ &\quad + T^3\tilde{z}^3(\epsilon-1)\left(3\epsilon^2 - \epsilon + 3\right) + O(\tilde{z}^4) \end{aligned} \tag{48}$$

We omit the higher order terms for now in anticipation of differentiating at $\tilde{z} = 0$ to use the implicit function theorem. Dividing by $\tilde{z}$, we have the equations

$$\begin{aligned} 0 &= 6\tilde{z}^2\epsilon - 2T - 3T\tilde{z}(\epsilon-1) - T\tilde{z}^2(\epsilon+2)(2\epsilon+1) + T^2 + \frac{7}{2}T^2\tilde{z}(\epsilon-1) \\ &\quad + \frac{1}{2}T^2\tilde{z}^2\left(9\epsilon^2 - 10\epsilon + 9\right) - \frac{1}{2}T^3\tilde{z}(\epsilon-1) - \frac{1}{2}T^3\tilde{z}^2\left(3\epsilon^2 - 4\epsilon + 3\right) + O(\tilde{z}^3) \end{aligned} \tag{49}$$

$$\begin{aligned} 0 &= -8\tilde{z}\epsilon - 12\tilde{z}^2(\epsilon-1)\epsilon + 4T(\epsilon-1) + 2T\tilde{z}\left(3\epsilon^2 - \epsilon + 3\right) + 4T\tilde{z}^2(\epsilon-1)\left(\epsilon^2 + 4\epsilon + 1\right) \\ &\quad - 2T^2(\epsilon-1) - T^2\tilde{z}\left(7\epsilon^2 - 8\epsilon + 7\right) - T^2\tilde{z}^2(\epsilon-1)\left(9\epsilon^2 - \epsilon + 9\right) + T^3\tilde{z}\left(\epsilon^2 - \epsilon + 1\right) \\ &\quad + T^3\tilde{z}^2(\epsilon-1)\left(3\epsilon^2 - \epsilon + 3\right) + O(\tilde{z}^3) \end{aligned} \tag{50}$$

We immediately see that $\tilde{z} = 0, T = 2$ solves both equations for all ϵ. Let $w(\epsilon, \tilde{z}, T)$ and $v(\epsilon, \tilde{z}, T)$ be the right hand sides of Equations 49 and 50 respectively. We have

$$\left.\frac{\partial w}{\partial T}\right|_{(0,0,2)} = 2, \quad \left.\frac{\partial v}{\partial T}\right|_{(0,0,2)} = 4 \tag{51}$$

In both cases the derivative is invertible. Therefore, $f_{\tilde{z},\epsilon}(\tilde{z})$ and $f_{T,\epsilon}(\tilde{z})$ are continuously differentiable in both $\tilde{z}$ and ϵ in some neighborhood of 0. In fact, since w and v are analytic in all three arguments, $f_{\tilde{z},\epsilon}(\tilde{z})$ and $f_{T,\epsilon}(\tilde{z})$ are analytic as well.

We can use the analyticity to solve for the low-order structure of the nullclines. One way to compute the values of the derivatives is to define the nullclines as formal power series:

$$f_{\tilde{z}}(\tilde{z}) = 2 + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}\epsilon^j\tilde{z}^k \tag{52}$$

$$f_T(\tilde{z}) = 2 + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}\epsilon^j\tilde{z}^k \tag{53}$$

We can then solve for the first few terms of the series using Equations 49 and 50. From this procedure, we have:

$$f_{\tilde{z},\epsilon}(\tilde{z}) = 2 + 2\left(1-\epsilon\right)\tilde{z} + 2\left(1 - \epsilon + \epsilon^2\right)\tilde{z}^2 + O(\tilde{z}^3) \tag{54}$$

$$f_{T,\epsilon}(\tilde{z}) = 2 - \frac{\left(2 - 3\epsilon + 2\epsilon^2\right)}{1-\epsilon}\tilde{z} + \frac{1}{2}\left(4 - \epsilon + 4\epsilon^2\right)\tilde{z}^2 + O(\tilde{z}^3) \tag{55}$$

The difference $f_{\Delta,\epsilon}(\tilde{z})$ between the two is:

$$f_\Delta(\tilde{z}) \equiv f_{\tilde{z}}(\tilde{z}) - f_T(\tilde{z}) = -\frac{\epsilon}{1-\epsilon}\tilde{z} - \frac{3}{2}\epsilon\tilde{z}^2 + O(\tilde{z}^3) \tag{56}$$

As ϵ decreases, for the low order terms the distance between the nullclines also decreases.

We can show that the difference goes as ϵ. The one-step dynamical equations for $\epsilon = 0$ are

$$\tilde{z}_{t+1} - \tilde{z}_t = -\tilde{z}_t T_t(0) + \frac{1}{2}\tilde{z}_t^2 T_t(0) \tag{57}$$

$$T_{t+1}(0) - T_t(0) = -2\tilde{z}_t T_t(0) + z_t^2 T_t(0) \tag{58}$$

Therefore, $\Delta\tilde{z} = 2\Delta T$. This means that both the one step AND two-step nullclines are identical. Since $f_{\tilde{z},0}(\tilde{z}) = f_{T,0}(\tilde{z})$, and both are differentiable with respect to ϵ, we have:

$$f_{\tilde{z},\epsilon}(\tilde{z}) - f_{T,\epsilon}(\tilde{z}) = \epsilon f_{\Delta,\epsilon}(\tilde{z}) \tag{59}$$

for some function $f_{\Delta,\epsilon}(\tilde{z})$ which is analytic in ϵ and $\tilde{z}$ in a neighborhood around $(0, 0)$.

B.3. Two-step dynamics of y

It is useful to define dynamical equations in coordinates $(\tilde{z}, y)$ where y is the difference between $T(0)$ and the $\tilde{z}$ nullcline:

$$y \equiv T(0) - f_{\tilde{z},\epsilon}(\tilde{z}) \tag{60}$$

To lowest order in $\tilde{z}$ and ϵ we have

$$y = T(0) - 2 - 2\left(1-\epsilon\right)\tilde{z} - 2\left(1-\epsilon+\epsilon^2\right)\tilde{z}^2 + O(\tilde{z}^3) \tag{61}$$

We note that $y = 0$, at $\tilde{z} = 0$ corresponds to $T(0) = 2$. y near but slightly less than 0 is equivalent to edge of stability behavior. For positive $\tilde{z}$, $y = 0$ implies $T(0) > 2$.

We can write the dynamics of $\tilde{z}$ and y. The dynamics for $\tilde{z}$ are given by:

$$\tilde{z}_{t+2} - \tilde{z}_t = p_0(\tilde{z}_t, \epsilon) + p_1(\tilde{z}_t, \epsilon)(y_t + f_{\tilde{z},\epsilon}(\tilde{z}_t)) + p_2(\tilde{z}_t, \epsilon)(y_t + f_{\tilde{z},\epsilon}(\tilde{z}_t))^2 + p_3(\tilde{z}_t, \epsilon)(y_t + f_{\tilde{z},\epsilon}(\tilde{z}_t))^3 \tag{62}$$

We know that the right hand side of this equation is analytic in $\tilde{z}$, ϵ, and (trivially) y as well. By evaluating the multiple continuous derivatives of f, we can write:

$$\tilde{z}_{t+2} - \tilde{z}_t = 2y_t\tilde{z}_t + y_t^2\tilde{z}_t f_{1,\epsilon}(\tilde{z}_t, y_t) + y_t\tilde{z}_t^2 f_{2,\epsilon}(\tilde{z}_t) \tag{63}$$

Here, $f_{1,\epsilon}$ and $f_{2,\epsilon}$ are analytic in $\tilde{z}$, ϵ, and y in some neighborhood around 0.

This means that we have the bounds

$$|f_{1,\epsilon}(\tilde{z}, y)| < F_1,\ |f_{2,\epsilon}(\tilde{z}, y)| < F_2 \tag{64}$$

for $(\tilde{z}, \epsilon, y) \in [-\tilde{z}_d, \tilde{z}_d] \times [0, \epsilon_d] \times [-y_d, y_d]$ for some non-negative constants F_1 and F_2. Note that this bound is independent of ϵ.

Now we consider the dynamics of y. We have:

$$y_{t+2} - y_t = T_{t+2}(0) - T_t(0) - f_{\tilde{z},\epsilon}(\tilde{z}_{t+2}) + f_{\tilde{z},\epsilon}(\tilde{z}_t) \tag{65}$$

Since $\lim_{\tilde{z}\to 0, y\to 0} \tilde{z}_{t+2} = 0$, $f_{\tilde{z},\epsilon}(\tilde{z}_{t+2})$ is analytic in some neighborhood of $(0,0,0)$. Therefore $y_{t+2} - y_t$ is analytic as well. Substituting, we have

$$\begin{aligned} y_{t+2} - y_t = q_0(\tilde{z}_t, \epsilon) + q_1(\tilde{z}_t, \epsilon)[y + f_{\tilde{z},\epsilon}(\tilde{z})] + q_2(\tilde{z}_t, \epsilon)[y + f_{\tilde{z},\epsilon}(\tilde{z})]^2 + q_3(\tilde{z}_t, \epsilon)[y + f_{\tilde{z},\epsilon}(\tilde{z})]^3 \\ - f_{\tilde{z},\epsilon}(\tilde{z}_t + 2y_t\tilde{z}_t + y_t^2\tilde{z}_t f_{1,\epsilon}(\tilde{z}_t, y_t) + y_t\tilde{z}_t^2 f_{2,\epsilon}(\tilde{z}_t)) + f_{\tilde{z},\epsilon}(\tilde{z}_t) \end{aligned} \tag{66}$$

If we write $f_{\tilde{z},\epsilon}(\tilde{z}) = f_{T,\epsilon}(\tilde{z}) + \epsilon f_{\Delta,\epsilon}(\tilde{z})$, then we can write:

$$\begin{aligned} y_{t+2} - y_t = q_0(\tilde{z}_t, \epsilon) + q_1(\tilde{z}_t, \epsilon)[f_{T,\epsilon}(\tilde{z})] + q_2(\tilde{z}_t, \epsilon)[f_{T,\epsilon}(\tilde{z})]^2 + q_3(\tilde{z}_t, \epsilon)[f_{T,\epsilon}(\tilde{z})]^3 \\ 2q_2(\tilde{z}_t, \epsilon)[f_{T,\epsilon}(\tilde{z})(y + \epsilon f_{\Delta,\epsilon}(\tilde{z}))] + 3q_3(\tilde{z}_t, \epsilon)[(f_{T,\epsilon}(\tilde{z}))(y + \epsilon f_{\Delta,\epsilon}(\tilde{z}))^2 + (f_{T,\epsilon}(\tilde{z}))^2(y + \epsilon f_{\Delta,\epsilon}(\tilde{z}))] \\ q_0(\tilde{z}_t, \epsilon) + q_1(\tilde{z}_t, \epsilon)[y + \epsilon f_{\Delta,\epsilon}(\tilde{z})] + q_2(\tilde{z}_t, \epsilon)[y + \epsilon f_{\Delta,\epsilon}(\tilde{z})]^2 + q_3(\tilde{z}_t, \epsilon)[y + \epsilon f_{\Delta,\epsilon}(\tilde{z})]^3 \\ - f_{\tilde{z},\epsilon}(\tilde{z}_t + 2y_t\tilde{z}_t + y_t^2\tilde{z}_t f_{1,\epsilon}(\tilde{z}_t, y_t) + y_t\tilde{z}_t^2 f_{2,\epsilon}(\tilde{z}_t)) + f_{\tilde{z},\epsilon}(\tilde{z}_t) \end{aligned} \tag{67}$$

By the definition of the nullclines, the first four terms vanish. Once again using the differentiability of the nullclines, as well as $f_{1,\epsilon}$ and $f_{2,\epsilon}$, we can rewrite the dynamics in terms of the expansion:

$$y_{t+2} - y_t = -2(4 - 3\epsilon + 4\epsilon^2)y_t\tilde{z}_t^2 - 4\epsilon\tilde{z}_t^2 + y_t^2\tilde{z}_t^2 g_{1,\epsilon}(\tilde{z}_t, y_t) + \epsilon\tilde{z}_t^3 g_{2,\epsilon}(\tilde{z}_t) \tag{68}$$

Here $g_{1,\epsilon}$ and $g_{2,\epsilon}$ are analytic near zero in $\tilde{z}$, y, and ϵ. We have the bounds

$$|g_{1,\epsilon}(\tilde{z}, y)| < G_1,\ |g_{1,\epsilon}(\tilde{z}, y)| < G_2 \tag{69}$$

for $(\tilde{z}, \epsilon, y) \in [-\tilde{z}_d, \tilde{z}_d] \times [0, \epsilon_d] \times [-y_d, y_d]$ for some non-negative constants G_1 and G_2. This bound is also independent of ϵ.

We can summarize these bounds in the following lemma:

Lemma 2 *Define $y = T - f_{\tilde{z}}(\tilde{z})$. The two step dynamics of $\tilde{z}$ and y are given by*

$$\tilde{z}_{t+2} - \tilde{z}_t = 2y_t\tilde{z}_t + y_t^2\tilde{z}_t f_{1,\epsilon}(\tilde{z}_t, y_t) + y_t\tilde{z}_t^2 f_{2,\epsilon}(\tilde{z}_t) \tag{70}$$

$$y_{t+2} - y_t = -2(4 - 3\epsilon + 4\epsilon^2)y_t\tilde{z}_t^2 - 4\epsilon\tilde{z}_t^2 + y_t^2\tilde{z}_t^2 g_{1,\epsilon}(\tilde{z}_t, y_t) + \epsilon\tilde{z}_t^3 g_{2,\epsilon}(\tilde{z}_t, y_t) \tag{71}$$

Where $f_{1,\epsilon}$, $f_{2,\epsilon}$, $g_{1,\epsilon}$, $g_{2,\epsilon}$ are all analytic in $\tilde{z}$, y, and ϵ. Additionally, there exist positive $\tilde{z}_c$, y_c, and ϵ_c such that

$$|f_{1,\epsilon}(\tilde{z}, y)| < F_1,\ |f_{2,\epsilon}(\tilde{z}, y)| < F_2,\ |g_{1,\epsilon}(\tilde{z}, y)| < G_1,\ |g_{1,\epsilon}(\tilde{z}, y)| < G_2 \tag{72}$$

for all $(\tilde{z}, \epsilon, y) \in [-\tilde{z}_d, \tilde{z}_d] \times [0, \epsilon_d] \times [-y_d, y_d]$, where F_1, F_2, G_1, and G_2 are all non-negative constants.

We can use this Lemma to analyze the dynamics for small fixed ϵ, for small initializations of $\tilde{z}$, y. The control of the higher order terms will allow for an analysis which focuses on the effects of the lower order terms.

B.4. Proof of Theorem 4

Using Lemma 2, the dynamics in $\tilde{z}$ and y can be written as:

$$\tilde{z}_{t+2} - \tilde{z}_t = 2y_t\tilde{z}_t + y_t^2\tilde{z}_t f_{1,\epsilon}(\tilde{z}_t, y_t) + y_t\tilde{z}_t^2 f_{2,\epsilon}(\tilde{z}_t) \tag{73}$$

$$y_{t+2} - y_t = -2(4 - 3\epsilon + 4\epsilon^2)y_t\tilde{z}_t^2 - 4\epsilon\tilde{z}_t^2 + y_t^2\tilde{z}_t^2 g_{1,\epsilon}(\tilde{z}_t, y_t) + \epsilon\tilde{z}_t^3 g_{2,\epsilon}(\tilde{z}_t, y_t) \tag{74}$$

Let $\epsilon < \epsilon_d$. Then we can use the bounds from Lemma 2 to control the contributions of the higher order terms to the dynamics:

Lemma 3 *Given constants $A > 0$ and $B > 0$, there exist $\tilde{z}_c$ and y_c such that for $\tilde{z} \in [0, 2\tilde{z}_c]$, $y \in [-y_c, y_c]$, we have the bounds:*

$$|y^2\tilde{z}f_{1,\epsilon}(\tilde{z}, y) + y\tilde{z}^2 f_{2,\epsilon}(\tilde{z})| \leq A|2y\tilde{z}| \tag{75}$$

$$|y^2\tilde{z}^2 g_{1,\epsilon}(\tilde{z}, y)| \leq \frac{B}{8}|2(4 - 3\epsilon + 4\epsilon^2)y\tilde{z}^2| \tag{76}$$

$$|\epsilon\tilde{z}^3 g_{2,\epsilon}(\tilde{z}, y)| \leq \frac{B}{4}|4\epsilon\tilde{z}^2| \tag{77}$$

Proof We begin by the following decomposition:

$$|y^2\tilde{z}f_{1,\epsilon}(\tilde{z}, y) + y\tilde{z}^2 f_{2,\epsilon}(\tilde{z})| \leq |y^2\tilde{z}f_{1,\epsilon}(\tilde{z}, y)| + |y\tilde{z}^2 f_{2,\epsilon}(\tilde{z})| \tag{78}$$

From Lemma 2, there exists a region $[-\tilde{z}_d, \tilde{z}_d] \times [0, \epsilon_d] \times [-y_d, y_d]$ where the magnitudes of $f_{1,\epsilon}$, $f_{2,\epsilon}$, $g_{1,\epsilon}$, and $g_{2,\epsilon}$ are bounded by F_1, F_2, G_1, and G_2 respectively.

$$|y^2\tilde{z}f_1(\tilde{z}, y) + y\tilde{z}^2 f_2(\tilde{z})| \leq F_1 y^2\tilde{z} + F_2 y\tilde{z}^2 \tag{79}$$

$$|y^2\tilde{z}^2 g_1(\tilde{z}, y)| \leq G_1 y^2\tilde{z}^2 \tag{80}$$

$$|\tilde{z}^3 g_2(\tilde{z}, y)| \leq G_2\tilde{z}^3 \tag{81}$$

Define $\tilde{z}_c$ and y_c as

$$y_c = \min(A/F_1, B/2G_1, y_d),\ \tilde{z}_c = \min(A/2F_2, B/2G_2, y_c) \tag{82}$$

The desired bounds follow immediately. ■

We consider an initialization $(\tilde{z}_0, y_0)$ such that $\tilde{z}_0 \leq \tilde{z}_c$ and $y_0 \leq y_c$, and $y_0 \leq \tilde{z}_0^2$. Armed with Lemma 3, we can analyze the dynamics. There are two phases; in the first phase, $\tilde{z}$ is increasing, and y is decreasing. The first phase ends when y becomes negative for the first time - reaching a value of $O(\epsilon)$. In the second phase, $\tilde{z}$ is decreasing, and y stays negative and $O(\epsilon)$.

B.4.1. PHASE ONE

Let t_{sm} be the time such that for $t \leq t_{sm}$, $\tilde{z}_t \leq 2\tilde{z}_0$. (We will later show that $\tilde{z}_t \leq 2\tilde{z}_0$ over the whole dynamics.) For $t \leq t_{sm}$, using Lemma 3, the change in $\tilde{z}$ can be bounded from below by

$$\tilde{z}_{t+2} - \tilde{z}_t \geq 2y_t\tilde{z}_t(1 - A) \tag{83}$$

Therefore at initialization, $\tilde{z}$ is increasing. It remains increasing until y_t becomes negative, or $\tilde{z}_t \geq 2\tilde{z}_0$. We want to show that y_t becomes negative before $\tilde{z}_t \geq 2\tilde{z}_0$.

For any $t \leq t_{sm}$, Lemma 3 gives the following upper bound on $y_{t+2} - y_t$:

$$y_{t+2} - y_t \leq -(8 - B)y_t\tilde{z}_t^2 - (4 - B)\epsilon\tilde{z}_t^2 \tag{84}$$

Let t_- be the first time that y_t becomes negative. Since $\tilde{z}_t$ is increasing for $t \leq t_-$, we have

$$y_{t+2} - y_t \leq -(8 - B)y_t\tilde{z}_0^2 - (4 - B)\epsilon\tilde{z}_0^2 \tag{85}$$

This gives us the following bound on y_t:

$$y_t \leq y_0 e^{-(8-B)\tilde{z}_0^2 t} \tag{86}$$

valid for $t \leq t_-$ and $t \leq t_{sm}$.

We will now show that $t_- < t_{sm}$. Suppose that $t_{sm} \leq t_-$. Then at $t_{sm} + 2$, $\tilde{z}_{t_{sm}+2} > 2\tilde{z}_0$ for the first time. Summing the bound in Equation 83, we have:

$$\tilde{z}_{t_{sm}+2} - \tilde{z}_0 \leq \sum_{t=0}^{t_{sm}} 2y_t\tilde{z}_t(1 + A) \leq 4\tilde{z}_0(1 + A)\sum_{t=0}^{t_{sm}} y_t \tag{87}$$

where the second bound comes from the definition of t_{sm}. Using our bound on y_t, we have:

$$\tilde{z}_{t_{sm}+2} - \tilde{z}_0 \leq 4\tilde{z}_0(1 + A)\sum_{s=0}^{t_{sm}} y_0 e^{-(8-B)\tilde{z}_0^2 s} \leq \frac{(1 + A)}{2}\frac{y_0}{\tilde{z}_0} \tag{88}$$

Since $y_0 \leq \tilde{z}_0^2$, $\tilde{z}_{t_{sm}+2} \leq 2\tilde{z}_0$. However, by assumption $\tilde{z}_{t_{sm}+2} > 2\tilde{z}_0$. We arrive at a contradiction; t_{sm} is not less than or equal to t_-.

There are three possibilities: the first is that t_- is well-defined, and $t_- < t_{sm}$. Another possibility is that t_- is not well-defined - that is, y_t never becomes negative. In this case the bounds we derived are valid for all t. Therefore using Equation 86, there exists some time t_ϵ where $y_{t_\epsilon} < (4 - B)\epsilon\tilde{z}_0^2$. Then, using Equation 85 we have $y_{t_\epsilon+2} < 0$. Therefore, we conclude that t_- is finite and less than t_{sm}.

Since the well defined value $t_- < t_{sm}$,, when y first becomes negative, $\tilde{z}_{t_-} \leq 2\tilde{z}_0$. This means that we can continue to apply the bounds from Lemma 3 at the start of the next phase. At $t = t_- - 2$, applying Lemma 3 and $\tilde{z}_{t_-} \leq 2\tilde{z}_0$, we have

$$y_{t_-} - y_{t_- - 2} \geq -4(8 + B)y_{t_- - 2}\tilde{z}_0^2 - 4(4 + B)\epsilon\tilde{z}_0^2 \tag{89}$$

which gives us $y_{t_-} \geq -4(4 + B)\epsilon\tilde{z}_0^2$. This concludes the first phase. To summarize we have

$$-4(4 + B)\epsilon\tilde{z}_0^2 < y_{t_-} \leq 0,\ \tilde{z}_{t_-} \leq 2\tilde{z}_0 \tag{90}$$

B.4.2. PHASE TWO

Now consider the second phase of the dynamics. We will show that y remains negative and $O(\epsilon)$, and $\tilde{z}$ decreases to 0. While y is negative, $\tilde{z}$ decreases. While $y \geq -y_0$, from Lemma 3 we have

$$\tilde{z}_{t+2} - \tilde{z}_t \leq (1 - A)2y_t\tilde{z}_t \tag{91}$$

Therefore as long as $-y_0 \leq y < 0$, $\tilde{z}_t$ is decreasing. If this is true for all subsequent t, $\tilde{z}_0$ will converge to 0.

We will now show that y remains negative and $O(\epsilon)$, concluding the proof. Let $y^* = -\frac{\epsilon}{2-(3/2)\epsilon+2\epsilon^2}$. We can re-write the dynamical equation for y as

$$y_{t+2} - y_t = -2(4 - 3\epsilon + 4\epsilon^2)\tilde{z}_t^2(y_t - y^*) + y_t^2\tilde{z}_t^2 g_1(\tilde{z}_t, y_t) + \tilde{z}_t^3 g_2(\tilde{z}_t, y_t) \tag{92}$$

Applying Lemma 3 to the higher order terms, we have:

$$y_{t+2} - y_t \leq -2(4 - 3\epsilon + 4\epsilon^2)\tilde{z}_t^2(y_t - y^*) + B(|y_t| + \epsilon)\tilde{z}_t^2 \tag{93}$$

$$y_{t+2} - y_t \geq -2(4 - 3\epsilon + 4\epsilon^2)\tilde{z}_t^2(y_t - y^*) - B(|y_t| + \epsilon)\tilde{z}_t^2 \tag{94}$$

These inequalities are valid as long as $|y_t| < y_c$.

At t_-, $y^* < y_t < 0$. When $y^* < y_t < 0$, then $|y_t| \leq |y^*|$. Note that $\epsilon < 2|y^*|$. From Equation 93, we have

$$y_{t+2} - y_t \leq -2(4 - 3\epsilon + 4\epsilon^2)\tilde{z}_t^2(y_t - y^*) + B(-y_t + \epsilon)\tilde{z}_t^2 \tag{95}$$

From this inequality we can conclude that

$$y_{t+2} \leq (1 - 2(4 - 3\epsilon + 4\epsilon^2)\tilde{z}_t^2 - B)y_t + \tilde{z}_t^2[2(4 - 3\epsilon + 4\epsilon^2)y^* + B\epsilon] \tag{96}$$

If $B < 1$, then both terms are negative. We can conclude that if $y^* < y_t < 0$, $y_{t+2} < 0$. In fact, from the last term we can conclude that $y_{t+2} < -4\epsilon\tilde{z}_t^2$.

Now we must show that when $y^* < y_t < 0$, y_{t+2} does not become too negative (namely, smaller than $-y_c$). Using Equation 94, we have:

$$y_{t+2} > y^*(1 + 3B\tilde{z}_0^2) \text{ if } y_t > y^* \tag{97}$$

This means that if y_t starts larger than y^*, it will be at most $3B\tilde{z}_0^2 y^*$ below y^* at the next step. Since $B < 1$, $y_{t+2} > -y_c$ if $y^* < y_t < 0$.

Finally, we will show that if $y^*(1+3B/(8-B)) < y_t < y^*$, $y^*(1+3B/(8-B)) < y_{t+2} < 0$. Since y_{t_-+2} fits this condition, we can conclude that y_t is negative for all $t > t_-$, with magnitude bounded from below by $y^*(1 + 3B/(8 - B))$, and complete the proof.

We will first show that $y^*(1 + 3B/(8 - B)) < y_t$ implies that $y^*(1 + 3B/(8 - B)) < y_{t+2}$. Let $y_t = (1+\delta_t)y^*$, for $\delta_t < 3B/(8-B)$. We will show that $\delta_{t+2} < 3B/(8-B)$. Using Equation 94, we have:

$$y_{t+2} \geq (1 + \delta_t)y^* - 8\tilde{z}_t^2\delta_t y^* - B\tilde{z}_t^2(\epsilon - (1 + \delta_t)y^*) \tag{98}$$

Substituting $y_{t+2} = (1 + \delta_{t+2})y^*$, and dividing both sides by y^* we have

$$\delta_{t+2} - \delta_t \leq -(8 - B)\tilde{z}_t^2\delta_t + 3B\tilde{z}_t^2 \tag{99}$$

If $\delta_t < 3B/(8-B)$, then we have $\delta_{t+2} < 3B/(8-B)$ as desired.

Finally, we will show that $0 < \delta_t < 3B/(8-B)$ implies that $\delta_{t+2} > -1$ - that is, $[1+3B/(8-B)]y^* < y_t < y^*$ implies $[1+3B/(8-B)]y^* < y_{t+2} < 0$. Equation 93 implies

$$y_{t+2} \leq (1+\delta_t)y^* - 8\tilde{z}_t^2\delta_t y^* + B\tilde{z}_t^2(\epsilon - (1+\delta_t)y^*) \tag{100}$$

which gives us

$$\delta_{t+2} - \delta_t \geq -(8-B)\tilde{z}_t^2\delta_t - 3B\tilde{z}_t^2 \tag{101}$$

If $\delta_t > 0$ implies

$$\delta_{t+2} > -3B\tilde{z}_t^2 \tag{102}$$

If $3B\tilde{z}_0^2 < 1$, then $\delta_{t+2} > -1$. This means that $y_{t+2} < 0$ if $[1+3B/(8-B)]y^* < y_t < y^*$.

Finally, we make some choices of B and $\tilde{z}_0$ to guarantee convergence. Choose $\tilde{z}_0^2 < 3/7$, and choose $B < \frac{1}{2}$. Then in summary, what we have shown for phase two is:

- At the start of the phase (time t_-), $y^* < y_{t_-} < 0$.
- If $y^* < y_t < 0$, $t > t_-$, $y^*(1+3B\tilde{z}_0^2) < y_{t+2} < -4\epsilon\tilde{z}_t^2$.
- If $[1+3B/(8-B)]y^* < y_t < y^*$, $t > t_-$, $[1+3B/(8-B)]y^* < y_{t+2} < 0$.

Through our choices of $\tilde{z}_0$ and B, we know that $[1+3B/(8-B)]y^* < y^*(1+3B\tilde{z}_0^2)$. Therefore, the entire trajectory for $t > t_-$ is accounted for by these regions, and $[1+3B/(8-B)]y^* < y_t < 0$ for all $t > t_-$. Additionally, we know that at least once every 2 steps, $y_t < -4\epsilon\tilde{z}_t^2$. This means that the dynamics of $\tilde{z}_t$ can be bounded from above by

$$\tilde{z}_{t+2} - \tilde{z}_t \leq -2\epsilon^2\tilde{z}_t^4 \tag{103}$$

From this we can conclude that $\tilde{z}_t$ converges to 0.

Therefore, for any positive initialization with $\tilde{z}_0 \leq \tilde{z}_c$, $y_0 \leq y_c$, and $y_0 \leq \tilde{z}_0^2$, we have:

$$\lim_{t\to\infty} \tilde{z}_t \to 0, \ \lim_{t\to\infty} y = -y_f \tag{104}$$

where $y_f = O(\epsilon)$.

Now we can prove the statement of Theorem 1. Given a model with $\epsilon \leq \epsilon_c$, there is a continuous mapping between $\boldsymbol{\theta} - \eta$ space and $\tilde{z} - y$ space. Since there is some neighborhood in $\tilde{z} - y$ space that displays edge of stability behavior ($T_t(0)$ converging to within $O(\epsilon)$ of 2), the inverse image of that neighborhood is a neighborhood in $\boldsymbol{\theta} - \eta$ space that displays edge of stability behavior. This concludes the proof.

B.5. Low order dynamics

In order to predict the final value of y, and understand the convergence to the fixed point, We can study the low order dynamics in $\tilde{z}$ and y. The low order dynamical equations are:

$$\tilde{z}_{t+2} - \tilde{z}_t = 2y_t\tilde{z}_t \tag{105}$$

$$y_{t+2} - y_t = -2(4 - 3\epsilon + 4\epsilon^2)y_t\tilde{z}_t^2 - 4\epsilon\tilde{z}_t^2 \tag{106}$$

For these reduced dynamics, we can show the following:

Theorem 4 *For the dynamics defined by Equations 105 and 106, for $\epsilon \ll 1$, for positive inititializations $\tilde{z}_0 \ll 1$, $y_0 \ll 1$ with the additional constraints $-\epsilon \log(\epsilon) \ll 16\tilde{z}_0^2$ and $y_0 < 2\tilde{z}_0^2$, we have*

$$\lim_{t\to\infty} \tilde{z}_t = 0, \quad \lim_{t\to\infty} y_t = -\epsilon/2 + O(\epsilon^2) \tag{107}$$

Proof The proof distinguishes two phases in the time evolution:

- Phase 1: $\tilde{z}$ starts positive and increases, y starts positive and decreases. At the end of the phase we want $\tilde{z}_t \leq 2\tilde{z}_0$ and y to be negative but bounded by $-16\tilde{z}_0^2\epsilon$.
- Phase 2: $\tilde{z}$ decreases slowly, and y settles to the fixed point (relatively) quickly, up to error $O(\epsilon^2)$.

Let $\epsilon \ll 1$. Consider an initialization $(\tilde{z}_0, y_0)$ where both variables are positive, such that $\tilde{z}_0 \ll 1$, $\epsilon \log(\epsilon) \ll \tilde{z}_0^2$, and $y_0 \ll \tilde{z}_0^2$. From Equations 105 and 106, we see that the dynamics of y will depend on the balance of the two terms.

Initially $\tilde{z}$ increases and y decreases. We analyze the dynamics of y assuming that $\tilde{z}$ is fixed, and then compute the corrections.

Phase 1. At initialization, the first term in the dynamics dominates, since by assumption $\epsilon\tilde{z}_t^2 \ll y_t\tilde{z}_t^2 \ll$. Since $\tilde{z}_0^2 \ll 1$, y initially decreases exponentially with decay rate bounded from above by $8\tilde{z}_0^2$. Therefore within $\log(-\epsilon/y_0)/8\tilde{z}_0^2$ steps, $y < \epsilon$.

At this point, the rate of change of y is at least $-4\epsilon\tilde{z}_0^2$. Therefore, in no more than $1/4\tilde{z}_0^2$ additional steps, y becomes negative. Let t_- be the first time that y becomes negative. We note that $y_{t_-} \geq -4\epsilon\tilde{z}_0^2$ under this analysis - the first term in Equation 106 is less than y_t in magnitude, so the smallest value that y_{t+2} can take if y_t is positive is $-4\epsilon\tilde{z}_0^2$.

We can now understand the corrections due to the change in $\tilde{z}$. We note that $e^{-8\tilde{z}_0^2 t}$ is an upper bound for y - since $\tilde{z}$ is increasing, and the $-4\epsilon\tilde{z}_t^2$ decreases y faster than exponential decay from the first term. Since $\tilde{z}$ is increasing, $y_t \geq e^{-8\tilde{z}_0^2 t}$ as long as y remains positive ($t < t_-$). Let t_{sm} be a time such that $\tilde{z}_{t_{sm}} < 2\tilde{z}_0$. We can bound the change in $\tilde{z}_t$ for $t < t_{sm}$. We know that $y_t \geq y_0 e^{-8\tilde{z}_0^2 t}$. The change in $\tilde{z}$ can be bounded by

$$\tilde{z}_{t_{sm}} - \tilde{z}_0 \leq \sum_{t=0}^{t_{sm}} 2z_t y_t \leq 4\tilde{z}_0 \sum_{t=0}^{t_{sm}} y_t \leq 4\tilde{z}_0 y_0 \sum_{t=0}^{t_{sm}} e^{-8\tilde{z}_0^2 t} \leq \frac{1}{2} \cdot \frac{y_0}{\tilde{z}_0}. \tag{108}$$

If $y_0 < 2\tilde{z}_0^2$, then the bound holds independent of the value of t_{sm}, as long as the bound on y is correct. We know that the bound on y is correct until time t_-; therefore, $t_{sm} \geq t_-$.

Phase 2. This proves that there exists a time t_-, such that $\tilde{z}_{t_-} \leq 2\tilde{z}$, and $-16\tilde{z}_0^2\epsilon \leq y_{t_-} \leq 0$. Now that y is negative, it will stay negative, and $\tilde{z}$ will decrease until it reaches 0. In order to understand the dynamics, we will use a change of coordinates. Consider solving Equation 106 for $y_{t+2} - y_t = 0$ for $\tilde{z}_t \neq 0$. We have

$$y^* = -\frac{\epsilon}{2 - 3/2\epsilon + 2\epsilon^2} \tag{109}$$

Consider now the coordinate δ_t defined by the equation

$$y_t = -(1 + \delta_t)\frac{\epsilon}{2 - 3/2\epsilon + 2\epsilon^2} \tag{110}$$

The dynamics of δ_t are given by

$$\delta_{t+2} = (1 - 2(4 - 3\epsilon + 4\epsilon^2)\tilde{z}_t^2)\delta_t \tag{111}$$

Since $\tilde{z}_t \ll 1$, δ_t is strictly decreasing in magnitude. We can bound δ_t from above by

$$|\delta_t| \leq \exp\left(-8\sum_{s=t_-}^{t} \tilde{z}_s^2\right)|\delta_{t_-}| \tag{112}$$

Since δ starts negative, and is decreasing in magnitude, we know that $y_t > -\frac{\epsilon}{2-3/2\epsilon+2\epsilon^2}$. This means that we can bound $\tilde{z}_t$ by

$$\tilde{z}_t \geq 2e^{-\epsilon t}\tilde{z}_0 \tag{113}$$

Substitution gives us the following bound on δ_t:

$$|\delta_t| \leq \exp\left(-8\sum_{s=t_-}^{t} 4e^{-2\epsilon s}\tilde{z}_0^2\right)|\delta_{t_-}| \tag{114}$$

Using the integral approximation for the sum, the bound becomes

$$|\delta_t| \leq \exp\left(-32\tilde{z}_0^2\int_0^t e^{-2\epsilon s}ds\right)\delta_{t_-} = \exp\left(-16\tilde{z}_0^2/\epsilon(1-e^{-2\epsilon t})\right)|\delta_{t_-}| \tag{115}$$

From our previous analysis, we know that $-1 \leq \delta_{t-} \leq 0$. In the limit of large t we have

$$\lim_{t\to\infty}|\delta_t| \leq \exp\left(-16\tilde{z}_0^2/\epsilon\right)|\delta_{t_-}| \tag{116}$$

If we have the condition

$$16\tilde{z}_0^2/\epsilon \geq -\log(\epsilon) \tag{117}$$

then $\lim_{t\to\infty}|\delta_t| \leq \epsilon^2$.

If we want $\lim_{t\to\infty} y_t = -\epsilon/2 + O(\epsilon^2)$, then we need the condition

$$16\tilde{z}_0^2 \geq -\epsilon\log(\epsilon) \tag{118}$$

or equivalently $-\epsilon\log(\epsilon) < 16\tilde{z}_0^2$. Under these conditions, $\lim_{t\to\infty}\tilde{z}_t = 0$ and $\lim_{t\to\infty} y_t = -\epsilon/2 + O(\epsilon^2)$. ■

This result can be confirmed numerically by running the dynamical equations from a variety of initializations, computing the median eigenvalue (restricted to the range $[1.9, 2.0]$), and plotting versus ϵ (Figure 6).We note that since the dynamics is slow, the ODE given by

$$\dot{\tilde{z}} = 2y\tilde{z} \tag{119}$$

$$\dot{y} = -2(4 - 3\epsilon + 4\epsilon^2)y\tilde{z}^2 - 4\epsilon\tilde{z}^2 \tag{120}$$

also obtains the same limit (Figure 6). The ODE suggests that the concentration relies on both the equal-orders in $\tilde{z}$ of the y^0 and y^1 terms, as well as a separation of timescales - $\tilde{z}$ converges to 0 at a rate of ϵ, while y converges to the fixed point at a rate $\tilde{z}_t^2$. In both cases, the deviation from $-\epsilon/2$ scales as $O(\epsilon^2)$ (Figure 6, right).

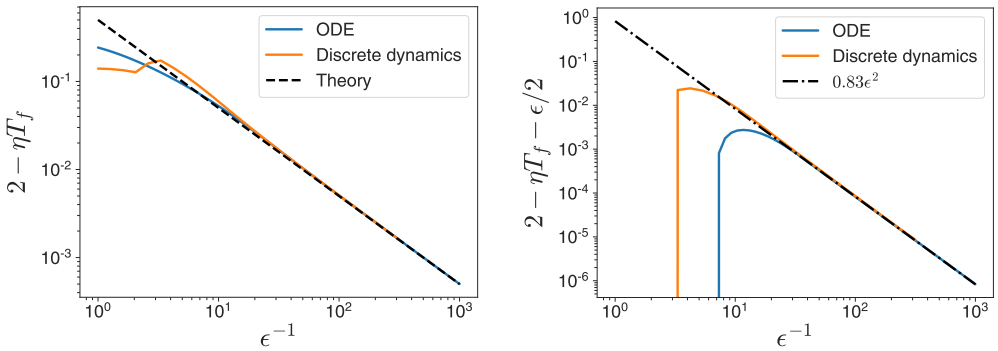

Figure 6: Final values of y, normalized deviation from critical value $T(0) = 2$, for discrete dynamics and ODE approximation. Deviation is well approximated by $\epsilon/2$ over a large range (left). Deviations from $\epsilon/2$ are $O(\epsilon^2)$ (right).

Appendix C. Quadratic regression model dynamics

In this section we analyze the quadratic regression model, particularly in the limit of large D and P.

C.1. Gradient flow dynamics

We will focus on training with squared loss $\mathcal{L}(\mathbf{z}) = \frac{1}{2}\sum_\alpha \mathbf{z}_\alpha^2$. We begin by considering the dynamics under gradient flow (GF):

$$\dot{\boldsymbol{\theta}} = -\frac{\partial \mathcal{L}(\mathbf{z})}{\partial \boldsymbol{\theta}} = -\mathbf{J}^\top \mathbf{z}\,. \tag{121}$$

We can write the dynamics in the output space $\mathbf{z}$ and the Jacobian $\mathbf{J}$ as

$$\dot{\mathbf{z}} = \mathbf{J}\dot{\boldsymbol{\theta}} = -\mathbf{J}\mathbf{J}^\top \mathbf{z},\ \dot{\mathbf{J}} = -\mathbf{Q}(\mathbf{J}^\top \mathbf{z}, \cdot) \tag{122}$$

When $\mathbf{Q} = \mathbf{0}$ (linearized/NTK regime), $\mathbf{J}$ is constant, the dynamics are then linear in $\mathbf{z}$, and are controlled by the eigenstructure of $\mathbf{J}\mathbf{J}^\top$, the empirical NTK. In this regime there is no EOS behavior.

We are interested in settings where progressive sharpening occurs under GF. We can study the dynamics of the maximum eigenvalue $\lambda_{\max}$ of $\mathbf{J}\mathbf{J}^\top$ at early times for random initializations. In Appendix C.3, we prove the following theorem:

Theorem 5 *Let $\mathbf{z}$, $\mathbf{J}$, and $\mathbf{Q}$ be initialized with i.i.d. elements with zero mean and variances σ_z^2, σ_J^2, and 1 respectively, with distributions invariant to rotation in data and parameter space, and have finite fourth moments. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{J}\mathbf{J}^\top$. In the limit of large D and P, with fixed ratio D/P, at initialization we have*

$$\mathrm{E}[\dot{\lambda}_{\max}(0)] = 0,\ \mathrm{E}[\ddot{\lambda}_{\max}(0)]/\mathrm{E}[\lambda_{\max}(0)] = \sigma_z^2 \tag{123}$$

where E denotes the expectation over $\mathbf{z}$, $\mathbf{J}$, and $\mathbf{Q}$ at initialization.

Much like in the $D = 1$ case, Theorem 5 suggests that it is easy to find initializations that show progressive sharpening - and increasing σ_z makes sharpening more prominent.

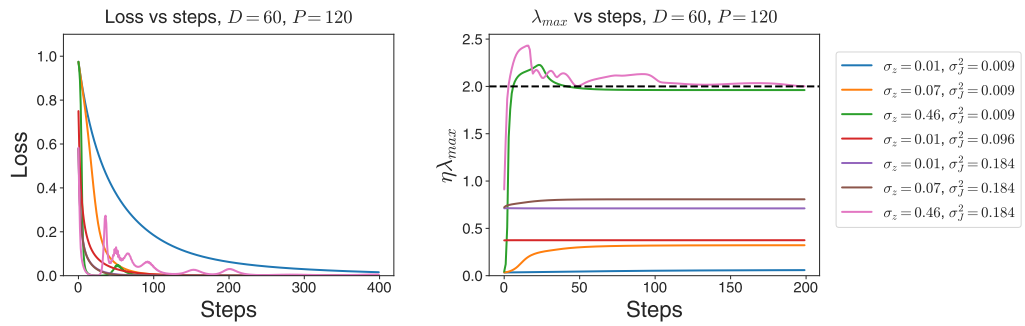

Figure 7: Gradient descent dynamics in the quadratic regression model. As $\mathbf{z}$ initialization variance σ_z^2 increases, so does the curvature $\lambda_{\max}$ upon convergence. As sharpening drives $\eta\lambda_{\max}$ near 2, larger σ_z allows for non-linear effects to induce edge of stability behavior (right). Resulting loss trajectories are non-monotonic but still converge to 0 (left).

C.2. Gradient descent dynamics

We now consider finite-step size gradient descent (GD) dynamics. The dynamics for $\boldsymbol{\theta}$ are given by:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \mathbf{J}_t^\top \mathbf{z}_t \,. \tag{124}$$

In this setting, the dynamic equations can be written as

$$\mathbf{z}_{t+1} - \mathbf{z}_t = -\eta \mathbf{J}_t \mathbf{J}_t^\top \mathbf{z}_t + \frac{1}{2}\eta^2 \mathbf{Q}(\mathbf{J}_t^\top \mathbf{z}_t, \mathbf{J}_t^\top \mathbf{z}_t) \tag{125}$$

$$\mathbf{J}_{t+1} - \mathbf{J}_t = -\eta \mathbf{Q}(\mathbf{J}_t^\top \mathbf{z}_t, \cdot) \,. \tag{126}$$

If $\mathbf{Q} = \mathbf{0}$, the dynamics reduce to discrete gradient descent in a quadratic potential - which converges iff $\lambda_{\max} < 2/\eta$.

One immediate question is: when does the η^2 in Equation 125 affect the dynamics? Given that it scales with higher powers of η and $\mathbf{z}$ than the first term, we can conjecture that the ratio of the magnitudes of the terms, r_{NL}, is proportional to $||\mathbf{z}||_2$ and η. A calculation in Appendix C.4 shows that, for the random rotationally invariant initialization, we have:

$$r_{NL} \equiv \left(\frac{\mathrm{E}[||\frac{1}{2}\eta^2 \mathbf{Q}(\mathbf{J}_0^\top \mathbf{z}_0, \mathbf{J}_0^\top \mathbf{z}_0)||_2^2]}{\mathrm{E}[||\eta \mathbf{J}_0 \mathbf{J}_0^\top \mathbf{z}_0||_2^2]} \right)^{1/2} = \frac{1}{2}\eta \sigma_z D \,, \tag{127}$$

where as before the expectation is taken over the initialization of $\mathbf{z}$, $\mathbf{J}$, and $\mathbf{Q}$. This suggests that increasing the learning rate increases the deviation of the dynamics from GF (which is obvious), but increasing $||\mathbf{z}||$ *also* increases the deviation from GF.

We can see this phenomenology in the dynamics of the GD equations (Figure 7). Here we plot different trajectories for random initializations of the type in Theorem 5 with $D = 60$, $P = 120$, and $\eta = 1$. As σ_z increases, so does the curvature $\lambda_{\max}$ (as suggested by Theorem 5), and when σ_z is $O(1)$, the dynamics is non-linear (as predicted by r_{NL}) and EOS behavior emerges. This suggests that the second term in Equation 125 is crucial for the stabilization of λ_{max}.

We can confirm this more generally by initializing over various η, D, P, σ_z, and σ_J over multiple seeds, and plotting the resulting phase diagram of the final $\lambda_{\max}$ reached. We can simplify the plotting with some rescaling of parameters and initializations. For example, in the rescaled variables

$$\tilde{\mathbf{z}} = \eta \mathbf{z}, \ \tilde{\mathbf{J}} = \eta^{1/2} \mathbf{J} \,, \tag{128}$$

the dynamics are equivalent to Equations 125 and 126 with $\eta = 1$. As in the $\tilde{z} - T(0)$ model of Equations 5–6, $\lambda_{\max}$ in the rescaled coordinates is equivalent to $\eta \lambda_{\max}$ in the unscaled coordinates. We can also define rescaled initializations for $\mathbf{z}$ and $\mathbf{J}$. If we set

$$\sigma_z = \tilde{\sigma}_z / D, \ \sigma_J = \tilde{\sigma}_J / \left(DP\right)^{1/4} \,, \tag{129}$$

then we have $r_{NL} = \tilde{\sigma}_z$ which allows for easier comparison across (D, P) pairs.

Using this initialization scheme, we can plot the final value of $\lambda_{\max}$ reached as a function of $\tilde{\sigma}_z$ and $\tilde{\sigma}_J$ for 100 independent random initializations for each $\tilde{\sigma}_z$, $\tilde{\sigma}_J$ pair (Figure 8). We see that the key is for $r_{NL} = \tilde{\sigma}_z$ to be $O(1)$ - corresponding to both progressive sharpening and non-linear dynamics near initialization. In particular, initializations with small $\tilde{\sigma}_J$ values which converge at the EOS correspond to trajectories which first sharpen, and then settle near $\lambda_{\max} = 2/\eta$. Large $\tilde{\sigma}_z$ and large $\tilde{\sigma}_J$ dynamics diverge. There is a small band of initial $\tilde{\sigma}_J$ over a wide range of $\tilde{\sigma}_z$ which have final $\lambda_{\max} \approx 2/\eta$; these correspond to models initialized near the EOS, which stay near it.

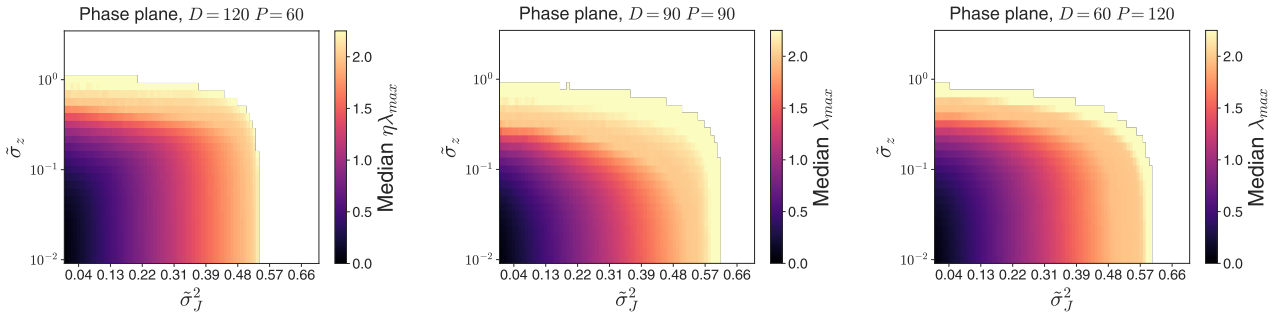

Figure 8: $\tilde{\sigma}_z / \tilde{\sigma}_J^2$ phase planes for quadratic regression models, for various D and P. Models were initialized with 100 random seeds for each $\tilde{\sigma}_z$, $\tilde{\sigma}_J$ pair and iterated until convergence. For each pair $\tilde{\sigma}_z, \tilde{\sigma}_J^2$ we plot the median $\lambda_{\max}$ of the NTK $\mathbf{J}^\top \mathbf{J}$. For intermediate $\tilde{\sigma}_z$, where both sharpening and non-linear $\mathbf{z}$ dynamics occur, trajectories tend to converge so $\lambda_{\max}$ of the NTK is near $2/\eta$ (EOS).

This suggests that progressive sharpening and edge of stability aren't uniquely features of neural network models, and could be a more general property of learning in high-dimensional, non-linear models.

C.3. Proof of Theorem 5

We use Einstein summation notation in this section - repeated indices on the right-hand-side of equations are considered to be summed over, unless they show up on the left-hand-side.

Let $\mathbf{z}$, $\mathbf{J}$, and $\mathbf{Q}$ be initialized with i.i.d. random elements with 0 mean and variance σ_z^2, σ_J^2, and 1 respectively. Furthermore, Let the distributions be invariant to rotations in both data space and parameter space, and have finite 4th moment.

In order to understand the development of the curvature at early times, we consider coordinates which convert $\mathbf{J}$ into its singular value form. In these coordinates, we can write:

$$\mathbf{J}_{\alpha i} = \begin{cases} 0 & \text{if } \alpha \neq i \\ \sigma_\alpha & \text{if } \alpha = i \end{cases} \tag{130}$$

The singular values σ_α are the square roots of the singular values of the NTK matrix. We assume that they are ordered from largest (σ_1) to smallest in magnitude. By assumption, under this rotation the statistics of $\mathbf{z}$ and $\mathbf{Q}$ are left unchanged.

The time derivatives at $t = 0$ can be computed directly in the singular value coordinates. The first derivative is given by

$$\frac{d}{dt}\sigma_\alpha^2 = 2\sigma_\alpha \dot{\sigma}_\alpha \tag{131}$$

Using the diagonal coordinate system, we have

$$\mathrm{E}\left[\frac{d}{dt}\sigma_\alpha^2\right] = \mathrm{E}[\mathbf{Q}_{\alpha\beta j}\mathbf{J}_{\beta j}\mathbf{z}_\beta] = 0 \tag{132}$$

However, the average second derivative is positive. Calculating, we have:

$$\frac{d^2}{dt^2}\sigma_\alpha^2 = 2(\dot{\sigma}_\alpha^2 + \sigma_\alpha \ddot{\sigma}_\alpha) \tag{133}$$

We can compute the average at initialization. We have:

$$\mathrm{E}[\dot{\sigma}_\alpha^2] = \mathrm{E}[\mathbf{Q}_{\alpha\beta j}\mathbf{J}_{\beta j}\mathbf{z}_\beta \mathbf{Q}_{\alpha\delta k}\mathbf{J}_{\delta k}\mathbf{z}_\delta] = \mathrm{E}[\delta_{\beta\delta}\delta_{jk}\mathbf{J}_{\beta j}\mathbf{J}_{\delta k}\mathbf{z}_\beta\mathbf{z}_\delta] \tag{134}$$

$$\mathrm{E}[\dot{\sigma}_\alpha^2] = \mathrm{E}[\mathbf{Q}_{\alpha\beta j}\mathbf{J}_{\beta j}\mathbf{z}_\beta \mathbf{Q}_{\alpha\delta k}\mathbf{J}_{\delta k}\mathbf{z}_\delta] = \sum_j \mathrm{E}[\mathbf{J}_{\beta j}^2\mathbf{z}_\beta^2] = DP\sigma_J^2\sigma_z^2 \tag{135}$$

To compute the second term, we compute $\ddot{\mathbf{J}}_{\alpha i}$:

$$\ddot{\mathbf{J}}_{\alpha i} = -\mathbf{Q}_{\alpha ij}(\mathbf{J}_{\beta j}\dot{\mathbf{z}}_\beta + \dot{\mathbf{J}}_{\beta j}\mathbf{z}_\beta) \tag{136}$$

Expanding, we have:

$$\ddot{\mathbf{J}}_{\alpha i} = \mathbf{Q}_{\alpha ij}(\mathbf{J}_{\beta j}\mathbf{J}_{\beta k}\mathbf{J}_{\delta k}\mathbf{z}_\delta + \mathbf{Q}_{\beta jk}\mathbf{J}_{\delta k}\mathbf{z}_\delta\mathbf{z}_\beta) \tag{137}$$

In the diagonal coordinates $\mathbf{J}_{\alpha\alpha} = \sigma_\alpha$. This gives us:

$$\mathrm{E}[\sigma_\alpha\ddot{\sigma}_\alpha] = \mathrm{E}[\sigma_\alpha\mathbf{Q}_{\alpha\alpha j}\mathbf{Q}_{\beta jk}\mathbf{J}_{\delta k}\mathbf{z}_\delta\mathbf{z}_\beta] \tag{138}$$

Averaging over the $\mathbf{Q}$, we get:

$$\mathrm{E}[\sigma_\alpha\ddot{\sigma}_\alpha] = P\mathrm{E}[\sigma_\alpha\delta_{\alpha\beta}\delta_{\alpha k}\mathbf{J}_{\delta k}\mathbf{z}_\delta\mathbf{z}_\beta] = \mathrm{E}[\sigma_\alpha\mathbf{z}_\alpha\mathbf{z}_\delta\mathbf{J}_{\delta\alpha}] \tag{139}$$

Which evaluates to:

$$\mathrm{E}[\sigma_\alpha\ddot{\sigma}_\alpha] = \sigma_z^2 P\mathrm{E}[\sigma_\alpha^2] \tag{140}$$

In the limit of large D and P, for fixed ratio D/P, the statistics of the Marchenko-Pastur distribution allow us to compute the derivative of the largest eigenmode as

$$\mathrm{E}[\sigma_0 \ddot{\sigma}_0] = \sigma_z^2 \sigma_J^2 P^2 D (1 + \sqrt{D/P})^2 \tag{141}$$

Taken together, this gives us

$$\mathrm{E}\left[\frac{d^2 \lambda_{max}}{dt^2}\right] = \sigma_z^2 \sigma_J^2 D P (P(1 + \sqrt{D/P})^2 + 1) \tag{142}$$

We confirm the prediction numerically in Figure 9.

That is, the second derivative of the maximum curvature is positive on average. If we normalize with respect to the eigenvalue scale, in the limit of large D and P we have:

$$\mathrm{E}\left[\frac{d^2 \lambda_{max}}{dt^2}\right] / \mathrm{E}[\lambda_{max}] = \sigma_z^2 \tag{143}$$

Therefore, increasing σ_z increases the relative curvature of the λ_{max} trajectory. This gives us the proof of Theorem 5.

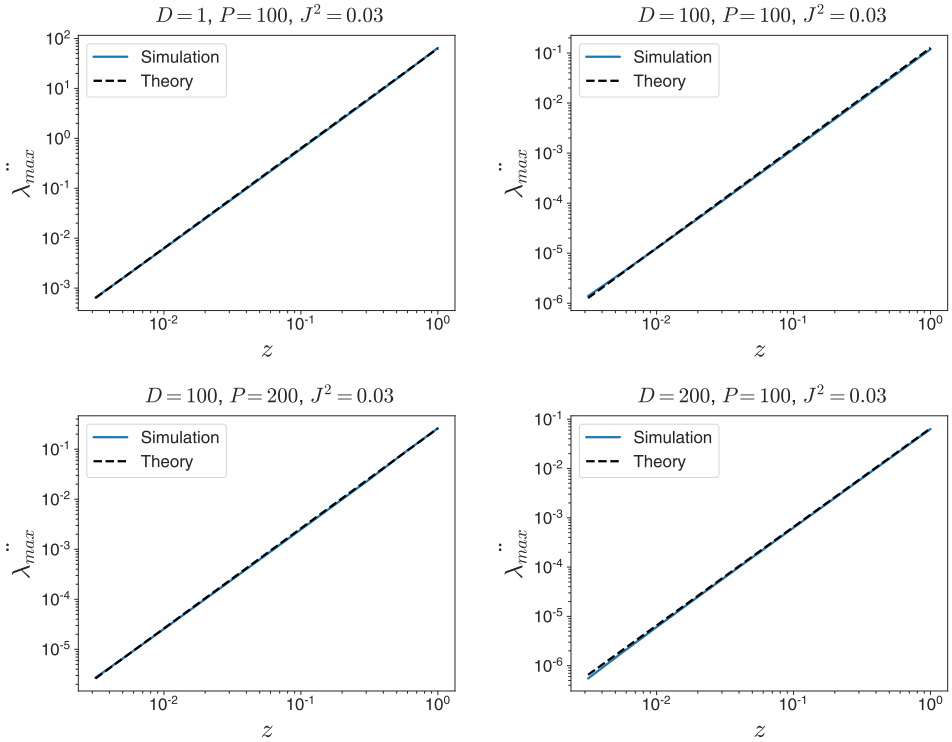

Figure 9: Average $\ddot{\lambda}_{\max}(0)$ versus σ_z, various D and P (100 seeds).

This result suggests that as σ_z increases, so does the degree of progressive sharpening. This can be confirmed by looking at GF trajectories (Figure 10). The trajectories with small σ_z don't change their curvature much, and the loss decays exponentially at some rate. However, when σ_z is larger, the curvature increases initially, and then stabilizes to a higher value, allowing for faster convergence to the minimum of the loss.

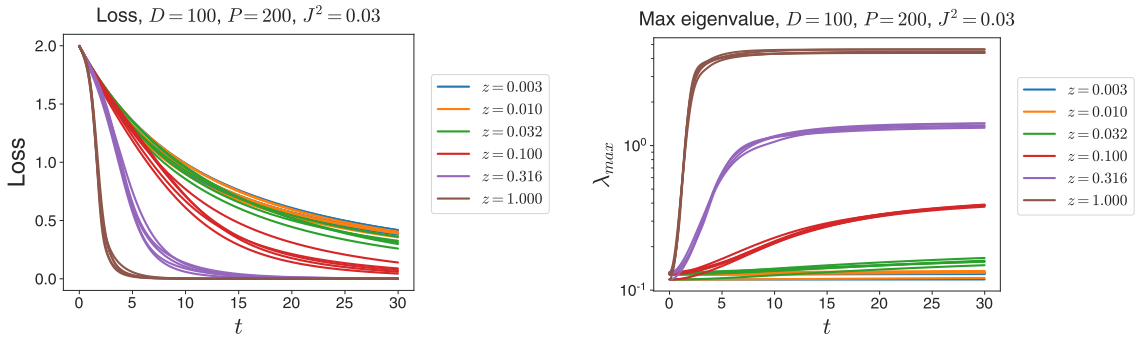

Figure 10: Gradient flow trajectories of loss and max NTK eigenvalues for quadratic regression models for varying σ_z. As σ_z increases, $\lambda_{\max}$ changes more quickly, and is generally increasing. Models with higher σ_z converge faster in GF dynamics.

C.4. Timescales for gradient descent

Consider a random initialization of $\mathbf{z}$, $\mathbf{J}$, and $\mathbf{Q}$, where the terms are i.i.d. with zero mean variances σ_z^2, σ_J^2, and 1 respectively, and finite fourth moments. Furthermore, suppose that $\mathbf{z}$, $\mathbf{J}$, and $\mathbf{Q}$ are rotationally invariant in both input and output space. Under these conditions, we hope to compute

$$r_{NL}^2 \equiv \frac{\mathrm{E}[||\frac{1}{2}\eta^2 \mathbf{Q}_{\alpha ij}(\mathbf{J}_{\beta i})_0(\mathbf{z}_\beta)_0(\mathbf{J}_{\delta j})_0(\mathbf{z}_\delta)_0||_2^2]}{\mathrm{E}[||\eta(\mathbf{J}_{\alpha i})_0(\mathbf{J}_{i\beta})_0(\mathbf{z}_\beta)_0||_2^2]} = \frac{1}{4}\eta^2\sigma_z^2 D^2 \tag{144}$$

at initialization, in the limit of large D and P.

The denominator is given by:

$$\mathrm{E}[\mathbf{J}_{\alpha i}\mathbf{J}_{\beta i}(\mathbf{z}_\beta)\mathbf{J}_{\alpha j}\mathbf{J}_{\delta j}(\mathbf{z}_\delta)] = \sigma_z^2 \mathrm{E}[\mathbf{J}_{\alpha i}\mathbf{J}_{\beta i}\mathbf{J}_{\alpha j}\mathbf{J}_{\delta j}\delta_{\beta\delta}] = \sigma_z^2 \mathrm{E}[\mathbf{J}_{\alpha i}\mathbf{J}_{\beta i}\mathbf{J}_{\alpha j}\mathbf{J}_{\beta j}] \tag{145}$$

Evaluation gives us:

$$\mathrm{E}[\mathbf{J}_{\alpha i}\mathbf{J}_{\beta i}(\mathbf{z}_\beta)\mathbf{J}_{\alpha j}\mathbf{J}_{\delta j}(\mathbf{z}_\delta)] = \sigma_z^2(\sigma_J^4(P(P-1)D) + C_4 DP) \tag{146}$$

where C_4 is the 4th moment of $\mathbf{J}_{\alpha i}$. To lowest order in D and P

$$\mathrm{E}[\mathbf{J}_{\alpha i}\mathbf{J}_{\beta i}(\mathbf{z}_\beta)\mathbf{J}_{\alpha j}\mathbf{J}_{\delta j}(\mathbf{z}_\delta)] = \sigma_z^2\sigma_J^4 DP^2 + O(DP) \tag{147}$$

Evaluating the numerator, we have:

$$\mathrm{E}[\mathbf{Q}_{\alpha ij}\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{Q}_{\alpha mn}\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu] = \mathrm{E}[\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu](\delta_{im}\delta_{jn} + (M_4 - 1)\delta_{ijmn}) \tag{148}$$

where M_4 is the $4th$ moment of $\mathbf{Q}_{\alpha ij}$. This gives us:

$$\begin{aligned}\frac{1}{D}\mathrm{E}[\mathbf{Q}_{\alpha ij}\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{Q}_{\alpha mn}\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu] = \mathrm{E}[\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{J}_{\gamma i}\mathbf{z}_\gamma\mathbf{J}_{\nu j}\mathbf{z}_\nu] + \\ (M_4 - 1)\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta i}\mathbf{z}_\delta\mathbf{J}_{\gamma i}\mathbf{z}_\gamma\mathbf{J}_{\nu i}\mathbf{z}_\nu]\end{aligned} \tag{149}$$

Next, we perform the $\mathbf{z}$ averages. We have

$$\begin{aligned}\frac{1}{D}\mathrm{E}[\mathbf{Q}_{\alpha ij}\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{Q}_{\alpha mn}\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu] &= \sigma_z^4\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta j}\mathbf{J}_{\gamma i}\mathbf{J}_{\nu j}](\delta_{\beta\delta}\delta_{\gamma\nu}+\delta_{\beta\gamma}\delta_{\delta\nu}+\delta_{\beta\nu}\delta_{\delta\gamma})\\ &+ (C_4-\sigma^4)\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta j}\mathbf{J}_{\gamma i}\mathbf{J}_{\nu j}]\delta_{\beta\delta\gamma\nu}\\ &+ (M_4-1)\sigma_z^4\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta i}\mathbf{J}_{\gamma i}\mathbf{J}_{\nu i}](\delta_{\beta\delta}\delta_{\gamma\nu}+\delta_{\beta\gamma}\delta_{\delta\nu}+\delta_{\beta\nu}\delta_{\delta\gamma})\\ &+ (M_4-1)(C_4-\sigma^4)\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta i}\mathbf{J}_{\gamma i}\mathbf{J}_{\nu i}]\delta_{\beta\delta\gamma\nu}\end{aligned} \tag{150}$$

where C_4 is the 4th moment of $\mathbf{z}$. Simplification gives us:

$$\begin{aligned}\frac{1}{D}\mathrm{E}[\mathbf{Q}_{\alpha ij}\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{Q}_{\alpha mn}\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu] &= \sigma_z^4(\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\beta j}\mathbf{J}_{\delta i}\mathbf{J}_{\delta j}]+\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta j}\mathbf{J}_{\beta i}\mathbf{J}_{\delta j}]+\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta j}\mathbf{J}_{\delta i}\mathbf{J}_{\beta j}])\\ &+ (C_4-\sigma_z^4)\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\beta j}\mathbf{J}_{\beta i}\mathbf{J}_{\beta j}]\\ &+ (M_4-1)\sigma_z^4(\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\beta i}\mathbf{J}_{\gamma i}\mathbf{J}_{\gamma i}]+\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta i}\mathbf{J}_{\beta i}\mathbf{J}_{\delta i}]+\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\delta i}\mathbf{J}_{\delta i}\mathbf{J}_{\beta i}])\\ &+ (M_4-1)(C_4-\sigma^4)\mathrm{E}[\mathbf{J}_{\beta i}\mathbf{J}_{\beta i}\mathbf{J}_{\beta i}\mathbf{J}_{\beta i}]\end{aligned} \tag{151}$$

For large D and P, the final three terms are asymptotically smaller than the first term. Evaluating the first term, to leading order we have:

$$\frac{1}{D}\mathrm{E}[\mathbf{Q}_{\alpha ij}\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{Q}_{\alpha mn}\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu] = \sigma_z^4\sigma_J^4(2DP^2+2D^2P+D^2P^2)+O(D^2P+DP^2) \tag{152}$$

$$\mathrm{E}[\mathbf{Q}_{\alpha ij}\mathbf{J}_{\beta i}\mathbf{z}_\beta\mathbf{J}_{\delta j}\mathbf{z}_\delta\mathbf{Q}_{\alpha mn}\mathbf{J}_{\gamma m}\mathbf{z}_\gamma\mathbf{J}_{\nu n}\mathbf{z}_\nu] = \sigma_z^4\sigma_J^4D^3P^2+O(D^3P+D^2P^2) \tag{153}$$

This gives us:

$$r_{NL}^2 = \frac{1}{4}\frac{\sigma_z^4\sigma_J^4D^3P^2}{\sigma_z^2\sigma_J^4DP^2} = \frac{1}{4}\sigma_z^2D^2 \tag{154}$$

to leading order, in the limit of large D and P.